ON THE SAITO BASIS AND THE TJURINA NUMBER FOR PLANE BRANCHES

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Abstract. We introduce the concept of good Saito basis for a plane curve and we explore it to obtain a formula for the minimal Tjurina number in a topological class. In particular, we give a lower bound for the Tjurina number in terms of the Milnor number that allow us to present a positive answer for a question of Dimca and Greuel.

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1. Introduction.

Let $S : \{f = 0\}$ be a germ of an irreducible analytic plane curve. An important analytic invariant of S is the Tjurina number $\tau(S) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{(f)+J(f)}$ where $J(f)$ denotes the Jacobian ideal of f.

In general, the computation of $\tau(S)$ is not easy. For instance, we can obtain it consider a Gröbner basis for the ideal $(f) + J(f)$, or alternatively, it is possible to compute τ by the dimension of $\frac{J(f):(f)}{J(f)}$ (see Theorem 1 in [8]) that is related with the $\mathbb{C}\{x, y\}$-module $\Omega^1(S)$ of all germs of 1-holomorphic forms

$$\omega \in \mathbb{C}\{x, y\}\mathrm{d}x + \mathbb{C}\{x, y\}\mathrm{d}y$$

such that f divides $\omega \wedge \mathrm{d}f$. More precisely, according to K. Saito [11], $\Omega^1(S)$ is freely generated by two elements $\{\omega_1, \omega_2\}$. It will be shown that $\tau(S)$ can be expressed from, among other invariants, the codimension of the ideal (g_1, g_2) where $\omega_i \wedge \mathrm{d}f = g_i f \mathrm{d}x \wedge \mathrm{d}y$.

If L denotes a topological class of a plane curve - for instante, given by the characteristic exponents - then the Milnor number $\mu = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{J(f)}$ is constant for any $S : \{f = 0\} \in L$ and $\tau_{\min} \leq \tau(S) \leq \mu$. Generically, an element $S \in L$ is such that $\tau(S) = \tau_{\min}$, so $\tau_{\min}$ can be express using the topological data that characterizes L. Delorme in [3], presented a formula to compute the generic dimension $d(\beta_0, \beta_1)$ of the moduli space for an irreducible plane curve with characteristic exponents (β_0, β_1). As $d(2, \beta_1) = 0$ and $d(\beta_0, \beta_1) = \frac{(\beta_0-3)(\beta_1-3)}{2} + \left[\frac{\beta_0}{\beta_1}\right] - 1 - \mu + \tau_{\min}$ (see

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[6]) we can compute the minimal Tjurina number for this topological class. On the other hand, Peraire in [10] developed an algorithm to compute $\tau_{\min}$ by means of a flag of $J(f)$.

In this paper we present a way to express the difference $\mu - \tau$ for a singular irreducible plane curve S when $\Omega^1(S)$ admits a basis $\{\omega_1, \omega_2\}$ of special kind, that we call a *good* Saito basis (see Definition 2).

More specifically, we present a formula (see Theorem 11) to compute the difference between $\mu(S) - \tau(S)$ and $\mu(\widetilde{S}) - \tau(\widetilde{S})$ where $\widetilde{S}$ denotes the strict transform of S.

If S is generic in L, then, according to [5], S admits a good basis and this fact allows us to obtain a formula to compute $\tau_{\min}$ in L by the sole topological data: the sequence of multiplicities in the canonical resolution or the characteristic exponents for instance. In particular, for irreducible plane curves, we are able to present a lower bound for the minimum Tjurina number in L in terms of the Milnor number that allow us to give an affirmative answer to a question of Dimca and Greuel [4] about the inequality $4\tau > 3\mu$ and obtained simultaneously by Alberich-Carramiñana *et al.* in [7] published in ArXiv a few days before the first version of this paper.

The paper is organized as follows. In the section 2 we present some general properties of a Saito basis. The concept of a good Saito basis is introduced in the section 3 and its properties as well. The section 4 is devoted to the formula for the minimal Tjurina number, a lower bound for the Tjurina number using the Milnor number and consequently an answer to the Dimca-Greuel question.

2. The Saito basis.

Let $S : \{f = 0\}$ be a germ of an analytic plane curve and consider the $\mathbb{C}\{x, y\}$-module $\Omega^1(S)$ of all germs of 1-holomorphic forms

$$\omega \in \mathbb{C}\{x, y\}\mathrm{d}x + \mathbb{C}\{x, y\}\mathrm{d}y$$

such that f divides $\omega \wedge \mathrm{d}f$. It is equivalent to require that the foliation induced by ω lets invariant S. Saito in [11] shows that $\Omega^1(S)$ is a free module of rank 2 and a basis of $\Omega^1(S)$ is called a Saito basis.

It is not trivial to obtain a Saito basis, but there is a simple criterion to verify if $\{\omega_1, \omega_2\}$ is a basis for $\Omega^1(S)$ (see Theorem, page 270 in [11]).

Theorem (Saito criterion). *The set $\{\omega_1, \omega_2\}$ is a Saito basis for $S : \{f = 0\}$ if and only if $\omega_1 \wedge \omega_2 = uf\mathrm{d}x \wedge \mathrm{d}y$, where u is a unit in $\mathbb{C}\{x, y\}$.*

This criterion can be interpreted as follows : $\{\omega_1, \omega_2\}$ is a basis for $\Omega^1(S)$ if the tangency locus between the two forms reduces to S.

Below, we present some examples of Saito basis for $S : \{f = 0\}$. All of them will illustrate, in the sequel, various sensitivities of the Saito basis with respect to small perturbations of the curve S. In the whole article, we will keep the same numbering of the examples for the convenience of the reader.

Example (1). The simplest case is when $f = y^p - x^q$, that is $S_1 : \{f = 0\}$ is quasi-homogeneous. In fact, if $\omega_1 = qy\mathrm{d}x - px\mathrm{d}y$ and $\omega_2 = \mathrm{d}f$, then

$$\omega_1 \wedge \omega_2 = pqf\mathrm{d}x \wedge \mathrm{d}y$$

and $\{\omega_1, \omega_2\}$ is a basis for $\Omega^1(S_1)$.

Example (2). Consider $f = (y^2 - x^3)^2 + x^5y$. The curve $S_2 : \{f = 0\}$ provides characteristic exponents $(4, 6, 7)$, and thus is not topologically quasi-homogeneous. One can show that

$$\omega_1 = \left(-3x^2y + \frac{5}{8}xy^2 - \frac{5}{24}x^4 - \frac{25}{48}x^3y\right)\mathrm{d}x + \left(-\frac{1}{6}y^2 + \frac{13}{6}x^3 - \frac{5}{12}x^2y + \frac{5}{16}x^4\right)\mathrm{d}y$$

$$\omega_2 = \left(-\frac{13}{4}y^2 + \frac{1}{4}x^3 - \frac{125}{288}xy^2\right)\mathrm{d}x + \left(2xy - \frac{5}{144}y^2 + \frac{11}{144}x^3 + \frac{25}{96}x^2y\right)\mathrm{d}y,$$

satisfy

$$\omega_1 \wedge \omega_2 = \left(-\frac{13}{14} - \frac{325}{3456}\right) f\mathrm{d}x \wedge \mathrm{d}y,$$

and consequently, the set $\{\omega_1, \omega_2\}$ is a Saito basis for $\Omega^1(S_2)$.

Example (3). If $f = y^5 - x^6 + x^4y^3$ then $S_3 : \{f = 0\}$ is topologically quasi-homogeneous, that is, S_3 presents characteristic exponents $(5, 6)$, but not analytically equivalent to $y^5 - x^6 = 0$. One can show that the set $\{\omega_1, \omega_2\}$ where

$$\begin{aligned}
\omega_1 &= \left(-6xy + \frac{16}{15}x^3y^2 - \frac{8}{5}xy^5\right)\mathrm{d}x + \left(5x^2 + \frac{4}{3}y^3 + \frac{4}{5}x^2y^4\right)\mathrm{d}y \\
\omega_2 &= \left(-6y^2 + \frac{8}{5}x^4 - \frac{12}{5}x^2y^3\right)\mathrm{d}x + \left(5xy + \frac{6}{5}x^3y^2\right)\mathrm{d}y
\end{aligned}$$

satisfy $\omega_1 \wedge \omega_2 = 8f\mathrm{d}x \wedge \mathrm{d}y$, so $\{\omega_1, \omega_2\}$ is a Saito basis for $\Omega^1(S_3)$.

Example (4). The curve $S_4 : \{f = 0\}$ with $f = y^5 - x^{11} + x^6y^3$ is topologically equivalent to the any curve with characteristic exponents $(5, 11)$ and its strict transform is S_3. The set $\{\omega_1, \omega_2\}$ where

$$\begin{aligned}
\omega_1 &= \left(605y^2 + 198xy^3 - 88x^6\right)\mathrm{d}x - \left(275xy + 66x^2y^2\right)\mathrm{d}y \\
\omega_2 &= \left(605x^4y + 150x^5y^2\right)\mathrm{d}x - \left(40y^3 + 275x^5 + 90x^6y\right)\mathrm{d}y
\end{aligned}$$

satisfy $\omega_1 \wedge \omega_2 = (-24200 - 7920xy)f\mathrm{d}x \wedge \mathrm{d}y$, so $\{\omega_1, \omega_2\}$ is a Saito basis for $\Omega^1(S_4)$.

Example (5). The class of curve with characteristic exponents the form $(n, n+1)$ has been extensively studied by Zariski [14]. The curve S_5 given by

$$f = y^7 - x^8 - 7x^6y^2 - \frac{147}{8}x^4y^4$$

that, belongs to the latter class, will be shown of a peculiar interest. The forms

$$\begin{aligned}
\omega_1 &= \left(8x^2y - \frac{147}{8}x^4 - \frac{3087}{4}x^2y^2 - \frac{21609}{16}y^4\right)\mathrm{d}x + \\
&\quad + \left(-7x^3 + \frac{7}{4}xy^2 + \frac{64827}{64}xy^3 + \frac{5145}{8}x^3y\right)\mathrm{d}y \\
\omega_2 &= \left(8xy^2 + \frac{1029}{8}x^3y\right)\mathrm{d}x + \left(-7x^2y + \frac{7}{4}y^3 - \frac{1029}{8}x^4\right)\mathrm{d}y.
\end{aligned}$$

produce a Saito basis for $\Omega^1(S_5)$ because $\omega_1 \wedge \omega_2 = -\frac{151263}{64}f\mathrm{d}x \wedge \mathrm{d}y$.

Example (6). Finally, the curve S_6 zero locus of $f = (y^2 - x^3)(x^2 - y^3)$ is a reducible one for which

$$\begin{aligned}\omega_1 &= (-12y^2 - 15x^3 + 27xy^3)\mathrm{d}x + (18yx - 18y^2x^2)\mathrm{d}y \\ \omega_2 &= (-18yx + 18y^2x^2)\mathrm{d}x + (12x^2 + 15y^3 - 27x^3y)\mathrm{d}y\end{aligned}$$

satisfy $\omega_1 \wedge \omega_2 = (180 - 405xy)f\mathrm{d}x \wedge \mathrm{d}y$ and $\{\omega_1, \omega_2\}$ is a Saito basis for $\Omega^1(S_6)$.

Given a 1-form

$$\omega = A\mathrm{d}x + B\mathrm{d}y$$

we denote by $\nu(\omega) = \min\{\nu(A), \nu(B)\}$ its algebraic multiplicity, where $\nu(H)$ indicates the multiplicity of $H \in \mathbb{C}\{x, y\}$ at $(0, 0) \in \mathbb{C}^2$.

Among all the possible basis $\{\omega_1, \omega_2\}$ for $\Omega^1(S)$ we choose some that maximizes the sum $\nu(\omega_1) + \nu(\omega_2)$ that, following the Saito criterion, cannot be bigger than $\nu = \nu(f) = \nu(S)$. For such basis we denote

$$\nu_1 := \nu(\omega_1) \qquad \nu_2 := \nu(\omega_2).$$

The following proposition is immediate and identify a new analytical invariant of S.

Proposition 1. *The couple (ν_1, ν_2), up to order, is an analytical invariant of S.*

Remark that the pair (ν_1, ν_2) is not a topological invariant. For instance, following the examples above, for S_1 with $p = 5$ and $q = 6$ we have $(\nu_1, \nu_2) = (1, 4)$. But the curve S_3 which is topological equivalent to S_1 has corresponding pair of multiplicities $(2, 2)$.

From now on, we consider $S : \{f = 0\}$ singular and irreducible (a plane branch) with a Saito basis $\{\omega_1, \omega_2\}$ such that

$$\omega_1 = A_1\mathrm{d}x + B_1\mathrm{d}y \quad \omega_2 = A_2\mathrm{d}x + B_2\mathrm{d}y.$$

In particular, we have

$$A_1B_2 - A_2B_1 = uf \tag{2.1}$$

$$A_i\frac{\partial f}{\partial y} - B_i\frac{\partial f}{\partial x} = g_i f \tag{2.2}$$

where $u(0, 0) \neq 0$ and $g_i \in \mathbb{C}\{x, y\}$ is called the cofactor of ω_i.

Applying a generic linear change of coordinates if necessary, we can suppose that for $i = 1, 2$, one has

$$\nu(A_i) = \nu(B_i) = \nu_i$$

and in this coordinates (x, y) the tangent cone of f, i.e. its ν-jet, is

$$f^{(\nu)} = (y + \epsilon x)^{\nu}.$$

Example (1). Consider the irreducible curve S_1. Suppose by symmetry that $p < q$, we have $\nu(A_1) = \nu(B_1) = \nu_1 = 1$ but $q - 1 = \nu(A_2) > p - 1 = \nu(B_2) = \nu_2$. Consider the change of coordinates $T(x, y) = (x, y - \epsilon x)$ with $\epsilon \neq 0$ we obtain $f_1 = T^*(f) = (y - \epsilon x)^p - x^q$ and the Saito basis $\eta_1 = T^*(\omega_1)$ and $\mathrm{d}f_1$

$$\begin{aligned}\eta_1 &= (q(y - \epsilon x) + \epsilon p x)\mathrm{d}x - px\mathrm{d}y \\ \mathrm{d}f_1 &= (-\epsilon p(y - \epsilon x)^{p-1} - qx^{q-1})\mathrm{d}x + p(y - \epsilon x)^{p-1}\mathrm{d}y\end{aligned}$$

satisfying the above condition. In addition,

$$\eta_1 \wedge \mathrm{d}f_1 = pqf_1\mathrm{d}x \wedge \mathrm{d}y,$$

that is, $g_1 = pq$ and $g_1 = 0$.

Example (2). For the curve S_2 we have $3 = \nu(A_1) > \nu(B_1) = \nu_1 = 2$ and $\nu(A_2) = \nu(B_2) = 2$. Considering $T(x, y) = (x, y + x)$ we get $f_1 = T^*(f) = (y^2 + 2xy + x^2 - x^3)^2 + x^5y + x^6$ and $\eta_i = T^*(\omega_i)$ for $i = 1, 2$ is given by

$$\begin{aligned}\eta_1 &= \left(-\frac{1}{6}y^2 - \frac{1}{3}xy - \frac{1}{6}x^2 - \frac{5}{8}x^3 - \frac{13}{6}x^2y + \frac{5}{8}xy^2 - \frac{5}{12}x^4 - \frac{25}{48}x^3y\right)\mathrm{d}x \\ &\quad + \left(-\frac{1}{6}x^2 - \frac{1}{3}xy - \frac{1}{6}y^2 + \frac{7}{4}x^3 - \frac{5}{12}x^2y + \frac{5}{16}x^4\right)\mathrm{d}y \\ \eta_2 &= \left(-\frac{473}{144}y^2 - \frac{329}{72}xy - \frac{185}{144}x^2 + \frac{11}{72}x^3 - \frac{125}{288}xy^2 - \frac{175}{288}x^2y\right)\mathrm{d}x \\ &\quad + \left(\frac{139}{72}xy + \frac{283}{144}x^2 - \frac{5}{144}y^2 + \frac{97}{288}x^3 + \frac{25}{96}x^2y\right)\mathrm{d}y.\end{aligned}$$

The cofactors are given by

$$\begin{aligned}\eta_1 \wedge \mathrm{d}f_1 &= \left(-\frac{21}{2}x^2 + \frac{5}{2}xy - \frac{25}{12}x^3\right)f_1\mathrm{d}x \wedge \mathrm{d}y \\ \eta_2 \wedge \mathrm{d}f_1 &= \left(-13x - 13y - \frac{35}{18}x^2 - \frac{125}{72}xy\right)f_1\mathrm{d}x \wedge \mathrm{d}y.\end{aligned}$$

Example (3). For the curve S_3, we have

$$\omega_1 \wedge \mathrm{d}f = (-30x - 8xy^4)f\mathrm{d}x \wedge \mathrm{d}y$$
$$\omega_2 \wedge \mathrm{d}f = (-30y - 12x^2y^2)f\mathrm{d}x \wedge \mathrm{d}y,$$

that is, $g_1 = -30x - 8xy^4$ and $g_2 = -30y - 12x^2y^2$.

Example (4). Considering the curve S_4, we have $\nu(A_1) = \nu(B_1) = 2$ but $5 = \nu(A_2) > \nu(B_2) = 3$. By the change of coordinates $T(x, y) = (x, x + y)$ we obtain $f_1 = T^*(f) = (y + x)^5 - x^{11} + x^6(y + x)^3$ and $\eta_i = T^*(\omega_i) = (A_i + B_i)\,\mathrm{d}x + B_i\mathrm{d}y$ with $\nu(A_1 + B_1) = \nu(B_1) = 2$ and $\nu(A_2 + B_2) = \nu(B_2) = 3$. In addition,

$$\eta_1 \wedge \mathrm{d}f = (3025(x + y) + 990x(y + x)^2)f_1\mathrm{d}x \wedge \mathrm{d}y$$
$$\eta_2 \wedge \mathrm{d}f = (3025x^4 + 990x^5(y + x))f_1\mathrm{d}x \wedge \mathrm{d}y,$$

consequently, $g_1 = 3025(x + y) + 990x(y + x)^2$ and $g_2 = 3025x^4 + 990x^5(y + x)$.

Example (5). Finally, for S_5 we find

$$\begin{aligned}\omega_1 \wedge \mathrm{d}f &= \left(56x^2 - \frac{151263}{16}y^3 - \frac{21609}{4}x^2y\right)f\mathrm{d}x \wedge \mathrm{d}y \\ \omega_2 \wedge \mathrm{d}f &= (56xy + 1029x^3)f\mathrm{d}x \wedge \mathrm{d}y.\end{aligned}$$

Notice that any generator ω_i in a Saito basis $\{\omega_1,\omega_2\}$ has an isolated singularity, that is, $\gcd(A_i,B_i)=1$. In addition, by (2.1), we have that $\nu(g_i)\geq \nu_i-1$.

3. Good Saito basis and the Tjurina number for S.

As we mentioned before, given a Saito basis $\{\omega_1,\omega_2\}$ for $\Omega^1(S)$ we get $\nu_1+\nu_2\leq\nu$. In [5], the first author shows the following theorem:

Theorem (Generic Basis Theorem)**.** *In a fixed topological class L, generically any curve S admits a Saito basis satisfying*

$$\nu_1=\nu_2=\tfrac{\nu}{2}\quad \textit{if } \nu=\nu(S) \textit{ is even}$$

$$\nu_1=\nu_2-1=\tfrac{\nu-1}{2}\quad \textit{if } \nu=\nu(S) \textit{ is odd.}$$

Notice that, generically $\nu_1+\nu_2$ is maximum. Of course, Example 1 shows that we can obtain $\nu_1+\nu_2=\nu$ in other cases. This motives the following definition.

Definition 2. We say that S (or $\Omega^1(S)$) admits a *good basis* if $\nu_1+\nu_2=\nu$.

This section is devoted to present some properties of a good basis. One of them is related with the index $\mathfrak{i}(S)$ we introduce in the sequel.

Let E be the standard blowing-up of the origin in $\mathbb{C}^2$ with coordinates (x,y) and suppose that, in the chart (x_1,y_1) such that $E(x_1,y_1)=(x_1,x_1y_1)$, the strict transform of S goes through $(0,y_1)$.

Definition 3. For any $\omega=A\mathrm{d}x+B\mathrm{d}y\in\Omega^1(S)$, we denote by $i(\omega)\in\mathbb{N}\cup\{\infty\}$ the valuation given by

$$i(\omega)=\nu_{y_1=-\epsilon}\left(A^{(\nu(\omega))}(1,y_1)+y_1B^{(\nu(\omega))}(1,y_1)\right)$$

where $\nu_{y_1=-\epsilon}(G)$ denotes de multiplicity of $G\in\mathbb{C}\{y_1\}$ at $-\epsilon\in\mathbb{C}$.

Moreover, we denote by $\mathfrak{i}(S)\in\mathbb{N}$ the integer

$$\mathfrak{i}(S)=\min_{\omega\in\Omega^1(S)} i(\omega).$$

The value $i(\omega)$ is nothing but the index $\mathrm{Ind}(\mathcal{F},\mathrm{C},0)$ introduced in [2] for a germ of foliation $\mathcal{F}$ having C as a smooth invariant curve.

Notice that for a given ω, the index $i(\omega)$ is infinite if and only if ω is dicritical, that is,

$$A^{\nu(\omega)}(1,y_1)+y_1B^{\nu(\omega)}(1,y_1)=0.$$

However, for any curve $\mathfrak{i}(S)$ is finite. Indeed, if f is a reduced equation for S then $\mathrm{d}f$ belongs to $\Omega^1(S)$ and it is not dicritical, thus $\mathfrak{i}(S)\leq i(\mathrm{d}f)<\infty$. In particular, if $\omega\in\Omega^1(S)$ is non dicritical, then $i(\omega)\leq\nu(\omega)+1$.

Example (1). For S_1 the considered Saito basis is a good basis. Moreover, $i(\omega_1)=1$ and $i(\omega_2)=p$.

Example (2). After the mixing change of coordinates, one can see that the Saito basis of S_2 introduced before is a good basis with $i(\omega_1) = 3$ and $i(\omega_2) = 2$.

Example (3). Having a good basis is a property sensitive to perturbation. Indeed, for instance, the basis of S_3 computed in the example is not good, and actually S_3 does not admit any good basis. Besides that, we have $i(\omega_1) = 1$ and $i(\omega_2) = 2$.

Example (4). Good basis is not preserved by blowing-up. In fact, S_4 has a good basis, but its strict transform is analytically equivalent to S_3 that does not admit good basis. For S_4 we have $i(\omega_1) = 2$ and $i(\omega_2) = 4$.

Example (5). Finally, S_5 does not have a good basis. We find $i(\omega_1) = 1$ and $i(\omega_2) = 2$.

The next result shows that if S admits a good basis, the index $\mathfrak{i}(S)$ is achieved for one of its elements.

Proposition 4. *If S admits a good basis $\{\omega_1, \omega_2\}$ then*

$$\mathfrak{i}(S) = \min\{i(\omega_1), i(\omega_2)\}.$$

Proof. By Saito criterion, one has

$$\omega_1 \wedge \omega_2 = uf$$

with $u(0,0) \neq 0$. Since $\nu_1 + \nu_2 = \nu$, one has

$$\omega_1^{(\nu_1)} \wedge \omega_2^{(\nu_2)} \neq 0,$$

where $\omega_i^{(\nu_i)} = A_i^{\nu_i}\mathrm{d}x + B_i^{\nu_i}\mathrm{d}y$. In particular, both forms ω_1 and ω_2 cannot be dicritical and therefore $\min\{i(\omega_1), i(\omega_2)\} < \infty$.

Now, consider any form $\omega = P_1\omega_1 + P_2\omega_2 \in \Omega^1(S)$ with $P_i \in \mathbb{C}\{x,y\}$ and $m_i = \nu(P_i)$. Since $P_1^{(m_1)}\omega_1^{(\nu_1)} + P_2^{(m_2)}\omega_2^{(\nu_1)}$ cannot identically vanish, it is the homogeneous part of smallest degree of ω. Therefore

$$\begin{aligned} i(\omega) = \nu_{y_1=-\epsilon} \Big(& P_1^{(m_1)}(1,y_1)\left(A_1^{\nu_1}(1,y_1) + y_1 B_1^{\nu_1}(1,y_1)\right) \\ & + P_2^{(m_2)}(1,y_1)\left(A_2^{\nu_2}(1,y_1) + y_1 B_2^{\nu_2}(1,y_1)\right)\Big) \\ \geq \min & \{i(\omega_1), i(\omega_2)\}. \end{aligned}$$

□

In the previous section, we remark that for an element ω_i in a Saito basis we get $\nu(g_i) \geq \nu_i - 1$ and $i(\omega_i) \leq \nu(\omega_i) + 1$. For good basis it is possible to obtain the following result.

Lemma 5. *Given a good basis $\{\omega_1, \omega_2\}$ for S, if $\nu(g_i) \geq \nu_i$ then $i(\omega_i) = \nu_i + 1$.*

Proof. By symmetry let us consider $i = 1$ and suppose that $\nu(g_1) \geq \nu_1$. The $(\nu_1 - 1 + \nu)$-jet of

$$A_1 \frac{\partial f}{\partial y} - B_1 \frac{\partial f}{\partial x} = g_1 f$$

is

$$A_1^{(\nu_1)}\nu\left(y+\epsilon x\right)^{\nu-1}-B_1^{(\nu_1)}\nu\epsilon\left(y+\epsilon x\right)^{\nu-1}=0,$$

thus $A_1^{(\nu_1)}=\epsilon B_1^{(\nu_1)}$. On the other hand the ν-jet of

$$A_1B_2-A_2B_1=uf$$

where $u(0,0)\neq 0$ reduces to

$$\begin{aligned}A_1^{(\nu_1)}B_2^{(\nu_2)}-A_2^{(\nu_2)}B_1^{(\nu_1)}&=u(0,0)\left(y+\epsilon x\right)^{\nu},\\ B_1^{(\nu_1)}\left(\epsilon B_2^{(\nu_2)}-A_2^{(\nu_2)}\right)&=u(0,0)\left(y+\epsilon x\right)^{\nu}.\end{aligned}$$

Thus, there exists some constant $c\neq 0$ such that

$$B_1^{(\nu_1)}=c\left(y+\epsilon x\right)^{\nu_1}.$$

Therefore, ω_1 can be written

$$\omega_1=\frac{c}{\nu_1+1}\mathrm{d}\left(\left(y+\epsilon x\right)^{\nu_1+1}\right)+\text{h.o.t.}$$

thus $i\left(\omega_1\right)=\nu_1+1$. □

Notice that the above proof ensures that the inequality $\nu\left(g_i\right)\geq\nu_i$ cannot hold for both elements in a good basis. Moreover, given a good basis for $\Omega^1(S)$ we can always get a good basis with some nice properties. To do this we present the following lemmas.

Lemma 6. *If $\Omega^1(S)$ admits a good basis $\{\omega_1,\omega_2\}$, then we can suppose that*

- $i\left(\omega_1\right)=\mathfrak{i}\left(S\right)$ *and*
- $\nu\left(g_1\right)=\nu_1-1$.

Proof. By symmetry we can suppose that $i\left(\omega_1\right)=\mathfrak{i}\left(S\right)$.

(1) If $i\left(\omega_2\right)=i\left(\omega_1\right)$, then, as mentioned above, for $i=1$ or 2, one has $\nu\left(g_i\right)=\nu_i-1$. Switching maybe the two forms, we can suppose that ω_1 satisfies the conclusion of the lemma.

(2) Suppose now that $i\left(\omega_1\right)<i\left(\omega_2\right)$.

(a) if $\nu_1\leq\nu_2$, we consider, the family

$$\left\{\omega_1,\overline{\omega_2}\right\},$$

where $\overline{\omega_2}=\omega_2+cx^{\nu_2-\nu_1}\omega_1$ and $c\in\mathbb{C}$. For a generic value of c, we still have a good basis for S. Moreover, the ν_2-jet of $\overline{\omega_2}$ is

$$\left(A_2^{(\nu_2)}+cx^{\nu_2-\nu_1}A_1^{(\nu_1)}\right)\mathrm{d}x+\left(B_2^{(\nu_2)}+cx^{\nu_2-\nu_1}B_1^{(\nu_1)}\right)\mathrm{d}y.$$

Thus, to evaluate its index, one writes

$$\begin{aligned}i\left(\overline{\omega_2}\right)&=\nu_{y=-\epsilon}\left(A_2^{(\nu_2)}\left(1,y\right)+cA_1^{(\nu_1)}\left(1,y\right)+y\left(B_2^{(\nu_2)}\left(1,y\right)+cB_1^{(\nu_1)}\left(1,y\right)\right)\right)\\&=\nu_{y=-\epsilon}\left(A_2^{(\nu_2)}\left(1,y\right)+yB_2^{(\nu_2)}\left(1,y\right)+c\left(A_1^{(\nu_1)}\left(1,y\right)+yB_1^{(\nu_1)}\left(1,y\right)\right)\right)\\&=i\left(\omega_1\right).\end{aligned}$$

Thus we are led to the previous case (1).

(b) Finally, if $\nu_1 > \nu_2$, suppose that $\nu(g_1) \geq \nu_1$, then by Lemma 5 we have $i(\omega_1) = \nu_1 + 1$. Consequently

$$i(\omega_1) > \nu_2 + 1$$

and then

$$i(\omega_2) > \nu_2 + 1.$$

If ω_2 is not dicritical, the inequality above leads to a contradiction, thus ω_2 is dicritical. Therefore, it can be seen that

$$\nu(g_2) = \nu_2 - 1.$$

Let us consider now

$$\overline{\omega}_1 = \omega_1 + x^{\nu_1 - \nu_2}\omega_2.$$

Then, the family $\{\overline{\omega}_1, \omega_2\}$ is still a good basis and one has

$$\begin{aligned} \overline{\omega}_1 \wedge \mathrm{d}f &= \overline{g}_1 f \mathrm{d}x \wedge \mathrm{d}y \qquad \text{with } \nu(\overline{g}_1) = \nu_1 - 1 \\ i(\overline{\omega}_1) &= i(\omega_1) = \mathfrak{i}(S). \end{aligned}$$

□

In addition, from a basis for $\Omega^1(S)$ we can get a basis satisfying the following lemma.

Lemma 7. *Given a basis $\{\omega_1, \omega_2\}$ for $\Omega^1(S)$ with $i(\omega_1) \leq i(\omega_2)$ we can suppose that*

$$\gcd\left(B_i, \frac{\partial f}{\partial y}\right) = 1, \quad \textit{for } i = 1, 2.$$

Proof. Suppose that $H = \gcd\left(B_1, B_2, \frac{\partial f}{\partial y}\right)$. Since by (2.1)

$$A_1 B_2 - A_2 B_1 = uf,$$

H would divide f. As $\frac{\partial f}{\partial y}$ and f are relatively prime, we get

$$\gcd\left(B_1, B_2, \frac{\partial f}{\partial y}\right) = 1. \tag{3.1}$$

Now consider the family

$$\{\overline{\omega_1} = \omega_1 + P_1\omega_2, \overline{\omega_2} = \omega_2 + P_2\omega_1\}$$

where $P_i \in \mathbb{C}\{x, y\}$ with $\nu(P_i) \gg 1$. Note that for P_i of algebraic multiplicity big enough, the forms

$$\overline{\omega_1} = (A_1 + P_1 A_2)\,\mathrm{d}x + (B_1 + P_1 B_2)\,\mathrm{d}y = \overline{A_1}\mathrm{d}x + \overline{B_1}\mathrm{d}y$$
$$\overline{\omega_2} = (A_2 + P_2 A_1)\,\mathrm{d}x + (B_2 + P_2 B_1)\,\mathrm{d}y = \overline{A_2}\mathrm{d}x + \overline{B_2}\mathrm{d}y$$

satisfy

$$\nu(\overline{\omega_i}) = \nu\left(\overline{A_i}\right) = \nu\left(\overline{B_i}\right) = \nu_i,$$

and

$$i(\omega_1) = i(\overline{\omega_1}) \leq i(\overline{\omega_2}).$$

Moreover, $\{\overline{\omega_1}, \overline{\omega_2}\}$ is a basis for $\Omega^1(S)$. Now the relation (3.1) ensures that for a generic choice of the $P_i's, i = 1, 2$ - in the sense of Krull -, one has

$$\gcd\left(\overline{B_i}, \frac{\partial f}{\partial y}\right) = 1.$$

□

As a consequence we obtain the following.

Corollary 8. *For any basis $\{\omega_1, \omega_2\}$ for $\Omega^1(S)$ satisfying the previous Lemma we have*

$$\gcd(B_i, g_i) = \gcd\left(\frac{\partial f}{\partial y}, g_i\right) = 1.$$

Proof. As $A_i \frac{\partial f}{\partial y} - B_i \frac{\partial f}{\partial x} = g_i f$, if $1 \neq H = \gcd(B_i, g_i)$ then H must divide $A_i \frac{\partial f}{\partial y}$. By the previous lemma, $\gcd(B_i, \frac{\partial f}{\partial y}) = 1$ so H divides A_i, a contradiction because ω_i has an isolated singularity.
Suppose $H' = \gcd\left(\frac{\partial f}{\partial y}, g_i\right)$, so H' divides $B_i \frac{\partial f}{\partial x}$. As $\gcd\left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x}\right) = \gcd(B_i, g_i) = 1$, we must have $H' = 1$. □

In particular, the above lemma allow us to consider a good Saito basis $\{\omega_1, \omega_2\}$ with $\mathfrak{i}(S) = i(\omega_1)$ and $\gcd\left(B_i, \frac{\partial f}{\partial y}\right) = \gcd(B_i, g_i) = \gcd\left(\frac{\partial f}{\partial y}, g_i\right) = 1$.

Lemma 9. *If $S : \{f = 0\}$ admits a good basis satisfying the previous conditions, then the intersection of the tangent cone of*

1. *g_1 and g_2,*
2. *B_i and g_i, for $i = 1, 2$,*
3. *B_i and $\frac{\partial f}{\partial y}$, for $i = 1, 2$*

is empty or equal to $y + \epsilon x = 0$.

Proof. The ν-jet of (2.1) is

$$A_1^{(\nu_1)} B_2^{(\nu_2)} - A_2^{(\nu_2)} B_1^{(\nu_1)} = c\,(y + \epsilon x)^{\nu}. \tag{3.2}$$

where $c \neq 0$ and $\epsilon \in \mathbb{C}$. Now, for $i = 1, 2$, both following relations

$$A_i^{(\nu_i)} - \epsilon B_i^{(\nu_i)} = 0$$

cannot be true all together since it would yield a contradiction with the relation (3.2). Suppose the relation above is not true for at least $i = 1$, then the cofactor relations ensures that

$$A_1^{(\nu_1)} - \epsilon B_1^{(\nu_1)} = \frac{1}{\nu} g_1^{(\nu(g_1))}\,(y + \epsilon x).$$

Combining the above relations yields

$$g_1^{(\nu(g_1))} B_2^{(\nu_2)} - g_2^{(\nu(g_2))} B_1^{(\nu_1)} = c\nu\,(y + \epsilon x)^{\nu - 1}, \qquad \text{or}$$
$$g_1^{(\nu(g_1))} B_2^{(\nu_2)} = c\nu\,(y + \epsilon x)^{\nu - 1}$$

from which is derived (1) and (2). The point (3) follows from the fact that the tangent cone of $\frac{\partial f}{\partial y}$ and f are the same. □

In what follows we denote by

$$I_P(G,H)$$

the intersection multiplicity of $G,H \in \mathbb{C}\{x,y\}$ at the point $P \in \mathbb{C}^2$. If $P=(0,0)$ then we write $I(G,H) := I_P(G,H)$, that is, $I(G,H) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{(G,H)}$.

An important topological invariant for $S : \{f=0\}$ is the Milnor number μ which can be computed by

$$\mu := I\left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x}\right) = \sum_{i=1}^{N} \nu_{(i)}(\nu_{(i)}-1) \tag{3.3}$$

where $\nu_{(i)}$; $i=1,\dots,N$ denote the sequence of multiplicities in the canonical resolution of S. In addition, by Zariski (see (2.4) in [14]), we have

$$I\left(\frac{\partial f}{\partial y}, f\right) = \mu + \nu - 1. \tag{3.4}$$

Combining the Lemma 6 and the above result we can obtain an expression for $I(g_1,g_2)$.

Lemma 10. *If g_1 and g_1 are the cofactors for a good basis for $\Omega^1(S)$, then $I(g_1,g_2)$ is finite and*

$$I(g_1,g_2) = I\left(\frac{\partial f}{\partial y}, B_1\right) - I(B_1,g_1) - \nu + 1.$$

Proof. By Lemma 6 we have $\nu(g_1) = \nu_1 - 1 < \nu$. As f is irreducible it follows that $\gcd(f,g_1)=1$ and $I\left(f\frac{\partial f}{\partial y}, g_1\right) < \infty$. So, from (2.1) that

$$\begin{aligned} I\left(f\frac{\partial f}{\partial y}, g_1\right) &= I\left(A_1B_2\frac{\partial f}{\partial y} - A_2B_1\frac{\partial f}{\partial y}, g_1\right) \\ &= I\left(A_1B_2\frac{\partial f}{\partial y} - A_2B_1\frac{\partial f}{\partial y} - B_2fg_1, g_1\right) \\ &= I\left(B_1B_2\frac{\partial f}{\partial x} - A_2B_1\frac{\partial f}{\partial y}, g_1\right) \\ &= I\left(B_1g_2f, g_1\right). \end{aligned}$$

Hence,

$$I(g_1,g_2) = I\left(\frac{\partial f}{\partial y}, g_1\right) - I(B_1,g_1). \tag{3.5}$$

The Corollary 8 insures that $\frac{\partial f}{\partial y}$ and g_1 are coprime. So, by (3.5) and using (3.4) we obtain

$$\begin{aligned}
I(g_1,g_2) &= I\left(\frac{\partial f}{\partial y}, g_1\right) + I\left(\frac{\partial f}{\partial y}, f\right) - I\left(\frac{\partial f}{\partial y}, f\right) - I(B_1, g_1)\\
&= I\left(\frac{\partial f}{\partial y}, g_1 f\right) - I\left(\frac{\partial f}{\partial y}, f\right) - I(B_1, g_1)\\
&= I\left(\frac{\partial f}{\partial y}, A_1\frac{\partial f}{\partial y} - B_1\frac{\partial f}{\partial x}\right) - (\mu+\nu-1) - I(B_1, g_1)\\
&= I\left(\frac{\partial f}{\partial y}, -B_1\frac{\partial f}{\partial x}\right) - \mu - \nu + 1 - I(B_1, g_1)\\
&= I\left(\frac{\partial f}{\partial y}, B_1\right) + \mu - \mu - \nu + 1 - I(B_1, g_1)\\
&= I\left(\frac{\partial f}{\partial y}, B_1\right) - \nu + 1 - I(B_1, g_1).
\end{aligned}$$

□

Let us consider the Tjurina number τ of a plane curve $S:\{f=0\}$, that is,

$$\tau := \dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{\left(f, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial x}\right)}.$$

Zariski (see Theorem 1 in [13]) considered the torsion submodule $T\Omega^1_{\mathcal{O}/\mathbb{C}}$ of the Kähler differential module $\Omega^1_{\mathcal{O}/\mathbb{C}}$ over $\mathcal{O} = \frac{\mathbb{C}\{x,y\}}{(f)}$ and he showed that

$$\tau = \dim_{\mathbb{C}} T\Omega^1_{\mathcal{O}/\mathbb{C}}.$$

On the other hand, Michler (Theorem 1 in [8]) proved that $T\Omega^1_{\mathcal{O}/\mathbb{C}}$ is isomorphic as $\mathcal{O}$-module, to $\frac{\left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x}\right):(f)}{\left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x}\right)}$. As $\left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x}\right):(f)$ is precisely the cofactor ideal of S, that is, (g_1, g_2), one has

$$\tau = \dim_{\mathbb{C}} \frac{(g_1,g_2)}{\left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x}\right)} = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{\left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x}\right)} - \dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{(g_1,g_2)} = \mu - I(g_1,g_2),$$

that is,

$$\mu - \tau = I(g_1, g_2).$$

The difference $\mu-\tau$ coincides with the dimension of the first de Rham cohomology group $H^1(\mathcal{O})$ of $\mathcal{O}$ (see Theorem 4.9 in [9]).

Berger, in [1], considering an algebroid curve S not necessarily plane, studies $\dim_{\mathbb{C}} T\Omega^1_{\mathcal{O}/\mathbb{C}} - \dim_{\mathbb{C}} T\Omega^1_{\widetilde{\mathcal{O}}/\mathbb{C}}$ where $\widetilde{\mathcal{O}} = \frac{\mathbb{C}\{x,y\}}{(\widetilde{f})}$ and $\widetilde{S}:\{\widetilde{f}=0\}$ is the strict transform of S. In our approach this is the same to consider $\tau - \widetilde{\tau}$ where $\widetilde{\tau}$ indicates the Tjurina number of $\widetilde{S}$.

Denoting $\widetilde{\mu}$ the Milnor number of $\widetilde{S}$ we provide in the next theorem a precise relation between $\mu-\tau$ and $\widetilde{\mu}-\widetilde{\tau}$ by means of the analytic invariants we have introduced previously for curves that admit a good basis.

Theorem 11. *If S admits a good basis, then*

$$\mu - \tau = \widetilde{\mu} - \widetilde{\tau} + (\nu_1 - 1)(\nu_2 - 1) + \mathfrak{i}(S) - 1.$$

Proof. By symmetry, one can suppose

$$\mathfrak{i}(S) = \min\{i(\omega_1), i(\omega_2)\} = i(\omega_1).$$

By Lemma 9 and the Max-Noether formula one has,

$$\mu - \tau = I(g_1, g_2) = I_{(0,-\epsilon)}(\tilde{g}_1, \tilde{g}_2) + \nu(g_1)\nu(g_2),$$

where $\widetilde{H} := E^*(H)$ and E denotes the standard blowing-up of the origin in $\mathbb{C}^2$. In addition, the previous lemma and Lemma 9, yield

$$\begin{aligned} I(g_1, g_2) &= I\left(\frac{\partial f}{\partial y}, B_1\right) - I(B_1, g_1) - \nu + 1 \\ &= I_{(0,-\epsilon)}\left(\frac{\partial \tilde{f}}{\partial y}, \tilde{B}_1\right) - I_{(0,-\epsilon)}\left(\tilde{B}_1, \tilde{g}_1\right) + \nu\left(\frac{\partial f}{\partial y}\right)\nu(B_1) - \nu(B_1)\nu(g_1) - \nu + 1. \end{aligned}$$

If $\tilde{\omega}_i = \frac{E^*\omega_i}{x^{\nu_i}}$, then the Saito criterion yields

$$x^{\nu_1}\tilde{\omega}_1 \wedge x^{\nu_2}\tilde{\omega}_2 = \widetilde{u}x^{\nu}\tilde{f}x\mathrm{d}x \wedge \mathrm{d}y.$$

Since we have a good basis, that is, $\nu_1 + \nu_2 = \nu$, one has

$$\tilde{\omega}_1 \wedge \tilde{\omega}_2 = u\tilde{f}x\mathrm{d}x \wedge \mathrm{d}y.$$

Locally around $(0, -\epsilon)$ we have

$$\begin{aligned} \tilde{\omega}_1 &= \left(A_1^{\nu_1}(1,y) + yB_1^{\nu_1}(1,y) + x(\cdots)\right)\mathrm{d}x + x\left(B_1^{\nu_1}(1,y) + (\cdots)\right)\mathrm{d}y \\ \tilde{\omega}_2 &= \left(A_2^{\nu_2}(1,y) + yB_2^{\nu_2}(1,y) + x(\cdots)\right)\mathrm{d}x + x\left(B_2^{\nu_2}(1,y) + (\cdots)\right)\mathrm{d}y. \end{aligned}$$

We notice that the form

$$\overline{\omega}_2 = \frac{1}{x}\left(\tilde{\omega}_2 - \frac{A_2^{\nu_2}(1,y) + yB_2^{\nu_2}(1,y)}{A_1^{\nu_1}(1,y) + yB_1^{\nu_1}(1,y)}\tilde{\omega}_1\right)$$

is holomorphic at $(0, -\epsilon)$ and $\{\tilde{\omega}_1, \overline{\omega}_2\}$ is a Saito basis for $\widetilde{S} : \{\tilde{f} = 0\}$. A computation shows that the cofactor associated to $\tilde{\omega}_1$ is written

$$g_1^{'} = \tilde{g}_1 + \nu\tilde{B}_1.$$

Moreover, one has

$$\tilde{\omega}_1 = \left(\tilde{A}_1 + y\tilde{B}_1\right)\mathrm{d}x + x\tilde{B}_1\mathrm{d}y = A'\mathrm{d}x + B'\mathrm{d}y.$$

Now,

$$\left(A_1^{\nu_1}(1,y) + yB_1^{\nu_1}(1,y) + x(\cdots)\right)\frac{\partial \tilde{f}}{\partial y} - x\tilde{B}_1\frac{\partial \tilde{f}}{\partial x} = g_1^{'}\tilde{f}.$$

If x divides $g_1^{'}$ then $\tilde{\omega}_1$ would be dicritical and this is not possible. Therefore,

$$I_{(0,-\epsilon)}\left(x, g_1^{'}\tilde{f}\right) = I_{(0-\epsilon)}\left(A_1^{\nu_1}(1,y) + yB_1^{\nu_1}(1,y), x\right) + I_{(0,-\epsilon)}\left(x, \frac{\partial \tilde{f}}{\partial y}\right).$$

and, by Corollary 8,

$$I_{(0,-\epsilon)}\left(x, g_1^{'}\right) = i(\omega_1) - 1 = \mathfrak{i}(S) - 1.$$

Notice that $\tilde{B}_1$ and $g_1^{'}$ cannot have a common divisor, since it would be a common divisor of $\tilde{g}_1$ and $\tilde{B}_1$ that is impossible by Lemma 7. So,

$$\begin{aligned} I_{(0,-\epsilon)}\left(\tilde{B}_1, \tilde{g}_1\right) &= I_{(0,-\epsilon)}\left(x\tilde{B}_1, g_1^{'}\right) - \mathfrak{i}\,(S) + 1 \\ &= I_{(0,-\epsilon)}\left(B_1^{'}, g_1^{'}\right) - \mathfrak{i}\,(S) + 1. \end{aligned}$$

Moreover,

$$\begin{aligned} I_{(0,-\epsilon)}\left(\frac{\partial \tilde{f}}{\partial y}, \tilde{B}_1\right) &= I_{(0,-\epsilon)}\left(\frac{\partial \tilde{f}}{\partial y}, B_1^{'}\right) - I_{(0,-\epsilon)}\left(\frac{\partial \tilde{f}}{\partial y}, x\right) \\ &= I_{(0,-\epsilon)}\left(\frac{\partial \tilde{f}}{\partial y}, B_1^{'}\right) - I_{(0,-\epsilon)}\left(\tilde{f}, x\right) + 1. \end{aligned}$$

So, as $\frac{\partial \tilde{f}}{\partial y} = \frac{\widetilde{\partial f}}{\partial y}$ and combining all the above relation yields

$$\begin{aligned} \mu - \tau = {} & I_{(0,-\epsilon)}\left(\frac{\partial \tilde{f}}{\partial y}, B_1^{'}\right) - I_{(0,-\epsilon)}\left(\tilde{f}, x\right) + 1 \\ & - \left(I_{y=-\epsilon}\left(B_1^{'}, g_1^{'}\right) - \mathfrak{i}\,(S) + 1\right) \\ & + \nu\left(\frac{\partial f}{\partial y}\right)\nu\left(B_1\right) - \nu\left(B_1\right)\nu\left(g_1\right) - \nu + 1 \\ = {} & I_{(0,-\epsilon)}\left(g_1^{'}, g_2^{'}\right) + (\nu - 1)\,\nu_1 - \nu_1\nu\left(g_1\right) - \nu + \mathfrak{i}\,(S)\,. \end{aligned}$$

As $I_{(0,-\epsilon)}\left(g_1^{'}, g_2^{'}\right) = \widetilde{\mu} - \widetilde{\tau}$ and $\nu\left(g_1\right) = \nu_1 - 1$, we obtain finally

$$\mu - \tau = \widetilde{\mu} - \widetilde{\tau} + \left(\nu_1 - 1\right)\left(\nu_2 - 1\right) + \mathfrak{i}\,(S) - 1.$$

□

Let us analyze the examples previously considered.

Example (1). For S_1 we have a good basis with $\nu_1 = 1$, $\nu_2 = p - 1$ and $\mathfrak{i}\,(S_1) = 1$, then $\mu - \tau = 0$ as classically known.

Example (2). The curve S_2 admits a good basis with $\mathfrak{i}\,(S) = 2$, $\nu_1 = \nu_2 = 2$ and $\widetilde{S}_2$ has multiplicity equal to 2, hence $\widetilde{\mu} - \widetilde{\tau} = 0$. Applying the previous theorem we obtain

$$2 = I(g_1, g_2) = \tau - \mu = 0 + (2-1)(2-1) + 2 - 1.$$

Example (3). Notice that for S_3 we have $\mathfrak{i}\,(S_3) = i\,(\omega_1) = 1$, $\nu_1 = \nu_2 = 2$ and $\widetilde{S}_3$ is regular, so $\widetilde{\mu} - \widetilde{\tau} = 0$. In this way,

$$1 = I(g_1, g_2) = \mu - \tau = 0 + (2-1)(2-1) + 1 - 1.$$

So, the formula in the previous theorem holds although S_3 does not admit any good basis.

Example (4). For S_4 we get $\mathfrak{i}\,(S_4) = i\,(\omega_1) = 2$, $\nu_1 = 2, \nu_2 = 3$ and $\widetilde{S}_4$ is analytically equivalent to S_3, so $\widetilde{\mu} - \widetilde{\tau} = 1$. In this way,

$$4 = I(g_1, g_2) = \mu - \tau = 1 + (2-1)(3-1) + 2 - 1.$$

Example (5). As we presented above, S_5 does not have a good basis. We have $\mathfrak{i}\,(S_5) = i\,(\omega_1) = 1$, $\nu_1 = \nu_2 = 3$ and $\widetilde{\mu} - \widetilde{\tau} = 0$, but in this case,

$$5 = I(g_1, g_2) = \mu - \tau \neq 4 = 0 + (3-1)(3-1) + 1 - 1.$$

A more detailed analysis shows that Lemma 9 is not valid in this case because the intersection of the tangent cone of g_1 and g_1 is $x = 0$ that is distinct to the tangent cone $y = 0$ of S_5.

4. The minimal Tjurina number and the Dimca-Greuel question for plane branches.

Given a curve S, we denote by $L = L(S)$ its topological class. Although the Milnor number is constant in L, the same is not true for the Tjurina number $\tau(S)$. On the other hand, as $\tau(S)$ is upper semicontinuous, the minimum value $\tau_{\min}$ for curves in L is achieved generically and it should be computed by the sole topological data (see Chapitre III, Appendice of [14] by Teissier).

There are several (all equivalent) data that characterize the topological class L: the characteristic exponents, the Puiseux pairs, the semigroup of values, the multiplicity sequence in a canonical resolution process, the proximity matrix, etc. For our purposes we consider the characteristic exponents $(\beta_0, \beta_1, \ldots, \beta_s)$ where $\beta_0 = \nu(S)$.

For a topological class L given by characteristic exponents (β_0, β_1), Delorme in [3] presented a formula for the dimension of the generic component of the Moduli space that allow us to compute $\tau_{\min}$. For an arbitrary topological class, Peraire (see [10]) presented an algorithm to compute the $\tau_{\min}$ using flag of the Jacobian ideal.

In this section, using the last theorem and results of [5], we give an alternative method to compute $\tau_{\min}$ in a fixed topological class L and as a bonus we are able to answer a question of Dimca-Greuel for the irreducible plane curves.

Recall that we can obtain directly the characteristic exponent of the strict transform $\widetilde{S}$ of S by $(\beta_0, \beta_1, \ldots, \beta_s)$ using the euclidian division (see Theorem 3.5.5 in [12]) and consequently we get the multiplicity sequence $\nu(S) = \nu_{(1)}, \nu_{(2)}, \ldots, \nu_{(N)} = 1, 1, \ldots$, that we use in (3.3) to describe the Milnor number and the topological class $\widetilde{L} = L(\widetilde{S})$ of $\widetilde{S}$.

If S admits a good basis we can not insure that the same is valid for $\widetilde{S}$ (see Example (4)). However, this property is true generically.

Theorem 12. *Let L the topological class of plane branch given by the characteristic exponents $(\beta_0, \beta_1, \ldots, \beta_s)$, $\tau_{\min}$ the minimal Tjurina number in L and $\widetilde{\tau}_{\min}$ the minimal Tjurina number in $\widetilde{L}$. If S is generic in L, then*

$$\mu - \tau_{\min} = \widetilde{\mu} - \widetilde{\tau}_{\min} + \left(\left[\frac{\beta_0}{2}\right] - 1\right)\left(\beta_0 - \left[\frac{\beta_0}{2}\right] - 1\right) + \mathfrak{i}\,(S) - 1. \tag{4.1}$$

Moreover, if $n = \left\lceil \frac{\beta_1}{\beta_1 - \beta_0} \right\rceil$, *then* $\mathfrak{i}(S) = \left[\frac{\beta_0}{2}\right] + 1 - p_1(S)$, *where* $p_1(S)$ *can be computed in the following way:*

- *if* β_0 *is even then* $p_1(S) = \begin{cases} 1 & \text{if } n = 2 \\ 1 & \text{if } \beta_1 \text{ is even} \\ \dfrac{n-1}{2} & \text{if } \beta_1 \text{ is odd and } n \text{ odd} \\ \dfrac{n-2}{2} & \text{if } \beta_1 \text{ is odd and } n \text{ is even} \end{cases}$
- *if* β_0 *is odd then* $p_1(S) = \begin{cases} 0 & \text{if } n = 2 \\ 1 & \text{if } \beta_1 \text{ is odd} \\ \dfrac{n-3}{2} & \text{if } \beta_1 \text{ is even and } n \text{ odd} \\ \dfrac{n-2}{2} & \text{if } \beta_1 \text{ is even and } n \text{ is even.} \end{cases}$

Proof. Suppose that $\beta_0 = \nu(S)$ is even. According to the Generic Basis Theorem, S admits a good basis $\{\omega_1', \omega_2'\}$ with $\nu(\omega_1') = \nu(\omega_2') = \frac{\beta_0}{2}$. For generic $\alpha_1, \alpha_2 \in \mathbb{C}$

$$\{\omega_1 = \omega_1' + \alpha_2\omega_2', \omega_2 = \omega_2' + \alpha_1\omega_1'\}$$

remain a good basis with $\nu_1 = \nu_2 = \frac{\beta_0}{2}$ and $i(\omega_1) = i(\omega_2)$.

Now, according to [5] - using the notations of the mentioned paper, it refers to the case $\delta_1 = 0$ and $\delta_2 = 1$ - we obtain

$$\nu_1 + 1 = \frac{\beta_0}{2} + 1 = \sum_{q \in \mathbb{P}^1} \mathrm{Ind}(\widetilde{\mathcal{F}}, C, q) = i(\omega_1) + p_1(S),$$

that is,

$$\mathfrak{i}(S) = i(\omega_1) = \nu_1 + 1 - p_1(S) = \frac{\beta_0}{2} + 1 - p_1(S) = \left[\frac{\beta_0}{2}\right] + 1 - p_1(S)$$

where $p_1(S)$ is described in [5].

Now, suppose β_0 is odd and let $\{\omega_1', \omega_2'\}$ be a Saito basis for $S \cup l$ with l a generic line that without loss of generality can be considered $x = 0$. As $\nu(S \cup l)$ is even, by the same above argument, we can suppose that

$$\nu(\omega_1') = \nu(\omega_2') = \frac{\beta_0 + 1}{2} = \left[\frac{\beta_0}{2}\right] + 1 \quad \text{and} \quad \mathfrak{i}(S \cup l) = i(\omega_1') = i(\omega_2').$$

Denoting

$$\omega_1' = (a_1(y) + x(\cdots))\,\mathrm{d}x + x(\cdots)\,\mathrm{d}y$$
$$\omega_2' = (a_2(y) + x(\cdots))\,\mathrm{d}x + x(\cdots)\,\mathrm{d}y$$

and considering generic $\alpha_1, \alpha_2 \in \mathbb{C}$ we obtain a good Saito basis

$$\{\omega_1 = \omega_1' + \alpha_2\omega_2', \omega_2 = \omega_2' + \alpha_1\omega_1'\}$$

such that $\nu(a_1(y) + \alpha_2 a_2(y)) = \nu(a_2(y) + \alpha_1 a_1(y))$,

$$i(\omega_1) = i(\omega_1') = i(\omega_2') = i(\omega_2) \text{ and } \nu(\omega_1) = \nu(\omega_1') = \nu(\omega_2') = \nu(\omega_2).$$

Now the family

$$\left\{\omega_1, \frac{1}{x}\left(\omega_2 - \frac{a_2(y)+\alpha_1 a_1(y)}{a_1(y)+\alpha_2 a_2(y)}\omega_1\right)\right\}$$

is a good Saito basis for S. Finally, since $i\left(\frac{1}{x}\left(\omega_2 - \frac{a_2(y)+\alpha_1 a_1(y)}{a_1(y)+\alpha_2 a_2(y)}\omega_1\right)\right) \geq i(\omega_1)$, one has

$$\mathfrak{i}(S) = i(\omega_1).$$

By the description of $p_1(S \cup l)$ given in [5] - using the notations of the article, it refers to the case $\delta_1 = 1$ and $\delta_2 = 1$ - we get

$$\mathfrak{i}(S) = i(\omega_1) = \frac{\nu(S)+1}{2} - p_1(S) = \left[\frac{\beta_0}{2}\right] + 1 - p_1(S).$$

Thus, the proof of the formula is a consequence of Theorem 11 noticing that by the Generic Basis Theorem we have $\nu_1 = \left[\frac{\beta_0}{2}\right]$ and $\nu_2 = \beta_0 - \left[\frac{\beta_0}{2}\right]$. □

Example (7). In [14], Zariski computed the generic Tjurina number for the topological class L with characteristic exponents $(p, p+1)$. Notice that any curve in L is smooth after one blowing-up and its Milnor number is $p(p-1)$. Moreover, the integer n in the previous theorem is equal to $p+1$. Considering a generic curve S, if p is even, $p_1(S) = \frac{p}{2}$, that is, $\mathfrak{i}(S) = 1$ and one obtain,

$$\tau_{\min} = p(p-1) - \left(\frac{p}{2} - 1\right)\left(\frac{p}{2} - 1\right) - 1 + 1 = \frac{3}{4}p^2 - 1.$$

If p is odd, then $p_1(S) = \frac{p-1}{2}$. So $\mathfrak{i}(S) = 1$ and

$$\tau_{\min} = p(p-1) - \left(\frac{p-1}{2} - 1\right)\left(p - \frac{p-1}{2} - 1\right) - 1 + 1 = \frac{3}{4}(p^2 - 1).$$

Both formulas coincide with the ones given in [14].

Example (8). In [3], Delorme proposed an algorithm to compute the dimension of the Moduli space for a curve with characteristic exponents (a, b) based upon the continuous fraction expansion of $\frac{b}{a}$ and consequently we can obtain the minimum Tjurina number for the topological class L determined by (a, b). For the generic curve with $(a, b) = (2k+1, 2k+3)$ with k even, we have $\widetilde{S}$ smooth, the integer n is equal to $k+2$, $p_1(S) = 1$ and $\mathfrak{i}(S) = k$, applying the formula (4.1) leads to

$$\tau_{\min} = 2k(2k+2) - (k-1)(2k+1-k-1) - k + 1 = 3k^2 + 4k + 1,$$

which coincide with the formula obtained from [3].

Example (9). In [10], Peraire computed the minimum Tjurina number for the topological class whose characteritic exponents are $(9, 12, 17)$. After five blowing-ups, we obtain a curve with multiplicity 2. The corresponding characteristics exponents of the sequence of blown-up curves are

$$(3, 14),\ (3, 11),\ (3, 8),\ (3, 5),\ (2, 3).$$

Applying inductively the formula (4.1), one accumulates contribution to the difference $\mu - \tau_{\min}$. Actually, it can be seen that the respective contributions are

$$15,\ 1,\ 1,\ 1,\ 0,\ 0.$$

Thus

$$\tau_{\min} = \mu - 18 = 98 - 18 = 80$$

which coincides with the computation of Peraire.

The last theorem allow us obtain a formula to compute the minimum Tjurina number in a topological class using the multiplicity sequence.

Corollary 13. *Let L a topological class of a singular plane branch determined by the multiplicity sequence $\nu_{(1)}, \nu_{(2)}, \ldots, \nu_{(N)}, \nu_{(N+1)} = 1, \ldots$. The minimal Tjurina number achieved in L is*

$$\tau_{\min} = \sum_{i=1}^{N} \left(\nu_{(i)}^2 + \left[\frac{\nu_{(i)}}{2} \right] \left(\left[\frac{\nu_{(i)}}{2} \right] - \nu_{(i)} - 1 \right) - 1 + p_1(S_{(i)}) \right)$$

where $S_{(i)}$ denote the curve with multiplicity $\nu_{(i)}$ in the canonical resolution process for a generic curve in L.

Proof. Applying inductively the formula presented in the last theorem and using that $\mathfrak{i}\left(S_{(i)}\right) = \left[\frac{\nu_{(i)}}{2}\right] + 1 - p_1(S_{(i)})$ yields

$$\tau_{\min} = \mu - \sum_{i=1}^{N} \left(\left(\left[\frac{\nu_{(i)}}{2} \right] - 1 \right) \left(\nu_{(i)} - \left[\frac{\nu_{(i)}}{2} \right] - 1 \right) - \left(\mathfrak{i}\left(S_{(i)}\right) - 1 \right) \right)$$

$$= \mu + \sum_{i=1}^{N} \left(\left[\frac{\nu_{(i)}}{2} \right] \left(\left[\frac{\nu_{(i)}}{2} \right] - \nu_{(i)} - 1 \right) + \nu_{(i)} - 1 + p_1(S_{(i)}) \right). \tag{4.2}$$

As $\mu = \sum_{i=1}^{N} \nu_{(i)} \left(\nu_{(i)} - 1\right)$, we get the proof. □

In [4], Dimca and Greuel present an interesting question about the Tjurina number for curves in a given topological class L. More specifically, they ask if $4\tau(S) > 3\mu(S)$ for any curve in L.

As the Tjurina number is semicontinuous in L and we have obtained a formula for the $\tau_{\min}$, we are able to given a lower bound for the Tjurina number in terms of the Milnor number and it answered positively the previous question for the irreducible case.

Corollary 14. *Let S be a singular irreducible plane curve. Then*

$$\tau(S) \geq \frac{3}{4}\mu(S) + \frac{\sqrt{1 + 4\mu(S)} - 1}{8}.$$

In particular, $4\tau(S) > 3\mu(S)$.

Proof. We denote $\mu = \mu(S)$. It is sufficient to show the inequality for the $\tau_{\min}$. By (4.2), the relation below holds

$$4\tau_{\min} - 3\mu = \mu + 4\sum_{i=1}^{N} \left(\left[\frac{\nu_{(i)}}{2} \right] \left(\left[\frac{\nu_{(i)}}{2} \right] - \nu_{(i)} - 1 \right) + \nu_{(i)} - 1 + p_1(\nu_{(i)}) \right).$$

Now, using that $\mu = \sum_{i=1}^{N} \nu_{(i)} \left(\nu_{(i)} - 1\right)$ and

$$4 \left[\frac{\nu_{(i)}}{2} \right] \left(\left[\frac{\nu_{(i)}}{2} \right] - \nu_{(i)} - 1 \right) = -\nu_{(i)}^2 - 2\nu_{(i)} + \delta_i$$

with

$$\delta_i = 0 \ \text{ if } \nu_{(i)} \text{ is even} \quad \textit{and} \quad \delta_i = 3 \ \text{ if } \nu_{(i)} \text{ is odd},$$

we obtain

$$4\tau_{\min} - 3\mu = \sum_{i=1}^{N} \left(\nu_{(i)} + \delta_i + 4(p_1(S_{(i)}) - 1)\right).$$

Now, by Theorem 12 we have that:

- if $\nu_{(i)}$ is even, then $p_1(S) \geq 1$ and $\nu_{(i)} + 0 + 4(p_1(\nu_{(i)}) - 1) \geq \nu_{(i)}$,
- if $\nu_{(i)}$ is odd, then $p_1(S) \geq 0$ and $\nu_{(i)} + 3 + 4(p_1(\nu_{(i)}) - 1) \geq \nu_{(i)} - 1$.

So, the following inequality follows

$$4\tau_{\min} - 3\mu \geq \sum_{i=1}^{N} (\nu_{(i)} - 1). \tag{4.3}$$

As $\mu = \sum_{i=1}^{N} (\nu_{(i)} - 1)^2 + \sum_{i=1}^{N} \left(\nu_{(i)} - 1\right)$ it follows that

$$4\tau_{\min} - 3\mu \geq \mu - \sum_{i=1}^{N} (\nu_{(i)} - 1)^2.$$

Using (4.3), that is, $-(4\tau_{\min} - 3\mu)^2 \leq -\left(\sum_{i=1}^{N} (\nu_{(i)} - 1)\right)^2 \leq -\sum_{i=1}^{N} (\nu_{(i)} - 1)^2$, we get

$$4\tau_{\min} - 3\mu \geq \mu - (4\tau_{\min} - 3\mu)^2$$

and consequently

$$\tau\,(S) \geq \tau_{\min} \geq \frac{3}{4}\mu + \left(\frac{-1 + \sqrt{1 + 4\mu}}{8}\right).$$

□

Example (10). Let us consider the topological class L determined by the characteristic exponents $(141, 142)$. The Milnor number for any curve in L is $\mu = (141 - 1)(142 - 1) = 19740$. Using the lower bound presented in the las result we obtain $\tau_{\min} \geq 14840$. For this topological class it follows by the Example 7 that $\tau_{\min} = 14910$.

While we submit the first version of this paper to Arxiv, we discover that, at the same time, a positive answer for the Dimca-Greuel question was obtained by Alberich-Carramiñana *et al.* and published in Arxiv [7] a few days before. Although the methods are a bit different, the key ingredient is still the formula for the generic dimension of the moduli space obtained in [5].

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