Analytical and formal classifications of quasi-homogeneous foliations in $\mathbb{C}^2$

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Abstract

We prove a result of classification for germs of formal and convergent quasi-homogeneous foliations in $\mathbb{C}^2$ with fixed separatrix. Basically, we prove that the analytical and formal class of such a foliation depend respectively only on the analytical and formal class of its representation of projective holonomy.

Key words: Dynamical system, Holomorphic foliations, Singularities

1 Introduction and main statements.

A *germ of foliation* $\mathcal{F}$ *with an isolated singularity in* $\mathbb{C}^2$ is given by an holomorphic 1-form up to a germ unity

$$\omega = a(x,y)dx + b(x,y)dy$$

where $a, b \in \mathbb{C}\{x,y\}$, $a(0,0) = b(0,0) = 0$ and $\gcd(a,b) = 1$. The foliation $\mathcal{F}$ is said to be *formal*, and then denoted by $\widehat{\mathcal{F}}$ when a and b are only formal functions in $\mathbb{C}[[x,y]]$. A *separatrix* of $\mathcal{F}$ is a leaf of the regular foliation given by $\omega|_{(\mathbb{C}^{2*},0)}$ whose closure in $(\mathbb{C}^2,0)$ is an irreducible analytical curve passing through the origin. A *formal separatrix* of $\widehat{\mathcal{F}}$ is a formal irreducible curve $\{\widehat{f} = 0\}$, $\widehat{f} \in \mathbb{C}[[x,y]]$ such that $\widehat{f}$ divides the product $d\widehat{f} \wedge \omega$. Obviously, any convergent separatrix seen as a formal curve is a formal separatrix of $\mathcal{F}$ seen as a formal foliation. C. Camacho and P. Sad show in [3] that $\mathcal{F}$ has at least one separatrix. Once the foliation is desingularized by a blowing-up morphism [20], i.e. a sequence of succesive standard blowing-up,

$$E : (\mathcal{M}, \mathcal{D}) \to (\mathbb{C}^2, 0), \quad \mathcal{D} = E^{-1}(0) \text{ exceptionnal divisor}$$

the pull-back foliation $E^*\mathcal{F}$ has only *reduced singularity*: any singularity of $E^*\mathcal{F}$ is written in some coordinates

(1) either $\omega = \lambda x dy + \beta y dx + \cdots$ with $\lambda/\beta \notin \mathbb{Q}^{-*}$,
(2) or $\omega = x dy + \cdots$

A singularity of *saddle-node* type corresponds to the second case. Now, $\mathcal{F}$ is said to be of *generalized curve type* when $E^*\mathcal{F}$ has no singularity of *saddle-node* type. Under the generalized curve assumption, the foliation $\mathcal{F}$ and its separatrix have the same desingularization [2]. All these definitions and results have their equivalent in the formal context: one has only to add the word *formal* in order to get adapted statements.

Following [14], the foliation $\mathcal{F}$ is said to be *quasi-homogeneous* when the union of its separatrix, denoted by $\mathrm{Sep}(\mathcal{F})$, is a germ of curve given in some coordinates by a quasi-homogeneous polynomial function:

$$\mathrm{Sep}(\mathcal{F}) = \left\{ \sum_{\alpha i + \beta j = \gamma} a_{ij} x^i y^j = 0 \right\}, \quad \alpha, \beta, \gamma \in \mathbb{N}^*, \quad \gcd(\alpha, \beta) = 1.$$

The couple (α, β) is called the *weight* of the curve. Suppose $\mathcal{F}$ to be a quasi-homogeneous foliation of generalized curve type. The exceptional divisor of its desingularization comes to be a unique chain of $\mathbb{CP}^2$ such that each component except the two extremal meets exactly two others. Appart from the extremal components, there is a unique component of the divisor, which meets the strict transform of S, i.e. the closure of $E^{-1}(\mathrm{Sep}(\mathcal{F})\backslash\{0\})$ in $\mathcal{M}$. In this article, we call the latter component the *central component*. As we will explain, the whole transversal structure of $\mathcal{F}$ is concentrated in the projective holonomy representation over the central component.

Let $\mathcal{F}_0$ and $\mathcal{F}_1$ be two germs of quasi-homogeneous foliations of generalized curve type. Let $E_0 : (\mathcal{M}_0, \mathcal{D}_0) \to (\mathbb{C}^2, 0)$ and $E_1 : (\mathcal{M}_1, \mathcal{D}_1) \to (\mathbb{C}^2, 0)$, be their respective desingularization. Suppose the separatrix analytically conjugated and denote by Φ a conjugacy. The conjugacy can be lifted-up in a conjugacy Φ^* from $\mathcal{M}_0$ to $\mathcal{M}_1$ sending the central component of $\mathcal{F}_0$ to that of $\mathcal{F}_1$. We say that $\mathcal{F}_0$ and $\mathcal{F}_1$ have *conjugated holonomy* if there is an interior automorphism ϕ of the group $\mathrm{Diff}(\mathbb{C}, 0)$

$$\phi(h) = g h g^{-1}, \text{ for some } g \text{ in } \mathrm{Diff}(\mathbb{C}, 0)$$

such that the projective holonomy representations satify the commutative diagram:

$$\begin{array}{ccc} \Pi_1\left(D_0 \backslash \mathrm{Sing}(E_0^* \mathcal{F}_0), x\right) & \xrightarrow{\mathrm{Hol}_{\mathcal{F}_0}} & \mathrm{Diff}(\mathbb{C}, 0) \\ \Phi^* \big\downarrow & & \phi \big\downarrow \\ \Pi_1\left(D_1 \backslash \mathrm{Sing}(E_1^* \mathcal{F}_1), \Phi^*(x)\right) & \xrightarrow{\mathrm{Hol}_{\mathcal{F}_1}} & \mathrm{Diff}(\mathbb{C}, 0) \end{array}$$

where D_0 and D_1 stand for the respective central components.

The aim of this article is to show the following theorem:

Theorem 1.1 *Let the foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ be two germs of (resp. formal) quasi-homogeneous foliations with analytically (resp. formally) conjugated separatrix. The foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ are analytically (resp. formally) conjugated if and only if they have conjugated holonomy (resp. formally conjugated holonomy).*

In [4], D. Cerveau and R. Moussu generalize a construction of the latter author in [17] and prove the above result in the particular case of a quasi-homogeneous foliation given by a 1-form with nilpotent linear part

$$ydy + \cdots .$$

In this case, the separatrix is the curve $y^2 + x^p = 0$ and, consequently, there is no separatrix whose strict transform is attached to the extremal components of the exceptionnal divisor of the desingularization. From this remark, their proof roughly consists in extending the conjugacy of the holonomy in a neighborhood of the divisor deprived of a few curves by lifting the paths with respect to some global transversal fibration. Since there exists a first integral for the foliation on the neighboorhood of the whole exceptionnal divisor deprived of a neighboorhood of the central component, the conjugacy is bounded and therefore well defined around the whole divisor. In the general case, the existence of a first integral does not hold and it is not clear wether the conjugacy one could obtain would be bounded. Actually, our point of view is completely different and based upon deformation techniques. The first section is devoted to proof the possibility of constructing some special deformations of foliations with prescribed underlying family of separatrix. The associated result is independent of the main goal of this article and should be of self interest. The second section provides some classical properties of quasi-homogeneous foliations and reduced singularities and, finally, a detailed proof of (1.1).

2 Isoholonomic deformations with prescribed separatrix.

This section is to devoted to the construction of some very special deformations of foliations, namely *isoholonomic deformations*, which are introduced hereafter. The main result is a realization kind theorem (2.2), which allows us to find an isoholonomic deformation of a foliation as soon as a deformation of its separatrix is given. This theorem specializes to a result of [6] if there is no parameter [1]. Since the proof of the latter result and the one we develop

[1] This case correponds to $K = \{0\}$ and $C = \emptyset$ in the theorem (1.1)

here are widely similar, we are about to only give the arguments one has to add to these presented in [6] in order to obtain a complete proof. However, for the convenience of the reader we repeat here the relevant material from [6] thus making our expostion almost self-contained. Nevertheless, the reader which would be only interested in the classification (1.1) could admit the main resullt of this section and begin its lecture with the last section.

2.1 Isoholonomic deformations.

Let $\mathcal{F}$ be a formal foliation in $\mathbb{C}^2$. Let K be a compact connected subset of $\mathbb{C}^p$. In this article, we call *transversally formal integrable deformation over* K *of* $\mathcal{F}$ any formal foliation $\mathcal{F}_K$ of codimension one in $(\mathbb{C}^{2+p}, 0\times K)$ with $0 \times K$ as singular locus such that the leaves are transversal to the fiber of the projection

$$\pi : (\mathbb{C}^{2+p}, 0 \times K) \to (\mathbb{C}^p, K), \quad \pi(x,t) = t.$$

and given by a 1-form

$$\Omega(x,y,t) = a(x,y,t)dx + b(x,y,t)dy + \sum_{i=1}^{p} c_i(x,y,t)dt_i,$$
$$(x,y) \in \mathbb{C}^2, t = (t_1,\dots,t_p) \in K$$

with

(1) $a, b, c_1, \dots, c_p$ lies in the ring $\mathcal{O}_K[[x,y]]$ of formal series with coefficients in the ring of holomorphic functions on K,
(2) Ω is integrable,
(3) $\Omega|_{t=0}$ defines $\mathcal{F}$,
(4) the set
$$\{a = 0, b = 0, c_1 = 0, \dots, c_p = 0\}$$
is equal to $0 \times K$ and the ideal $(c_1, \dots, c_p)$ is a sub-ideal of $\sqrt{(a,b)}$.

The latter condition is equivalent to the transversality of the leaves and the fibers of π. If the coefficients come to be convergent series the deformation is simply called *integrable deformation*. In any case, the foliation $i_t^*\mathcal{F}$ where $i_t,\ t \in K$ is the embedding $i_t(x) = (x,t),\ x \in \mathbb{C}^2$, is a formal foliation in usual sense. More generally, for any subset J of K, we denote by $\mathcal{F}_J$ the tranversally formal integrable deformation over J induced by restriction of $\mathcal{F}_K$ on $\pi^{-1}(J)$. A transversally formal integrable deformation is said *equisingular* if the induced family of foliation $\{i_t^*\mathcal{F}\}_{t\in K}$ admits a desingularization *in family*. We refer to [16] for a precise definition of this notion.

Let us first recall a result due Mattei, which is the foundation of our method. Let $\mathcal{F}$ be any germ of analytic foliation in $(\mathbb{C}^2,0)$ and $E:(\mathcal{M},\mathcal{D})\to(\mathbb{C}^2,0)$ its desingularization. We denote by $\mathrm{Fix}(\mathcal{F})$ the sheaf over $\mathcal{D}$ whose fiber is the group of germs of automorphisms ϕ in $(\mathcal{M},\mathcal{D})\times(\mathbb{C},0)$ such that

- $\phi|_{\mathcal{M}\times\{0\}}=\mathrm{Id}$
- ϕ commutes with the projection on the space of parameter

$$\Pi:(\mathcal{M},\mathcal{D})\times(\mathbb{C},0)\to(\mathbb{C},0)$$

- ϕ lets invariant each local leaf and $\phi^*(\mathcal{F}\times(\mathbb{C},0))=\mathcal{F}\times(\mathbb{C},0)$

Basically, the flows of vector field X tangent to the foliation $\mathcal{F}\times(\mathbb{C},0)$ with $D\Pi(X)=0$ are sections of $\mathrm{Fix}(\mathcal{F})$.

Theorem 2.1 (Mattei,[13]) *There is a bijection between the moduli space of germs of equisingular integrable deformations of $\mathcal{F}$ with parameters in $\mathbb{C}$ and $H^1(\mathcal{D},\mathrm{Fix}(\mathcal{F}))$.*

In what follows, we are deeply going to make the most of this cohomological interpretation of equisingular integrable deformations. Since the sections of $\mathrm{Fix}(\mathcal{F})$ act in the local leaf, the holonomy pseudo-group of $i_t^*\mathcal{F}_{(\mathbb{C},0)}$ does not depend on t along any equisingular integrable deformation. Hence, we adopt the following defintion:

Definition 2.1 *An (resp. transversally formal) isoholonomic deformation of foliation is an equisingular (resp. transversally formal) integrable deformation.*

For simplicity, the article is written from now on in the convergent context. However, there is no special difficulty for transposing the proofs and the results in the transversally formal context.

2.2 Existence of isoholonomic deformation with prescribed separatrix.

To make a precise statement, we have to introduce some more definitions. Let $\mathcal{F}_K$ be an isoholonomic deformation with parameter in $K\subset\mathbb{C}^p$. When $i_t^*\mathcal{F}_K$ is of generalized curve type for some $t\in K$, the whole deformation $\mathcal{F}_K$ is naturally said to be of generalized curve type. This definition is coherent since, along an isoholonomic deformation, the property holds for any foliation $i_t^*\mathcal{F}_K$ as soon as it holds for some $t\in K$. A *separatrix* of $\mathcal{F}_K$ is an invariant hypersurface of the regular foliation $\mathcal{F}|_{\mathbb{C}^{2*}\times K}$ whose closure is an irreducible analytical germ of hypersurface along $0\times K$. When $\mathcal{F}_K$ has only a finite number of separatrix, we denote by $\mathrm{Sep}(\mathcal{F}_K)$ their union and $\mathcal{F}_K$ is said *non-dicritical*. Since, $\mathcal{F}_K$ is equisingular in the foliated meaning, the family $\{\mathrm{Sep}(\mathcal{F}_K)\}_{t\in K}$ is an equisingular deformation of curves.

From now on, we assume the compact of parameters K to have a fundamental system of open connected neighborhoods, which are Stein open sets. Let C be an analytical subset of K given by the zeros of holomorphic functions.

Theorem 2.2 (Cobordism theorem) *Let $\mathcal{F}^0$ be a non-dicritical isoholonomic deformation over K of generalized curve type and let us denote by S^0 the separatrix of $\mathcal{F}^0$. Let S^1 be any equisingular family of germs of curves at the origin of $\mathbb{C}^2$ with*

(1) S^1 and S^0 are topologically equivalent as families,
(2) $S^1|_C$ and $S^0|_C$ are analytically equivalent as families.

There exists an isoholonomic deformation $\mathcal{F}^1$ over K such that

(1) $\mathrm{Sep}(\mathcal{F}^1) = S^1$,
(2) $\mathcal{F}^1|_C$ and $\mathcal{F}^0|_C$ are analytically equivalent.

Moreover, the deformation $\mathcal{F}_1$ and $\mathcal{F}_0$ are embedded in an isoholonomic deformation $\mathcal{F}$ over $K \times \overline{\mathbb{D}}$

$$\mathcal{F}|_{K\times\{0\}} = \mathcal{F}_0 \quad \text{and } \mathcal{F}|_{K\times\{1\}} = \mathcal{F}_1$$

which is trivial above C: the deformation $\mathcal{F}|_{C\times\overline{\mathbb{D}}}$ is analytically equivalent to $\mathcal{F}|_{C\times\{0\}} \times \overline{\mathbb{D}}$.

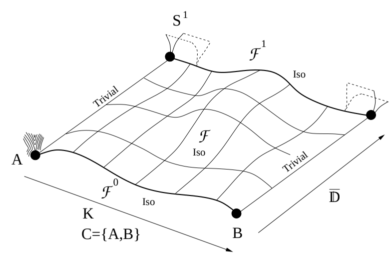

Fig. 1. Theorem (2.2)

As mentionned before, in this section, we are going to follow the proof performed in [6] and give only the arguments related to the difficulties appearing with the parameter context. One of the interests of the formalism introduced in [6] is the easy way arguments are extended to this more general context. Actually, the main difficulties arising with the parameters are removed thanks to the Stein property of K.

In the formal context, there is a similar result: replace convergent objects by formal objects and isoholonomic deformations by transversally formal isoholonomic deformations.

Let us describe a short plan of the proof. In the section (2.3), we fix a manifold $\mathcal{M}$ built from a family of blowing-up process and we define a very special class of manifolds denoted by $\mathrm{Glu}_0^C(\mathcal{M},\mathcal{U},Z)$ related to $\mathcal{M}$. In the section (2.4), $\mathcal{M}$ is supposed to be foliated by an isoholonomic deformation $\mathcal{F}$. A property of cobordism type is pointed out , which allows us to detect the existence of an isoholonomic deformation on any element of $\mathrm{Glu}_0^C(\mathcal{M},\mathcal{U},Z)$. This deformation will be automatically linked to $\mathcal{F}$ by isoholonomic deformations. In sections (2.4.1), (2.4.2) and (2.4.3), we show the theorem (2.6) which states that, under the generical hypothesis, the cobordism property holds for any element of $\mathrm{Glu}_0^C(\mathcal{M},\mathcal{U},Z)$. In the third section, we deduce the theorem (2.2) from this cobordism property.

2.3 *The category* $\mathrm{Glu}_n^C(\mathcal{M},Z,\mathcal{U})$.

A blowing-up process over K is a commutative diagram

$$\begin{array}{llllll}
\mathcal{M}^h \overset{E^h}{\to} \dots & \mathcal{M}^j \overset{E^j}{\to} & \dots \overset{E^1}{\to} & \mathcal{M}^0 & = (\mathbb{C}^{2+p}, 0\times K) \overset{\pi}{\to} (\mathbb{C}^p, K) & \\
\cup & \cup & & \cup & & \\
\Sigma^h \to \dots & \Sigma^j \to & \dots \to & \Sigma^0 & = 0\times K & (1)\\
\cup & \cup & & \cup & & \\
S^h \to \dots & S^j \to & \dots \to & S^0 & = 0\times K &
\end{array}$$

where $\mathcal{M}^j$ is an analytical manifold of dimension 2+p; Σ^j is a closed analytical subset of dimension p called the j^{th} singular locus; $S^j \subset \Sigma^j$ is an analytical smooth sub-variety with a finite number of connected components called the j^{th} blowing-up center. Futhermore, denoting

$$E_j := E^0 \circ \cdots \circ E^j, \quad \pi_j := \pi \circ E_j, \quad \text{et} \quad \mathcal{D}^j := E_j^{-1}(S^0),$$

we require that: π is the projection on the second factor; each E_{j+1} is the standard blowing-up centered at S^j; each S^j is a union of irreducible components of Σ^j; each Σ^j is a smooth subset of the divisor $\mathcal{D}^j$. Moreover, the maps $\pi_j|_{\Sigma^j}$ and $\pi_j|_{S^j}$ are etale over K. The set of irreducible components of $\mathcal{D}^j$ is denoted by $\mathrm{Comp}(\mathcal{D}^j)$. The integer h is called *height* of the blowing-up process and $\left(\mathcal{M}^h, \mathcal{D}^h, \Sigma^h, \pi_h\right)$ *the top* of the process. The composed map E_h is called the *total morphism* of the process.

More generally, we call *tree over* K a triplet $(\mathcal{M},\mathcal{D},\Sigma,\pi)$ such that there exists a blowing-up process over K whose $\left(\mathcal{M}^h, \mathcal{D}^h, \Sigma^h, \pi_h\right)$ is the top and a biholomorphism Φ between $\mathcal{M}$ and $\mathcal{M}^h$ such that

$$\Phi(\mathcal{D}) = \mathcal{D}^h, \quad \Phi(\Sigma) = \Sigma^h \quad \text{and} \quad \pi_h \circ \Phi = \pi.$$

Finally, for any $J \subset K$ we denote by $(\mathcal{M}_J, \mathcal{D}_J, \Sigma_J, \pi_J)$ the tree over J obtained by simple restriction.

2.3.1 The sheaves $\mathcal{G}_Z^{Cn}$, $n \geq 0$.

From now on, we fix a marked tree $(\mathcal{M}, \mathcal{D}, \Sigma, \pi)$. In order to get through a technical difficulty, the tree is enhanced with a *cross*: let E be the total morphism of $(\mathcal{M}, \mathcal{D}, \Sigma, \pi)$.

Definition 2.2 (Cross) *A cross on $\mathcal{M}$ is the strict transform $Z = E^* Z_0$ of a single $Z_0 = \{Z_1\}$ or of a couple $Z_0 = \{Z_1, Z_2\}$ of germs of smooth transversal hypersurfaces along $0 \times K$. We assume that any component Z meets a unique irreducible component of $\mathcal{D}$.*

We consider $\mathrm{Aut}^C(\mathcal{M}, Z)$ the sheaf over $\mathcal{D}$ of germs of automorphism defined in a neighborhood of $\mathcal{D}$ such that

$$\pi \circ \Phi = \pi, \quad \Phi|_{\mathcal{D}} = \mathrm{Id}, \quad \Phi|_Z = \mathrm{Id} \quad \text{et} \quad \Phi|_{\pi^{-1}(C)} = \mathrm{Id}.$$

Let $\mathcal{O}_{\mathcal{M}}$ be the sheaf over $\mathcal{D}$ of germs of functions on $\mathcal{M}$. Let us consider the subsheaf $\mathfrak{M} \subset \mathcal{O}_{\mathcal{M}}$ generated by the pull-back of the ideal of functions vanishing along $0 \times K$. It is easily seen that one has the following decomposition

$$\mathfrak{M} = \mathcal{O}\left(- \sum_{D \in \mathrm{Comp}(\mathcal{D})} \nu(D) D \right)$$

where $\nu(D)$ is an integer called the *multiplicity of the component* D. We consider a filtration of $\mathcal{O}_{\mathcal{M}}$ defined by $\mathfrak{M}_Z^n := I_Z \cdot \mathfrak{M}^n$, $n \geq 1$. The sheaf $\mathfrak{M}^{C}{}_Z^n$ is a sub-sheaf of $\mathfrak{M}_Z^n$ whose sections vanish along $\pi^{-1}(C)$. We also have to consider the sheaf $\mathfrak{I}_Z \subset \mathcal{O}_{\mathcal{M}}$ defined by

$$\mathfrak{I}_Z := \mathcal{O}\left(-Z - \sum_{D \in \mathrm{Comp}(\mathcal{D})} D \right).$$

We call n^{th} *infinitesimal neighboorhood* the analytical space

$$\mathcal{M}^{[n],Z} := \left(\mathcal{D}, \mathcal{O}_{\mathcal{M}} / \mathfrak{M}_Z^n \right).$$

The neighborhood of order 0 is $\mathcal{M}^{[0],Z} := \left(\mathcal{D}, \mathcal{O}_{\mathcal{M}} / \mathfrak{I}_Z \right)$. We also consider the following ringed spaces: $\mathcal{M}^{\underline{n},Z} := \left(\mathcal{D}, \mathfrak{I}_Z / \mathfrak{I}_Z \mathfrak{M}_Z^n \right)$ and $\mathcal{M}^{\underline{0},Z} := \left(\mathcal{D}, \mathfrak{I}_Z / \mathfrak{I}_Z^2 \right)$.

Definition 2.3 *We denote by $\mathrm{Aut}_n^C(\mathcal{M}, Z)$ the subsheaf of $\mathrm{Aut}^C(\mathcal{M}, Z)$ of* *germs that coincide with* Id *when restricted to the infinitesimal neighborhood of order n.*

Let us have a close look at the form of the sections of $\mathrm{Aut}_n^C(\mathcal{M}, Z)$ in order to define a special morphism of sheaves. In any following expressions, the used coordinates are naturally adapted to the situtation: the coordinates (x, y) stand for the local components of $\mathcal{D}$ or Z and t for the parameter in K.

At a regular point c of $\mathcal{D} \cup Z$: let p be the multiplicity of the component containing c. The elements of $\mathrm{Aut}_n^C(\mathcal{M}, Z)_c$ can be written $\phi(x, y, t) = (x + x^{pn}A, y + x^{pn}B, t)$, where A, B belong to $\mathbb{C}\{x, y, t\}$ and vanishing along $\pi^{-1}(C)$. Let $\mathcal{J}_n$ be the function defined by

$$\phi(x, y, t) = (x + x^{pn}A, y + x^{pn}B, t) \in \mathrm{Aut}_n^C(\mathcal{M}, Z)_c \\ \xmapsto{\mathcal{J}_n} x^{pn-1}A \in \left(\mathcal{O}_{\mathcal{M}^{[n]}, Z}\right)_c .$$

One can see that $\mathcal{J}_n$ is a morphism of groups that doesn't depend on the adapted coordinates.

At a singular point s of $\mathcal{D}$: let p and q be the multiplicities of the local components. The elements of $\mathrm{Aut}_n^C(\mathcal{M}, Z)_s$ are those of the form $\phi(x, y, t) = (x + x^{pn}y^{qn}A, y + x^{pn}y^{qn}B, t)$. In the same way, we define an intrisic group morphism by

$$\phi(x, y, t) = (x + x^{pn}y^{qn}A, y + x^{pn}y^{qn}B, t) \in \mathrm{Aut}_n^C(\mathcal{M}, Z)_s \\ \xmapsto{\mathcal{J}_n} x^{pn-1}y^{qn-1}\left(yA + xB\right) \in \left(\mathcal{O}_{\mathcal{M}^{[n]}, Z}\right)_s .$$

At an attachment point z of Z: the multiplicity of the local component is 1 since the components of Z are smooth curves downstairs at the origin. The elements of $\mathrm{Aut}_n^C(\mathcal{M}, Z)_z$ are of the form $\phi(x, y, t) = (x + x^n yA, y + x^n yB, t)$ and the morphism is defined by

$$\phi(x, y, t) = (x + x^n yA, y + x^n yB, t) \in \mathrm{Aut}_n^C(\mathcal{M}, Z)_z \\ \xmapsto{\mathcal{J}_n} x^{n-1}\left(yA + xB, t\right) \in \left(\mathcal{O}_{\mathcal{M}^{[n]}, Z}\right)_z .$$

Finally, we get a morphism of sheaves defined by its previous local description $\mathrm{Aut}_n^C(\mathcal{M}, Z) \xmapsto{\mathcal{J}_n} \mathcal{O}_{\mathcal{M}^{[n]}, Z}$. We have likewise a morphism of sheaves $\mathcal{J}_0$ defined, for example, near a regular point by

$$\phi(x, y, t) = (x + xA, y + xB, t) \in \mathrm{Aut}_0^C(\mathcal{M}, Z)_c \xmapsto{\mathcal{J}_0} A \in \left(\mathcal{O}_{\mathcal{M}^{[0]}, Z}\right)_c .$$

Definition 2.4 *We denote $\mathcal{G}_Z^{C^n}$ the subsheaf of $\mathrm{Aut}_n^C(\mathcal{M}, Z)$ kernel of the morphism $\mathcal{J}_n$.*

2.3.2 The tree gluing.

Thanks to the sheaf $\mathrm{Aut}^C(\mathcal{M}, Z)$, we are going to introduce a process called *gluing* on $\mathcal{M}$. This construction will allow us to define a large class of trees with same divisor analytical type. These trees will inherit a canonical cross.

Let us define a particular type of open covering of the divisor. Let $\mathcal{U} = \{U_i\}_{i\in\mathbb{I}=\mathbb{I}_0\cup\mathbb{I}_1}$ be the covering of $\mathcal{D}$ constituted of two kinds of open sets: if i belongs to $\mathbb{I}_0$, U_i is the trace on $\mathcal{D}$ of a neighborhood of a unique singular locus conformally equivalent to a polydisc; if i belongs to $\mathbb{I}_1$, U_i is an irreducible component of $\mathcal{D}$ deprived of the singular locus of $\mathcal{D}$. Such a covering is called *distinguished* when there is no 3-intersection. Distinguished coverings contain Stein open sets having fundamental systems of Stein neighborhood. From now on, a covering denoted by $\mathcal{U}$ will always supposed to be distinguished.

Thanks to distinguished covering, we are able to glue the open sets of that covering by identifying points with respect to a 1-cocycle in $\mathrm{Aut}^C(\mathcal{M}, Z)$. Let (ϕ_{ij}) be a 1-cocycle in $\mathcal{Z}^1\left(\mathcal{U}, \mathrm{Aut}^C(\mathcal{M}, Z)\right)$. We define:

$$\mathcal{M}[\phi_{ij}] = \bigcup_i \mathcal{U}_i \times \{i\}/_{\left(\{x\}\times\{i\}\sim\{\phi_{ij}(x)\}\times\{j\}\right)},$$

where $\mathcal{U}_i$ is a neigborhood of U_i in $\mathcal{M}$ such that ϕ_{ij} is an automorphism of $\mathcal{M}$ along $U_i \cap U_j$. The manifold we get comes with an embedding

$$\mathcal{D} \hookrightarrow \mathcal{M}[\phi_{ij}] \tag{2}$$

whose image is denoted by $\mathcal{D}[\phi_{ij}]$ and $\mathcal{M}[\phi_{ij}]$ is considered as a germ of neighborhood of $\mathcal{D}[\phi_{ij}]$.

Definition 2.5 *The manifold germ $\mathcal{M}[\phi_{ij}]$ is called gluing of $\mathcal{M}$ along $\mathcal{U}$ by the cocycle (ϕ_{ij}).*

By construction of the sheaf $\mathrm{Aut}^C(\mathcal{M}, Z)$, we have the following isomorphism of tree

$$\mathcal{M}[\phi_{ij}]|_C \simeq \mathcal{M}|_C$$

The gluing of a crossed tree comes naturally with a cross: it is the direct image of Z by the quotient map for gluing relation. Such a tree and cross are respectively denoted by

$$(\mathcal{M}[\phi_{ij}], \mathcal{D}[\phi_{ij}], \Sigma[\phi_{ij}])\,, \text{ and } Z[\phi_{ij}].$$

We associate to any gluing the data of morphisms on infinitesimal neigboorhoods generalizing the embedding (2). Actually, the description of $\mathcal{G}_Z^{C^n}$ sections reveals the following property

Proposition 2.1 *Let n be an integer and $\mathcal{N} = \mathcal{M}[\phi_{ij}]$ be a gluing of $\mathcal{M}$ by a cocycle in $\mathcal{Z}^1(\mathcal{U}, \mathcal{G}_Z^{Cn})$. There are canonical isomorphisms of analytical and ringed spaces*

$$\begin{aligned}\rho_{\mathcal{N}}^{[n]} : \mathcal{M}^{[n],Z} &\xrightarrow{\sim} \mathcal{N}^{[n],Z[\phi_{ij}]},\\ \rho_{\mathcal{N}}^{\underline{n}} : \ \mathcal{M}^{\underline{n},Z} &\xrightarrow{\sim} \mathcal{N}^{\underline{n},Z[\phi_{ij}]}.\end{aligned}$$

Loosely speaking, since the cocyle (ϕ_{ij}) is tangent to Id at order n along the divisor $\mathcal{D} \subset \mathcal{M}$, the n^{th} infinitesimal neighboorhood of the glued tree $\mathcal{N} = \mathcal{M}[\phi_{ij}]$ is the same as $\mathcal{M}$ in a canonical way.

2.3.3 The $\mathrm{Glu}_n^C(\mathcal{M}, Z, \mathcal{U})$ *categories.*

Let p be an integer. Let us consider the crossed tree built by a succession of gluings

$$\mathcal{M}[\phi_{ij}^1][\phi_{ij}^2][\ldots][\phi_{ij}^p] \tag{3}$$

where

- $\left(\phi_{ij}^1\right)$ is a 1-cocyle of $\mathcal{G}_Z^n$;
- for $k = 2, \ldots, p$, $\left(\phi_{ij}^k\right) \in \mathcal{Z}^1\left(\mathcal{U}[\phi_{ij}^1]\ldots[\phi_{ij}^{k-1}], \mathcal{G}^{Cn}{}_{Z[\phi_{ij}^1]\ldots[\phi_{ij}^{k-1}]}\right)$ with

$$\mathcal{G}^n_{Z[\phi_{ij}^1]\ldots[\phi_{ij}^{k-1}]} \subset \mathrm{Aut}^C\left(\mathcal{M}[\phi_{ij}^1]\ldots[\phi_{ij}^{k-1}], Z[\phi_{ij}^1]\ldots[\phi_{ij}^{k-1}]\right).$$

Following the proposition (2.1), we have canonical isomorphisms

$$\rho^{[n]}_{\mathcal{M}[\phi_{ij}^1][\phi_{ij}^2][\ldots][\phi_{ij}^p]} : \mathcal{M}^{[n],Z} \xrightarrow{\sim} \mathcal{M}[\phi_{ij}^1][\phi_{ij}^2][\ldots][\phi_{ij}^p][n], Z[\phi_{ij}^1][\phi_{ij}^2][\ldots][\phi_{ij}^p], \tag{4}$$

$$\rho^{\underline{n}}_{\mathcal{M}[\phi_{ij}^1][\phi_{ij}^2][\ldots][\phi_{ij}^p]} : \mathcal{M}^{\underline{n},Z} \xrightarrow{\sim} \mathcal{M}[\phi_{ij}^1][\phi_{ij}^2][\ldots][\phi_{ij}^p]^{\underline{n},Z[\phi_{ij}^1][\phi_{ij}^2][\ldots][\phi_{ij}^p]}. \tag{5}$$

Definition 2.6 *The* $\mathrm{Glu}_n^C(\mathcal{M}, Z, \mathcal{U})$ *category is the category whose objects are crossed trees built as (3) with the data of the isomorphisms (4) and (5). Arrows are biholomorphisms of trees that commute with these isomorphisms. If $\mathcal{M}$ and $\mathcal{N}$ are isomorphic in* $\mathrm{Glu}_n^C(\mathcal{M}, Z, \mathcal{U})$, *we denote*

$$\mathcal{M} \overset{\mathtt{G}_n}{\simeq} \mathcal{N}.$$

From now on, we assume the tree $\mathcal{M}$ to be the support of an isoholonomical deformation $\mathcal{F}$. We are going to define a cobordism property in order to detect on any element of $\mathrm{Glu}_0^C(\mathcal{M}, Z, \mathcal{U})$ the existence of an isoholonomical deformation linked to $\mathcal{F}$ by isoholonomical deformations. The next purpose will be to prove that this property holds for any element of $\mathrm{Glu}_0^C(\mathcal{M}, Z, \mathcal{U})$.

Definition 2.7 (Cross adapted to $\mathcal{F}$) *Let Z be a cross on $\mathcal{M}$. Z is said to be adapted to $\mathcal{F}$ when each component Z_i is either a separatrix of $\mathcal{F}$ or is attached at a regular point of $\mathcal{F}$. In the latter case, Z_i will be transversal to the leaves of the isoholonomical deformation.*

Now, we consider the sheaf $\mathfrak{X}^C_{S,Z}$ over $\mathcal{D}$ which is a subsheaf of the sheaf of holomorphic vector field. A section X of $\mathfrak{X}^C_{S,Z}$ is supposed to vanish on $\pi^{-1}(C)$, to be tangent to $\mathcal{D}$, to the separatrix and to the cross. We also assume that X is *vertical*:

$$D\pi(X) \equiv 0.$$

The sub-sheaf $\mathfrak{X}^C_{\mathcal{F},Z} \subset \mathfrak{X}^C_{S,Z}$ is the sheaf of vector fields tangent to the deformation $\mathcal{F}$. From now on, e^{tX} refers to the flow of the vector field X at time t.

Definition 2.8 (Cobordism) *Let $\mathcal{N}$ be in $\mathrm{Glu}_0^C(\mathcal{M}, Z, \mathcal{U})$. $\mathcal{N}$ is said to be $\mathcal{F}$-cobordant to $\mathcal{M}$ if there exists a finite sequence of 1-cocyles $\left(T^k_{ij}\right)_{k=1,\dots,N}$ such that the two following conditions are verified:*

(1) for any $p = 0, \dots, N-1$, let $\mathfrak{X}^C_{\mathcal{F}_p,Z_p}$ be the sheaf over $\mathcal{D}[e^{T^1_{ij}}][\cdots][e^{T^p_{ij}}]$ of germs of vector field defined as above adapted to the tree $\mathcal{M}[e^{T^1_{ij}}][\cdots][e^{T^p_{ij}}]$, to the foliation $\mathcal{F}_p = \mathcal{F}[e^{T^1_{ij}}][\cdots][e^{T^p_{ij}}]$, and to the cross $Z_p = Z[e^{T^1_{ij}}][\cdots][e^{T^p_{ij}}]$. We assume $\left(T^{p+1}_{ij}\right)$ is a 1-cocycle with values in $\mathfrak{X}^C_{\mathcal{F}_p,Z_p}$.

(2) $\mathcal{N} \overset{\mathtt{G}_0}{\simeq} \mathcal{M}[e^{T^1_{ij}}][\cdots][e^{T^N_{ij}}]$.

We summarize this definition with the following notation:

$$\mathcal{M} \overset{\mathcal{F}_1,Z_1}{\dashrightarrow} \mathcal{M}_2 \overset{\mathcal{F}_2,Z_2}{\dashrightarrow} \cdots \overset{\mathcal{F}_{N-1},Z_{N-1}}{\dashrightarrow} \mathcal{M}_N \overset{\mathtt{G}_0}{\simeq} \mathcal{N}.$$

Here, $\mathcal{N}$ inherit a canonical isoholonomical deformation embedded in an isoholonomical deformation of $\mathcal{F}$. For example if the cobordism is elementary, i.e $N = 1$, the isoholonomic deformation is constructed in the following way

$$\coprod_i \mathcal{F}|_{\mathcal{U}_i} \times \overline{\mathbb{D}} \Big/ (x,t) \sim (e^{(t)T_{ij}}x, t).$$

The glue is well defined for the map $x \mapsto e^{(t)T_{ij}}x$ acts in the local leaves of $\mathcal{F}$.

Let us explain what motivates the introduction of such a formalism. Roughly speaking, the proof of the theorem (2.2) is going to be performed firstly on the infinitesimal neighboorhood of the divisor at any order. The result will be deduced on the whole neighboorhood by a stability kind argument. Hence, the infinitesimal step (2.4.1) is going to be more or less the key of the proof. Now, infinitesimal deformations of a tree are given by cocycles with values in the sheaves of holomorphic vector fields tangent to the divisor of the tree. In view of the process of the proof, we choose a filtration of the sheaves of holomorphic vector fields related to the vanishing order along the divisor. However, this filtration must have a vanishing property at the cohomological level (2.1) to ensure that any infinitesimal deformation of the tree can be lifted-up in an infinitesimal isoholonomic deformation of the foliation. It appears that very few filtration admit such a property. The most natural one in our context is precisely $\left\{\mathfrak{M}_Z^n \mathfrak{X}_{S,Z}^C\right\}_{n\geq 0}$. Now an easy computation in local coordinates shows that the flow of any section of $\mathfrak{M}_Z^n \mathfrak{X}_{S,Z}^C$ is a section of $\mathcal{G}_Z^{C^n}$: this property is the main reason why we introduce these sheaves of automorphisms. Finally, following our strategy, we filtered the category of trees with a family of adapted categories (2.6) and prove the cobordism property by induction with respect to this family. During the induction, the role of the isomorphisms (4) and (5) is critical: they ensure in an intrinsic way a correspondance beetween the geometrical and cohomological relations two trees may have. Actually, if these two isomorphisms are removed of the definition (2.6), then two trees may be isomorphic in the normal sense but not isomorphic in the category.

2.4 Proof of the existence theorem.

2.4.1 First step : the infinitesimal level.

Let $\mathcal{F}_0$ be the isoholonomical deformation over K such that $E^*\mathcal{F}_0 = \mathcal{F}$. In view of the definition , the deformation $\mathcal{F}_0$ is given by a 1-form Ω_0. Let us denote F_0 a reduced equation of the separatrix of $\mathcal{F}_0$. Since the deformation $\mathcal{F}$ is locally trivial, one can reproduce with parameter the computation done in [6] in the non-parameter case to prove the following lemma based upon the generalized curve hypothesis:

Proposition 2.2 *There exists an exact sequence of sheaves*

$$0 \longrightarrow \mathfrak{M}_Z^n \mathfrak{X}_{\mathcal{F},Z}^C \longrightarrow \mathfrak{M}_Z^n \mathfrak{X}_{S,Z}^C \xrightarrow{E^*\Omega_0(.)} \mathfrak{M}_Z^{C^n}(F_0 \circ E) \longrightarrow 0$$

where $(F_0 \circ E)$ *is the sub-sheaf of* $\mathcal{O}_{\mathcal{M}}$ *generated by the function* $F_0 \circ E$ *and* $\mathfrak{M}_Z^{C^n}$ *the sheaf of ideals* $\mathfrak{M}_Z^n \cdot \mathcal{O}(-(\pi \circ E)^{-1}C)$

In order to establish an equivalent of the infinitesimal cobordism result in [6] in the present parameter context, we only have to show the

Lemma 2.1

$$H^1(\mathcal{D}, \mathfrak{M}_Z^{C^n}) = 0.$$

Proof: Let us consider $\mathcal{M}_{\mathcal{W}}$ a neighborhood of $\mathcal{D}$ and the associted fibration

$$\Pi : \mathcal{M}_{\mathcal{W}} \longmapsto \mathcal{K} = \pi \circ E$$

where $K \subset \mathcal{K}$ is a Stein open set. The spectral sequence of [7] associated to the sheaf $\mathfrak{M}_Z^n$ and the fibration Π induce an exact sequence

$$H^1\left(\mathcal{K}, \Pi_* \mathfrak{M}_Z^{\mathcal{C}\,n}\right) \to H^1\left(\mathcal{M}_{\mathcal{W}}, \mathfrak{M}_Z^{\mathcal{C}\,n}\right) \to H^0\left(\mathcal{K}, \mathcal{R}^1\Pi_* \mathfrak{M}_Z^{\mathcal{C}\,n}\right) \to H^2\left(\mathcal{K}, \Pi_* \mathfrak{M}_Z^{\mathcal{C}\,n}\right).$$

Since π is proper and $\mathfrak{M}_Z^{\mathcal{C}}$ coherent, $\Pi_* \mathfrak{M}_Z^{\mathcal{C}\,n}$ is a coherent sheaf [8]. As $\mathcal{K}$ is Stein, each extremal term of the sequence vanishes [9]. Hence, we have

$$H^1\left(\mathcal{M}_{\mathcal{W}}, \mathfrak{M}_Z^{\mathcal{C}\,n}\right) \simeq H^0\left(K, \mathcal{R}^1\Pi_* \mathfrak{M}_Z^{\mathcal{C}\,n}\right). \tag{6}$$

The fiber of the derived sheaf satisfies:

$$\left(\mathcal{R}^1\Pi_* \mathfrak{M}_Z^{\mathcal{C}\,n}\right)_x \simeq H^1\left(\Pi^{-1}(x), \mathfrak{M}_Z^{\mathcal{C}\,n}|_{\Pi^{-1}(x)}\right). \tag{7}$$

Let us denote by $\mathfrak{M}_x \subset \left(\mathcal{O}_{\mathcal{K}}\right)_x$ the ideal of germ of function vanishing at x. Thanks to a distinguished covering of $\Pi^{-1}(x)$, one can see that

$$H^1\left(\Pi^{-1}(x), \mathfrak{M}_Z^{\mathcal{C}\,n}\right) \otimes_{\mathcal{O}_{\mathcal{K}x}} \mathcal{O}_{\mathcal{K}x}/_{\mathfrak{M}_x} \simeq H^1\left(\Pi^{-1}(x), \mathfrak{M}_Z^{\mathcal{C}\,n} \otimes_{\mathcal{O}_{\mathcal{K}x}} \mathcal{O}_{\mathcal{K}x}/_{\mathfrak{M}_x}\right).$$

Let i_x be the embedding $\Pi^{-1}(x) \subset \mathcal{M}$; a simple local computation ensures that

$$\mathfrak{M}_Z^{\mathcal{C}\,n} \otimes_{\mathcal{O}_{\mathcal{K}x}} \mathcal{O}_{\mathcal{K}x}/_{\mathfrak{M}_x} \simeq i_x^* \mathfrak{M}_Z^{\mathcal{C}\,n}.$$

Now, in view of the non-parameter computation in [6], the cohomology of $i_x^* \mathfrak{M}_Z^{\mathcal{C}}$ satisfies

$$H^1\left(\Pi^{-1}(x), i_x^* \mathfrak{M}_Z^{\mathcal{C}\,n}\right) = 0.$$

Hence, one gets

$$H^1\left(\Pi^{-1}(x), \mathfrak{M}_Z^{\mathcal{C}\,n}\right) \otimes_{\mathcal{O}_{\mathcal{K}x}} \mathcal{O}_{\mathcal{K}x}/_{\mathfrak{M}_x} = 0.$$

The Nakayama's lemma [10] and the relation (7) ensure that the derived sheaf $\mathcal{R}^1\Pi_* \mathfrak{M}_Z^{\mathcal{C}\,n}$ is the trivial sheaf. Using (6), we have

$$H^1\left(\mathcal{M}_{\mathcal{W}}, \mathfrak{M}_Z^{\mathcal{C}\,n}\right) = 0.$$

Finally, the lemma is obtained by taking the inductive limit on a familly of Stein neighborhood of K. □

The long exact sequence of cohomology associated to the short sequence (2.2) and to the covering $\mathcal{U}$ give us the infinitesimal cobordism property

Proposition 2.3 (Infinitesimal cobordism) *The canonical map*

$$H^1\left(\mathcal{D}, \mathfrak{M}_Z^n \mathfrak{X}_{\mathcal{F},Z}^C\right) \longrightarrow H^1\left(\mathcal{D}, \mathfrak{M}_Z^n \mathfrak{X}_{S,Z}^C\right)$$

is onto.

2.4.2 Second step : cobordism in $\mathrm{Glu}_1^C(\mathcal{M}, Z, \mathcal{U})$.

This section is devoted to prove the following proposition:

Proposition 2.4 *Any element of* $\mathrm{Glu}_1^C(\mathcal{M}, Z, \mathcal{U})$ *is elementary* $\mathcal{F}$*-cobordant to* $\mathcal{M}$.

The proof performed in [6] using an algorithm of Newton type can be reproduced in the $\mathrm{Glu}_1^C(\mathcal{M}, Z, \mathcal{U})$ category. Basically, we write $\mathcal{N} = \mathcal{M}[\phi_{ij}] \in \mathrm{Glu}_1^C(\mathcal{M}, Z, \mathcal{U})$ and *normalized* the n-jet of the cocycle (ϕ_{ij}) for n big enough using an induction and (2.3). The last argument is a stability property for neighboorhoods of exceptional divisors. Hence, we have only to establish an equivalent of the latter result in our context.

Proposition 2.5 (Stability property) *For n big enough, any tree in the image of the natural embedding*

$$\mathrm{Glu}_n^C(\mathcal{M}, Z, \mathcal{U}) \hookrightarrow \mathrm{Glu}_1^C(\mathcal{M}, Z, \mathcal{U})$$

is isomorphic to $\mathcal{M}$ *in the category* $\mathrm{Glu}_1^C(\mathcal{M}, Z, \mathcal{U})$.

Roughly speaking, if the cocyle ϕ_{ij} is tangent enough to Id along the divisor, then the trees $\mathcal{M}$ and $\mathcal{M}[\phi_{ij}]$ are isomorphic in the category $\mathrm{Glu}_1^C(\mathcal{M}, Z, \mathcal{U})$.

Proof: Let p and n be integers and $\mathcal{N} = \mathcal{M}[\phi_{ij}]$ with (ϕ_{ij}) in $\mathcal{Z}^1\left(\mathcal{U}, \mathrm{Aut}_p^C(\mathcal{M}, Z)\right)$. For p big enough, the image of the natural morphism

$$H^1(\mathcal{D}, \mathrm{Aut}_p(\mathcal{M}, Z)) \to H^1(\mathcal{D}, \mathrm{Aut}_n(\mathcal{M}, Z))$$

is trivial [16]. Since $\mathrm{Aut}_p^C(\mathcal{M}, Z) \subset \mathrm{Aut}_p(\mathcal{M}, Z)$, one can take a trivialisation

$$\phi_{ij} = \phi_i \circ \phi_j^{-1},\ (\phi_i) \in \mathcal{Z}^0\left(\mathcal{U}, \mathrm{Aut}_n(\mathcal{M}, Z)\right).$$

By definition of the sheaf $\mathrm{Aut}_p^C(\mathcal{M}, Z)$, the restricted components of the cocycle satisfy

$$\phi_i|_{\pi^{-1}(C)} = \phi_j|_{\pi^{-1}(C)}.$$

Hence, the familly (ϕ_i) defines a global section over C of $\mathrm{Aut}_n(\mathcal{M}, Z)$. This section induces a germ of biholomorphism ϕ along $0 \times C \subset \mathbb{C}^2 \times \mathbb{C}^p$ which commutes with the projection and lets fixed each point of the cross. In some adapted coordinates $(x, y, t),\ (x, y) \in (\mathbb{C}^2, 0),\ t \in K$, ϕ is written

$$\phi(x, y, t) = (x, y, t) + H(x, y, t)\left(\sum_{i,j\geq\nu} a_{ij}(t)x^i y^j, \sum_{i,j\geq\nu} b_{ij}(t)x^i y^j, 0\right)$$

where H is a reduced equation of the cross over K and a_{ij}, b_{ij} holomorphic functions on C. Since K is Stein, in view of [5] there exist holomorphic functions A_{ij} and B_{ij} on K such that

$$A_{ij}|_C = a_{ij} \quad \text{et} \quad B_{ij}|_C = b_{ij}.$$

Then, the map

$$\Phi(x,y,t) = (x,y,t) + H(x,y,t)\left(\sum_{i,j} A_{ij}(t)x^i y^j, \sum_{i,j} B_{ij}(t)x^i y^j, 0\right) \tag{8}$$

is a germ of automorphism along $0 \times K \subset \mathbb{C}^2 \times K$, which extends ϕ, commutes with the projection and fixes the cross. Morevover, if one chooses $A_{ij} = 0$ et $B_{ij} = 0$ as soon as $a_{ij} = 0$ and $b_{ij} = 0$, then the tangency order to the identity of the extension is the same as ϕ. Hence, for p big enough, the biholomorphism (8) can be lifted up in a global section of $\mathrm{Aut}_n(\mathcal{M}, Z)$ over K with $\Phi|_C = \phi$. Hence, the 0-cocycle $(\psi_i) = (\phi_i \circ \Phi^{-1}|U_i)$ is a trivialisation of (ϕ_{ij}) with values in $\mathrm{Aut}_n^C(\mathcal{M}, Z)$.

Now, since K is Stein, the curve Z_0 defining the cross $Z = E^*Z_0$ can be straightened along $0 \times K$ and in some coordinates (x,y,t), $(x,y) \in \mathbb{C}^2, t \in K$, Z_0 admits $xy = 0$ for equation. The latter coordinates induce two canonical systems of coordinates along the components of $Z = Z_1 \cup Z_2$. The total morphism E is now written $E(x_1, y_1, t) = (x_1, y_1 x_1^{N_1}, t)$ and $E(x_2, y_2, t) = (x_2 y_2^{N_2}, y_2, t)$. Let us denote by ψ_1 and ψ_2 the components of (ψ_i) defined on the open set of the covering which contains the components of Z. Since each component of Z_0 is smooth, the automorphisms ψ_1 and ψ_2 can be written

$$\begin{aligned}
\psi_1(x_1,y_1,t) &= (x_1 + x_1^n y_1 U_1(x_1,y_1,t), y_1 + x_1^n y_1 V_1(x_1,y_1,t), t) \\
\psi_2(x_2,y_2,t) &= (x_2 + y_2^n x_2 U_2(x_2,y_2,t), y_2 + y_2^n x_2 V_2(x_2,y_2,t), t)\,.
\end{aligned}$$

Let ψ be the germ of biholomorphism along $0 \times K$ defined by

$$\psi(x,y,t) = (x(1 + y^n U_2(0,y,t)), y(1 + x^n V_1(x,0,t)), t)\,.$$

For n big enough, Ψ can be lifted up on $\mathcal{M}$ in an automorphism Ψ that fixes each point of $\mathcal{D}$ and Z. Now, if one takes a closed look to the expression of ψ, one can verify that, for any point x on $\mathcal{D}$ and any component ψ_i of (ψ_i) defined near x,

$$(\mathfrak{J}_1)_x(\psi_i) = (\mathfrak{J}_1)_x(\Psi).$$

Hence, the 0-cocycle $(\psi_i \circ \Psi^{-1}|_{U_i})$ is a trivialisation of (ϕ_{ij}) in $\mathcal{G}_Z^1$. Since, the tree $\mathcal{N}$ is the gluing $\mathcal{M}[\phi_{ij}]$, $\mathcal{N}$ is isomorphic to $\mathcal{M}$ in the category $\mathrm{Glu}_1^C(\mathcal{M}, Z, \mathcal{U})$. $\square$

2.4.3 *Third step: cobordism in* $\mathrm{Glu}_0^C(\mathcal{M}, Z, \mathcal{U})$.

The cobordism in $\mathrm{Glu}_0^C(\mathcal{M}, Z, \mathcal{U})$ is related to the following result:

Proposition 2.6 *Any element in* $\mathrm{Glu}_0^C(\mathcal{M}, Z, \mathcal{U})$ *is* $\mathcal{F}$*-cobordant to* $\mathcal{M}$.

The proof done in [6] for the category $\mathrm{Glu}_0(\mathcal{M}, Z, \mathcal{U})$ can be repeated here without any change. The main tools are an induction on the height of the trees and the cobordism result for $\mathrm{Glu}_1^C(\mathcal{M}, Z, \mathcal{U})$.

2.4.4 *Fourth step: preparation of a cocyle.*

Let $(\mathcal{M}', \mathcal{D}', \Sigma', \pi')$ be a tree over K topologically equivalent to $(\mathcal{M}, \mathcal{D}, \Sigma, \pi)$. We suppose that over C the trees $\mathcal{M}'|_C$ and $\mathcal{M}|_C$ are conjugated.

Proposition 2.7 *There exists an isoholonomic deformation* $\mathcal{F}'$ *on* $\mathcal{M}'$ *such that* $\mathcal{F}'|_C$ *and* $\mathcal{F}|_C$ *are analytically equivalent and* $\mathcal{F}$ *and* $\mathcal{F}'$ *are embedded in an isoholonomic deformation over* $K \times \overline{\mathbb{D}}$ *which is trivial above* C *in the sense of the theorem (1.1).*

In order to prove the latter proposition, we prepare a 1-cocycle such that the tree $\mathcal{M}'$ becomes a gluing of $\mathcal{M}$ in a category $\mathrm{Glu}_0(\mathcal{M}, Z, \mathcal{U})$. This cocycle must well behave with respect to the condition $\mathcal{M}'|_C \simeq \mathcal{M}|_C$.

In view of [19], one can find a first isoholonomic deformation $\tilde{\mathcal{F}}$ over $\overline{\mathbb{D}} \times K$ such that:

(1) $\tilde{\mathcal{F}}|_{-1\times K}$ is equal to $\mathcal{F}$,
(2) $\tilde{\mathcal{F}}|_{1\times K}$ is a deformation on a tree $(\tilde{\mathcal{M}}, \tilde{\mathcal{D}}, \tilde{\Sigma}, \tilde{\pi})$ with $\mathcal{D}'$ and $\tilde{\mathcal{D}}$ analytically equivalent,
(3) the deformations $\tilde{\mathcal{F}}|_{1\times C}$ and $\mathcal{F}|_C$ are analytically equivalent.

Notice that, even if the divisors $\mathcal{D}'$ and $\tilde{\mathcal{D}}$ are analytically equivalent, there is no reason for the trees $\mathcal{M}'$ and $\tilde{\mathcal{M}}$ to be also analytically equivalent. Let us denote by θ a conjugacy between $\mathcal{D}'$ and $\tilde{\mathcal{D}}$ and ϕ a conjugacy between $\mathcal{M}'|_C$ and $\tilde{\mathcal{M}}|_C$. Since K is Stein, the tubular neighborhood of any irreducible component of $\mathcal{D}$ is a trivial deformation over K. Therefore, for such any component D, there exists a biholomorphism Θ_D from a tubular neighborhood $T(D)$ of D to a tubular neighborhood of $\theta(D)$ extending $\theta|_D$ such that $\Theta_D(\tilde{\mathcal{D}}) = \mathcal{D}'$. The automorphism of $T(D)|_C$ defined by

$$\Theta_D|_C \circ \phi^{-1}|_{T(D)}$$

is well-defined on a neigborhood of $D|_C$, lets invariant each component of $\mathcal{D}|_C$ and commutes with the projection.

Lemma 2.2 *There exists an automorphism Φ_D of a neighborhood of D extending $\Theta_D|_C \circ \phi^{-1}|_{T(D)}$ over K, which lets invariant $\mathcal{D}$ and commutes with π.*

Proof: Let us denote by h_D the automorphism $\Theta_D|_C \circ \phi^{-1}|_{T(D)}$. Since K is Stein, there exists a system of coordinates $(x,t,s),\ x \in \mathbb{P}^1,\ t \in (\mathbb{C},0),\ s \in K$ in a neighborhood of D such that: $\{x=0\}$ is a local equation of D; the components of $\mathcal{D}$ transversal to D have equations of the form $\{t = f_i(s), t=\infty\}_{i=1,\ldots,N}$; the fibration π is $\pi : (x,t,s) \mapsto s$. In view of all its properties, h_D is written

$$(x,t,s) \mapsto \left(xA(x,t,s), t + xB(x,t,s) \prod_{i=1,\ldots,N} (t - f_i(s)), s \right), \quad s \in C$$

where A and B are holomorphic functions. Since h_D is a germ of automorphism, we have $A(0,t,s) \neq 0$. Moreover as h_D is global and extendable along $\{t=\infty\}$, if A and B are written

$$A(x,t,s) = \sum_{ij} a_{ij}(s) x^i t^j, \quad B(x,t,s) = \sum_{ij} b_{ij}(s) x^i t^j, \quad s \in C$$

the functions a_{ij} and b_{ij} vanish as soon as $ip - j < 0$, p refering to the self-intersection of D. Particulary, we find $A(0,t,s) = a_{00}(s)$ for any s,t. Since K is Stein and compact, a_{00} can be extended in a non-vanishing holomorphic function on K and any other function a_{ij} or b_{ij} can be extended too. If one carefully chooses to extend by the zero function as soon as a_{ij} or b_{ij} is the zero function, one gets an extension on h_D satisfying all the properties. □

Now for any component D, let us consider $\Lambda_D = \phi_D^{-1} \circ \Theta_D$. If we glue the familly of tubular neighborhoods with respect to the familly of automorphisms $\Lambda_{DD'} = \Lambda_D^{-1} \circ \Lambda_{D'}$, we find:

$$\mathcal{M}' \simeq \coprod_{D \in \mathrm{Comp}(\tilde{\mathcal{D}})} T(D) \Big/ (x \sim \Lambda_{DD'}(x))_{(D,D') \in \mathrm{Comp}(\tilde{\mathcal{D}})^{\check{2}}}, \tag{9}$$

$$\Lambda_{DD'}|_C = \mathrm{Id}. \tag{10}$$

From now on, we make two operations on the familly $(\Lambda_{DD'})$ which leads us to a 1-cocyle taking its values in $\mathcal{G}^{C^0}{}_Z$. Using an analogous with parameter of the lemma (3.3) in [6] , one can first suppose $\Lambda_{DD'}$ to be tangent to the identity along the singular locus $D \cap D'$. Then, by taking a distinguished covering $\mathcal{U}$ of $\mathcal{D}$ finer than the tubular neighborhood as in (3.2) of [6], one can build an element (ϕ_{ij}) related to $\Lambda_{DD'}$, which belongs to $\mathcal{Z}^1(\mathcal{U}, \mathcal{G}^{C^0}{}_Z)$ for a cross Z well chosen. Moreover, in this construction, one keeps the property $\mathcal{M}' \simeq \tilde{\mathcal{M}}[\phi_{ij}]$. Hence, the tree $\mathcal{M}'$ is conjugated to an element of $\mathrm{Glu}_0(\mathcal{M}, Z, \mathcal{U})$. Therefore, the proposition (2.7) is a consequence of (2.6).

2.4.5 Fifth and last step: a finite determinacy argument.

If E is the total morphism of a blowing-up process, let us denote by $\mathrm{Att}(E,S)$ the locus of intersection between the exceptional divisor and the strict transform of S by E. Let us prove now the theorem (2.2). Let the tree $(\mathcal{M}^1,\mathcal{D}^1,\Sigma^1,\pi^1)$ be the top of the desingularization process of the equisingular family S^1. For n, let us denote by $(\mathcal{M}^1_n,\mathcal{D}^1_n,\Sigma^1_n,\pi^1_n)$ the top of the process given by the following diagram

$$\begin{array}{ccccc} \mathcal{M}^1_n & \stackrel{E^n}{\to}\ldots & \mathcal{M}^1_j & \stackrel{E^j}{\to}\ \ldots \stackrel{E^1}{\to} & \mathcal{M}^1 \\ \cup & & \cup & & \cup \\ \mathrm{Att}(E_n,S^1) & \to \ldots & \mathrm{Att}(E_j,S^1) & \to\ \ldots \to & S^1 \\ \cup & & \cup & & \cup \\ \mathrm{Att}(E_n,S^1) & \to \ldots & \mathrm{Att}(E_j,S^1) & \to\ \ldots \to & S^1 \end{array} \qquad (11)$$

where E_j refers to the total morphism of $\mathcal{M}^1_{j-1}$. For any integer n, the proposition (2.7) ensures the existence of an isoholonomic deformation $\mathcal{F}^1_n$ on $\mathcal{M}^1_n$ such that $\mathcal{F}^1_n|_C$ and $\mathcal{F}^0|_C$ are analytically equivalent. For n big enough, one can suppose that each component of $\mathcal{D}^1_n$ meets at most one irreducible component of $E^*_nS^1$. Moreover, S^1 and S^0 are topologically equivalent as well as $\mathcal{F}^1_n$ and $E^*_n\mathcal{F}^0$. Hence, each component of $\mathcal{D}^1_n$, which meets a component of $E^*_nS^1$, meets exactly one component of the separatrix of $\mathcal{F}^1_n$. In view of [12], a property of finite determinacy ensures that for n big enough, the separatrix of $\mathcal{F}^1_n$ and $E^*_nS^1$ are analytically equivalent. Therefore, $\mathcal{F}^1_n$ can be pulled down in an isoholonomical deformation $\mathcal{F}^1$ over K satisfiying $\mathrm{Sep}(\mathcal{F}^1)=S^1$ and $\mathcal{F}^1|_C\simeq\mathcal{F}^0|_C$. Moreover, by construction, $\mathcal{F}^0$ and $\mathcal{F}^1$ are embedded in some isoholonomical deformations over $K\times\overline{\mathbb{D}}$ satisfying the checked properties.

3 Quasi-homogeneous foliations.

3.1 Desingularization of quasi-homogeneous foliations.

We are interested first in the desingularization of quasi-homogeneous foliation of generalized curve type.

Let Ω be an open set in $\mathbb{C}^2$, p a point in Ω and S a germ of smooth curve. For any proper morphism $E:X\to\Omega$, we call the *strict transform of S by E* the closure in X of the analytical set $E^{-1}(S\backslash 0)$. The intersection of the strict transform of S and the divisor $E^{-1}(0)$ is called the attaching locus of S with the respect to E.

We define a special morphism with a finite number of successive blowing-up centered at point $\mathfrak{E}_d^n(p,S) : (\mathcal{M}_d^n, \mathcal{D}_d^n) \to (\Omega, p)$ by

$$\mathfrak{E}_d^n(p,S) = E_1 \circ \cdots \circ E_n$$

where E_i is the blowing-up centered at the attaching locus of S with respect to $E_1 \circ \cdots \circ E_{i-1}$. The exceptional divisor $\mathcal{D}_d^n = \mathfrak{E}_d^n(p,S)^{-1}(0)$ is a chain of n irreducible components $\{D_i\}_{1\leq i\leq n}$ such that

$$(E_1 \circ \ldots E_i)^{-1}(p) = D_1 \cup \ldots \cup D_i.$$

The component D_i is called the i-component of $\mathfrak{E}_d^n(p,S)$

Let f be a reduced quasi-homogeneous polynomial function defined in some coordinates (u,v)

$$f = \left\{ \sum_{\alpha i + \beta j = \gamma} a_{ij} u^i v^j = 0 \right\}, \quad \alpha, \beta, \gamma \in \mathbb{N}^*, \quad \mathrm{pgcd}(\alpha,\beta) = 1, \ \alpha < \beta.$$

Let us write the Euclide algorithm for the couple (α,β):

$$r_0 = \beta, \ r_1 = \alpha \quad \begin{cases} r_0 = q_1 r_1 + r_2 \\ \quad \ldots \\ r_i = q_{i+1} r_{i+1} + r_{i+2} \\ \quad \ldots \\ r_N = q_{N+1} r_{N+1} + 0 \end{cases} . \tag{12}$$

We are now able to give a precise description of the morphism of desingularization of f and the following classical result:

Proposition 3.1 *The morphism of desingularization of $f^{-1}(0)$ is the composition*

$$\mathfrak{E}_d^{q_1}(p_1,S_1) \circ \mathfrak{E}_d^{q_2}(p_2,S_2) \circ \cdots \circ \mathfrak{E}_d^{q_{N+1}}(p_{N+1},S_{N+1}) \tag{13}$$

where

(1) $p_1 = 0, \ S_1 = \{v = 0\}$.
(2) p_{i+1} is the intersection point of the (q_i)-component and the $(q_i - 1)$-component of $\mathfrak{E}_d^{q_i}(p_i,S_i)$.
(3) S_{i+1} is the germ of smooth curve defined by the q_{i-1}-component of $\mathfrak{E}_d^{q_i}(p_i,S_i)$.

Proof: The proof is an induction on the length of the Euclid algorithm. If the length is 1, since α and β are relatively prime, the algorithm is reduced to

$$r_0 = \beta, \ r_1 = \alpha = 1 \quad r_0 = r_0 \times r_1.$$

In the coordinates (u,v), the morphism $\mathfrak{E}_d^{r_0}(0,\{v=0\})$ is locally written

$$\mathfrak{E}_d^{r_0}(0,\{v=0\})(u_1,v_1)=(u_1,v_1u_1^{r_0}).$$

Hence, the pull-back function $\mathfrak{E}_d^{r_0}(0,\{v=0\})^*f$ is expressed by

$$\mathfrak{E}_d^{r_0}(0,\{v=0\})^*f(u_1,v_1)=\sum_{i+r_0j=\gamma}a_{ij}u_1^{i+r_0j}v_1^j=u_1^\gamma\sum_{i+r_0j=\gamma}a_{ij}v_1^j$$

The curve $\sum_{i+r_0j=\gamma}a_{ij}v_1^j$ is the union of a finite number of smooth curves transversal to the exceptional divisor. One can verify that in any other canonical coordinates for the blowing-up morphism $\mathfrak{E}_d^{r_0}(0,\{v=0\})$, the function $\mathfrak{E}_d^{r_0}(0,\{v=0\})^*f$ does not vanish along any curve transversal to the exceptional divisor except maybe on the first component: the latter case only occurs when the axe $\{u=0\}$ in the considered coordinates is an irreducible component of $f^{-1}(0)$. In any case, the curve $f^{-1}(0)$ is desingularized by $\mathfrak{E}_d^{r_0}(0,\{v=0\})$. If the Euclid algorithm is of length n, using the notation of (12), one considers the morphism $\mathfrak{E}_d^{q_1}(p_1,S_1)$. As above, the pull-back equation is written

$$\begin{aligned}f_1(u_1,v_1)=\mathfrak{E}_d^{q_1}(0,\{v=0\})^*f(u_1,v_1)&=\sum_{\alpha i+\beta j=\gamma}a_{ij}u_1^{i+q_1j}v_1^j\\&=\sum_{\alpha(i+q_1j)+r_2j=\gamma}a_{ij}u_1^{i+q_1j}v_1^j.\end{aligned}$$

To conclude, one observes that the last equation defines a quasi-homogeneous curve of weight (α,r_2). Since the Euclid algorithm for the couple (α,r_2) is of length $n-1$, the induction hypothesis ensures that the desingularization of the latter curve admits a description of type (13). As the desingularization of $f^{-1}(0)$ is the composition of $\mathfrak{E}_d^{q_1}(0,\{v=0\})$ and of the desingularization of $f_1^{-1}(0)$, the proposition is proved. □

Now, if $\mathcal{F}$ is a quasi-homogeneous curve of generalized curve type, its desingularization admits the same description as (13). In particular, $\mathcal{F}$ has a dual tree of the form depicted on Figure 2 : the non-extremal vertex with some ar-

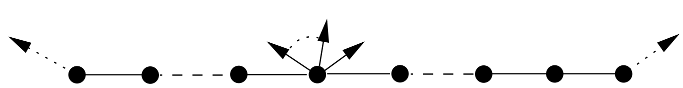

Fig. 2. Dual tree of a quasi-homogeneous foliation

rows corresponds to the central component. The extremal components carry an arrow if and only if, in the (u,v) coordinates, at least one of the axes is an irreducible component of the separatrix.

In order to prove that there is no *transversal obstruction* for quasi-homogeneous foliations with same holonomy to be conjugated and in view of the cohomological interpretation (2.1), we first recall some classical facts about automorphisms of reduced singularities.

Let $\mathcal{F}$ be a germ of reduced singularity with two non-vanishing eigen-values. In some coordinates [15], the foliation $\mathcal{F}$ is given by an holomorphic 1-form ω where

$$\omega = x(1 + A(x, y))d + y(\lambda + B(x, y))dy \qquad (14)$$

with $A(0, 0) = B(0, 0) = 0$. Hence, the axes $\{x = 0\}$ and $\{y = 0\}$ are both separatrix. In fact, these are the sole separatrix. Let us denote by S and S' the respective axes. For any open set U in S, the notation $\mathrm{Aut}(\mathcal{F}, U)$ refers to the group of local automorphism germs ϕ along U, which let globally invariant the foliation and satisfy

$$\phi|_U = \mathrm{Id}.$$

Precisely, a germ of automorphism ϕ is in $\mathrm{Aut}(\mathcal{F}, U)$ if and only if:

$$\phi^*\omega \wedge \omega = 0, \quad \phi|_U = \mathrm{Id}.$$

Let us denote by $\mathrm{Fix}(\mathcal{F}, U)$ the sub-group of $\mathrm{Aut}(\mathcal{F}, U)$, which lets invariant each local leaf. In [1], D. Cerveau and R. Meziani call $\mathrm{Aut}(\mathcal{F}, U)$ the *isotropy group* of the singularity. They give a description of all elements in $\mathrm{Fix}(\mathcal{F}, U)$ when U is a neighborhood of 0 in the separatrix S. To be more specific, any element of $\mathrm{Fix}(\mathcal{F}, U)$ can be written

$$(x, y) \mapsto e^{(\tau(x,y))X}$$

Here, X is a germ of tangent vector field and $\tau(x, y)$ a germ of holomorphic function vanishing along $\{x = 0\}$. The notation $e^{(t)X}$ refers to the flow of X at time t. This kind of description persists if one considers a punctured neighborhood of 0 in S or even any corona around 0.

Let T be a germ of curve transversal to S and $\mathrm{Hol}_T \in \mathrm{Diff}(T, T \cap S)$ be the holonomy automorphism of $\mathcal{F}$ computed on T. Let U be either a neighborhood of 0 in S or a corona around 0 in S such that $T \cap S \in U$. Let ϕ be any element of $\mathrm{Aut}(\mathcal{F}, U)$. In a small neigborhood of $T \cap S$, the foliation $\mathcal{F}$ can be straightened: precisely, there exists local coordinates (u, v) such that $S = \{v = 0\}$, $T = \{u = 0\}$ and $\mathcal{F} = \{v = \mathrm{cst}\}$. The curve $\phi(T)$ is transversal to S and meets S at $S \cap T$. Hence, there exists a germ of biholomorphism ρ_ϕ such that

$$\rho_\phi : (0, v) \in (T, S \cap T) \mapsto (\alpha(v), v) \in (\phi(T), S \cap T).$$

where α is a germ of holomorphic function with $\alpha(0) = 0$ and $\alpha'(0) \neq 0$. By definition, the holonomy map $\mathrm{Hol}_{\phi(T)}$ computed as a germ in $\mathrm{Diff}(\phi(T), T \cap S)$ satisfies the following relation

$$\mathrm{Hol}_T = \rho_\phi^{-1} \circ \mathrm{Hol}_{\phi(T)} \circ \rho_\phi$$

Let us consider a path γ in the leaf passing through $(0, v)$ from the point $(0, v)$ to the point $(0, \mathrm{Hol}_T(v))$ making exactly one turn around S'. Since ϕ is the identity when restricted to S, for v small enough, $\phi(\gamma)$ is a path in the leaf from $(0, \phi(v))$ to $(0, \phi(\mathrm{Hol}_T(v)))$ making exactly one turn around S' with the same orientation as γ. Hence, by definition of the holonomy,

$$\mathrm{Hol}_{\phi(T)}(\phi(v)) = \phi(\mathrm{Hol}_T(v)).$$

With the above relations, we find

$$(\rho_\phi^{-1}\phi) \circ \mathrm{Hol}_T = \mathrm{Hol}_T \circ (\rho_\phi^{-1}\phi)$$

Hence, we build a morphism defined by

$$\phi \in \mathrm{Aut}(\mathcal{F}, U) \longmapsto \left[\rho_\phi^{-1}\phi\right] \in \mathrm{Cent}(\mathrm{Hol}_T)/ < \mathrm{Hol}_T > .$$

It is a morphism of groups with values in the quotient of the centralisator $\mathrm{Cent}(\mathrm{Hol}_T)$ by the abelian sub-group generated Hol_T. Moreover, we have the following result:

Lemma 3.1 *If U is a small enough neighborhood of the singularity or a corona around it then the following sequence*

$$0 \longrightarrow \mathrm{Fix}(\mathcal{F}, U) \longrightarrow \mathrm{Aut}(\mathcal{F}, U) \longrightarrow \mathrm{Cent}(\mathrm{Hol}_T) /_{<\mathrm{Hol}_T>} \longrightarrow 0 \qquad (15)$$

is exact.

In the formal context, since one can define the holonomy of a transversally formal foliation along an irreducible component of the divisor, the lemma (3.1) can be reproduced by using transversally formal vocabulary.

3.3 Proof of the classification result (1.1).

One can clearly suppose that $\mathcal{F}_0$ and $\mathcal{F}_1$ are two quasi-homogeneous generalized curves with

$$\mathrm{Sep}(\mathcal{F}_0) = \mathrm{Sep}(\mathcal{F}_1) = S.$$

As each foliation is of generalized curve type, their desingularizations are both equal to the desingularization of S [2]. Let us denote by $E : (\mathcal{M}, \mathcal{D}) \to (\mathbb{C}^2, 0)$ the morphism of desingularization where $\mathcal{D}$ refers to the exceptional divisor

$E^{-1}(0)$. Since the projective holonomy of $E^*\mathcal{F}_0$ and $E^*\mathcal{F}_1$ over the central component D_0 are conjugated, the restricted foliation $E^*\mathcal{F}_0$ and $E^*\mathcal{F}_1$ on a tubular neighborhood of D_0 are conjugated: in order to prove the latter fact, let us consider a curve T_0 transversal to D_0, on which one computes the projective holonomy representations. Let the set $\{s_1, \ldots, s_n\}$ refer to the singularities of $E^*\mathcal{F}_0$ along D_0. In view of the hypothesis, there exists a germ of automorphism $h_0 : T_0 \to T_0$ so that for any $[\gamma] \in \Pi_1(D_0 \backslash \{s_1, \ldots, s_n\})$ and $x \in T_0$, one has

$$h_0([\gamma]_{\mathcal{F}_0} x) = [\gamma]_{\mathcal{F}_1} h_0(x) \tag{16}$$

where $[\gamma]_{\mathcal{F}_0}$ and $[\gamma]_{\mathcal{F}_1}$ refer to the image of the path γ through the respective projective holonomy representations. In view of the desingularization process of $\mathcal{F}_0$, there exists a fibration π over D_0, for which the irreducible components of the strict transform of S, which are attached to D_0, and the two components of $\mathcal{D}$ transversal to D_0 are some fibers. Let us call them the *special* fibers. One can choose the fibration π such that any fiber different from the special fibers, is transversal to the leaves of the foliations $E^*\mathcal{F}_0$ and $E^*\mathcal{F}_1$, at least on a little neighborhood of D_0. Let us choose for T_0 any fiber of π different from the special fibers. For any point x in a neighborhood of D_0 deprived of the special fibers, one can consider a path $\gamma(t),\ t \in [0,1]$, which links x to some point of T_0 in the leaf. The point $H_0(x)$ is now defined as the extremity of the lifting path in the leaf passing by $\gamma(1)$ with respect to the fibration π. The relation (16) ensures that this construction does not depend on the path γ; hence, $x \to H_0(x)$ is well defined. Moreover, by construction H_0 is bounded near the special fibers. Hence, H_0 can be holomorphically extended on a neighborhood of D_0. One can check that H_0 sends any local leaf of $E^*\mathcal{F}_0$ on a local leaf of $E^*\mathcal{F}_1$. We can observe that, by construction, the restriction of H_0 on the component D_0 is the identity.

Let us denote by D_1 an irreducible component of the divisor meeting D_0 at the point s_{01}. Since D_1 has only two singular points, the holonomy representations are morphisms of the form

$$k \in \mathbb{Z} = \Pi_1\Big(D_1 \backslash \mathrm{Sing}(\mathcal{D})\Big) \to [\gamma_i]^k \in \mathrm{Diff}(\mathbb{C}, 0)$$

where γ_i is the holonomy of a path in D_1 making one turn around D_0 for the foliation $E^*\mathcal{F}_i$. As the two foliations are analytically equivalent near s_{01}, the holonomy maps $[\gamma_0]$ and $[\gamma_1]$ are conjugated by an interior automorphism of $\mathrm{Diff}(\mathbb{C}, 0)$. Hence, the whole projective holonomy representations over D_1 of $E^*\mathcal{F}_0$ and $E^*\mathcal{F}_1$ are conjugated. In view of the geometry of the desingularization process, one can repeat the argument for any component of the divisor. Then thanks to the same construction as before, one can extend any conjugacy of the holonomy over D on a tubular neighborhood of D denoted by $T(D)$. Hence, we get a germ of biholomorphism H_D along each D such that

$$H_D^* E^* \mathcal{F}_0|_{T(D)} = E^* \mathcal{F}_1|_{T(D)} \quad \text{and} \quad H_D|_D = \mathrm{Id}$$

Of course, there is no chance for the family $\{H_D\}_{D\in \text{Comp}(\mathcal{D})}$ to induce a global biholomorphism, i.e to verify the condition

$$H_D = H_{D'}, \text{ on a neighborhood of } D \cap D'.$$

However, we are going to introduce a finer covering than $\{T(D)\}_{D\in \text{Comp}(\mathcal{D})}$, which allows us to *twist* the family $\{H_D\}_{D\in \text{Comp}(\mathcal{D})}$ and to build a new family $\left\{\tilde{H}_i\right\}_{i\in\mathbb{I}}$ satisfying

$$\tilde{H}_i \circ \tilde{H}_j^{-1} \text{ acts in the local leaf of } E^*\mathcal{F}_0.$$

Precisely, let us consider a covering $\{U_i\}_{i\in\mathbb{I}=\mathbb{I}_0\cup\mathbb{I}_1}$ of $\mathcal{D}$ defined by

$$\begin{cases} i \in \mathbb{I}_0 = \text{Comp}(\mathcal{D}), \;\; U_D = D\backslash \text{Sing}(E^*\mathcal{F}_0) \\ i \in \mathbb{I}_1 = \text{Comp}(\mathcal{D})^2, \; U_{(D,D')} = T(D) \cap T(D') \cap \mathcal{D} \end{cases}$$

In view of the form of the dual graph of $\mathcal{F}_0$ or $\mathcal{F}_1$, we use the following clear notation for the components of $\mathcal{D}$

$$\text{Comp}(\mathcal{D}) = \{D_{-m}, D_{-m-1}, \ldots, D_{-1}, D_0, D_1, \ldots, D_{p-1}, D_p\}$$

where D_0 refers to the central component. We consider the filtration $\mathbb{I}_n$ of $\mathbb{I}$ defined by

$$\mathbb{I}_n = \{D_{-n}, \ldots, D_n\} \bigcup \{D_{-n}, \ldots, D_n\}^2$$

Using the special geometry of the dual tree of quasi-homogenous foliation, we establish the following lemma:

Lemma 3.2 *For any integer n, there exists a family $\{\phi_i\}_{i\in I_n}$ such that*

- *for all $D \in \mathbb{I}_n$, ϕ_D belongs to* $\text{Aut}(E^*\mathcal{F}_0, U_D)$
- *for all $(D, D') \in \mathbb{I}_n$, $\phi_{(D,D')}$ belongs to* $\text{Aut}(E^*\mathcal{F}_0, U_{(D,D')})$

and such that for all $(D, (D, D')) \in \mathbb{I}_0 \times \mathbb{I}_1$ the two maps

$$\phi_D^{-1} \circ H_D^{-1} \circ H_{D'} \circ \phi_{(D,D')}$$
$$\text{and} \quad \phi_{D'}^{-1} \circ \phi_{(D,D')}$$

act in the local leaf of $E^*\mathcal{F}_0$.

Proof: The proof is an induction on the integer n. For $n = 0$, the lemma is trivial: since the condition is empty, on can choose $\Phi_{D_0} = \text{Id}$. Let us suppose the result true for n. The automorphism $\phi_{D_n}^{-1} \circ H_{D_n}^{-1} \circ H_{D_{n+1}}$ is an automorphism of the foliation defined in a neighborhood of $U_{D_n} \cap U_{(D_n,D_{n+1})}$. By construction, the foliation $E^*\mathcal{F}_0$ restricted to a neighborhood of $U_{(D_n,D_{n+1})}$ has exactly one reduced isolated singularity. Hence, one can consider Hol_n its holonomy map computed on any curve transversal to D_n and attached to a point in $U_{(D_n,D_{n+1})}$. In view of the lemma (3.1), there exists $\phi_{(D_n,D_{n+1})}$ in the group $\mathrm{Aut}(E^*\mathcal{F}_0, U_{(D_n,D_{n+1})})$ such that

$$\phi_{D_n}^{-1} \circ H_{D_n}^{-1} \circ H_{D_{n+1}} \quad \text{and} \quad \phi_{(D_n,D_{n+1})}$$

have same images in $\mathrm{Cent}(\mathrm{Hol}_n) /_{<\mathrm{Hol}_n>}$. Hence, the automorphism

$$\phi_{D_n}^{-1} \circ H_{D_n}^{-1} \circ H_{D_{n+1}} \circ \phi_{(D_n,D_{n+1})}$$

acts in the local leaf. In the same way, the map $\phi_{(D_n,D_{n+1})}$ is an automorphism of the restricted foliation in an open neighborhood of the set $U_{D_{n+1}} \cap U_{(D_n,D_{n+1})}$. Since the open set $U_{D_{n+1}}$ is conformally equivalent to a corona, there exists $\phi_{D_{n+1}}$ in $\mathrm{Aut}(E^*\mathcal{F}_0, U_{D_{n+1}})$ such that

$$\phi_{D_{n+1}}^{-1} \circ \phi_{(D_n,D_{n+1})}$$

acts in the local leaf. Hence, we have obtained the induction hypothesis at rank $n+1$. □

Let us now consider the following family of maps

$$\begin{cases} i \in \mathbb{I}_0 = \mathrm{Comp}(\mathcal{D}), & \psi_D = \phi_D \circ H_D \\ i \in \mathbb{I}_1 = \mathrm{Comp}(\mathcal{D})^2, & \psi_{(D,D')} = \phi_{(D,D')} \circ H_D \end{cases}$$

In view of the construction, the automorphism $\psi_i^{-1} \circ \psi_j,\ i,j \in \mathbb{I} = \mathbb{I}_0 \cup \mathbb{I}_1$ acts in the local leaf. In view of [1], there exists a family of tangent vector fields $\{X_{ij}\}_{i,j\in\mathbb{I}}$ and a family of holomorphic functions $\{t_{ij}\}_{i,j\in\mathbb{I}}$ such that

$$\psi_i^{-1} \circ \psi_j = \Phi_{X_{ij}}^{t_{ij}},$$

where Φ_X^t refers to the flow of the vector field X at t time. Let us denote by $\mathcal{U}_i$ a neighborhood of U_i in $\mathcal{M}$. The deformation defined by the identification

$$s \longmapsto \coprod_{i\in\mathbb{I}} E^*\mathcal{F}_0|_{\mathcal{U}_i} \Big/ _{x \sim \Phi_{X_{ij}}^{s\cdot t_{ij}} x}, \quad s \in \overline{\mathbb{D}}$$

is well defined since the automorphisms of identification $x \to \Phi_{X_{ij}}^{s\cdot t_{ij}} x$ act in the local leaf. Moreover, in view of the cohomological interpretation of isoholonomic deformation, this deformation is precisely an isoholonomic deformation of foliation. The fiber of this deformation at $0 \in \overline{\mathbb{D}}$ is the foliation $\mathcal{F}_0$ and the fiber at $1 \in \overline{\mathbb{D}}$ is a foliation analytically equivalent to $\mathcal{F}_1$. Now, the theorem (2.2) applied with $C = \{0,1\} \subset \overline{\mathbb{D}}$ ensures the existence of an isoholonomic deformation $\mathcal{R}$ over $\overline{\mathbb{D}}$ from $\mathcal{F}_0$ to $\mathcal{F}_1$ such that the underlying deformation of separatrix is the trivial deformation $S \times \overline{\mathbb{D}}$. Since $\mathcal{F}_0$ is quasihomogeneous, the deformation $\mathcal{R}$ is locally trivial [14]. Hence, $\mathcal{F}_0$ and $\mathcal{F}_1$ are analytically conjugated. This ends the proof in the convergent context. The transposition to the formal context is let to the reader.

Recently, E. Paul generalized a result of F. Loray [11] and found unique formal normal forms for any formal quasi-homogeneous vector fields [18]. Once the separatrix are given, the theorem (1.1) ensures the equivalence between the data of such a formal normal form and the choice of a point in the space of representation of a free group in $\widehat{\mathrm{Diff}}(\mathbb{C},0)$. However, the meaning of such a correspondance is still to be worked out.

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