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Number of moduli for a union of smooth curves in $(\mathbb{C}^2, 0)$

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ABSTRACT

In this article, we provide an algorithm to compute the number of moduli of a germ of curve which is a union of germs of smooth curves in the complex plane.

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0. Introduction

The problem of the determination of the number of moduli of a germ of complex plane curve was addressed by Oscar Zariski in his famous notes Zariski (1986), where he focused on the case of a curve with only one irreducible component. The *number of moduli* refers to the number of analytical invariants that remain once the topological class of S is given. The topological classification of an irreducible curve S is well known and relies on a semi-group of integers extensively studied by Zariski himself in the 70s. However, at this time, even in the case of an irreducible curve, the analytical classification was a widely open question. Since then, a lot of progress has been made, and, up to our knowledge, the initial question can be considered as mostly solved by the combination of the works of Hefez and Hernandes (2009, 2011, 2013) and these of the author Genzmer (2022): the firsts provided a family of algorithms that describes a sharp stratification of the moduli space of S, the second a formula to compute the dimension of its generic stratum, and thus the number of moduli of S.

In this article, we propose to go beyond the irreducible case and to study the case of a union of smooth curves, one of the simplest situations once the irreducibility hypothesis is dropped. To do so, we follow some methods introduced in Genzmer (2022): from the study the module of vector

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fields tangent to a curve S, which we refer to as the Saito module of S, we propose an algorithm to compute the number of moduli of S that can be easily implemented. The associated algorithm is built upon the desingularization process of S, for which we have already at our disposal, some classical and available routines on many symbolic computation softwares.

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1. Notation

Let S be a germ of curve in the complex plane. According to Zariski (1932), there exists a minimal process of desingularization E that consists in a sequence of elementary blowing-ups of points. We denote it by

$$E = E_1 \circ E_2 \circ \cdots \circ E_N : \left(\widetilde{\mathbb{C}^2}, D\right) \to \left(\mathbb{C}^2, 0\right).$$

Here, $D = E^{-1}(0)$ is the *exceptional divisor* of E and $\widetilde{\mathbb{C}^2}$ stands for the germ of non singular neighborhood of D obtained from the successive blowing-ups over $\left(\mathbb{C}^2, 0\right)$. The strict transform of S by any process of blowing-ups F will be referred to as $F^\star S$. The decomposition of D in irreducible components is written

$$D = \bigcup_{i=1}^{N} D_i$$

where D_i is the exceptional divisor of the elementary blowing-up E_i.

Let $\{t_2, \ldots, t_M\} \subset D_1$ be the tangency locus between $E_1^\star S$ and D_1. For any $k = 2, \ldots, M$, S_k stands for the germ of the curve $E_1^\star S$ at t_k. Doing inductively the same construction for each curve S_k, we finally obtain a family of curves

$$(S_k)_{k=2,\ldots,N},$$

whose numbering is chosen so that E_k is the blowing-up centered at the tangency locus between S_k and the exceptional divisor. By extension, we set $S_1 = S$. For $k = 2, \ldots, M$ the desingularization of the curve $S_k \cup D_1$ is a composition of blowing-ups that we denote

$$E_1^k \circ E_2^k \circ \cdots \circ E_{N_k}^k.$$

Each E_j^k is a certain elementary blowing-up $E_{\sigma^k(j)}$ appearing in the decomposition of the initial process E. This correspondence defines an injective map σ^k,

$$\sigma^k : \{1, \ldots, N_k\} \to \{2, \ldots, N\}.$$

Notice that by construction, for any $k = 2, \ldots, M$, $\sigma^k(1) = k$ and the images

$$\left(\mathrm{Im}\left(\sigma^i\right)\right)_{i=2,\ldots,M}$$

provide a partition of the set $\{2, \ldots, N\}$.

Subsequently, the notation $\nu(\square)$ will stand for the standard valuation of the object $\square$:

- if S is a germ of curve, then $\nu(S) = \nu(f)$ is the algebraic multiplicity of any reduced local equation $f = 0$ of S.
- if X is a germ of vector field written in some coordinates $X = a\partial_x + b\partial_y$, then

$$\nu(X) = \min(\nu(a), \nu(b)).$$

If any confusion is possible, we will precise the point p where the valuation is evaluated. The associated notation will be $\nu_p(\square)$.

Since $E^\star S$ is smooth and transverse to the exceptional divisor, one can consider for any component D_k, the number n_k^S of components of $E^\star S$ attached to D_k.

We say that D_i is *in the neighborhood of* D_k if $i \neq k$ and $D_i \cap D_k \neq \emptyset$. The set of all indexes $i \in \{1, \ldots, N\}$ such that D_i is in the neighborhood of D_k will be denoted by $\mathfrak{N}(k)$.

For $i \geq 2$, the component D_i is the blowing-up of a point which belongs to, either a single component D_j or to a couple of components D_j and D_k. The associated set of indexes $\{j\}$ or $\{j,k\}$ is called the set of *parents* of D_i and will be denoted by $\mathfrak{P}(i)$. By extension, we set $\mathfrak{P}(1) = \emptyset$. Notice that for any $i = 2, \ldots, M$, one has

$$\mathfrak{P}(i) = \{1\}.$$

Finally, we introduce the following notation: for $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$

$$\left\lfloor \begin{array}{l} a \\ b \end{array} \right._n = \begin{cases} a & \text{if } n \text{ is even ;} \\ b & \text{if } n \text{ is odd.} \end{cases}$$

2. Topological class of S and number of moduli

We recall *the proximity matrix* of S as defined in Wall (2004, p. 52).

Definition 1. The proximity matrix of S is the $N \times N$ matrix P^S whose entries are

$$\left(P^S\right)_{i,j} = \begin{cases} 1 & \text{if } i = j \\ -1 & \text{if } i \in \mathfrak{P}(j) \\ 0 & \text{otherwise} \end{cases}.$$

Given the numbering of the components of the exceptional divisor, the matrix P^S is an upper triangular matrix.

The data of P^S (or of the topological type of E) and the integers $\left(n_k^S\right)_{k=1,\ldots,N}$ characterize the topological class of S. More precisely, following Zariski (1932), two curves S and S' are topologically equivalent if and only if there exists a permutation ϕ of $\{1, \ldots, N\}$ such that

$$\forall i, j \leq N, \; P^{S'}_{\phi(i)\phi(j)} = P^S_{ij}$$

and

$$\forall i \leq N, \; n^{S'}_{\phi(i)} = n^S_i.$$

Now, if S is a union of K smooth germs of curves then S admits a reduced equation of the form

$$f_1 f_2 \cdots f_K = 0$$

where f_K is a germ of analytic function with a non trivial linear part:

$$f_j = f_j^{00} + f_j^{01} x + f_j^{10} y + f_j^{20} x^2 + f_j^{11} xy + \cdots$$

with $f_j^{00} = 0$ and $f_j^{01} \neq 0$ or $f_j^{10} \neq 0$. Being in a fixed topological class translates into a finite set of algebraic conditions depending on a finite number of complex variables f_j^{kl}. Thus, there exists a complex constructible subset[1] $\Sigma(S) \subset \left(\mathbb{C}^{\mathbb{N}^2}\right)^K$ and a surjective map

[1] Here, complex constructible subset means a subset of $\left(\mathbb{C}^{\mathbb{N}^2}\right)^K$ that is a finite union of finite intersections of sets of the form $\{Q = 0 \text{ and } R \neq 0\}$ where Q and R are polynomial functions.

$$\Sigma(S) \twoheadrightarrow \mathrm{Top}(S)$$

where $\mathrm{Top}(S)$ stands for the set of germs of curves topologically equivalent to S. A property is going to be said true *for a generic curve in its topological class* if it is true for the image of a Zariski open set in $\Sigma(S)$, i.e., the intersection of $\Sigma(S)$ with the complementary of the zero locus of a finite list of polynomial functions on $\left(\mathbb{C}^{\mathbb{N}^2}\right)^K$. In the present article, most of the results will assume that *the curve S is generic in its topological class*, implying that the stated results will be true only for a curve generic in its topological class.

When S is irreducible, Ebey (1965) constructed a non Hausdorff complex structure on *the moduli space* of S, that is the standard quotient of $\mathrm{Top}(S)$ up to analytical equivalence relation. This quotient happens to be the quotient of a complex constructible set of finite dimension by the action of a connected solvable algebraic group. By definition, the generic dimension of this quotient is *the number of moduli* of S. The extension of this construction to the general non irreducible case is a work in progress. However, even in the general case, one can still give a suitable definition of the number of moduli by a local approach: a curve S being generic in its topological class, one can consider a miniversal deformation of S (Zariski, 1986), i.e., a commutative diagram

$$\begin{array}{ccc} (S,0) & \hookrightarrow & (\Sigma,0) \\ \downarrow & & \downarrow \\ \{0\} & \hookrightarrow & (\mathbb{C}^L,0) \end{array}$$

through which factorizes any germ of deformation of $(S,0)$. In the parameter space $(\mathbb{C}^L,0)$, there is a smooth stratum along which the topological type is constant and equal to the one of S. The dimension of this stratum is by definition the number of moduli of S. In the last section of this article, we will mention a cohomological description of the tangent space to this stratum due to Mattei (2000), upon which our computation is based.

3. Saito vector field

In this section, S is any germ of curve - not necessarily a union of smooth curves.

3.1. Definition of a Saito vector field for a curve

Let $\mathrm{Der}(S)$ be the set of germs of vector fields X tangent to S, i.e., such that for a reduced equation f of S, one has

$$X \cdot f \in (f).$$

According to Saito (1980), $\mathrm{Der}(S)$ is a free $\mathcal{O}_2$-module of rank 2 and any basis $\{X_1, X_2\}$ of $\mathrm{Der}(S)$ will be said a *Saito basis* for S. The *number of Saito* of S is

$$\mathfrak{s}(S) = \min_{X \in \mathrm{Der}(S)} \nu(X) = \min(\nu(X_1), \nu(X_2)).$$

A vector field $X \in \mathrm{Der}(S)$ is said to be *optimal* for S if

$$\nu(X) = \mathfrak{s}(S).$$

If E is any process of blowing-up, we denote by X^E the *divided pull-back* vector field of X by E. It is a family a vector field parametrized by the point of the exceptional divisor: for any $c \in D$, $\left(X^E\right)_c$ is written $\frac{Y}{u^a}$ (or $\frac{Y}{u^a v^b}$) where Y projects onto X with respect to E and u^a is the maximal power of u that divides Y, where u (or $uv = 0$) is a local equation of D at c. An alternative way to construct X^E is the following: the vector field X induces a saturated foliation $\mathcal{F}$ at the origin of $\mathbb{C}^2$. The foliation

$\mathcal{F}$ can be pulled-back by E in $E^{\star}\mathcal{F}$ which defines a saturated foliation in the neighborhood of D. The vector field $\left(X^{E}\right)_{c}$ is any generator of the latter at c.

The vector field X is said to be *dicritical* if X^{E_1} is generically transversal to the exceptional divisor D_1.

Below, we recall some material established in Genzmer (2020).

Theorem 2. *Let S be a curve* generic *in its topological class. Then there exists a Saito basis $\{X_1, X_2\}$ for S with one of the following forms*

- *if $\nu(S)$ is even*
  $(\mathfrak{E})$ *: $\nu(X_1) = \nu(X_2) = \frac{\nu(S)}{2}$, X_1 and X_2 are non dicritical.*
  $(\mathfrak{E}_d)$ *: $\nu(X_1) = \nu(X_2) - 1 = \frac{\nu(S)}{2} - 1$, X_1 and X_2 are dicritical.*
- *if $\nu(S)$ is odd*
  $(\mathfrak{O})$ *: $\nu(X_1) = \nu(X_2) - 1 = \frac{\nu(S)-1}{2}$, X_1 and X_2 are non dicritical.*
  $(\mathfrak{O}_d)$ *: $\nu(X_1) = \nu(X_2) = \frac{\nu(S)-1}{2}$, X_1 and X_2 are dicritical.*

In particular, the Saito number of S is equal to

$$\mathfrak{s}(S) = \frac{\nu(S)}{2} - \left\lfloor \frac{1-\Delta}{2} \right\rfloor_{\nu(S)}$$

where $\Delta = \begin{cases} 1 & \text{if } S \text{ is of type } (\mathfrak{O}) \text{ or } (\mathfrak{E}) \\ 0 & \text{else} \end{cases}$.

The curve S being of type $(\mathfrak{E}_d)$ or $(\mathfrak{O}_d)$, there exists a basis of the following form

$\left(\mathfrak{E}'_d\right)$ *: $\nu(X_1) = \nu(X_2) - 2 = \frac{\nu(S)}{2} - 1$, X_1 is dicritical but not X_2.*
$\left(\mathfrak{O}'_d\right)$ *: $\nu(X_1) = \nu(X_2) - 1 = \frac{\nu(S)-1}{2}$, X_1 is dicritical but not X_2.*

If and only if S has no free point *- see below.*

By definition, the tangency locus $\mathrm{Tan}\left(E_1^{\star}S, D_1\right)$ is the set of points

$$\{t_2, \ldots, t_M\} \subset D_1.$$

Since $X_1^{E_1}$ leaves invariant $E_1^{\star}S$, the locus of tangency $\mathrm{Tan}\left(X_1^{E_1}, D_1\right)$ between the vector field $X_1^{E_1}$ and D_1 contains $\mathrm{Tan}\left(S^{E_1}, D_1\right)$. Following Genzmer (2020), we recall that S is said to have *no free point* if and only if

$$\mathrm{Tan}\left(X_1^{E_1}, D_1\right) = \mathrm{Tan}\left(S^{E_1}, D_1\right).$$

The number of free points is, by definition, the number of elements of the difference

$$\mathrm{Tan}\left(X_1^{E_1}, D_1\right) \setminus \mathrm{Tan}\left(S^{E_1}, D_1\right).$$

The number of free points depends only on the topological type of S. In Table 1, we present an example of curve for each type of Saito bases. In this table, the notation $\sharp f$ stands for the vector field

$$\partial_x f \partial_y - \partial_y f \partial_x$$

obviously tangent to $f = 0$.

A basis of the Saito module of S is said to *be adapted* if it has one of the first four types described in Theorem 2. An adapted basis behaves well with respect to the blowing-up: indeed, in any case, if $\{X_1, X_2\}$ is an adapted basis for S then for any $c \in D_1$, the family

Table 1
Examples of different types of Saito bases.

S	$f=x$	$f=xy$	$f=xy(x+y)$	$f=xy\left(x^2-y^2\right)$
$\nu(S)$	1	2	3	4
$X_1,\ X_2$	$\partial_x,\ x\partial_y$	$x\partial_x,\ y\partial_y$	$x\partial_x+y\partial_y,\ \sharp f$	$x\partial_x+y\partial_y,\ \sharp f$
$\nu(X_1),\ \nu(X_2)$	$0,\ 1$	$1,\ 1$	$1,\ 2$	$1,\ 3$
Type	$(\mathfrak{O})$	$(\mathfrak{E})$	$(\mathfrak{O}'_d)$	$(\mathfrak{E}'_d)$

S	$f=xy\left(x^3-y^3+\cdots\right)$	$f=xy\left(x^2-y^2\right)(x+2y+\cdots)(x+3y+\cdots)$
$\nu(S)$	5	6
$X_1,\ X_2$	$x\left(x\partial_x+y\partial_y\right)+\cdots$ $y\left(x\partial_x+y\partial_y\right)+\cdots$	$\left(x+\frac{29}{15}y\right)\left(x\partial_x+y\partial_y\right)+\cdots$ $x^2\left(x\partial_x+y\partial_y\right)+\cdots$
$\nu(X_1),\ \nu(X_2)$	$2,\ 2$	$2,\ 3$
Type	$(\mathfrak{O}_d)$ - 1 free point	$(\mathfrak{E}_d)$ - 1 free point

$$\left\{\left(X_1^{E_1}\right)_c,\left(X_2^{E_1}\right)_c\right\}$$

is a Saito basis for $\left(E_1^\star S\right)_c$ or $\left(E_1^\star S\cup D_1\right)_c$ depending on the type of the basis. Notice that this property does not hold for any Saito basis and that the basis above may not be adapted.

For the sake of simplicity, we will say that S is of class $\mathfrak{1}$ if S admits a Saito basis $\{X_1,X_2\}$ of type $(\mathfrak{E})$ or $(\mathfrak{O}_d)$. Otherwise, we will say that S is of class $\mathfrak{2}$. The main difference between the two classes is that the vector fields of an adapted basis for a curve of class $\mathfrak{1}$ share the same valuations, whereas they are different for a curve of class $\mathfrak{2}$.

To keep track of the type of the successive blowing-ups of the curve S, we introduce the notion of *relative strict transform of S*.

Definition 3. The *relative strict transform* of S by E, denoted by S^E, is the following union of curves

$$S^E=E^\star S\cup\bigcup_{i\in J\subset\{1,\ldots,N\}}D_i$$

where J is inductively defined as follows:

$$i\in J\iff S_i\cup\textstyle\bigcup_{j\in\mathfrak{P}(i)\cap J}D_j\text{ is of type }(\mathfrak{E})\text{ or }(\mathfrak{O}).$$

A *branch* of the process E is a sequence of integers $\left(i_1,\cdots,i_j\right)$ such that the blowing-up E_{i_k} is centered at a point c_k which belongs to exceptional divisor of $E_{i_{k-1}}$. We will denote by E^{i_a,i_b} the composition

$$E^{i_a,i_b}=E_{i_a}\circ E_{i_{a+1}}\circ\cdots\circ E_{i_b}.$$

Finally, we are able to introduce the main object of interest here.

Definition 4. A germ of vector field is said to be *Saito* for S if for any branch $(1,\cdots,k)$ of E and $k\le N$, the vector field $X^{E^{1,k}}$ is optimal for $S^{E^{1,k}}$.

In other words, a vector field is said Saito for S if it is optimal for S and if this property propagates all along the process of desingularization of S.

Given the definition, there is apparently no reason for such a vector field to exist in general. However, we will see that this is actually the case for unions of germs of smooth curves generic in their topological class.[2]

[2] We conjecture that such vector field exists for any curve S generic in its topological class.

Example 5. Let us consider the curve S defined by

$$S=\left\{f=x\left(x+f_0y^2+\cdots\right)y\left(y+f_1x^2+\cdots\right)=0\right\},$$

with $f_0\neq 0$ and $f_1\neq 0$. The two latter constructible conditions and the form of the function f fix the topological type of S. The dots in the above expression stand for higher order terms. It can be checked that S is of type $(\mathfrak{C})$ and that

$$X_1=x^2\partial_x+2xy\partial_y+\cdots$$
$$X_2=2xy\partial_x+y^2\partial_y+\cdots$$

is an adapted basis. In particular, the Saito number of S is

$$\mathfrak{s}(S)=2.$$

Moreover, for $\alpha\in\mathbb{C}\setminus\{0\}$, after one blowing-up, $X_1+\alpha X_2$ is given, in the chart (x,t) for which

$$E_1(x,t)=(x,tx)$$

by

$$(X_1+\alpha X_2)^{E_1}=x(2t+\alpha)\partial_x+t(-t+\alpha)\partial_t+\cdots$$

which, at $(x,t)=(0,0)$ – the singular point of S_2 - is of multiplicity 1 and tangent to the radial vector field at order 1. In the other chart, the same occurs for the singular point of S_3. Therefore, $X_1+\alpha X_2$ is Saito for S. Notice that, the vector field $(X_1+\alpha X_2)^{E_1}$ admits an other singular point whose coordinates are $(0,\alpha)$ in the coordinates of the chart above. At $(0,\alpha)$, the linear part is not trivial and has two non vanishing eigenvalues whose quotient is not a non negative rational number. In particular, according to Dulac (1904), $X_1+\alpha X_2$ admits a smooth invariant curve that is neither contained in S nor tangent to a component of S. Finally, although X_1 is optimal for S, X_1 is not Saito for S since the vector field $X_1^{E_1}$ is written

$$X_1^{E_1}=2xt\partial_x-t^2\partial_t+\cdots$$

and its multiplicity is 2 at $(0,0)$ and thus not optimal for $S_2\cup D_1$.

3.2. Numerical properties of a Saito vector field

Let us investigate the topological properties of a Saito vector field.

First, let us recall some results from Hertling (2000).

Let $\mathfrak{M}$ be the sheaf generated by the global functions $h\circ E$ with $h\in\mathcal{O}_2$ and $h(0)=0$. It is a simple matter to get the following decomposition of sheaves

$$\mathfrak{M}=\mathcal{O}\left(-\sum_{i=1}^{N}\rho_i^E D_i\right)$$

where the integers ρ_i^E are known as *the multiplicities of* D. The number ρ_i^E is also the multiplicity of a curve whose strict transform by E is smooth and attached to a regular point of D_i.

The valence $\mathrm{val}(D_i)$ of D_i is the number of components D attached to D_i, i.e., the cardinal of $\mathfrak{N}(i)$. The integer $\mathrm{val}_X(D_i)$ refers to *the non-dicritical valence* of D_i with respect to the vector field X, which is the number of X^E-invariant components of D that are in the neighborhood of D_i.

The following definitions are proposed in Hertling (2000).

Definition 6. Let X be a germ of vector field at p given by

$$X=a(x,y)\partial_x+b(x,y)\partial_y$$

(1) Let (S, p) be a germ of smooth invariant curve. If, in some coordinates, S is the curve $\{x = 0\}$ and p the point $(0, 0)$, then the integer $\nu(b(0, y))$ is called *the index* of X at p with respect to S and it is denoted by

$$\operatorname{Ind}(X, S, p).$$

(2) Let (S, p) be a germ of smooth non-invariant curve. If, in some coordinates, S is the curve $\{x = 0\}$ and p the point $(0, 0)$, then the integer $\nu(a(0, y))$ is called *the tangency order* of X with respect to S and it is denoted by

$$\operatorname{Tan}(X, S, p).$$

The equality below is proved in Hertling (2000) and specializes to a classical result of Camacho et al. (1984) if X^E leaves invariant D.

Proposition 7. *The multiplicity of X satisfies the equality*

$$\nu(X) + 1 = \sum_{i=1}^{N} \rho_i^E \epsilon_i(X, E)$$

where

(1) if D_i is invariant by X^E,

$$\epsilon_i(X, E) = -\mathrm{val}_X(D_i) + \sum_{c \in D_i} \operatorname{Ind}\left(X^E, D_i, c\right);$$

(2) if D_i is non invariant by X^E,

$$\epsilon_i(X, E) = 2 - \mathrm{val}_X(D_i) + \sum_{c \in D_i} \operatorname{Tan}\left(X^E, D_i, c\right).$$

Beyond the integers $\epsilon_i(X, E)$ which describe partially the topology of the vector field X, we will introduce additional numerical invariants to control its topology. Besides, when X is Saito for S, we will establish a relation between the latter and the integers $\epsilon_i(X, E)$.

The curve S be given, let $\Delta^S = \left(\Delta_i^S\right)_{i=1,\ldots,N}$ be any element in $\{0, 1\}^N$. Denote by δ_k^S the integer

$$\delta_k^S = \operatorname{card}\left\{ i \in \mathfrak{P}(k) \mid \Delta_i^S = 1 \right\}.$$

We consider the following vector of integers

$$\mathfrak{S}^S = \left(\frac{\nu(S_k) - \delta_k^S}{2} + \left\lfloor \begin{matrix} \Delta_k^S \\ \frac{1}{2} \end{matrix} \right. \right)^{t}_{\nu(S_k)-\delta_k^S \quad k=1,\ldots,N}$$

Below, Theorem 8 will provide some geometric interpretations of the invariants above: actually, $\Delta_k^S = 1$ will indicate that the vector field X^E leaves invariant the component D_k and thus, δ_k^S will be the number of X^E-invariant parents of D_k in the process E. In particular, for any k, $\delta_k^S \in \{0, 1, 2\}$.

We introduce the system of equations $\left(\mathcal{H}^S\right)$ whose unknown variables are the vectors $\mathcal{E}^S = \begin{pmatrix} \epsilon_1^S \\ \vdots \\ \epsilon_N^S \end{pmatrix} \in \mathbb{N}^N$ and $\Delta^S = \left(\Delta_i^S\right)_{i=1,\ldots,N}$ defined by

$$\left(\mathcal{H}^S\right): \ \left(P^S\right)^{-1}\mathcal{E}^S = \mathfrak{S}^S$$

A solution $\left(\mathcal{E}^S, \Delta^S\right)$ is *admissible* if it satisfies the *compatibility conditions*: for any $k = 1, \dots, N$

$$(\star): \begin{cases} \Delta_k^S = 1 \implies \epsilon_k^S \geq n_k^S \\ \Delta_k^S = 0 \implies \epsilon_k^S \geq 2 - \sum_{i\in\mathfrak{N}(k)} \Delta_i^S \end{cases}. \tag{3.1}$$

From Proposition 7, it can be seen that the compatibility conditions are necessary so that a solution of $\left(\mathcal{H}^S\right)$ is induced by the numerical data $(\epsilon_i(X, E))_{i=1,\dots,N}$.

Actually, as a consequence of Proposition 7, a Saito vector field for S provides an admissible solution to the system $\left(\mathcal{H}^S\right)$.

Theorem 8. *If X is a Saito vector field for S then setting*

$$\mathcal{E}^S = (\epsilon_i(X, E))_{i=1,\dots,N}$$

and $\Delta^S = \left(\Delta_i^S\right)_{i=1,\dots,N}$ *such that*

$$\Delta_i^S = \begin{cases} 1 & \text{if } D_i \text{ is invariant by } X^E \\ 0 & \text{else} \end{cases}$$

yields an admissible solution $\left(\mathcal{E}^S, \Delta^S\right)$ *of* $\left(\mathcal{H}^S\right)$.

Proof. For $k = 1, \dots, N$, let E' be the intermediate process of blowing-ups that leads to S_k and E^k such that

$$E = E' \circ E^k.$$

Let us denote by p the point of attachment of S_k to the exceptional divisor of E'. The vector field $X^{E'}$ being optimal for $\left(S^{E'}\right)_p$, we have

$$\begin{aligned} \nu_p\left(X^{E'}\right) + 1 &= \sum_{i=k}^{N} \rho_i^{E^k} \epsilon_i\left(X^{E'}, E^k\right) = \mathfrak{s}\left(\left(S^{E'}\right)_p\right) + 1 \\ &= \frac{\nu\left(\left(S^{E'}\right)_p\right)}{2} + \left\lfloor \begin{matrix} \Delta_k^S \\ \frac{1}{2} \end{matrix} \right._{\nu\left(\left(S^{E'}\right)_p\right)} . \end{aligned} \tag{3.2}$$

Now, it can be seen that for $i \neq k$

$$\epsilon_i\left(X^{E'}, E^k\right) = \epsilon_i(X, E)$$

and that

$$\epsilon_k\left(X^{E'}, E^k\right) = \epsilon_k(X, E) + \delta_k^S.$$

Since $\rho_k^{E^k} = 1$ and $\nu\left(\left(S^{E'}\right)_p\right) = \nu(S_k) + \delta_k^S$, the relation (3.2) is written

$$\sum_{i=k}^{N} \rho_i^{E^k} \epsilon_i(X, E) = \frac{\nu(S_k) - \delta_k^S}{2} + \left\lfloor \begin{matrix} \Delta_k^S \\ \frac{1}{2} \end{matrix} \right._{\nu(S_k) - \delta_k^S} .$$

Now following Wall (2004), the matrix defined by

$$\left(\rho_i^{E^k}\right)_{N\geq i\geq k\geq 1}$$

is an upper triangular invertible matrix and its inverse is the proximity matrix P^S. Thus, the vectors $\mathcal{E}^S$ and Δ^S as defined in the statement provide a solution to the system $\left(\mathcal{H}^S\right)$. Moreover, if $\Delta_k^S=0$, then

$$\begin{aligned}\epsilon_k(X,E)-2+\sum_{i\in\mathfrak{N}(k)}\Delta_i^S &= \epsilon_k(X,E)-2+\mathrm{val}_X(D_k)\\ &= \sum_{c\in D_k}\mathrm{Tan}\left(X^E,D_k,c\right)\geq 0\end{aligned}$$

and if $\Delta_k^S=1$ then

$$\begin{aligned}\epsilon_k(X,E) &= -\mathrm{val}_X(D_k)+\sum_{c\in D_k}\mathrm{Ind}\left(X^E,D_k,c\right)\\ &= -\sum_{i\in\mathfrak{N}(k)}\Delta_i^S+\sum_{c=D_k\cap D_i,\ i\in\mathfrak{N}(k)}\mathrm{Ind}\left(X^E,D_k,c\right)\\ &\quad+\sum_{c\neq D_k\cap D_i,\ i\in\mathfrak{N}(k)}\mathrm{Ind}\left(X^E,D_k,c\right)\end{aligned}$$

It can be seen that if $\Delta_i^S=1$ then $\mathrm{Ind}\left(X^E,D_k,D_k\cap D_i\right)\geq 1$. Moreover, for any regular component of S^E attached to D_k at c, one has

$$\mathrm{Ind}\left(X^E,D_k,c\right)\geq 1.$$

Thus,

$$\sum_{c\neq D_k\cap D_i,\ i\in\mathfrak{N}(k)}\mathrm{Ind}\left(X^E,D_k,c\right)\geq n_k^S.$$

Finally, we are led to

$$\epsilon_k(X,E)\geq n_k^S.$$

Therefore, the solution $\left(\mathcal{E}^S,\Delta^S\right)$ is admissible. □

If we restrict ourselves to the case where S is a union of germs of smooth curves, we can prove that an admissible solution exists and is unique. We postpone the proof of the proposition below in a final appendix.

Proposition 9. *If S is a union of germs of smooth curves, then there exists a unique choice of Δ^S such that the associated solution of $\left(\mathcal{H}^S\right)$ is admissible.*

Example 10. The proximity matrix of Example (5) is

$$P^S=\begin{pmatrix}1&-1&-1\\0&1&0\\0&0&1\end{pmatrix}$$

and one has

$$\nu(S_1)=4,\ \nu(S_2)=2,\nu(S_3)=2,\ n_1^S=0,\ n_2^S=2,\ n_3^S=2.$$

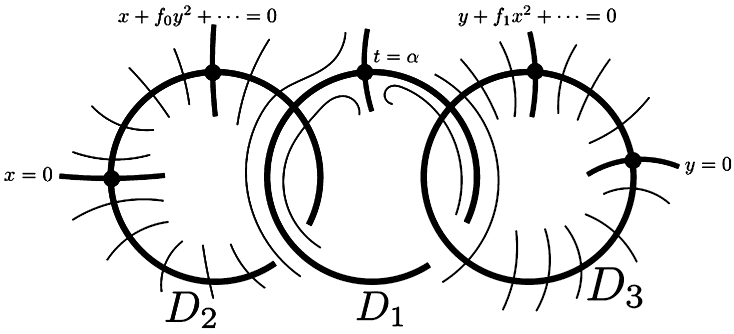


Fig. 3.1. Topology of the leaves of the vector field $(X_1 + X_2)^E$.

In addition, we have

$$\epsilon_1^S = 1,\ \epsilon_2^S = \epsilon_3^S = 1 \quad \Delta_1^S = 1,\ \Delta_2^S = \Delta_3^S = 0,$$

as illustrated in Fig. 3.1. Thus, we obtain that $\delta_1^S = 0,\ \delta_2^S = \delta_3^S = 1$. Finally, one can check that

$$\left(P^S\right)^{-1} \mathcal{E}^S = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{4-0}{2} + 1 \\ \frac{2-1}{2} + \frac{1}{2} \\ \frac{2-1}{2} + \frac{1}{2} \end{pmatrix} = \mathfrak{S}^S$$

3.3. Existence of a Saito vector field

Below, we establish the existence of a Saito vector field for a union of germs of smooth curves.

Theorem 11. *Let S be a union of smooth germs of curves. Suppose that S is generic in its topological class. Then*

(1) there exists a vector field X Saito for S.
(2) there exists l a germ of smooth curve such that $S \cup l$ has no Saito basis of type $\left(\mathfrak{E}_d'\right)$.

The proof of Theorem 11 relies deeply on the form of a Saito basis for S whose description is given in Theorem 2: this description can be only made in a suitable way for a curve generic in its topological class.

Proof. The proof is by induction on the maximal length of a branch in the desingularization of S.

If this length is zero, then S is a smooth curve. In some coordinates (x, y) such that $S = \{x = 0\}$, the vector field $X = \partial_x$ is Saito for S. Moreover, if l is the line $\{y = 0\}$, then the family $\left\{x\partial_x, y\partial_y\right\}$ is an adapted basis for S which is not of type $\left(\mathfrak{E}_d'\right)$.

We suppose now that the maximal length of a branch in the desingularization of S is strictly positive. Let us consider $\{X_1, X_2\}$ an adapted basis for S and

$$(1, \cdots, p_0, p_0 + 1, \cdots, p_1, p_1 + 1, \cdots)$$

a branch of E such that for each k, the curve $\left(S^{E^{1,p_k-1}}\right)_{c_{p_k}}$ is of type $\mathtt{2}$ whereas in the open interval of integers $]p_k, p_{k+1}[$, it is of type $\mathtt{1}$:

$$\underset{\mathtt{1}}{c_1} \leftarrow \cdots \leftarrow \underset{\mathtt{1}}{c_{p_0-1}} \leftarrow \underset{\mathtt{2}}{c_{p_0}} \leftarrow \underset{\mathtt{1}}{c_{p_0+1}} \leftarrow \cdots \leftarrow \underset{\mathtt{1}}{c_{p_1-1}} \leftarrow \underset{\mathtt{2}}{c_{p_1}} \leftarrow \underset{\mathtt{1}}{c_{p_0+1}} \leftarrow \cdots .$$

Notice that the index p_0 may be equal to 1, so that, the first curve S_1 is of class $\mathtt{2}$. Hence, the above description of a branch covers actually the general case.

Any family $\left\{X_1^{E^{1,k}}, X_2^{E^{1,k}}\right\}$ for $k \leq p_0-1$ is a Saito basis for $\left(S^{E^{1,k}}\right)_{c_{k+1}}$: indeed, the curve $\left(S^{E^{1,k}}\right)_{c_{k+1}}$ being of class $\mathtt{1}$ for $k \leq p_0-2$, the Saito basis $\left\{X_1^{E^{1,k}}, X_2^{E^{1,k}}\right\}$ is also adapted. Considering a convenient combination of X_1 and X_2, we can suppose that in the basis $\left\{X_1^{E^{1,p_0-1}}, X_2^{E^{1,p_0-1}}\right\}$, one has

$$\nu_{c_{p_0}}\left(X_1^{E^{1,p_0-1}}\right) \leq \nu_{c_{p_0}}\left(X_2^{E^{1,p_0-1}}\right), \tag{3.3}$$

and $X_1^{E^{1,p_0-1}}$ is optimal for $\left(S^{E^{1,p_0-1}}\right)_{c_{p_0}}$. Since, the latter is a curve of class $\mathtt{2}$, there exists $c \in \mathbb{C}$ such that

$$\nu_{c_{p_0}}\left(X_1^{E^{1,p_0-1}}\right) < \nu_{c_{p_0}}\left(\underbrace{X_2^{E^{1,p_0-1}} + cX_1^{E^{1,p_0-1}}}_{\tilde{X}_2}\right)$$

making of the basis $\left\{X_1^{E^{1,p_0-1}}, \tilde{X}_2\right\}$ an adapted basis for $\left(S^{E^{1,p_0-1}}\right)_{c_{p_0}}$. Keeping on blowing-up along the branch until the point c_{p_1}, we get a succession of adapted bases. Now, at the point c_{p_1}, we have to prove that the vector field $X_1^{E^{1,p_1-1}}$ satisfies the inequality

$$\nu_{c_{p_1}}\left(X_1^{E^{1,p_1-1}}\right) \leq \nu_{c_{p_1}}\left(\tilde{X}_2^{E^{p_0,p_1-1}}\right). \tag{3.4}$$

Indeed, if the above inequality does not hold then there is no hope to obtain a vector field Y which satisfies both inequalities (3.3) and (3.4), which means optimal for both curves $\left(S^{E^{1,p_0-1}}\right)_{c_{p_0}}$ and $\left(S^{E^{1,p_1-1}}\right)_{c_{p_1}}$. However, we can establish the lemma below adapted to a branch along which the successive blown-up curves have the following type

$$\underset{\mathtt{2}}{c_{p_0}} \leftarrow \underset{\mathtt{1}}{c_{p_0+1}} \leftarrow \underset{\mathtt{1}}{c_{p_0+2}} \leftarrow \cdots \leftarrow \underset{\mathtt{1}}{c_{p_1-1}} \leftarrow \underset{\mathtt{2}}{c_{p_1}}.$$

Lemma 12. *There exists a vector field Y optimal for $\left(S^{E^{1,p_0-1}}\right)_{c_{p_0}}$ such that $Y^{E^{p_0,p_1-1}}$ is optimal for $\left(S^{E^{1,p_1-1}}\right)_{c_{p_1}}$.*

Proof. Applying inductively the property (2) of Theorem 11 to $\left(S^{E^{1,p_1-1}}\right)_{c_{p_1}}$ yields a germ of smooth curve l such that

$$\left((S \cup l)^{E^{1,p_1-1}}\right)_{c_{p_1}}$$

is not of type $(\mathfrak{E}'_d)$. Since $\left(S^{E^{1,p_0-1}}\right)_{c_{p_0}}$ is of class $\mathtt{2}$ then $\left((S \cup l)^{E^{1,p_0-1}}\right)_{c_{p_0}}$ is of class $\mathtt{1}$. Let us consider an adapted basis $\{Y_1, Y_2\}$ for $\left((S \cup l)^{E^{1,p_0-1}}\right)_{c_{p_0}}$. Since the latter is of class $\mathtt{1}$, one has

$$\nu_{c_{p_0}}(Y_1) = \nu_{c_{p_0}}(Y_2).$$

The vector fields Y_1 and Y_2 leave invariant the smooth curve $l^{E^{1,p_0-1}}$. Thus there exists a germ of analytic function ϕ such that the vector field

$$Y_1 - \phi Y_2$$

can be divided by a reduced equation L of the curve $l^{E^{1,p_0-1}}$. According to the criterion of Saito (1980), $\{Y_1, Y_2\}$ being a Saito basis for $S^{E^{1,p_0-1}}$, the vector fields Y_1 and Y_2 satisfy

$$\det(Y_1, Y_2) = uF,$$

where u is a unity and F is a local equation of $S^{E^{1,p_0-1}}$. Therefore, one has

$$\det\left(\frac{Y_1 - \phi Y_2}{L}, Y_2\right) = u\frac{F}{L},$$

and, still following the criterion of Saito, the curve $\left(S^{E^{1,p_0-1}}\right)_{c_{p_0}}$ admits the family

$$\left\{\tilde{Y}_1 = \frac{Y_1 - \phi Y_2}{L}, Y_2\right\}$$

as Saito basis. Notice that Y_2 is still tangent to $l^{E^{1,p_0-1}}$ and that

$$\nu_{c_{p_0}}\left(\tilde{Y}_1\right) < \nu_{c_{p_0}}(Y_2).$$

In particular, $\tilde{Y}_1$ is optimal for $S^{E^{1,p_0-1}}$. Now suppose that

$$\nu_{c_{p_1}}\left(\tilde{Y}_1^{E^{p_0,p_1-1}}\right) \geq \nu_{c_{p_1}}\left(Y_2^{E^{p_0,p_1-1}}\right) + 1.$$

Then, multiplying by $L^{E^{p_0,p_1-1}}$ leads to

$$\nu_{c_{p_1}}\left(L^{E^{p_0,p_1-1}}\tilde{Y}_1^{E^{p_0,p_1-1}}\right) \geq \nu_{c_{p_1}}\left(Y_2^{E^{p_0,p_1-1}}\right) + 2 \tag{3.5}$$

since $\nu_{c_{p_1}}\left(L^{E^{p_0,p_1-1}}\right) = 1$. The family

$$\left\{L^{E^{p_0,p_1-1}}\tilde{Y}_1^{E^{p_0,p_1-1}}, Y_2^{E^{p_0,p_1-1}}\right\}$$

is a Saito basis for $\left((S \cup l)^{E^{1,p_1-1}}\right)_{c_{p_1}}$. However, the inequality (3.5) implies that the latter curve is of type $(\mathfrak{E}'_d)$, which is a contradiction with the choice of l. Therefore, one has

$$\nu_{c_{p_1}}\left(\tilde{Y}_1^{E^{p_0,p_1-1}}\right) \leq \nu_{c_{p_1}}\left(Y_2^{E^{p_0,p_1-1}}\right)$$

and $\tilde{Y}_1$ satisfies the lemma. $\square$

The property established in the lemma is also satisfied by $X_1^{E^{1,p_1-1}}$ since one can write

$$X_1^{E^{1,p_1-1}} = a\tilde{Y}_1 + bY_2$$

where $a, b \in \mathcal{O}_2$ with a a unity. Thus, $X_1^{E^{1,p_1-1}}$ is optimal for $\left(S^{E^{1,p_1-1}}\right)_{c_{p_1}}$ and, repeating the arguments along the whole branch, we can see that the optimality property propagates.

Finally, for any branch B, we consider a vector field X_B optimal along the branch B and a generic combination of the form

$$\sum \alpha_B X_B, \ \alpha_B \in \mathbb{C}$$

The latter is a Saito vector field for S, which finishes the proof of property (1).

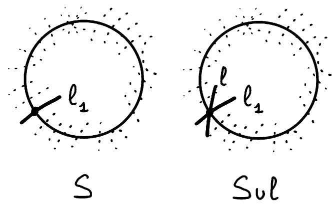


Fig. 3.2. Topology of the Saito vector fields of S and $S \cup l$.

Now, let us prove the second statement of Theorem 11. The vector field X_1 being Saito, Theorem 8 ensures that his topological data provide an admissible solution of the system $\left(\mathcal{H}^S\right)$. Since S is a union of smooth curves, for any $i = 2, \ldots, M$,

$$\delta_i^S = \Delta_1^S.$$

In particular, the following relation holds

$$\epsilon_1^S = \frac{n_1^S}{2} + \left\lfloor \begin{matrix} \Delta_1^S \\ \frac{1}{2} \end{matrix} \right._{\nu_1} + \frac{M-1}{2}\Delta_1^S - \sum_{k=2}^{M} \left\lfloor \begin{matrix} \Delta_k^S \\ \frac{1}{2} \end{matrix} \right._{\nu_k - \Delta_1^S} . \tag{3.6}$$

Suppose that $\nu(S_1)$ is even, then for any smooth curve l, the valuation $\nu(S_1 \cup l)$ is odd, thus $S \cup l$ cannot be of type $\left(\mathfrak{E}'_d\right)$. Hence, we may suppose $\nu(S_1)$ odd. If S is of type $(\mathfrak{O}_d)$ then for any generic smooth curve $S \cup l$ is of type $(\mathfrak{E}_d)$ and not of type $\left(\mathfrak{E}'_d\right)$. If S is of type $(\mathfrak{O})$ then $S \cup l$ is of type $(\mathfrak{E})$. Thus, we can also suppose that S is of type $\left(\mathfrak{O}'_d\right)$. It remains a couple of cases to investigate

Case 1. $n_1^S > 0$. Let l_1 be some irreducible component of S such that l_1^E is attached to D_1. Let l be any germ of smooth curve such that l^E and $l_1^{E_1}$ are transverse but attached to the same point of D_1. Notice that S and $S \cup l$ are not in the same topological class, but both can be supposed as generic as necessary in their own topological class. We assert that $S \cup l$ cannot be of type $\left(\mathfrak{E}'_d\right)$. Indeed, if it was so, then the multiplicity of its Saito vector field $X_1^{S\cup l}$ would be equal to

$$\nu\left(X_1^{S\cup l}\right) = \frac{\nu(S_1)+1}{2} - 1 = \frac{\nu(S_1)-1}{2} = \nu(X_1) \tag{3.7}$$

which is exactly the multiplicity of a Saito vector field for S. However, one can obtain the topology of $X_1^{S\cup l}$ from the one of X_1 provided that S is of type $\left(\mathfrak{O}'_d\right)$ and $S \cup l$ is of type $\left(\mathfrak{E}'_d\right)$. As depicted in Fig. 3.2 it consists in replacing the invariant smooth curve l_1 by two tangent smooth curves l and l_1 that are transverse after the first blowing-up. In the process, it can be seen that the valuation of the associated vector field increases by one, which contradicts the equality (3.7).

Case 2. $n_1^S = 0$. Since S is of type $\left(\mathfrak{O}'_d\right)$, it follows from Genzmer (2020) that one has

$$\epsilon_1^S = 2 - \sum_{k=2}^{M} \Delta_k^S.$$

Combining with the relation (3.6) yields

$$\sum_{k=2}^{M} \left\lfloor \begin{matrix} 0 \\ \Delta_k^S - \frac{1}{2} \end{matrix} \right._{\nu_k} = \frac{3}{2}. \tag{3.8}$$

Let us consider l a generic germ of smooth curve such that l^{E_1} is attached to S_2 and suppose that $S \cup l$ is of type $\left(\mathfrak{E}'_d\right)$. Applying the same arguments as above leads to

$$\sum_{k=3}^{M} \left\lfloor \begin{matrix} 0 \\ \Delta_k^{S\cup l} - \frac{1}{2} \end{matrix} \right._{\nu_k} + \left\lfloor \begin{matrix} 0 \\ \Delta_2^{S\cup l} - \frac{1}{2} \end{matrix} \right._{\nu_2+1} = 2.$$

Since l is attached to S_2, the curve $S\cup l$ satisfies

$$\forall k \neq 2,\ (S\cup l)_k = S_k.$$

Now Proposition 8 ensures the unicity of the family of integer $\left(\Delta_k^S\right)$, which provides a compatible solution to $\left(\mathcal{H}^S\right)$. Therefore, for any $k\neq 2$, one has

$$\Delta_k^{S\cup l} = \Delta_k^S.$$

Combining the two relations above yields

$$\left\lfloor \begin{matrix} 0 \\ \Delta_2^{S\cup l} - \frac{1}{2} \end{matrix} \right._{\nu_2+1} - \frac{1}{2} = \left\lfloor \begin{matrix} 0 \\ \Delta_2^{S} - \frac{1}{2} \end{matrix} \right._{\nu_2}.$$

In particular, if ν_2 is odd, then $\Delta_2^S = 0$. Thus, if for any generic germ of smooth curve l such that l^{E_1} is attached to S_i for $i = 2, \ldots, k$, the curve $S\cup l$ is of type $\left(\mathfrak{E}'_d\right)$, then we have the following alternative: either ν_i is even or $\Delta_i^S = 0$. But the latter contradicts (3.8). $\square$

4. Number of moduli of S

According to Mattei (2000), if S is any curve generic in its topological class, its number of moduli, denoted by $\mathbb{M}^S$, is equal to

$$\mathbb{M}^S = \dim_{\mathbb{C}} H^1\left(D, \mathfrak{X}_S|_D\right)$$

where $\mathfrak{X}_S$ is the sheaf of germs of vector fields on $\widetilde{\mathbb{C}^2}$ tangent to the total transform $E^{-1}(S)$ of S by E. Indeed, the first group of cohomology of the sheaf $\mathfrak{X}_S|_D$ can be identified as the tangent space to the space of parameters of any miniversal deformation of S. This dimension can be *inductively* computed along the desingularization of S following the result below.

Theorem *(Genzmer, 2022). For **any** germ of curve S, the number of moduli $\mathbb{M}^S$ is written*

$$\mathbb{M}^S = \dim_{\mathbb{C}} H^1\left(D_1, \mathfrak{X}_S^1|_{D_1}\right) + \sum_{k=2}^{M} \mathbb{M}^{S_k\cup D_1}$$

where $\mathfrak{X}_S^1$ is the sheaf of germs of vector fields on the total space of E_1 tangent to the total transform $E_1^{-1}(S)$.

This theorem is the one upon which our whole strategy is based.

Following Genzmer (2020) and setting

$$\sigma(S) = \dim_{\mathbb{C}} H^1\left(D_1, \mathfrak{X}_S^1|_{D_1}\right),$$

one has: if S is *generic* in its topological class then

$$\sigma\left(S\right)=\begin{cases}\dfrac{\left(\nu\left(S\right)-2\right)\left(\nu\left(S\right)-4\right)}{4} & \text{if } S \text{ is of type } (\mathfrak{C})\\ \dfrac{\left(\nu\left(S\right)-3\right)^2}{4} & \text{if } S \text{ is of type } (\mathfrak{D})\\ \dfrac{\left(\nu\left(S\right)-2\right)\left(\nu\left(S\right)-4\right)}{4}-1+\epsilon_1^S+\displaystyle\sum_{k=2}^{M}\Delta_k^S & \text{if } S \text{ is of type } (\mathfrak{C}_d)\\ \dfrac{\left(\nu\left(S\right)-3\right)^2}{4}-2+\epsilon_1^S+\displaystyle\sum_{k=2}^{M}\Delta_k^S & \text{if } S \text{ is of type } (\mathfrak{D}_d)\end{cases} \tag{4.1}$$

Thus the expression of $\sigma\left(S\right)$ depends firstly, on the type of the curve S, secondly, on some topological data associated to S and its Saito vector field. When S is a union of germs of smooth curves, these data can be obtained from an admissible solution of $\left(\mathcal{H}^S\right)$ since Proposition 14 and Theorem 8 assert that this solution is unique and given precisely by the topological data of a Saito foliation for S.

The formula (4.1) recovers the number of moduli computed by Granger for the curve $\Sigma_n : x^n + y^n = 0$ in Granger (1979), $n \geq 2$. Indeed, this curve is desingularized by one blowing-up and it can be seen that for $n \geq 5$, it is of type $(\mathfrak{C}_d)$ or $(\mathfrak{D}_d)$ with

$$\epsilon_1^{\Sigma_n} = \left\lfloor \frac{\frac{n}{2}}{\frac{n+1}{2}} \right.\\ n \; .$$

Therefore, one has

$$\mathbb{M}^{\Sigma_n} = \sigma\left(\Sigma_n\right) = \begin{cases}\frac{(n-2)^2}{4} & \text{if } n \text{ is even}\\ \frac{(n-1)(n-3)}{4} & \text{if } n \text{ is odd}\end{cases},$$

which is in accordance with the results of Granger.

Algorithm 1 computes an admissible solution of $\left(\mathcal{H}^S\right)$. The proof of Algorithm 1 is given in Appendix. Algorithm 2 computes the number of moduli of S.

Example 13. Let us consider the curve $S_{4,2,4}$ defined by the proximity matrix

$$P^{S_{4,2,4}} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

and the integers $n_1^S = 4,\ n_2^S = 2,\ n_3^S = 4$. The curve $S_{4,2,4}$ is topologically equivalent to the one given in some coordinates (x, y) by

$$\begin{aligned} y\left(y+x\right)\left(y+2x\right)\left(y+3x\right)\times\cdots \\ x\left(x+y^2\right)\times\cdots \\ \left(x-y^2\right)\left(x-y^2+y^3\right)\left(x-y^2+2y^3\right)\left(x-y^2+3y^3\right)=0 \end{aligned}$$

Then we get the data summarized in Table 2. Therefore, the number of moduli of $S_{4,2,4}$ is equal to $13+4+2=19$.

We implemented, among other procedures, the Algorithm 1 and Algorithm 2 on Sage 9.* to compute the number of moduli. See the routine *Courbes.Planes* following the link

https://perso.math.univ-toulouse.fr/genzmer/

Algorithm 1 Algorithm to compute an admissible solution of $\left(\mathcal{H}^S\right)$.

INPUT : $P^S,\ \left(n_1^S, \cdots, n_N^S\right)$
IF $P^S = (1)$:
IF $n_1^S = 1$: RETURN $(1, 1)$.
IF $n_2^S = 2$: RETURN $(2, 1)$.
IF $n_3^S \geq 3$: RETURN $\left(\frac{n_1^S}{2} + \left\lfloor \begin{matrix} 0 \\ \frac{1}{2} \end{matrix}, 0\right.\right)_{n_1^S}$.
IF $P^S \neq (1)$:
FOR $k = 2, \ldots, M$:
Extract from P^S the proximity matrix of P^{S_k}.
Apply Algorithm 1 to the input

$$\left(P^{S_k}, \left(n^{S_k}_{\sigma^k(1)}, \cdots, n^{S_k}_{\sigma^k(N_k)}\right)\right)$$

to get a family of admissible solutions $\left\{\left(\mathcal{E}^{S_k}, \Delta^{S_k}\right)\right\}_{k=2,\ldots,M}$.
SET

$$\Delta_k^{S,0} = \begin{cases} 0 & \text{if } k = 1 \\ \Delta_j^{S_i} & \text{if } k = \sigma^i(j). \end{cases}$$

and the associated vector $\mathfrak{S}^{S,0}$.
Apply Algorithm 1 to the input

$$\left(P^{S_k}, \left(n^{S_k}_{\sigma^k(1)} + 1, \cdots, n^{S_k}_{\sigma^k(N_k)}\right)\right)$$

to get a family of admissible solutions $\left\{\left(\mathcal{E}^{S_k \cup D_1}, \Delta^{S_k \cup D_1}\right)\right\}_{k=2,\ldots,M}$.
SET

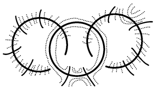
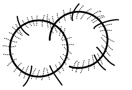
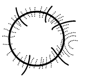
$$\Delta_k^{S,1} = \begin{cases} 1 & \text{if } k = 1 \\ \Delta_j^{S_i} & \text{if } k = \sigma^i(j). \end{cases}$$

and the associated vector $\mathfrak{S}^{S,1}$.
RETURN : **Exactly one** solution of $\left(\mathcal{H}^S\right)$ defined by $\left(\left(P^S\right)^{-1}\mathfrak{S}^{S,0}, \Delta^{S,0}\right)$ and $\left(\left(P^S\right)^{-1}\mathfrak{S}^{S,1}, \Delta^{S,1}\right)$ is admissible.

Algorithm 2 Algorithm to compute the number of moduli of S.

INPUT : $P^S,\ \left(n_1^S, \cdots, n_N^S\right)$
IF $P^S = (1)$:
RETURN : $\begin{cases} \frac{\left(n_1^S - 2\right)^2}{4} & \text{if } n_1^S \text{ is even} \\ \frac{\left(n_1^S - 1\right)\left(n_1^S - 3\right)}{4} & \text{if } n_1^S \text{ is odd} \end{cases}$.
IF $P^S \neq (1)$:
Compute an admissible solution for $\left(\mathcal{H}^S\right)$ by Algorithm 1.
Determine the type of the curve S and the values of ϵ_1^S and $\left(\Delta_k^S\right)_{k=2,\ldots,M}$.
Compute $\sigma(S)$.
FOR $k = 2, \ldots, M$:
Compute inductively the number $\mathbb{M}^{S_k \cup D_1}$.
RETURN : $\mathbb{M}^S = \sigma(S) + \sum_{k=2}^M \mathbb{M}^{S_k \cup D_1}$.

Table 2
Number of moduli of $S_{4,2,4}$.

	S_1	$S_2 \cup D_1$	$S_3 \cup D_2$
Type	$(\mathfrak{E}_d)$	$(\mathfrak{O}_d)$	$(\mathfrak{O}_d)$
Saito Picture			
P^S	$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$	(1)
$\nu(S_k)$	$10,\ 6,\ 4$	$7,\ 4$	5
n_i^S	$4,\ 2,\ 4$	$3,\ 4$	5
Δ_i^S	$(0,1,0)$	$(0,0)$	(0)
δ_i^S	$(0,0,1)$	$(0,0)$	(0)
$\mathfrak{S}^S$	$\begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 4 \\ 2 \end{pmatrix}$	(3)
$\mathcal{E}^S$	$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 2 \end{pmatrix}$	(3)
$\sigma(S)$	13	4	2

5. Appendix

Once a proximity matrix P^S and a family of integers $\left(n_1^S, \cdots, n_k^S\right)$ are given, one can choose an arbitrary vector Δ^S, compute the associated integers δ_k^S and obtain the vector $\mathfrak{S}^S$. Then, the invertible system $\left(\mathcal{H}^S\right)$

$$\left(\mathcal{H}^S\right): \ \left(P^S\right)^{-1} \mathcal{E}^S = \mathfrak{S}^S$$

provides a unique corresponding solution $\mathcal{E}^S$. However, there is no reason for this solution $\left(\mathcal{E}^S, \Delta^S\right)$ to be admissible. Nevertheless, we will prove that for any union of germs of smooth curves - not necessarily generic in its topological class - we have the following proposition.

Proposition 14. *If S is a union of germs of smooth curves, there exists a unique choice of Δ^S such that the associated solution of $\left(\mathcal{H}^S\right)$ is admissible.*

To prove the above proposition, first, let us establish a lemma that describes the behavior of the system $\left(\mathcal{H}^S\right)$ when one goes from S to $S \cup l$ where l is somehow a generic smooth curve.

Lemma 15. *Let l be a smooth curve such that the strict transform $E_1^\star l$ is attached at D_1 at no point of attachment of any component of $E_1^\star S$.*

- *If there exists Δ^S with $\Delta_1^S = 0$, such that the solution of $\left(\mathcal{H}^S\right)$ is admissible, then the same Δ^S provides an admissible solution for the system $\left(\mathcal{H}^{S\cup l}\right)$.*
- *If $\nu(S)$ is odd and there exists Δ^S with $\Delta_1^S = 1$, such that the solution of $\left(\mathcal{H}^S\right)$ is admissible, then the same Δ^S provides an admissible solution for the system $\left(\mathcal{H}^{S\cup l}\right)$.*

Proof. The lemma above can be seen on the behavior of the system $\left(\mathcal{H}^S\right)$ when n_1^S is increased by 1: indeed the hypothesis on l ensures that E is still the desingularization process of $S \cup l$ and that

$$\begin{cases} n_1^{S\cup l} = n_1^S + 1 \\ n_k^{S\cup l} = n_k^S & \forall k \neq 1 \end{cases}$$

Since S is a union of smooth curves, its proximity matrix P^S is written

$$P^S = \begin{pmatrix} 1 & -1 & \cdots & -1 & 0 & \cdots & 0 \\ \vdots & & & \vdots & & & \vdots \end{pmatrix}$$

with the number -1 repeated M times on the first line. Therefore, one can expand the expression of ϵ_1^S as below

$$\epsilon_1^S = \frac{\nu(S_1) - \delta_1^S}{2} + \left\lfloor \begin{matrix} \Delta_1^S \\ \frac{1}{2} \end{matrix} \right._{\nu(S_1)-\delta_1^S} - \sum_{k=2}^{M} \left(\frac{\nu(S_k) - \delta_k^S}{2} + \left\lfloor \begin{matrix} \Delta_k^S \\ \frac{1}{2} \end{matrix} \right._{\nu(S_k)-\delta_k^S} \right) \tag{5.1}$$

By construction, $\delta_1^S = 0$. Now, if $\Delta_1^S = 0$, since the solution $(\mathcal{E}^S, \Delta^S)$ is admissible, one has $\epsilon_1^S \geq 2 - \sum_{i=2}^{M} \Delta_i^S$. If n_1^S is increased by one, it does not affect $\nu(S_k)$ for $k \geq 2$ but it changes $\nu(S_1)$ into $\nu(S_1) + 1$. However, if $\nu(S_1)$ is even then

$$\frac{\nu(S_1 \cup l)}{2} + \left\lfloor \begin{matrix} 0 \\ \frac{1}{2} \end{matrix} \right._{\nu(S_1\cup l)} = \frac{\nu(S_1)}{2} + \left\lfloor \begin{matrix} 0 \\ \frac{1}{2} \end{matrix} \right._{\nu(S_1)} + 1$$

and if $\nu(S_1)$ is odd then

$$\frac{\nu(S_1 \cup l)}{2} + \left\lfloor \begin{matrix} 0 \\ \frac{1}{2} \end{matrix} \right._{\nu(S_1\cup l)} = \frac{\nu(S_1)}{2} + \left\lfloor \begin{matrix} 0 \\ \frac{1}{2} \end{matrix} \right._{\nu(S_1)}$$

Thus setting $\epsilon_1^{S\cup l} = \epsilon_1^S$ or $\epsilon_1^S + 1$ depending on $\nu(S_1)$ being odd or even and

$$\begin{cases} \Delta_i^{S\cup l} = \Delta_i^S & i = 1, \ldots, N \\ \epsilon_i^{S\cup l} = \epsilon_i^S & i \neq 1 \end{cases}$$

yields an admissible solution of the system $(\mathcal{H}^{S\cup l})$.

Now, if $\nu(S_1)$ is odd and $\Delta_1^S = 1$, then one has

$$\epsilon_1^S = \frac{\nu(S_1) + 1}{2} - \sum_{k=2}^{M} \left(\frac{\nu(S_k) - \delta_k^S}{2} + \left\lfloor \begin{matrix} \Delta_k^S \\ \frac{1}{2} \end{matrix} \right._{\nu(S_k)-\delta_k^S} \right)$$

and $\epsilon_1^S \geq n_1^S$. The multiplicity $\nu(S_1)$ being odd, one has

$$\frac{\nu(S_1 \cup l)}{2} + \left\lfloor \begin{matrix} 1 \\ \frac{1}{2} \end{matrix} \right._{\nu(S_1\cup l)} = \frac{\nu(S_1) + 1}{2} + 1.$$

Thus setting $\epsilon_1^{S\cup l} = \epsilon_1^S + 1$ yields an admissible solution of the system for $S \cup l$ since

$$\epsilon_1^{S\cup l} = \epsilon_1^S + 1 \geq n_1^S + 1 = n_1^{S\cup l}. \quad \square$$

Proof of the proposition. The proof is by induction on the length of the desingularization of S. First, let us prove the proposition for a curve S desingularized after one blowing-up. The system $(\mathcal{H}^S)$ reduces to the sole equation

$$\epsilon_1^S = \frac{\nu(S_1)}{2} + \left\lfloor \frac{\Delta_1^S - \frac{1}{2}}{\nu(S_1)} \right\rfloor \tag{5.2}$$

If $\nu(S_1) = n_1^S = 1$ or 2 then $\Delta_1^S = 1$ is the unique admissible choice since ϵ_1^S is respectively equal to 1 and 2, which are all bigger than the respective n_1^S, whereas if $\Delta_1^S = 0$ one finds always 1 which is smaller than $2 = 2 - \sum_{i \in \mathfrak{N}(1)} \Delta_i^S$. If $\nu(S_1) = n_1^S \geq 3$ then $\epsilon_1^S < n_1^S$ thus $\Delta_1^S = 1$ is excluded. However, $\Delta_1^S = 0$ brings an admissible solution to the equation (5.2) since $\epsilon_1^S \geq 2$.

For the inductive step, let us consider the unique choice $\left(\Delta_k^{S_i,0}\right)_{k=1,\ldots,N_i}$ provided by the inductive application of the proposition to each S_i. In the same way, consider the unique choice $\left(\Delta_k^{S_i,1}\right)_{k=1,\ldots,N_i}$ obtained when the proposition is applied to each $S_i \cup D_1$. Then we set

$$\Delta^{S,0}\colon \Delta_k^{S,0} = \begin{cases} 0 & \text{if } k = 1 \\ \Delta_j^{S_i,0} & \text{if } k = \sigma^i(j), \end{cases}$$

and

$$\Delta^{S,1}\colon \Delta_k^{S,1} = \begin{cases} 1 & \text{if } k = 1 \\ \Delta_j^{S_i,1} & \text{if } k = \sigma^i(j). \end{cases}$$

Notice that the above values of $\Delta_k^{S_i,\star}$ are well defined since for any $k \neq l$, one has

$$\mathrm{Im}\sigma^k \cap \mathrm{Im}\sigma^l = \emptyset.$$

From these data, we can compute the integers $\delta_k^{S,\star}$, $\star = 0, 1$ and the vectors of integers $\mathfrak{S}^{S,\star}$, $\star = 0, 1$ respectively associated to $\Delta^{S,0}$ and $\Delta^{S,1}$. Then, we can consider the associated solutions

$$\left(\mathcal{E}^{S,\star}, \Delta^{S,\star}\right) \tag{5.3}$$

of the invertible system $\left(\mathcal{H}^S\right)$.

For any $k = 2, \ldots, M$, $\delta_k^{S,0} = 0$. Thus, the vectors $\mathfrak{S}^{S,0}$ and $\mathfrak{S}^{S_k}$ satisfy

$$\left(\mathfrak{S}^{S,0}\right)_k = \left(\mathfrak{S}^{S_k}\right)_1, \quad k = 2, \ldots, M.$$

Moreover, for any $k = 2, \ldots, M$, $\delta_k^{S,1} = 1$ and therefore

$$\left(\mathfrak{S}^{S_k \cup D_1}\right)_1 = \frac{\nu(S_k \cup D_1)}{2} + \left\lfloor \frac{\Delta_1^{S_k \cup D_1} - \frac{1}{2}}{\nu(S_k \cup D_1)} \right\rfloor = \frac{\nu(S_k) - 1}{2} + \left\lfloor \frac{\Delta_k^{S,1} - \frac{1}{2}}{\nu(S_k) - 1} \right\rfloor + 1 = \left(\mathfrak{S}^{S,1}\right)_k + 1.$$

In particular, for $k = 2, \ldots, M$, one has

$$\epsilon_1^{S_k \cup D_1} = \left(\mathcal{E}^{S_k \cup D_1}\right)_1 = \left(\mathcal{E}^{S,1}\right)_k + 1 = \epsilon_k^{S,1} + 1.$$

Since

$$n_1^{S_k \cup D_1} = n_k^S + 1,$$

we conclude that both solutions (5.3) for $\star = 0, 1$ satisfy the compatibility conditions (3.1) for any $k = 2, \ldots, M$ and also for any $k \geq M + 1$, since for $k \geq M + 1$,

$$\begin{array}{l} \left(\mathfrak{S}^{S,0}\right)_k = \left(\mathfrak{S}^{S,1}\right)_k = \left(\mathfrak{S}^{S_i}\right)_j \\ n_k^{S,0} = n_k^{S,1} = n_j^{S_i} \end{array} \quad \text{where } k = \sigma^i(j).$$

To finish the proof, we are going to see that exactly one of the solutions (5.3) satisfies the compatibility condition for $k=1$.

Following (5.1) one has

$$\epsilon_1^{S,1}+\epsilon_1^{S,0}=\nu(S_1)+\underbrace{\left\lfloor \begin{matrix} 1 \\ \frac{1}{2} \end{matrix}\right._{\nu(S_1)}+\left\lfloor \begin{matrix} 0 \\ \frac{1}{2} \end{matrix}\right._{\nu(S_1)}}_{=1}$$

$$-\sum_{i=2}^{M}\left(\frac{\nu(S_i)-1}{2}+\left\lfloor \begin{matrix} \Delta_i^{S,1} \\ \frac{1}{2} \end{matrix}\right._{\nu(S_i)-1}\right)-\sum_{i=2}^{M}\left(\frac{\nu(S_i)}{2}+\left\lfloor \begin{matrix} \Delta_i^{S,0} \\ \frac{1}{2} \end{matrix}\right._{\nu(S_i)}\right).$$

Observe that $\nu(S_1)-\sum_{i=2}^{M}\nu(S_i)=n_1^S$. Thus, the relation above reduces to

$$\begin{aligned}\epsilon_1^{S,1}+\epsilon_1^{S,0}&=n_1^S+1-\sum_{i=2}^{M}\left(\frac{-1}{2}+\left\lfloor \begin{matrix} \Delta_i^{S,1} \\ \frac{1}{2} \end{matrix}\right._{\nu(S_i)-1}+\left\lfloor \begin{matrix} \Delta_i^{S,0} \\ \frac{1}{2} \end{matrix}\right._{\nu(S_i)}\right)\\ &=n_1^S+1-\sum_{i=2}^{M}\left\lfloor \begin{matrix} \Delta_i^{S,0} \\ \Delta_i^{S,1} \end{matrix}\right._{\nu(S_i)}.\end{aligned}$$

If $\nu(S_i)$ is even then $\left\lfloor \begin{matrix} \Delta_i^{S,0} \\ \Delta_i^{S,1} \end{matrix}\right._{\nu(S_i)}=\Delta_i^{S,0}$. If $\nu(S_i)$ is odd, then

$$\left\lfloor \begin{matrix} \Delta_i^{S,0} \\ \Delta_i^{S,1} \end{matrix}\right._{\nu(S_i)}=\Delta_i^{S,1}.$$

But Lemma 15 ensures that

$$\begin{aligned}\Delta_i^{S,0}=0 &\implies \Delta_i^{S,1}=0\\ \nu(S)\text{ odd and }\Delta_i^{S,0}=1 &\implies \Delta_i^{S,1}=1\end{aligned}$$

Thus whether $\nu(S_i)$ is odd or even, one has

$$\left\lfloor \begin{matrix} \Delta_i^{S,0} \\ \Delta_i^{S,1} \end{matrix}\right._{\nu(S_i)}=\Delta_i^{S,0}.$$

Finally, we are led to the relation

$$\epsilon_1^{S,1}+\epsilon_1^{S,0}=n_1^S+1-\sum_{i=2}^{M}\Delta_i^{S,0}. \tag{5.4}$$

Hence, one of the following inequalities holds

$$\epsilon_1^{S,1}\geq n_1^S$$

or

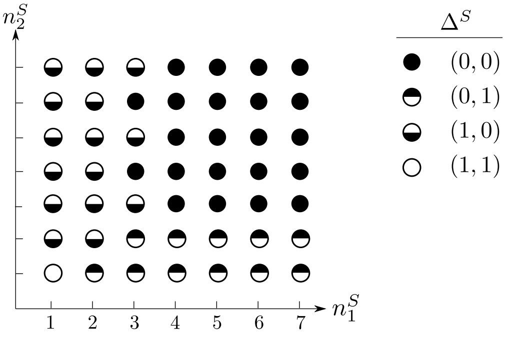


Fig. 5.1. Unique admissible choice of $\Delta^S = (\Delta_1^S, \Delta_2^S)$.

$$\epsilon_1^{S,0} \geq 2 - \sum_{i=2}^{M} \Delta_i^{S,0}$$

the two being mutually exclusive according to (5.4). By induction, this concludes the proof. □

Example 16. Suppose that S is desingularized after two successive blowing-ups, then its proximity matrix is written

$$P^S = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

In Fig. 5.1, we present the unique choice of $\Delta^S = (\star, \star) \in \{0,1\}^2$ depending on n_1^S and n_2^S that leads to an admissible solution of $(\mathcal{H}^S)$.

For instance, if $n_1^S = 3$ and $n_2^S = 5$ then $\nu(S_1) = 8$ and $\nu(S_2) = 5$. Setting $\Delta^S = (1,0)$ yields

$$\mathcal{E}^S = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

One can check that

$$\left(P^S\right)^{-1} \mathcal{E}^S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{8-0}{2} + 1 \\ \frac{5-1}{2} \end{pmatrix} = \mathfrak{S}^S$$

and

$$\epsilon_1^S = 3 \geq n_1^S = 3 \qquad \epsilon_2^S = 2 \geq 2 - 0$$

so the solution $\left(\mathcal{E}^S, \Delta^S\right)$ is admissible.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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