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Dimension of the Moduli Space of a Germ of Curve in $\mathbb{C}^2$

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In this article, we prove a formula that computes the generic dimension of the moduli space of a germ of irreducible curve in the complex plane. It is obtained from the study of the Saito module associated to the curve, which is the module of germs of holomorphic 1-forms leaving the curve invariant.

Introduction

In 1973, in his lecture [27], Zariski started the systematic study of the analytic classification of *the branches* of the complex plane, which are germs of irreducible curves at the origin of $\mathbb{C}^2$. The general purpose was to describe as accurately as possible the moduli space of S that is the quotient of the topological class of S by the action of the group $\mathrm{Diff}\left(\mathbb{C}^2,0\right)$,

$$\mathbb{M}(S) = \left\{S' \middle| S' \text{ topologically equivalent to } S\right\} \Big/ \mathrm{Diff}\left(\mathbb{C}^2,0\right).$$

The Puiseux parametrization of a branch $S = \{\gamma(t) | t \in (\mathbb{C},0)\}$ written

$$\gamma : \begin{cases} x = t^p \\ y = t^q + \sum_{k>q} a_k t^k \end{cases}, \ p < q,\ p \nmid q,\ t \in (\mathbb{C},0) \tag{0.1}$$

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highlights two basic topological invariants, namely the integers p and q. In the whole article, we will denote them by $p(S)$ and $q(S)$, or simply p and q when no confusion is possible. The integer $p(S)$ corresponds to the algebraic multiplicity of the branch S. This is also the algebraic multiplicity at $(0,0)$ of any irreducible function $f \in \mathbb{C}\{x,y\}$ that vanishes along S. Actually, Zariski proved that the whole topological classification depends on a sub-semigroup Γ_S of $\mathbb{N}$ defined by

$$\Gamma_S = \{\nu(f \circ \gamma) | f \in \mathbb{C}\{x,y\}, f(0) = 0\},$$

where ν is the standard valuation of $\mathbb{C}\{t\}$.

Beyond the topological classification, Zariski proposed in [27] various approaches to achieve the analytical classification, introducing in particular the set Λ_S of valuations of Kähler differential forms for S

$$\Lambda_S = \left\{\nu\left(\gamma^*\omega\right) + 1 \middle| \omega \in \Omega^1\left(\mathbb{C}^2, 0\right)\right\} \supset \Gamma_S \setminus \{0\}.$$

Fixing the topological type—and thus the semigroup Γ_S above—Zariski gave a precise description of the associated moduli space for, for instance,

$$\Gamma_S = \langle 2,3\rangle, \ \langle 4,5\rangle \text{ or} \left\langle 4,6,\beta_2\right\rangle$$

and more generally for $\langle n, n+1\rangle$ and $\langle n, hn+1\rangle$. According to him, of special interest is *the generic component* of the moduli space: a finite determinacy property ensures that γ is analytically equivalent to a parametrization whose Taylor expansion is truncated at an order depending on the sole topological class. Having so a finite dimension family of branches, the theory of geometric invariants provides an open set of orbits of same dimension under the action of $\text{Diff}(\mathbb{C},0) \times \text{Diff}\left(\mathbb{C}^2,0\right)$, see [27, Chapter VI] or [6]. The image of this open set in the moduli space is the generic component studied by Zariski. In some sense, its dimension is the minimal number of parameters on which a universal family for the deformation of S depends. In the particular cases mentioned above, Zariski found an explicit formula of this dimension.

In fact, as far as we know, the 1st example of computation of the dimension of the generic component of the moduli space of a branch goes back to Ebey [6] who, anticipating in 1965 some ideas of Zariski, described not only the generic component, but also the whole moduli space of the branch whose semigroup is $\langle 5,9\rangle$. Hereafter, we will give some more details about the construction of Ebey. In 1978, Delorme [5] studied extensively the case of one Puiseux pair—$\Gamma_S = \langle m, n\rangle$ with $m \wedge n = 1$—and established some formulas to compute the generic dimension. In 1979, Granger [11] and later, in 1988, Briançon et al. [2] produced an algorithm to compute the generic dimension of the moduli space of a non irreducible quasi-homogeneous curve defined by $x^m+y^n = 0$ first, for m and n relatively prime, and then in the general case. The common denominator of the two previous works is the algorithmic approach based upon arithmetic properties of the continuous fraction expansion associated to the pair (m, n) . In 1988, Laudal et al. in [18], improved the work of Delorme and gave an explicit description of a universal family for S with $\Gamma_S = \langle m, n\rangle$, $m \wedge n = 1$ and a stratification of the moduli space. Finally, in 1998, Peraire exhibited an algorithm in [23] to compute the Tjurina number for a curve in its generic component when $\Gamma_S = \langle m, n\rangle$, $m \wedge n = 1$, which is linked to the dimension of the generic component.

From 2009, in a series of papers [14–16], Hefez and Hernandes achieved a impressive breakthrough in the problem of Zariski. They completed the analytical classification of irreducible germs of curves thanks to the set of valuations of Kähler differential forms. Moreover, they built an algorithm that describes very precisely the stratification of the moduli space in terms of the possible Λ_S for a given topological class, computes the dimension of each stratum and produces some normal forms corresponding to each stratum. One could consider that these works gave a definitive answer to the initial problem addressed by Zariski. Nevertheless, the disadvantage of the algorithmic approach is twofold: first, the high complexity of the algorithm—based upon Groebner basis routine—prevents its actual effectiveness as soon as the degree of the curve is big. Second, it is difficult to extract general geometric informations or formulas from it.

In 2010 and 2011, in [9, 10], Paul and the author described the moduli space of a topologically quasi-homogeneous curve S as the spaces of leaves of an algebraic foliation defined on the moduli of a foliation whose analytic invariant curve is precisely S. These works initiated an approach based upon the theory of foliations, which is at stake here.

In this article, we propose a construction relying basically, on one hand, on the desingularization of the curve S, on the other hand, on technics from the framework of the theory of holomorphic foliations. We intend to obtain an explicit formula for the *generic* dimension of the moduli space—the dimension of the generic stratum—that can be performed *by hand.*

The generic stratum of the moduli space

At first, let us give a definition of the moduli space of a germ of curve and its generic component in line with the ideas of Ebey [6].

Let $\mathrm{Top}\,(S)$ be the set of all germs of curves in $\left(\mathbb{C}^2,0\right)$ that are topologically equivalent to S. The group of changes of coordinates $\mathrm{Diff}\left(\mathbb{C}^2,0\right)$ acts naturally on $\mathrm{Top}\,(S)$ by

$$\phi\cdot S'=\phi\left(S'\right)\qquad \phi\in\mathrm{Diff}\left(\mathbb{C}^2,0\right),\ S'\in\mathrm{Top}\,(S)\,.$$

This action leads to a formal quotient and the following definition

Definition. The moduli space of S, denoted by $\mathbb{M}\,(S)$, is the quotient of $\mathrm{Top}\,(S)$ by the action of $\mathrm{Diff}\left(\mathbb{C}^2,0\right)$.

Notice that here the quotient is understood as a purely quotient in the category of sets: in particular, it does not have, a priori, any particular structure. Nevertheless, in [6], Ebey proved the following.

Theorem ([6], Theorems 4 and 6). Suppose that S is irreducible. Then, there exists a constructible subset A of $\mathbb{C}^P$ for some $P\in\mathbb{N}$ and an action of a connected solvable algebraic group G on A such that there exists a bijection

$$\Psi: A\to\mathrm{Top}\,(S)$$

that is equivariant for the action of G and $\mathrm{Diff}\left(\mathbb{C}^2,0\right)$. Moreover, there exists a constructible set $X\subset A$ such that each orbit of G on A meets exactly one point of X.

In other words, X is a transversal for the action of G on A. The set X being constructible, it can be written

$$X=\bigcup_i X_i,$$

where the above union is a finite union of open sets in affine set. Since X is a transversal, there is a unique i_0 such that the orbit of X_{i_0} under G is dense. The image of $X_{i_0}\setminus\mathrm{Sing}\left(X_{i_0}\right)$ by Ψ in the quotient $\mathbb{M}\,(S)$ is called *the generic component of* $\mathbb{M}\,(S)$. By construction, it is a smooth manifold and its dimension defines the generic dimension of $\mathbb{M}\,(S)$, denoted by $\dim_{\mathrm{gen}}\mathbb{M}\,(S)$. A curve will be said *generic* if its class in $\mathbb{M}\,(S)$ belongs to the generic component.

The proof of the theorem of Ebey is based upon the fact that an analytical class $[S']$ of $\mathbb{M}(S)$ is characterized by the class of its ring of functions $\mathcal{O}_{S'}$, which can be embedded in $\mathbb{C}\{t\}$ for some uniformizing parameter t. Ebey showed that the set of all sub-rings of $\mathbb{C}\{t\}$ that arise as a ring $\mathcal{O}_{S'}$ where $S' \in \mathrm{Top}(S)$ can be parametrized by some constructible set in some $\mathbb{C}^P$, $P \in \mathbb{N}$. Moreover, the action of $\mathrm{Diff}(\mathbb{C}^2, 0)$ on $\mathrm{Top}(S)$ reduces to the action on the sub-rings of $\mathbb{C}\{t\}$ of the group of finite jets of order Q, $Q \in \mathbb{N}$, $J^Q(\mathrm{Diff}(\mathbb{C}, 0))$ of all diffeomorphisms written

$$t \to \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_Q t^Q, \qquad \alpha_1 \neq 0.$$

Here, both integers P and Q depend only on the topological class of S.

Example. Let S be the curve $y^4 - x^5 = 0$. Following Ebey, the set of all $\mathcal{O}_{S'}$ where $S' \in \mathrm{Top}(S)$ can be parametrized by

$$\mathbb{C}\left[t^4, t^5 + at^6\right], \qquad a \in A = \mathbb{C}$$

and the action of $G = J^1(\mathrm{Diff}(\mathbb{C}, 0)) = \mathbb{C}^\star$ on $A = \mathbb{C}$ is written

$$\lambda \cdot a = \lambda a.$$

Thus, the transversal X of the theorem of Ebey is here $X = \{0, 1\}$ and the orbit of $\{1\}$ is dense in A. Its image in $\mathbb{M}(S)$ is the generic component whose dimension is 0. Finally, the moduli space $\mathbb{M}(S)$ consists in two points $\mathbb{M}(S) = \{\overline{0}, \overline{1}\}$ such that $\overline{0}$ is a dense point and $\overline{1}$ is a closed and open point. In particular, the topology of $\mathbb{M}(S)$ is not Hausdorff.

To compute a stratification of $\mathbb{M}(S)$ induced by the above construction, by no means, is easy. Hefez and Hernandes constructed an algorithm based upon Gröbner routines that describes very precisely a stratification of $\mathbb{M}(S)$ by the set of valuations of the Kähler differentials forms.

Let us describe an example of this stratification for the curve S defined by $y^4 - x^9 = 0$. This example is extracted from [14]. The set of all rings $\mathcal{O}_{S'}$ for S' in $\mathrm{Top}(S)$ can be parametrized by

$$\mathbb{C}\left[t^4, t^9 + at^{10} + bt^{11} + ct^{15} + dt^{19}\right], \qquad (a, b, c, d) \in \mathbb{C}^4.$$

The stratification is given in Table 1. The equation of each stratum is given in the coordinates (a, b, c, d). The symbol $\mathcal{S} \Subset \mathcal{S}'$ means the stratum $\mathcal{S}'$ contains $\mathcal{S}$ in its closure.

Table 1 Stratification of $\mathbb{M}\left(y^4 - x^9 = 0\right)$

$a=1$ $b=\frac{19}{18}$ $c\in\mathbb{C}$ $d=0$	$\Supset$	$a=1$ $b\neq\frac{19}{18}$ $c=0$ $d=0$	$\Subset$	$a=0$ $b=1$ $c=0$ $d=0$	$\Subset$	$a=0$ $b=0$ $c=1$ $d=0$	$\Subset$	$a=0$ $b=0$ $c=0$ $d=1$	$\Subset$	$a=0$ $b=0$ $c=0$ $d=0$

The 2nd stratum is the generic one. Its dimension is 1. The last one is the stratum reduced to the curve $\left(t^4, t^9\right)$, which is a point whose closure is the whole moduli space.

The dimension of the generic stratum

Let S be a germ of irreducible curve in the complex plane.

Theorem (Main Theorem). Let $E = E_1 \circ \cdots \circ E_N$ be the minimal desingularization of S. Let c_i be the center of E_i. Then

$$\dim_{\text{gen}} \mathbb{M}(S) = \sum_{i=1}^{N} \sigma\left(\nu_{c_i}\left(\left(E_1 \circ \cdots \circ E_{i-1}\right)^{-1}(S)\right)\right),$$

where $\nu_\star$ is the algebraic multiplicity at $\star$ and $\sigma(k) = \begin{cases} \frac{(k-3)^2}{4} & \text{if } k \text{ is odd} \\ \frac{(k-2)(k-4)}{4} & \text{else} \end{cases}$.

Notice that this formula depends only on some topological invariants of the curve S: in particular, it is not necessary to exhibit a curve in the generic component of the moduli space of S—that is in general difficult—to perform the computation above. One can take any curve in the topological class of S to compute the multiplicities involved in Theorem 2.

Remark. Actually, the proof performed here will lead us to a slightly more general result where the formula keeps on being the same but appears to be correct for any germ of curve of the form

$$S \cup d,$$

where d will be called a *direction* for S and will be defined later in the article. This trick will be helpful for the whole induction structure of the proof. However, for the sake of simplicity, we do not mention it directly in the theorem.

Example. In [27], Zariski showed that the dimension of the generic component of the moduli space of $S = \left\{y^n - x^{n+1} = 0\right\}$ is $\sigma(n)$. After one blowing-up E_1, the strict transform of S by E_1 is a smooth curve tangent to the exceptional divisor, thus for any $i \geq 2$, the multiplicity satisfy

$$\nu_{c_i}\left(\left(E_1 \circ \cdots \circ E_{i-1}\right)^{-1}(S)\right) \leq 3.$$

Example. More generally, for the semi-group $\Gamma_S = \langle n, nh+1\rangle$ with $h \geq 1$, the desingularization of S consists first in h successive blowing-ups, after which the curve is smooth. The algebraic multiplicity of the curve S is n. After $k < h$ blowing-ups, the strict transform of S is a curve whose topological class is given by the semi-group $\langle n, n(h-k)+1\rangle$ that is transverse to the exceptional divisor. Thus, according to the Main Theorem, one has

$$\begin{aligned}\dim_{\mathrm{gen}} \mathbb{M}\left(S_{\langle n,nh+1\rangle}\right) &= \sigma(n) + \underbrace{\sigma(n+1) + \cdots + \sigma(n+1)}_{h-1} + \sigma(3) + \cdots \\ &= \sigma(n) + (h-1)\,\sigma(n+1).\end{aligned}$$

This formula coincides with the one in [27].

Example. Let us consider the following Puiseux parametrization

$$S : \begin{cases} x &= t^8 \\ y &= t^{20} + t^{30} + t^{35} \end{cases}.$$

Its semigroup is $\langle 8, 20, 50, 105\rangle$ and its Puiseux pairs are $(2, 5)$, $(2, 15)$, and $(2, 35)$. Thus, S is not topologically quasi-homogeneous. The successive multiplicities $\nu_{c_i}\left(\left(E_1 \circ \cdots \circ E_{i-1}\right)^{-1}(S)\right)$ are

$$8,\ 9,\ 5,\ 6,\ 5,\ 5,\ 3.$$

Thus, the generic dimension of the moduli space is

$$\sigma(8) + \sigma(9) + \sigma(5) + \sigma(6) + \sigma(5) + \sigma(5) = 20,$$

which is confirmed by the algorithm of Hefez and Hernandes [15].

The Saito module of a germ of curves in $\left(\mathbb{C}^2, 0\right)$

The inductive form of the formula in the Main Theorem comes naturally from the inductive structure of the desingularization. At each step, the theory of foliations is involved through the theory of logarithmic vector fields or forms introduced by Saito in 1980 in [24]. Let us consider the set $\Omega^1(S)$ of germs of holomorphic one forms ω that leave invariant S, that is $\gamma^*\omega = 0$. Saito proved that $\Omega^1(S)$ is a free $\mathcal{O}_2$−module of rank 2. If f is a reduced equation of S, then $\frac{\omega}{f}$ is logarithmic in the original sense of Saito, see [4, Chapter II]. Adapting the criterion of Saito for the existence of a basis, the family $\left\{\omega_1, \omega_2\right\}$ is a basis of $\Omega^1(S)$ if and only if there exists a germ of unity $u \in \mathcal{O}$, $u(0) \neq 0$ such that the exterior product of ω_1 and ω_2 is written

$$\omega_1 \wedge \omega_2 = uf\,\mathrm{d}x \wedge \mathrm{d}y.$$

In other words, the tangency locus between ω_1 and ω_2 is reduced to the sole curve S. Beyond this characterization, very few is known about these two generators. At first glance, we can say the following: among all the possible basis $\left\{\omega_1, \omega_2\right\}$, there are some for which the sum of the algebraic multiplicities

$$\nu\left(\omega_1\right) + \nu\left(\omega_2\right) \tag{0.2}$$

is maximal. According to the Saito criterion,

$$\nu\left(\omega_1\right) + \nu\left(\omega_2\right) \leq \nu\left(\omega_1 \wedge \omega_2\right) = \nu(f) = \nu(S).$$

Thus, the sum (0.2) cannot exceed $\nu(S)$. However, for a given analytical class S it can be strictly smaller. Notice that for a basis $\left\{\omega_1, \omega_2\right\}$ being defined as above with $\nu\left(\omega_1\right) \leq \nu\left(\omega_2\right)$, one has

$$\nu\left(\omega_1\right) = \min_{\omega \in \Omega^1(S)} \nu(\omega)$$

since for any $a,\ b \in \mathbb{C}\{x, y\}$, the following inequality holds

$$\nu\left(\omega_1\right) \leq \nu\left(a\omega_1 + b\omega_2\right).$$

It can be seen furthermore that

Proposition. The couple of multiplicities $\left(\nu\left(\omega_1\right), \nu\left(\omega_2\right)\right)$ such that $\nu\left(\omega_1\right) \leq \nu\left(\omega_2\right)$ that maximizes its sum is an analytic invariant of S.

Proof. Let us first prove that this couple is well defined. Consider another basis $\left\{\eta_1, \eta_2\right\}$ of $\Omega^1(S)$ such that $\nu\left(\eta_1\right) \leq \nu\left(\eta_2\right)$ and

$$\nu\left(\eta_1\right)+\nu\left(\eta_2\right)=\nu\left(\omega_1\right)+\nu\left(\omega_2\right).$$

Since, $\nu\left(\eta_1\right)=\min_{\omega\in\Omega^1(S)} \nu(\omega)$ then, $\nu\left(\eta_1\right)=\nu\left(\omega_1\right)$. Thus, $\nu\left(\eta_2\right)=\nu\left(\omega_2\right)$.

Now, let S' be a germ of curve such that $\phi\left(S'\right)=S$ where ϕ belongs to $\mathrm{Diff}\left(\mathbb{C}^2, 0\right)$. Let $\left\{\omega_1, \omega_2\right\}$ be a basis of $\Omega^1(S)$ such that $\nu\left(\omega_1\right) \leq \nu\left(\omega_2\right)$ and

$$\nu\left(\omega_1\right)+\nu\left(\omega_2\right)$$

is maximal among all the possible basis of $\Omega^1(S)$. According to the Saito criterion, $\left\{\phi^\star\omega_1, \phi^\star\omega_2\right\}$ is a basis of $\Omega^1\left(S'\right)$. Since the algebraic multiplicity is an analytic invariant one has,

$$\nu\left(\phi^\star\omega_1\right)=\nu\left(\omega_1\right) \text{and} \nu\left(\phi^\star\omega_2\right)=\nu\left(\omega_2\right).$$

Suppose that there exists a basis $\left\{\omega_1', \omega_2'\right\}$ of $\Omega^1\left(S'\right)$ with $\nu\left(\omega_1'\right) \leq \nu\left(\omega_2'\right)$ and $\nu\left(\omega_1'\right)+\nu\left(\omega_2'\right)>\nu\left(\phi^\star\omega_1\right)+\nu\left(\phi^\star\omega_2\right)$. Then the family $\left\{\left(\phi^\star\right)^{-1}\omega_1', \left(\phi^\star\right)^{-1}\omega_2'\right\}$ would be a basis of $\Omega^1(S)$ whose sum of algebraic multiplicities is strictly bigger than $\nu\left(\omega_1\right)+\nu\left(\omega_2\right)$, which is impossible. Thus, $\left\{\phi^\star\omega_1, \phi^\star\omega_2\right\}$ is a basis of $\Omega^1\left(S'\right)$ whose sum of algebraic multiplicities is maximal among all the basis of $\Omega^1(S)$. ■

The two integers $\nu\left(\omega_1\right)$ and $\nu\left(\omega_2\right)$ as well as their sum are analytical invariants. However, they are not topologically invariants and in the topological class of a given curve, they may vary widely.

Example. Let S be the curve $y^p-x^q=0$ where $p<q$ and $\gcd(p, q)=1$. Then the family

$$\left\{px\mathrm{d}y-qy\mathrm{d}x, \mathrm{d}\left(y^p-x^q\right)\right\}$$

is a basis of the Saito module since

$$(px\mathrm{d}y-qy\mathrm{d}x)\wedge \mathrm{d}\left(y^p-x^q\right)=-pq\left(y^p-x^q\right)\mathrm{d}x\wedge \mathrm{d}y.$$

In that case, the couple of valuation is $(1, p-1)$ whose sum is exactly p.

Example. However, perturbing a bit S, when for instance $p = 6$ and $q = 7$ leads to different values of the multiplicities. For instance, if S is the curve $y^6 - x^7 + x^4y^4 = 0$ which is topologically but not analytically equivalent to $y^6 = x^7$, one can show that the couple

$$\begin{aligned}\omega_1 &= \frac{5}{3}x^4\mathrm{d}x - \frac{20}{21}x^2y^3\mathrm{d}y + \left(\frac{8}{21}xy^3 + y\right)(6x\mathrm{d}y - 7y\mathrm{d}x)\\ \omega_2 &= \frac{20}{21}x^3y^3\mathrm{d}x + \left(\frac{10}{7}y^4 - \frac{80}{147}xy^6\right)\mathrm{d}y + \left(x^2 + \frac{32}{147}y^6\right)(6x\mathrm{d}y - 7y\mathrm{d}x)\end{aligned}$$

is a basis for $\Omega^1(S)$. The multiplicities are respectively 2 and 3 whose sum is strictly smaller than the multiplicity of S. This sum cannot be made bigger: indeed, the initial parts of ω_1 and ω_2 are written

$$\begin{aligned}\omega_1 &= y\,(6x\mathrm{d}y - 7y\mathrm{d}x) + \cdots\\ \omega_2 &= x^2\,(6x\mathrm{d}y - 7y\mathrm{d}x) + \cdots,\end{aligned}$$

so if $\nu\left(a\omega_1 + b\omega_2\right) > 2$ then $a(0) = 0$. Therefore, any changes of basis

$$\left\{a\omega_1 + b\omega_2, c\omega_1 + d\omega_2\right\},$$

—where necessarily $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}\{x,y\})$—yields a new basis with associated couple of multiplicities equal to $(2,2)$ or $(2,3)$.

Example. Finally, if S is given by $y^6 - x^7 + y^2x^5 = 0$, another perturbation of $y^6 - x^7 = 0$, then it can be seen that S admits a basis $\left\{\omega_1, \omega_2\right\}$ with

$$\left(\nu\left(\omega_1\right), \nu\left(\omega_2\right)\right) = (3,3) \neq (1,5).$$

This example leads us to introduce the following class of curves.

Definition. A curve S, reducible or not, is said to admit *a balanced basis* if there exists a basis $\left\{\omega_1, \omega_2\right\}$ of $\Omega^1(S)$ with

- $\nu\left(\omega_1\right) = \nu\left(\omega_2\right) = \frac{\nu(S)}{2}$ if $\nu(S)$ is even,
- $\nu\left(\omega_1\right) = \nu\left(\omega_2\right) - 1 = \frac{\nu(S)-1}{2}$ else.

In some sense, when S admits a balanced basis, the multiplicity of S is *shared* equally between the two generators of $\Omega^1(S)$, which is why we call it *balanced.*

To overcome a technical issue, we introduce below the notion of *direction.*

Definition. A *direction* d is either

- an empty set $d = \emptyset$,
- a smooth germ of curve $d = l$, or
- the union of two transverse smooth curves, $d = l_1 \cup l_2$.

The structure of many proofs here relies on an induction on the length of the reduction of singularities of S. Along this process, the strict transform of S can be attached to a regular point of the exceptional divisor or a singular one. Thus, the local trace of the total transform of S is either the union of S and a germ of smooth curve or the union of S and a germ of a couple of transverse smooth curves. This is why we introduce *a direction.* In the sequel, for any direction d, we will denote by S_d the union $S \cup d$. The following result will be the key to prove the formula in the Main Theorem.

Theorem 1. For a *generic* irreducible curve S and any direction d, one has

$$\min_{\omega \in \Omega^1(S_d)} \nu(\omega) = \left[\frac{\nu(S_d)}{2}\right],$$

where $[\cdot]$ stands for the integer part function. Moreover, for any direction d, the curve S_d admits a balanced basis.

This result will be a consequence of a construction of a very particular element in the Saito module of S_d. This construction will be based upon an arithmetic property of the reduction of singularities following some results of Wall [26] and a recipe to produce foliations with desired invariant curves inspired by [8, 19].

Theorem 2. If $\nu(S_d)$ is even or if d is empty or reduced to one component, then there exists a $1-$form ω of multiplicity $\left[\frac{\nu(S_d)}{2}\right]$ in $\Omega^1(S_d)$ whose induced foliation is not *dicritical* along the exceptional divisor of a single blowing-up of its singularity, which means that the strict transform of ω by E_1 leaves invariant $E_1^{-1}(0)$.

Structure of the article

The structure of the proof of the Main Theorem is

$$\text{Theorem } 2 \Longrightarrow \text{Theorem } 1 \Longrightarrow \text{Main Theorem}$$

The 1st section of this article is devoted to the proof of the 2nd implication. The 2nd and the 3rd ones focus on the proof of Theorem 2. Finally, the last one contains the proof of the 1st implication.

The author wishes to express his gratitude to the referees for several helpful comments concerning, among other things, the structure of this paper.

1 Dimension of the moduli space & Theorem 1$\Longrightarrow$ Main Theorem

To describe the contribution of the deformation theory, let us introduce first some notations that will be used all along the article.

Let E be the minimal resolution of singularities of S. We denote it by

$$E : (\mathcal{M}, D) \to \left(\mathbb{C}^2, 0\right).$$

The map E is a finite sequence of elementary blowing-ups of points

$$E = E_1 \circ E_2 \circ \cdots \circ E_N.$$

If Σ is a germ of curve at $\left(\mathbb{C}^2, 0\right)$ or a divisor, Σ^E will stand for the strict transform of Σ by E, that is, the closure in $\mathcal{M}$ of $E^{-1}\left(\Sigma \setminus \{0\}\right)$.

The exceptional divisor of E, $D = E^{-1}(0)$, is a union of a finite number of exceptional smooth rational curves intersecting transversely

$$D = \bigcup_{i=1}^{N} D_i, \qquad D_i \simeq \mathbb{P}^1(\mathbb{C}).$$

The components are numbered such that D_i appears exactly after i blowing-ups. Finally, let us denote E^j the truncated process

$$E^j = E_j \circ E_{j+1} \circ \cdots \circ E_N \text{ and } D^j = \bigcup_{i=j}^{N} D_i.$$

The initial lemma is the following:

Lemma 3. Let TS be the a sheaf of base D whose stalk at a point $x \in D$ is the set of germs of tangent vector fields to the total transform of S by E. Then the generic dimension of the moduli space of S is

$$\dim_{\mathbb{C}} H^1\left(D, TS_{\text{gen}}\right),$$

where S_{gen} is a curve in the generic component of the moduli space of S.

Proof. In [27], Zariski proved that the dimension of the generic component is equal to the dimension of the space of parameters of a semi-universal deformation of any curve S_{gen} in the generic component of the moduli space of S. On the other hand, Mattei proved in [21] that any curve S admits a semi-universal deformation whose base space is $\left(\mathbb{C}^{\dim_{\mathbb{C}} H^1(D, TS_{\text{gen}})}, 0\right)$, which conclude the proof. ■

Let S be a curve (irreducible or not), E_1 be the standard blow-up, and $D_1 = E_1^{-1}(0)$.

Proposition 4. If the module of Saito $\Omega^1(S)$ admits a basis $\{\omega_1, \omega_2\}$ with

$$\nu(\omega_1) + \nu(\omega_2) = \nu(S).$$

Then

$$\dim_{\mathbb{C}} H^1(D_1, TS) = \frac{(\nu_1 - 1)(\nu_1 - 2)}{2} + \frac{(\nu_2 - 1)(\nu_2 - 2)}{2}$$

with $\nu_i = \nu(\omega_i)$.

Proof. Since $\{\omega_1, \omega_2\}$ is a basis of $\Omega^1(S)$, the criterion of Saito ensures that

$$\omega_1 \wedge \omega_2 = uf\, dx \wedge dy \tag{1.1}$$

for some unity u and some reduced equation f of S. Let X_1 and X_2 be the two vector fields defined by

$$X_i = \omega_i^{\sharp},$$

where $\omega_i = i_{X_i}(\mathrm{d}x \wedge \mathrm{d}y)$ and $i_\star$ is the inner product. Let us consider the standard covering of D_1 by two open sets U_1 and U_2 and two charts (x_1, y_1) and (x_2, y_2) with

$$y_2 = y_1 x_1 \qquad x_2 = \frac{1}{y_1} \qquad E_1(x_1, y_1) = (x_1, y_1 x_1).$$

The pull-back of (1.1) by E_1 is written in the 1st chart

$$E_1^\star \omega_1 \wedge E_1^\star \omega_2 = E_1^\star u E_1^\star f x_1 \mathrm{d}x_1 \wedge \mathrm{d}y_1.$$

Dividing by $x^\nu = x^{\nu_1+\nu_2}$ yields the relation

$$\left(\tilde{X}_1^1\right)^\sharp \wedge \left(\tilde{X}_2^1\right)^\sharp = E_1^* u \tilde{f} x_1 \mathrm{d}x_1 \wedge \mathrm{d}y_1,$$

where $\tilde{X}_i^1 = \frac{E_1^* X_i}{x_1^{\nu_i - 1}}$. The two vector fields $\tilde{X}_1^1$ and $\tilde{X}_2^1$ are tangent to the exceptional divisor. Obviously, they are also tangent to $\tilde{f} = 0$. According to the Saito criterion, at any point c of the exceptional divisor, the germ of $\left\{\tilde{X}_1^1, \tilde{X}_2^1\right\}$ at c is a basis of the module $(TS)_c$. The computation works the same in the 2nd chart (x_2, y_2) of the blow-up.

The open sets U_1 and U_2 are Stein. Thus, following [25], they admit a system of Stein neighborhoods. Since TS is coherent, by inductive limit, we deduce that the covering $\{U_1, U_2\}$ is acyclic for TS. Therefore, one can compute the cohomology using this covering and thus

$$H^1(D_1, TS) = H^1(\{U_1, U_2\}, TS) = \frac{H^0(U_1 \cap U_2, TS)}{H^0(U_1, TS) \oplus H^0(U_2, TS)}.$$

Now, the spaces of global sections on U_1, U_2 and the intersection can be described as follows:

$$\begin{aligned}
H^0(U_1 \cap U_2, TS) &= \left\{\phi_{12}\tilde{X}_1^1 + \psi_{12}\tilde{X}_2^1 \middle| \phi_{12}, \psi_{12} \in \mathcal{O}(U_1 \cap U_2)\right\} \\
H^0(U_1, TS) &= \left\{\phi_1\tilde{X}_1^1 + \psi_1\tilde{X}_2^1 \middle| \phi_1, \psi_1 \in \mathcal{O}(U_1)\right\} \\
H^0(U_2, TS) &= \left\{\phi_2\tilde{X}_1^2 + \psi_2\tilde{X}_2^2 \middle| \phi_2, \psi_2 \in \mathcal{O}(U_2)\right\}.
\end{aligned}$$

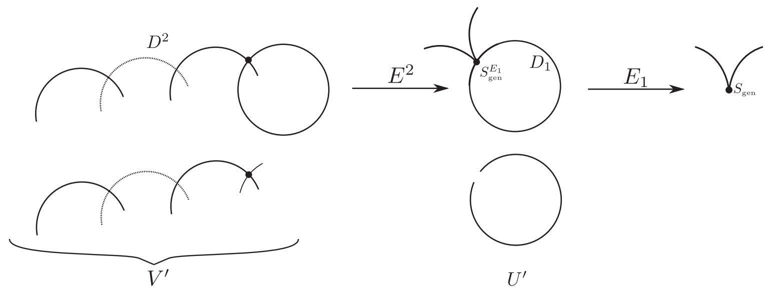

Fig. 1.1. Covering of D adapted to the Mayer–Vietoris argument.

Thus, the cohomological equation is written

$$\begin{aligned}\phi_{12}\tilde{X}_1^1 + \psi_{12}\tilde{X}_2^1 &= \phi_1\tilde{X}_1^1 + \psi_1\tilde{X}_2^1 - \left(\phi_2\tilde{X}_1^2 + \psi_2\tilde{X}_2^2\right)\\ &= \left(\phi_1 - \phi_2 y_1^{-\nu_1+1}\right)\tilde{X}_1^1 + \left(\psi_1 - \psi_2 y_1^{-\nu_2+1}\right)\tilde{X}_2^1.\end{aligned}$$

Since, $\left\{\tilde{X}_1^1, \tilde{X}_2^1\right\}$ is a basis of $\mathcal{O}$-module, the above leads to the system

$$\begin{cases}\phi_{12} &= \phi_1 - \phi_2 y_1^{-\nu_1+1}\\ \psi_{12} &= \psi_1 - \psi_2 y_1^{-\nu_2+1}\end{cases}.$$

Writing these equations using Taylor expansions leads to the checked number of obstructions. ∎

Finally, the proof of

Theorem 1 $\Longrightarrow$ Main Theorem

goes as follows. Consider the covering $\{U, V\}$ of D_1 where V is a very small ball around the singular point of $S_{\text{gen}}^{E_1}$ and $U = D_1 \setminus \text{Sing}\left(S_{\text{gen}}^{E_1}\right)$.

The set

$$\left\{U' = \left(E^2\right)^{-1}(U), V' = \left(E^2\right)^{-1}(V)\right\}$$

consists in a covering of D and V' is a neighborhood of D^2 as shown in Figure (1.1). The Mayer–Vietoris sequence associated to this covering and applied to the sheaf TS_{gen} leads to the following long exact sequences in cohomology

$$0 \to N \to H^1\left(D, TS_{\text{gen}}\right) \to H^1\left(V', TS_{\text{gen}}\right) \oplus H^1\left(U', TS_{\text{gen}}\right) \to H^1\left(V' \cap U', TS_{\text{gen}}\right),$$

where N is given by the exact sequence

$$H^0\left(V', TS_{\text{gen}}\right) \oplus H^0\left(U', TS_{\text{gen}}\right) \to H^0\left(V' \cap U', TS_{\text{gen}}\right) \to N.$$

Since $V' \cap U'$ and U' are Stein, one has

$$\begin{aligned} H^1\left(V' \cap U', TS_{\text{gen}}\right) &= 0 \\ H^1\left(U', TS_{\text{gen}}\right) &= 0. \end{aligned}$$

By inductive limit on the neighborhood of $\mathrm{Sing}\left(S_{\text{gen}}^{E_1}\right)$, one can show that

$$H^1\left(V', TS_{\text{gen}}\right) \simeq H^1\left(D^2, TS_{\text{gen}}\right).$$

Moreover, E^2 induces the following isomorphisms:

$$\begin{aligned} \left(E^2\right)^* &: H^0\left(U', TS_{\text{gen}}\right) \to H^0\left(U, TS_{\text{gen}}\right) \\ \left(E^2\right)^* &: H^0\left(V' \cap U', TS_{\text{gen}}\right) \to H^0\left(U \cap V, TS_{\text{gen}}\right), \\ \left(E^2\right)^* &: H^0\left(V', TS_{\text{gen}}\right) \to H^0\left(V, TS_{\text{gen}}\right). \end{aligned}$$

In the two 1st cases, E^2 is an isomorphism itself on the involved neighborhoods. In the 3rd case, this is a consequence of Hartogs extension lemma noticing that E^2 is an isomorphism from a neighborhood of $(E^2)^{-1}\left(V \setminus \mathrm{Sing}\left(S_{\text{gen}}^{E_1}\right)\right)$ to its image. The Mayer–Vietoris sequence finally decomposes $H^1\left(D, TS_{\text{gen}}\right)$ along the desingularization of S_{gen} through the following isomorphism of $\mathbb{C}-$vector spaces,

$$H^1\left(D, TS_{\text{gen}}\right) \simeq H^1\left(D_1, TS_{\text{gen}}\right) \bigoplus H^1\left(D^2, TS_{\text{gen}}\right).$$

The curve S_{gen} admits a balanced basis according to Theorem 1. Hence, the Main Theorem is an inductive application of Proposition 4 noticing that in that case

$$\dim H^1\left(D_1, TS_{\text{gen}}\right) = \sigma\left(\nu\left(S_{\text{gen}}\right)\right).$$

As a corollary, the formula gives a straightforward proof of the following result contained in [14].

Corollary. A germ of irreducible curve S is generically rigid if and only if

- $\nu(S) \in \{1,2,3\}$ or
- $\nu(S) = 4$ and its Puiseux pairs are $(4,5)$, $(4,7)$ or $((2,3),(2,2k+1))$ with $k \geq 3$.

Indeed, one can check that the cases above are the only one for which the formula in Main Theorem yields 0.

2 A Remarkable Element in $\Omega^1(S_d)$

The proof of Theorems 1 and 2 starts with the construction of a particular element in $\Omega^1(S_d)$.

For any basis $\{\omega_1, \omega_2\}$ of $\Omega^1(S_d)$, the criterion of Saito ensures that

$$\nu(\omega_1) + \nu(\omega_2) \leq \nu(S_d).$$

Thus, at least one of these multiplicities is smaller or equal to $\left[\frac{\nu(S_d)}{2}\right]$, which proves one part of the equality in Theorem 1. However, to obtain the whole equality we will need some more informations about these generators. In this section, we are going to construct quite explicitly an element of $\Omega^1(S_d)$ with multiplicity $\left[\frac{\nu(S_d)}{2}\right]$.

We recall that a foliation $\mathcal{F}$ is said to be *dicritical along a divisor* Σ if and only if $\mathcal{F}$ is generically transverse to Σ.

Let us give a sketch of the proof of Theorem 2. First, we construct an auxiliary foliation $\mathcal{F}[S_d]$ tangent to some curve $\mathfrak{S}$ topologically equivalent to S_d—but not necessarily analytically equivalent to S_d—with the desired algebraic multiplicity. Then, we study the deformations of $\mathcal{F}[S_d]$ by means of cohomological tools. In particular, considering a deformation linking $\mathfrak{S}$ to S_d, we prove that it can be followed by a deformation of $\mathcal{F}[S_d]$ that preserves the algebraic multiplicity. The resulting foliation is tangent to S_d with $\left[\frac{\nu(S_d)}{2}\right]$ as algebraic multiplicity. Among other properties, we obtain Theorem 2.

2.1 The auxiliary foliation $\mathcal{F}[S_d]$

In this section, we are going to construct a foliation associated to S_d, denoted by $\mathcal{F}[S_d]$, thanks to a result of Lins Neto [19, 20] that is a kind of *recipe* to construct germs of singular foliations in the complex plane.

Let E be the minimal desingularization of S. We denote it by

$$E : (\mathcal{M}, D) \to \left(\mathbb{C}^2, 0\right).$$

Recall that E is a finite sequence of elementary blowing-ups of points

$$E = E_1 \circ E_2 \circ \cdots \circ E_N.$$

We can encode the map E in a square matrix $\mathcal{E}$ of size N called by Wall the *proximity matrix* (see [26, p. 52]). Let $\mathcal{E}$ be the matrix $N \times N$ whose entries are

$$\mathcal{E}_{i,j} = \begin{cases} 1 & \text{if } i = j \\ -1 & \text{if the } j^{th} \text{ blow-up of } E \text{ is centered at a point of } D_i \\ 0 & \text{otherwise} \end{cases}.$$

Notice that for any $i \leq N-1$, $\mathcal{E}_{i,i+1} = -1$ and that for any $i > j$, $\mathcal{E}_{i,j} = 0$.

Let S_i be the strict transform of S by $E_1 \circ \cdots \circ E_{i-1}$ for $i \geq 2$ and $S_1 = S$. The map E^i is the minimal desingularization of the total transform of S_1 by $E_1 \circ \cdots \circ E_{i-1}$. In the sequel, we will denote by $p(S_i)$ and $q(S_i)$, $p(S_i) < q(S_i)$ the 1st two characteristic exponents of the curve S_i: $p(S_i)$ is nothing but the multiplicity of S_i and $q(S_i)$ the 1st exponent appearing in a Puiseux parametrization of S_i

$$\begin{cases} x = t^{p(S_i)} \\ y = t^{q(S_i)} + \cdots \end{cases} \qquad p(S_i) \nmid q(S_i).$$

Definition 5. Let $E : (\mathcal{M}, D) \to \left(\mathbb{C}^2, 0\right)$ be a process of blow-ups $E = E_1 \circ \cdots \circ E_N$. Let us write $D = E^{-1}(0) = \bigcup_{i=1}^{N} D_i$. Let $\mathfrak{M}$ be the maximal ideal at $\left(\mathbb{C}^2, 0\right)$ and $\mathcal{I}$ the sheaf over D of ideals generated locally by the functions of the form $g \circ E$ where $g \in \mathfrak{M}$. Then $\mathcal{I}$ can be decomposed the following way

$$\mathcal{I} = \prod_{i=1}^{N} \mathcal{I}_{D_i}^{n(E, D_i)},$$

where $\mathcal{I}_{D_i}$ is the sheaf of functions vanishing on D_i and $n\left(E, D_i\right)$ are some integers depending only on E and D. The integer $n\left(E, D_i\right)$ is called the *multiplicity* of D_i with respect to E.

The following lemma is in [26, p. 53, Lemma 3.5.3]

Lemma 6. The inverse of the proximity matrix $\mathcal{E}^{-1}$ has the following form

$$\begin{pmatrix} 1 & & & \\ 0 & \ddots & e_{kl} & \\ & \ddots & 1 & \\ 0 & & 0 & 1 \end{pmatrix},$$

where $e_{kl} = n\left(E^k, D_l\right)$. Moreover, the coefficient e_{kN} is the algebraic multiplicity of the S_k. Finally, the matrix $-\mathcal{E}\left({}^t\mathcal{E}\right)$ is the intersection matrix of D.

Example 7. Let us consider $S = \left\{y^5 = x^{13}\right\}$. Then the proximity matrix $\mathcal{E}$ is written

$$\mathcal{E} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The inverse matrix is written

$$\mathcal{E}^{-1} = \begin{pmatrix} 1 & 1 & 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The exceptional divisors of the associated sequence of processes of blowing-ups $\left\{E^k\right\}_{k=1..5}$ are presented in Figure (2.1).

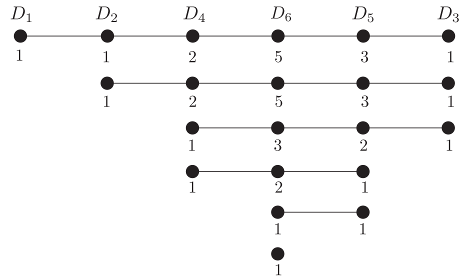

Fig. 2.1. Exceptional divisors of the sequence of processes of blowing-ups associated to the desingularization of $y^5 - x^{13} = 0$.

Notice that, as soon as S is singular, for any direction d, S and S_d share the same reduction. The next proposition is the one upon which the construction of the auxiliary foliation $\mathcal{F}\left[S_d\right]$ is based.

Proposition 8. Let $\delta_1 \in \{0, 1, 2\}$ be the number of components of the direction d. In the same way, consider the number δ_i of branches of the total transform of d by $E_1 \circ \cdots \circ E_{i-1}$ that meets S_i for $2 \leq i \leq N$. Let us denote $\kappa_i - 1$ the number of -1 on the i-th row of $\mathcal{E}$. Notice that, for $i \geq 2$, $\delta_i \in \{1, 2\}$.

Let us consider the vector of integers defined by

$$\begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_N \end{pmatrix} = \mathcal{E} \begin{pmatrix} \left[\frac{\nu(S_1)-\delta_1}{2}\right] + 1 \\ \left[\frac{\nu(S_2)-\delta_2}{2}\right] + 1 \\ \vdots \\ \left[\frac{\nu(S_N)-\delta_N}{2}\right] + 1 \end{pmatrix}. \tag{2.1}$$

Then,

(1) any integer p_i is bigger or equal to -1. The case $p_i = -1$ occurs if and only if
   (a) either, $\kappa_i = 2$, $\delta_i = 2$, $\delta_{i+1} = 1$ and $\nu\left(S_i\right) = p\left(S_i\right)$ is odd, or
   (b) $\kappa_i = 3$, $\delta_i = 2$, $\delta_{i+1} = 1$, $\nu\left(S_i\right)$ is odd and $q\left(S_i\right)$ is even.

(2) If $D_i \cap D_j \neq \emptyset$ then one cannot have both $p_i = -1$ and $p_j = -1$.

(3) Let us consider $\overline{D}$ the exceptional divisor D deprived of D_N and of the components D_i for which $p_i = -1$. Then in each connected component of $\overline{D}$, there exists at least one component D_j for which, either $p_j > 0$ or, that meets a component of d^E.

(4) $p_N = 0$.

In the proposition above, the integer $-\kappa_i$ is also equal to the self-intersection of the component D_i.

Proof. The proof is an induction on the length of the desingularization of S. Let us consider that $\mathcal{E}$ is written

$$\mathcal{E} = \begin{pmatrix} 1 & -1 & -1 & \cdots & -1 & 0 & \\ & 1 & -1 & & & & \\ & & 1 & & & & \cdots \\ & & & \ddots & -1 & & \\ & & & & 1 & -1 & \\ & & & & & 1 & \\ & & \vdots & & & & \ddots \end{pmatrix}$$

Expanding the expression of p_1, we find

$$p_1 = \left[\frac{\nu(S_1) - \delta_1}{2}\right] + 1 - \sum_{j=2}^{\kappa} \left(\left[\frac{\nu\left(S_j\right) - \delta_j}{2}\right] + 1\right),$$

where for the sake of simplicity, we denote κ_1 simply by κ. Consider a Puiseux parametrization of $S_1 = S$,

$$S_1 : \begin{cases} x = t^p \\ y = t^q + \cdots \end{cases}$$

with $p = p(S_1) < q = q(S_1)$ and $\gcd(p, q) = 1$. According to the desingularization of S_1, encoded in the proximity matrix, the multiplicities and the δ_i's satisfy

$$\begin{aligned} \nu(S_1) &= p \\ \nu\left(S_j\right) &= q - p \text{ for } 2 \leq j \leq \kappa - 1 \\ \nu(S_\kappa) &= (\kappa - 1)p - (\kappa - 2)q \\ \delta_1 &\in \{0, 1, 2\} \\ \delta_2 &\in \{1, 2\} \\ \delta_j &= 2 \text{ for } 3 \leq j \leq \kappa. \end{aligned}$$

Table 2 Values of p_1 depending on p being odd or even.

	p is odd	p is even
$\delta_1 = 0,\ \delta_2 = 1$	0	1
$\delta_1 = 1,\ \delta_2 = 1$	0	0
$\delta_1 = 1,\ \delta_2 = 2$	1	0
$\delta_1 = 2,\ \delta_2 = 1$	-1	0
$\delta_1 = 2,\ \delta_2 = 2$	0	0

Table 3 Values of p_1 depending on κ being even or odd.

	p and q both odd	p and q both even	p even q odd	p odd q even
$\delta_1 = 0,\ \delta_2 = 1$	1	1	$\frac{\kappa-2}{2},\ \frac{\kappa-1}{2}$	$\frac{\kappa-2}{2},\ \frac{\kappa-3}{2}$
$\delta_1 = 1,\ \delta_2 = 1$	1	0	$\frac{\kappa-4}{2},\ \frac{\kappa-3}{2}$	$\frac{\kappa-2}{2},\ \frac{\kappa-3}{2}$
$\delta_1 = 1,\ \delta_2 = 2$	1	0	$\frac{\kappa-2}{2},\ \frac{\kappa-1}{2}$	$\frac{\kappa}{2},\ \frac{\kappa-1}{2}$
$\delta_1 = 2,\ \delta_2 = 1$	0	0	$\frac{\kappa-4}{2},\ \frac{\kappa-3}{2}$	$\frac{\kappa-4}{2},\ \frac{\kappa-5}{2}$
$\delta_1 = 2,\ \delta_2 = 2$	0	0	$\frac{\kappa-2}{2},\ \frac{\kappa-1}{2}$	$\frac{\kappa-2}{2},\ \frac{\kappa-3}{2}$

Thus, the integer p_1 is written

$$p_1 = \left[\frac{p-\delta_1}{2}\right] - \sum_{j=2}^{\kappa-1}\left[\frac{q-p-\delta_j}{2}\right] - \left[\frac{(\kappa-1)\,p-(\kappa-2)\,q-\delta_\kappa}{2}\right] - \kappa + 2.$$

The following lemma is straightforward.

Lemma 9. If $\kappa = 2$, then the values of $p_1 = \left[\frac{p-\delta_1}{2}\right] - \left[\frac{p-\delta_2}{2}\right]$ are given in Table 2.

If $\kappa \geq 3$ then the values of p_1 are given in Table 3. When the value depends on κ, it is the precise value of p_1 if κ is even or odd. In particular, $p_1 = -1$ if and only if one of the following case occurs:

- $\kappa = 2$, $\delta_1 = 2$, $\delta_2 = 1$ and p is odd;
- $\kappa = 3$, $\delta_1 = 2$, $\delta_2 = 1$ and p is odd and q is even.

∎

Now, we are able to study the general behavior of p_1 and to prove Proposition 8.

The Property (1) can be seen by reading inductively Lemma 9.

Property (2) is proved as follows. Suppose that $p_1 = -1$. According to property (1) , two cases may occur:

- if $\kappa = 2$, $\delta_1 = 2$ and $\delta_2 = 1$, then D_1 meets D_2 in D. Since $\delta_2 = 1$, p_2 cannot be equal to -1. Proposition 8 applied inductively to S_2 yields the proposition for S_2.
- if $\kappa = 3$, $\delta_1 = 2$, $\delta_2 = 1$, p is odd and q is even, then D_1 meets D_3 and $\delta_3 = 2$. Suppose that $\delta_4 = 1$ then S_3 is neither tangent to D_1 nor to D_2. Looking at the Puiseux parametrization of S_3 yields

$$q - p = 2p - q,$$

which is impossible since p is odd. Thus, $\delta_4 = 2$, and p_3 cannot be equal to -1. We conclude by induction.

Let us now focus on property (3).

- Suppose first that $\delta_1 = 2$.
  - If $p_1 > 0$, then the connected component of D_1 in $\overline{D}$ contains D_1 as component with $p_1 > 0$. Applying inductively Proposition 8 to S_2 with the sequence of δ's equal to

$$\delta_2, \ \delta_3, \ \ldots$$

yields the proposition for S_1 with the sequence of δ 's equal to $\delta_1, \ \delta_2, \ \ldots$.
  - If $p_1 = 0$, since at least one of the component of d^E is attached to D_1, the same argument as before ensures the proposition.
  - If $p_1 = -1$, then two cases may occur:
    - ○ if $\kappa = 2$ then $\nu(S_2) = \nu(S_1)$ is odd and $\delta_2 = 1$. Applying inductively Proposition 8 to S_2 with the sequence of δ's equal to

$$0, \ \delta_3, \ \delta_4, \ldots$$

yields the result: indeed, one has

$$\left[\frac{\nu(S_2) - 0}{2}\right] = \left[\frac{\nu(S_2)}{2}\right] = \left[\frac{\nu(S_2) - 1}{2}\right] = \left[\frac{\nu(S_2) - \delta_2}{2}\right]$$

and for $j \geq 3$, since $\kappa = 2$, one has $\delta_j' = \delta_j$ where the δ_j' would be the sequence obtained following the desingularization of S_2 with $\delta_2' = 0$.

- If $\kappa = 3$, then $\delta_2 = 1$, $\nu(S_1) = p$ is odd and q is even. Moreover, since $\kappa = 3$, one has $\delta_3 = 2$. Following the desingularization of S_1, one has $\nu(S_2) = q-p > 0$ that is odd and $\nu(S_3) = 2p-q$ that is even. Applying inductively Proposition 8 to S_2 with the sequence of δ's equal to

$$0,\ 1,\ \delta_4, \ldots$$

yields the result: indeed, one has

$$\left[\frac{\nu(S_2) - 0}{2}\right] = \left[\frac{\nu(S_2) - 1}{2}\right] = \left[\frac{\nu(S_2) - \delta_2}{2}\right],$$
$$\left[\frac{\nu(S_3) - 1}{2}\right] = \left[\frac{\nu(S_3) - 2}{2}\right] = \left[\frac{\nu(S_2) - \delta_3}{2}\right],$$

and for $j \geq 4$, since $\kappa = 3$, one has $\delta_j' = \delta_j$ where the δ_j' would be the sequence obtained following the desingularization of S_2 with $\delta_2' = 0$ and $\delta_3' = 1$.

- Suppose now that $\delta_1 = 1$. Then according to property (1), $p_1 \geq 0$. If $\delta_2 = 1$ then the component of d^E meets D_1. So applying inductively Proposition 8 to S_2 with the sequence $\delta_2, \delta_3, \cdots$ yields the proposition. Let us suppose that $\delta_2 = 2$. If $p_1 > 0$, then inductively the proposition is proved. If $p_1 = 0$ then according to Lemma 9 two cases may occur
  - if $\kappa = 2$ and $\nu(S_2) = \nu(S_1)$ is even, then D_2 meets D_1 in D and p_2 cannot be equal to -1. Applying inductively Proposition 8 to S_2 with the sequence

$$1,\ \delta_3, \ldots$$

yields the result. The arguments are the same as before.
  - if $\kappa \geq 3$, then p and q are even and the curve S cannot be topologically quasi-homogeneous. While $\delta_i \neq 1$, no component D_j with $p_j = -1$ can appear. If at some point, one has $\delta_j = 1$ then the multiplicity of $\nu(S_j)$ is written $\alpha p+\beta q$ for some α, β in $\mathbb{Z}$. Thus, it is even and p_j cannot be equal

to -1. Therefore, D_2 and D_1 belong to the same connected component $\overline{D}$, which inductively proves the proposition since d^E is attached to D_2.

- Suppose finally that $\delta_1 = 0$. One has $\delta_2 = 1$. If $p_1 > 0$ then the proposition is proved inductively. If not, two cases may occur:
  - If $\kappa = 2$ then $\nu(S_2) = \nu(S_1)$ is odd. The proposition is proved applying it inductively to S_2 with the sequence

$$0, \delta_3, \ldots.$$

    The arguments are the same as above noticing that

$$\left[\frac{\nu(S_2)}{2}\right] = \left[\frac{\nu(S_2) - \delta_2}{2}\right].$$

  - If $\kappa \geq 3$ and $p_1 = 0$ then $\kappa = 3$, p is odd and q is even. The proposition is proved applying it inductively to S_2 with the sequence

$$0,\ 1,\ \delta_4 \ldots.$$

    Again, the arguments are the same as before.

Now, we introduce a foliation associated to S_d prescribing some topological data.

Definition 10. The numbered dual tree $\mathbb{A}[\mathcal{F}]$ of a foliation $\mathcal{F}$ is a numbered graph constructed as follows. Let E be the minimal desingularization of $\mathcal{F}$. The vertices of $\mathbb{A}[\mathcal{F}]$ are in one-to-one correspondence with the irreducible components of the exceptional divisor of E. There is an edge between D_i and D_j if and only if $D_i \cap D_j \neq \emptyset$. Each vertex is numbered following the next rules:

- if $E^*\mathcal{F}$ is dicritical along D_i, then D_i is numbered $+\infty$
- else it is numbered by the number of irreducible invariant curves of $E^*\mathcal{F}$ intersecting D_i transversely.

Now, the proposition below produces the checked foliation.

Proposition 11. Let $\mathbb{A}$ be the dual tree of S_d and $p_1, \ldots, p_N$ be the integers given by Proposition 8. We number $\mathbb{A}$ the following way:

- if $p_i = -1$ then D_i is numbered ∞.
- otherwise, D_i is numbered $p_i + \left(\text{the number of component of } d^E \text{ meeting } D_i\right)$
- D_N is numbered $+\infty$.

Then, there exists a foliation $\mathcal{F}\left[S_d\right]$ whose singularities are linearizable and such that

$$\mathbb{A}\left[\mathcal{F}\left[S_d\right]\right] = \mathbb{A}.$$

Proof. We use a result of Lins Neto [19] whose statement is also mentioned in [20] and written in a more compact way. For the arguments to come, we will refer to the latter version.

The statement of Lins Neto is quite long to enunciate because the hypothesis require that we prescribe all the local and semi-local data attached to the desired foliation. Below, to be the most specific as possible, we will follow the numbering of the hypothesis in [20, p. 151]. We require that

- **Hypothesis** (1): the desingularization of $\mathcal{F}\left[S_d\right]$ has the same topology as the desingularization of S_d. For the sake of simplicity, we keep denoting by $D = \bigcup_{i=1}^{N} D_i$ the exceptional divisor of its desingularization.
- **Hypothesis** (2): $\mathcal{F}\left[S_d\right]$ is dicritical and regular along D_N. If $p_i = -1$, then $\mathcal{F}\left[S_d\right]$ is dicritical and regular along D_i. Otherwise, D_i is invariant.
- **Hypothesis** (5): At each corner point of D that does not meet a dicritical component, $\mathcal{F}\left[S_d\right]$ admits a linear singularity written in some local coordinates (x, y)

$$\lambda x\mathrm{d}y + y\mathrm{d}x,\ \lambda \notin \mathbb{Q}^-, \tag{2.2}$$

  where $xy = 0$ is a local equation of D.
- **Hypothesis** $(4), (6)$: For each D_i with $p_i \geq 0$, $\mathcal{F}\left[S_d\right]$ admits p_i more linear singularities along D_i that can be written in some local coordinates (x, y)

$$\lambda x\mathrm{d}y + y\mathrm{d}x,\ \lambda \notin \mathbb{Q}^-, \tag{2.3}$$

  where $x = 0$ is a local equation of D_i. The local analytic class of the singularities added above depends on the value of λ which is called the *Camacho–Sad index* [3] of the singularity s along D. It is denoted by

$$-\lambda = CS_s\left(\mathcal{F}\left[S_d\right], D\right),$$

where s is the singularity. Finally, for each component of d^E attached to D_j with $p_j \geq 0$, $\mathcal{F}[S_d]$ admits one more linear singularity along D_j.

The remaining hypothesis control the projective representations of holonomy of the desired foliation: this part is irrelevant for our construction and can be chosen arbitrarily.

The above data must satisfy some compatibility conditions stated in the theorem of Lins Neto:

- two dicritical components cannot meet which is ensured by the 2nd property of Proposition 8.
- the Camacho–Sad indexes of the singularities along a given component D_j have to satisfy a relation known as *the Camacho–Sad relation*

$$\sum_{s\in D_j} CS_s\left(\mathcal{F}[S_d], D_j\right) = D_j \cdot D_j.$$

  The 3rd property in Proposition 8 allows us to choose the Camacho–Sad indices of the linear singularities added at (2.2) and at (2.3) in order to ensure the Camacho–Sad relation for any component D_j.

According to the theorem of Lins Neto, there exists a germ of foliation $\mathcal{F}[S_d]$ defined at the origin of $(\mathbb{C}^2, 0)$ that realizes all the above prescription. In particular, by construction, one has

$$\mathbb{A}[\mathcal{F}[S_d]] = \mathbb{A}.$$

■

A lot of foliations can be constructed as above, prescribing freely the projective representations of holonomy. Hence, there is a big number of non analytically equivalent choices. However, all the foliations built the way above share some properties. In any case, $\mathcal{F}[S_d]$ is dicritical along D_N. Its singularities are all linearizable and thus $\mathcal{F}[S_d]$ is of *2nd kind* as defined in [7, 21]. Its desingularization has the same topological type as the desingularization of S_d. Moreover, the foliation $\mathcal{F}[S_d]$ is tangent to some curve $\mathfrak{S}$ topologically equivalent to S_d since $\mathfrak{S}$ and S_d share the same process of desingularization. Finally, the algebraic multiplicity is the desired one. Indeed, one has the following result:

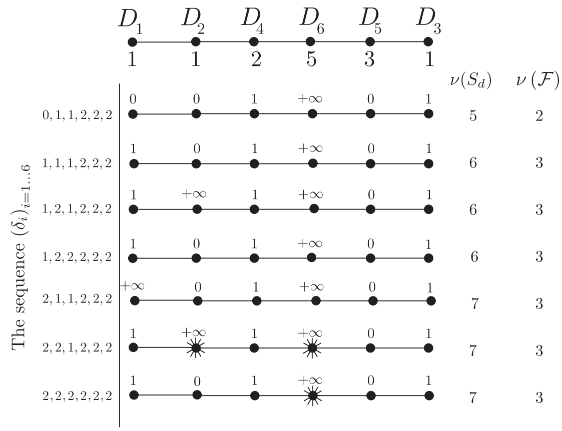

Fig. 2.2. Dual numbered tree $\mathbb{A}\left[\mathcal{F}\left[S_d\right]\right]$ for $S=\left\{y^5=x^{13}\right\}$ and any direction d.

Lemma 12. Regardless the foliation $\mathcal{F}\left[S_d\right]$ constructed as above, one has

$$\nu\left(\mathcal{F}\left[S_d\right]\right)=\left\lfloor\frac{\nu\left(S_d\right)}{2}\right\rfloor.$$

Proof. A formula of Hertling in [17]—see Theorem 3.(a)—gives us

$$\nu\left(\mathcal{F}\left[S_d\right]\right)=\sum_{i=1}^{N-1}p_ie_{1i}+\delta_1-1.$$

In the notations of the Hertling's formula, one has $\rho_i=e_{1i}$ and $\epsilon_i^{(k)}=p_i+$ (the number of component of d^E meeting D_i). Since $\nu\left(S_N\right)=1$ and $\delta_N=2$, one has $p_N=0$. Using the expression of $\mathcal{E}^{-1}$ to invert the formula (2.1), the 1st row yields

$$\sum_{i=1}^{N-1}p_ie_{1i}+\delta_1-1=\left\lfloor\frac{\nu\left(S_1\right)-\delta_1}{2}\right\rfloor+\delta_1=\left\lfloor\frac{\nu\left(S_d\right)}{2}\right\rfloor.$$

■

As an example, Figure 2.2 presents the numbered dual tree $\mathbb{A}\left[\mathcal{F}\left[S_d\right]\right]$ for the curve $S=\left\{y^5-x^{13}\right\}$ and for all the possible topological types of direction d. Hence, for instance, if $\delta_1=1$, $\delta_2=2$, and $\delta_3=1$ then $\nu\left(S_d\right)=5+1$ and $\nu\left(\mathcal{F}\right)=3$. Besides, in the latter case, the topological type of the foliation $E^\star\mathcal{F}\left[S_d\right]$ prescribed by $\mathbb{A}\left[\mathcal{F}\left[S_d\right]\right]$ is depicted in Figure 2.3

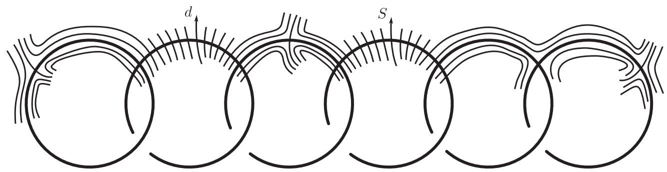

Fig. 2.3. Topology of $E^{\star}\mathcal{F}$.

3 Deformations of $\mathcal{F}\left[S_d\right]$ & Proof of Theorem 2.

In this section, we are interested in the deformations of foliations with a cohomological approach.

3.1 Basic vector fields and deformations

Let ω be a germ of 1−form and X be a germ of vector field. The vector field X is said to be *basic* for ω if and only if

$$\left(L_X\omega\right)\wedge\omega = d\left(\omega\left(X\right)\right)\wedge\omega-\omega\left(X\right)d\omega = 0.$$

The property of being basic for the 1−form ω depends only on the foliation induced by ω, since for any function f, one has

$$L_X\left(f\omega\right)\wedge f\omega = f^2\left(L_X\omega\right)\wedge\omega.$$

Lemma 13. [4, p. 36] Let X be a germ of vector field. It is basic for ω if and only if for any $t\in(\mathbb{C},0)$, the flow at time t of X, denoted by $e^{[t]X}$, is an automorphism of the foliation defined by ω, that is,

$$\left(\left(e^{[t]X}\right)^*\omega\right)\wedge\omega = 0.$$

In particular, the flow $e^{[t]X}$ preserves the set of leaves of the foliation but may permute them.

More generally, a germ of automorphism of ω or *basic automorphism for* ω—or for the foliation induced by ω—is a germ of diffeomorphism ϕ such that

$$\left(\phi^*\omega\right)\wedge\omega = 0.$$

If ϕ is tangent to Id, then there exists aformal basic vector field X such that $e^{[1]X} = \phi$. In what follows, we will simply denote the flow at time 1 of X by e^X. If X is singular at p, then the flow e^X is convergent in a neighborhood of p.

Thanks to basic automorphisms, we can describe a surgery construction that produces many non-equivalent germs of foliations from a given one. Consider the desingularization $E : (\mathcal{M}, D) \to (\mathbb{C}^2, 0)$ of some singular foliation $\mathcal{F}$ at $(\mathbb{C}^2, 0)$. For any covering $\{U_i\}_{i\in I}$ of a neighborhood of D in $\mathcal{M}$ and for any 2-intersection $U_{ij} = U_i \cap U_j$, we consider ϕ_{ij} a basic automorphism of $E^*\mathcal{F}$, which is the identity map along $U_{ij} \cap D$. We suppose that the family $\left\{\phi_{ij}\right\}_{i,j}$ satisfies the cocycle relation: on any 3-intersection U_{ijk}, one has

$$\phi_{ij} \circ \phi_{jk} \circ \phi_{ki} = \mathrm{Id}.$$

We construct a manifold with the following gluing

$$\mathcal{M}\left[\phi_{ij}\right] = \coprod_i U_i/_{(x,i)\sim(\phi_{ij}(x),j)},$$

which is a neighborhood of some divisor isomorphic to D. This manifold is foliated by a foliation $\mathcal{F}'$ obtained by gluing with the same collection of maps the family of restricted foliations $\left\{E^*\mathcal{F}|_{U_i}\right\}_i$.

Lemma 14. There exists a germ of singular foliation at the origin of $(\mathbb{C}^2, 0)$ denoted by $\mathcal{F}\left[\phi_{ij}\right]$ and a process of blowing-ups E' such that $(E')^* \mathcal{F}\left[\phi_{ij}\right]$ is analytically equivalent to $\mathcal{F}'$.

Proof. The manifold $\mathcal{M}\left[\phi_{ij}\right]$ is an open neighborhood of a divisor whose intersection matrix is the same as the one of D since the gluing maps ϕ_{ij} leave invariant the trace of the divisor $U_{ij} \cap D$. In particular, its intersection matrix is definite negative. Following the Grauert's contraction result [12], there exists a process of blowing-ups $E' : (\mathcal{M}', D') \to (\mathbb{C}^2, 0)$ such that $\mathcal{M}'$ is analytically equivalent to $\mathcal{M}\left[\phi_{ij}\right]$. Being analytically equivalent to $\mathcal{M}\left[\phi_{ij}\right]$, the manifold $\mathcal{M}'$ is foliated. Since E' is an isomorphism between $\mathcal{M}' \setminus D'$ and $(\mathbb{C}^2, 0) \setminus \{0\}$, there exists a foliation in $(\mathbb{C}^2, 0) \setminus \{0\}$ whose pull-back by E' coincides with the foliation of $\mathcal{M}'$ on $\mathcal{M}' \setminus D'$. The Hartogs's extension result allows us to extend this foliation in $(\mathbb{C}^2, 0)$. The obtained foliation is $\mathcal{F}\left[\phi_{ij}\right]$. ■

A foliation built the way above is said to be a *basic surgery* of $\mathcal{F}$. Our goal is to study the basic surgeries of $\mathcal{F}\left[S_d\right]$ and in particular to prove the following

Proposition 15. For any curve $\mathfrak{S}$ topologically equivalent to S_d, there is a 1-form $\omega \in \Omega^1(\mathfrak{S})$ defining a foliation obtained from a basic surgery of $\mathcal{F}[S_d]$.

The proof is based upon the study of deformations of $\mathcal{F}[S_d]$ with a cohomological point of view based upon the notion of *balanced equation of separatricies*. This is developed below.

3.2 The balanced equation of separatricies

Roughly speaking, a balanced equation of the separatricies of a foliation is a meromorphic function that plays the role of the standard equation of the separatricies associated to a foliation admitting only a finite number of them. Here, in general $\mathcal{F}[S_d]$ admits an infinite number of separatricies. Thus, the classical notion is not adapted. Basically, the zeros of F contain the set of isolated separatricies—which are always in finite number—and the poles contain a finite number of curvets attached to the dicritical components of the exceptional divisor with valence bigger than 3. Below we reproduce some material from [9], which we refer to for more details.

We denote respectively by

$$\mathrm{Dic}(D) \text{ and } \overline{\mathrm{Dic}}(D)$$

the set of dicritical and non dicritical components of D with respect to $\mathcal{F}$. A germ of irreducible separatrix S of $\mathcal{F}$ is said to be *isolated* if its strict transform S^E does not meet a dicritical component. We denote by $\mathrm{Iso}(\Delta)$ the set of isolated separatrices of $\mathcal{F}$ attached to the component $\Delta \in \overline{\mathrm{Dic}}(D)$. A germ of irreducible separatrix whose strict transform crosses a dicritical component is called a *curvet*. The set of all curvets associated to $\Delta \in \mathrm{Dic}(D)$ is denoted by $\mathrm{Curv}(\Delta)$. Finally, $\mathrm{val}(\Delta)$ stands for the *valence* of Δ, defined as the number of components of D intersecting Δ.

Definition 16. A *balanced equation of separatrices* for $\mathcal{F}$ is a meromorphic function F whose divisor is written

$$(F)_0 - (F)_\infty = \sum_{\Delta \in \overline{\mathrm{Dic}}(D)} \sum_{C \in \mathrm{Iso}(\Delta)} a_{\Delta,C}(C) + \sum_{\Delta \in \mathrm{Dic}(D)} \sum_{C \in \mathrm{Curv}(\Delta)} a_{\Delta,C}(C),$$

where

- for every non dicritical component $\Delta \in \overline{\mathrm{Dic}}(D)$, the coefficient $a_{\Delta,C}$ is equal to 1;

- for every dicritical component $\Delta \in \mathrm{Dic}(D)$, the coefficients $a_{\Delta,C} \in \{-1, 0, 1\}$ are zero except for finitely many and satisfy the following equality

$$\sum_{C \in \mathrm{Curv}(\Delta)} a_{D,C} = 2 - \mathrm{val}\,(\Delta)\,.$$

Since $a_{\Delta,C}$ belongs to $\{-1, 0, 1\}$, the function F has reduced zeros and poles without multiplicities. If D has no dicritical component, then a balanced equation is nothing but any equation of the finite set of separatricies.

The next lemma follows from the construction of $\mathcal{F}\left[S_d\right]$ in Proposition 11.

Lemma 17. Any balanced equation of the separatricies of $\mathcal{F}\left[S_d\right]$ satisfies that for all $i = 1, \ldots, N$

$$\sum_{C} a_{D_i,C} = p_i + \left(\text{the number of component of } d^E \text{ meeting } D_i\right).$$

3.3 The sheaf TS_d

In the desingularization $E : (\mathcal{M}, D) \to \left(\mathbb{C}^2, 0\right)$, let us consider the sheaf TS_d, with D as basis, of vector fields tangent to D and to S^E that vanish along the strict transform d^E.

For any divisor $\Sigma = \sum n_i \Sigma_i$ in $\mathcal{M}$, we denote by $\Omega^2\,(\Sigma)$ the sheaf with D as basis, of 2−forms ω such that the multiplicity of ω along Σ_i satisfies

$$\nu_{\Sigma_i}\,(\omega) \geq -n_i.$$

Let F be a balanced equation of the separatricies of $\mathcal{F}\left[S_d\right]$. First, we prove the following proposition.

Proposition 18. In Cech cohomology, one has

$$H^1\left(D, \Omega^2\left(2\,(F)^E - S_d^E + \overline{D}\right)\right) = 0,$$

where the divisor $(F)^E$ is $(F = 0)^E - (F = \infty)^E$ and $\overline{D}$ is the divisor D deprived of D_N and of the components D_i for which $p_i = -1$.

The proof is an induction on the length of the desingularization E. The 1st step is the following lemma.

Let us consider a germ of divisor Σ at the origin of $(\mathbb{C}^2, 0)$ and $E_1 : (\mathcal{M}_1, D_1) \to (\mathbb{C}^2, 0)$ the single blowing-up of the origin. Consider the standard coordinates of the blowing-up together with its standard covering.

$$U_1 : \begin{cases} y = y_1 x_1 \\ x = x_1 \end{cases} \qquad U_2 : \begin{cases} y = y_2 \\ x = y_2 x_2 \end{cases} .$$

Lemma 19. For any $n \geq 0$, the following statements are equivalent:

- The multiplicity of Σ at the origin satisfies $\nu(\Sigma) \geq n$.
- The 1st cohomology group of $\Omega^2\left(\Sigma^{E_1} + nD_1\right)$ on D_1 vanishes

$$H^1\left(D_1, \Omega^2\left(\Sigma^{E_1} + nD_1\right)\right) = 0. \tag{3.1}$$

More precisely, let us denote $l_1 = \frac{l \circ E_1}{x_1^{\nu(\Sigma)}}$ where l is an equation of Σ. If $a(x_1, y_1)$ is a Laurent series in y_1

$$a(x_1, y_1) = \sum_{i \in \mathbb{N}, j \geq -N} a_{ij} x_1^i y_1^j$$

then

$$\begin{cases} \left[a \frac{dx_1 \wedge dy_1}{x_1^n l_1}\right] = 0 \in H^1\left(D_1, \{U_1, U_2\}\, \Omega^2\left(\Sigma^E + nD_1\right)\right) \\ \text{and} \\ a_{0-1} \neq 0 \end{cases} \Longrightarrow \nu(\Sigma) \geq n.$$

Proof. The global sections of $\Omega^2\left(\Sigma^{E_1} + nD_1\right)$ on each associated open sets are written

$$\begin{aligned}
\Omega^2\left(\Sigma^{E_1} + nD_1\right)(U_1) &= \left\{ f(x_1, y_1) \frac{1}{l_1 x_1^n} dx_1 \wedge dy_1 \middle| f \in \mathcal{O}(U_1) \right\} \\
\Omega^2\left(\Sigma^{E_1} + nD_1\right)(U_2) &= \left\{ g(x_2, y_2) \frac{1}{l_2 y_2^n} dx_2 \wedge dy_2 \middle| g \in \mathcal{O}(U_2) \right\} \\
\Omega^2\left(\Sigma^{E_1} + nD_1\right)(U_1 \cap U_2) &= \left\{ h(x_1, y_1) \frac{1}{l_1 x_1^n} dx_1 \wedge dy_1 \middle| h \in \mathcal{O}(U_1 \cap U_2) \right\},
\end{aligned}$$

where $l_1 = \frac{l\circ E_1}{x_1^{\nu(\Sigma)}}$, $l_2 = \frac{l\circ E_1}{y_2^{\nu(\Sigma)}}$. Since the covering $\{U_1, U_2\}$ is acyclic, one has the following isomorphism

$$H^1\left(D_1, \Omega^2\left(\Sigma^{E_1} + nD_1\right)\right) \simeq \frac{\Omega^2\left(\Sigma^{E_1} + nD_1\right)\left(U_1 \cap U_2\right)}{\Omega^2\left(\Sigma^{E_1} + nD_1\right)\left(U_1\right) \oplus \Omega^2\left(\Sigma^{E_1} + nD_1\right)\left(U_2\right)}.$$

Therefore, the dimension of (3.1) is the number of obstructions to the following cohomological equation

$$\begin{aligned} h\left(x_1, y_1\right) \tfrac{1}{l_1 x_1^n} \mathrm{d}x_1 \wedge \mathrm{d}y_1 &= g\left(x_2, y_2\right) \tfrac{1}{l_2 y_2^n} \mathrm{d}x_2 \wedge \mathrm{d}y_2 \\ &\quad - f\left(x_1, y_1\right) \tfrac{1}{l_1 x_1^n} \mathrm{d}x_1 \wedge \mathrm{d}y_1, \end{aligned}$$

which is equivalent to

$$h\left(x_1, y_1\right) = -f\left(x_1, y_1\right) - \frac{1}{y_1^{-\nu(\Sigma)+n+1}} g\left(\frac{1}{y_1}, y_1 x_1\right). \tag{3.2}$$

Let $h = x_1^{i_0} y_1^{j_0}$. Then h is an obstruction to (3.2) if and only if $j_0 < 0$ and the following system cannot be solved in $\mathbb{N}^2$

$$\begin{cases} i_0 = j \\ j_0 = j - i + \nu\left(\Sigma\right) - n - 1 \end{cases} \iff \begin{cases} j = i_0 \\ i = i_0 - j_0 + \nu\left(\Sigma\right) - n - 1. \end{cases}$$

Thus, $\nu\left(\Sigma\right) \geq n$ if and only if there is no obstruction. The 2nd part of the lemma follows from the fact that if the above system has a solution $(i, j) \in \mathbb{N}^2$ for $\left(i_0, j_0\right) = (0, -1)$ then $\nu\left(\Sigma\right) \geq n$. ∎

Now let us prove Proposition 18.

Proof. The proof of the proposition is an induction on the length of the desingularization of S_d. Let us write

$$E = E_1 \circ E^2.$$

Let U_1 be $D_1 \setminus \mathrm{Sing}\left(S_2\right)$ and U_2 a very small neighborhood of $\mathrm{Sing}\left(S_2\right)$. We defined the following open sets

$$\mathcal{U}_1 = \left(E^2\right)^{-1}\left(U_1\right) \qquad \mathcal{U}_2 = \left(E^2\right)^{-1}\left(U_2\right). \tag{3.3}$$

The system $\{\mathcal{U}_1, \mathcal{U}_2\}$ is an open covering of D. The associated Mayer–Vietoris sequence for the sheaf $\Omega^2\left(2(F)^E - S_d^E + \overline{D}\right)$ is written

$$H^0\left(\mathcal{U}_1, \Omega^2\left(2(F)^E - S_d^E + \overline{D}\right)\right) \bigoplus H^0\left(\mathcal{U}_2, \Omega^2(\cdots)\right) \to H^0\left(\mathcal{U}_1 \cap \mathcal{U}_2, \Omega^2(\cdots)\right) \to \mathcal{N} \to 0 \quad (3.4)$$

and

$$0 \to \mathcal{N} \to H^1\left(D, \Omega^2(\cdots)\right) \to H^1\left(\mathcal{U}_1, \Omega^2(\cdots)\right) \bigoplus H^1\left(\mathcal{U}_2, \Omega^2(\cdots)\right). \quad (3.5)$$

We are going to identify each term of the above exact sequences.

The manifold $D_1 \setminus \mathrm{Sing}(S_2)$ is isomorphic to $\mathbb{C}$. Thus, it is Stein. Since, the sheaf $\Omega^2(\cdots)$ is coherent, its cohomology vanishes on $\mathcal{U}_1$ (see [13]) and, in (5), the following relation holds,

$$H^1\left(\mathcal{U}_1, \Omega^2\left(2(F)^E - S_d^E + \overline{D}\right)\right) = 0.$$

Let $\mathcal{F}_2$ be defined by the germ of foliation $E_1^*\mathcal{F}[S_d]$ at $\mathrm{Sing}(S_2)$. By construction, the foliation $\mathcal{F}_2$ leaves invariant S_2. Let F_2 be a balanced equation of $\mathcal{F}_2$. Let h be a local equation of D_1 at $\mathrm{Sing}(S_2)$. Two cases have to be considered

- If D_1 is invariant for $\mathcal{F}[S_d]$, then, following [9], F_2 can be chosen so that

$$(F_2)^{E^2} = (h)^{E^2} + (F)^E\Big|_{\mathcal{U}_2}$$

Thus, if the direction d_2 of S_2 is chosen to be the local trace at $\mathrm{Sing}(S_2)$ of the union of d^{E_1} and D_1, then the next equalities hold

$$\begin{aligned}\left(2(F)^E - S_d^E + \overline{D}\right)\Big|_{\mathcal{U}_2} &= 2\left((F_2)^{E^2} - (h)^{E^2}\right) - S_d^E\Big|_{\mathcal{U}_2} + \overline{D}\big|_{\mathcal{U}_2} \\ &= 2(F_2)^{E^2} - 2(h)^{E^2} - S_d^E\Big|_{\mathcal{U}_2} + \overline{D^2} + (h)^{E^2} \\ &= 2(F_2)^{E^2} - S_{2,d_2}^{E^2} + \overline{D^2}\end{aligned}$$

- If D_1 is not invariant for $\mathcal{F}[S_d]$ then F_2 can be chosen so that

$$(F_2)^{E^2} = (F)^E .$$

Thus, setting for the direction d_2 of S_2 the local trace at $\mathrm{Sing}\left(S_2\right)$ of the sole d^{E_1} still yields

$$\left(2\left(F\right)^E - S_d^E + \overline{D}\right)\Big|_{\mathcal{U}_2} = 2\left(F_2\right)^{E^2} - S_{2,d_2}^{E^2} + \overline{D^2}$$

since here $\overline{D}\big|_{\mathcal{U}_2} = \overline{D^2}$.

In any case, applying inductively Proposition 18 to S_2 and to the associated divisor $2\left(F_2\right)^{E^2} - S_{2,d_2}^{E^2} + \overline{D^2}$ ensures that, in (5), one has

$$H^1\left(\mathcal{U}_2, \Omega^2\left(2\left(F\right)^E - S_d^E + \overline{D}\right)\right) = H^1\left(\mathcal{U}_2, \Omega^2\left(2\left(F_2\right)^{E^2} - S_{2,d_2}^{E^2} + \overline{D^2}\right)\right) = 0.$$

The map E^2 induces isomorphisms in cohomology

$$\begin{aligned} H^0\left(\mathcal{U}_1, \Omega^2\left(2\left(F\right)^E - S_d^E + \overline{D}\right)\right) &\simeq H^0\left(U_1, \Omega^2\left(2\left(F\right)^{E_1} - S_d^{E_1} + \overline{D_1}\right)\right) \\ H^0\left(\mathcal{U}_1 \cap \mathcal{U}_2, \Omega^2\left(\cdots\right)\right) &\simeq H^0\left(U_1 \cap U_2, \Omega^2\left(\cdots\right)\right). \end{aligned} \tag{3.6}$$

Let us prove that E^2 induces also an isomorphism on the set of global sections along U_2 and $\mathcal{U}_2$. If η is a global section of $\Omega^2\left(2\left(F\right)^E - S_d^E + \overline{D}\right)$ on $\mathcal{U}_2$ then the push-forward of η by E^2 can be extended analytically at $\mathrm{Sing}\left(S_2\right)$ by Hartogs's extension result. It induces naturally a section of $\Omega^2\left(2\left(F\right)^{E_1} - S_d^{E_1} + \overline{D_1}\right)$ on U_2. Thus, E^2 induces a injective map

$$H^0\left(\mathcal{U}_2, \Omega^2\left(2\left(F\right)^E - S_d^E + \overline{D}\right)\right) \stackrel{E^2}{\hookrightarrow} H^0\left(U_2, \Omega^2\left(2\left(F\right)^{E_1} - S_d^{E_1} + \overline{D_1}\right)\right). \tag{3.7}$$

By induction, it is enough to prove that (3.7) is onto when E^2 is the simple blowing-up of $\mathrm{Sing}\left(S_2\right)$ and D reduced to $D_1 \cup D_2$. Let η be a section of $\Omega^2\left(2\left(F\right)^{E_1} - S_d^{E_1} + \overline{D_1}\right)$ on U_2.

- If D_1 is not dicritical for $\mathcal{F}\left[S_d\right]$ then η is written in coordinates

$$\eta = hl\frac{\mathrm{d}x \wedge \mathrm{d}y}{x},$$

where x is a local equation of D_1, l is any meromorphic function whose local divisor is $S_d^{E_1} - 2\left(F\right)^{E_1}$ and h is any holomorphic function. If $\delta_2 = 1$ then the possible component of d meets D_1 at a different point from $\mathrm{Sing}\left(S_2\right)$. The family of integers p_i satisfies the system (2.1) of Proposition 8. Thus, one has

$$\mathcal{E}^{-1}\begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_N \end{pmatrix} = \begin{pmatrix} \left[\frac{\nu(S_1)-\delta_1}{2}\right]+1 \\ \left[\frac{\nu(S_2)-\delta_2}{2}\right]+1 \\ \vdots \\ \left[\frac{\nu(S_N)-\delta_N}{2}\right]+1 \end{pmatrix}$$

According to Proposition 8, p_N is equal to 0. Moreover, $\nu\left(S_2\right)=e_{2N}$. Thus, writing the 2nd line of this system leads to

$$\sum_{i=2}^{N-1} p_i e_{2i} = \left[\frac{e_{2N}-1}{2}\right] - 1.$$

By construction of $\mathcal{F}\left[S_d\right]$ and its balanced equation of separatricies F [9], the multiplicity of l is equal to

$$\nu\left(l\right) = e_{2N} - 2\sum_{i=2}^{N-1} p_i e_{2i} = e_{2N} - 2\left[\frac{e_{2N}-1}{2}\right] - 2 \geq -1.$$

Now, after the blowing-up E^2, which is written in adapted coordinates $E^2\left(x,t\right)=\left(x,tx\right)$, the pull back of η is written

$$E^{2*}\eta = E^{2*}hE^{2*}l\mathrm{d}x\wedge\mathrm{d}t.$$

Thus, the multiplicity of $E^{2*}\eta$ along D_2 is at least -1. The exceptional divisor of E^2 cannot be dicritical for $\mathcal{F}\left[S_d\right]$ since $\delta_2=1$. Therefore, $E^{2*}\eta$ is a section of $\Omega^2\left(2\left(F\right)^{E^2}-S_d^{E^2}+\overline{D_1\cup D_2}\right)$ along $D_1\cup D_2$. Now, if $\delta_2=2$ then one of the components of d^{E_1}, say $d_1^{E_1}$, meets $\mathrm{Sing}\left(S_2\right)$. Whether or not the component d_1^E meets a dicritical component, the multiplicity of l is at least

$$\nu\left(l\right) \geq e_{2N} - 2\sum_{i=2}^{N-1} p_i e_{2i} - 1 = e_{2N} - 2\left[\frac{e_{2N}}{2}\right] - 1.$$

If the exceptional divisor of E^2 is dicritical then e_{2N} is odd and $\nu\left(l\right)\geq 0$. If not, $\nu\left(l\right)\geq -1$. Thus, wether the exceptional divisor of E^2 is dicritical or not, $E^{2*}\omega$ is a section of $\Omega^2\left(2\left(F\right)^{E^2}-S_d^{E^2}+\overline{D_1\cup D_2}\right)$ along $D_1\cup D_2$.

- if D_1 is dicritical then $\delta_2 = 1$. Moreover, η is written

$$\eta = hl\mathrm{d}x \wedge \mathrm{d}y, \qquad E^{2*}\eta = E^{2*}hE^{2*}lx\mathrm{d}x \wedge \mathrm{d}t,$$

where

$$\nu(l) + 1 = e_{2N} - \sum_{i=2}^{N-1} p_i e_{2i} + 1 = e_{2N} - 1 - 2\left[\frac{e_{2N}-1}{2}\right] \geq 0.$$

Hence, $E^{2*}\omega$ is still a section of $\Omega^2\left(2(F)^{E^2} - S_d^{E^2} + \overline{D_1 \cup D_2}\right)$ along $D_1 \cup D_2$.

By induction on the length of E^2, the isomorphism (3.7) is proved. Thus, the isomorphisms (3.1) and the exact sequence (4) identify $\mathcal{N}$ with the cohomology group

$$H^1\left(D_1, \Omega^2\left(2(F)^{E_1} - S_d^{E_1} + \overline{D_1}\right)\right).$$

Let us prove that the latter vanishes. If $p_1 = -1$, then D_1 is dicritical and $\delta_1 = 2$ and $\delta_2 = 1$. Therefore,

$$\nu\left(2(F)^{E_1} - S_d^{E_1}\right) = 2\sum_{i=2}^{N-1} p_i e_{1i} - e_{1N} = 2\left(\left[\frac{e_{1N}-2}{2}\right] + 2\right) - e_{1N} = 1$$

since e_{1N} is odd. If $p_1 \neq -1$, then

$$\nu\left(2(F)^{E_1} - S_d^{E_1}\right) = 2\sum_{i=1}^{N-1} p_i e_{1i} + \delta_1 - e_{1N} = 2\left[\frac{e_{1N}-\delta_1}{2}\right] + 2 + \delta_1 - e_{1N} \geq 1.$$

Therefore, according to Lemma 19, $\mathcal{N}$ vanishes, which completes the proof of Proposition 18. ■

To compare the deformations of $\mathcal{F}[S_d]$ and of the underlying curve S_d, we introduce the following operator.

Definition 20. The operator of basic vector fields for $\mathcal{F}[S_d]$ is a morphism of sheaves defined by

$$\mathcal{B} : X \in T\mathcal{S}_d \mapsto L_X E^* \frac{\omega}{F} \wedge E^* \frac{\omega}{F} \in \Omega^2, \tag{3.8}$$

where ω is any $1-$form with an isolated singularity defining $\mathcal{F}\left[S_d\right]$ and F any balanced equation of $\mathcal{F}\left[S_d\right]$.

The operator of basic vector fields may behave quite wildly around the singular point of $\mathcal{F}\left[S_d\right]$. Indeed, one can check that the description of its local image at singular points may involve the phenomenon known as *small divisors*. However, for our construction, we can disregard what happens exactly at the singular points, since *we control everything happening around.* To take into account this remark, we introduce the following notation:

Notation. For any sheaf $\mathfrak{F}$ of basis D, we denote by $\mathfrak{F}^\circ$ the sheaf whose stalk satisfies that for all $x \in D \setminus \operatorname{Sing}\left(\mathcal{F}\left[S_d\right]\right)$, $(\mathfrak{F})_x = \left(\mathfrak{F}^\circ\right)_x$ and for all $x \in \operatorname{Sing}\left(\mathcal{F}\left[S_d\right]\right)$, $\left(\mathfrak{F}^\circ\right)_x = 0$.

The interest of the above notation relies on the following lemma:

Lemma 21. For any $i \geq 1$, one has

$$H^i\left(D, \mathfrak{F}\right) = H^i\left(D, \mathfrak{F}^\circ\right).$$

Proof. Indeed, there is a direct sum of skyscraper sheaves $\mathfrak{F}_\circ$ such that $\mathfrak{F}^\circ = \mathfrak{F}/\mathfrak{F}_\circ$. The long exact sequence of sheaves associated to the short sequence

$$0 \to \mathfrak{F}_\circ \to \mathfrak{F} \to \mathfrak{F}/\mathfrak{F}_\circ \to 0$$

and the fact that the cohomology of $\mathfrak{F}_\circ$vanishes in degree more than 1 ensure the lemma. ∎

Proposition 22. Let $\mathcal{B}_n\left(\mathcal{F}\left[S_d\right]\right)$ be the sheaf defined by the kernel

$$\mathcal{B}_n\left(\mathcal{F}\left[S_d\right]\right) = \ker\left(\mathcal{B}|_{\mathfrak{M}^n \cdot TS_d}\right),$$

where $\mathfrak{M}^n$ is the n^{th} power of the sheaf of $\mathcal{O}-$module generated by the functions E^*f with $f(0) = 0$. There is an exact sequence of sheaves

$$0 \to \mathcal{B}_n\left(\mathcal{F}\left[S_d\right]\right)^\circ \to \mathfrak{M}^n \cdot TS_d^\circ \to \mathfrak{M}^n \cdot \Omega^2\left(2\,(F)^E - S_d^E + \overline{D}\right)^\circ \to 0. \tag{3.9}$$

In particular, extracted from the long exact in cohomology associated to (3.9), there is an exact sequence

$$H^1\left(D, \mathcal{B}_n\left(\mathcal{F}\left[S_d\right]\right)^\circ\right) \to H^1\left(D, \mathfrak{M}^n \cdot TS_d^\circ\right) \to 0. \tag{3.10}$$

Proof. The 1st part of the proposition is a computation in local coordinates. We describe the image of $\mathfrak{M}^n \cdot TS_d$ by the operator $\mathcal{B}$. Since, $\mathcal{F}\left[S_d\right]$ has only linearizable singularities, the multiplicities of $\mathcal{F}\left[S_d\right]$ and of the balanced equation F along any irreducible component D_i of the exceptional divisor satisfy [7, 9, Proposition 3.7]

- $\nu_{D_i}\left(\mathcal{F}\left[S_d\right]\right) = \nu_{D_i}\left(E^*F\right)$ if D_i is dicritical
- $\nu_{D_i}\left(\mathcal{F}\left[S_d\right]\right) = \nu_{D_i}\left(E^*F\right) + 1$ else.

Let $p \in D_i$, for some i, be a regular point of $\mathcal{F}\left[S_d\right]$ where the foliation is tangent to exceptional divisor. In some local coordinates (x, y) around p, the pull-back $E^*\frac{\omega}{F}$ is written

$$E^*\frac{\omega}{F} = u\frac{\mathrm{d}x}{x},$$

where x is a local equation of D_i. Now, a local section X of $\mathfrak{M}^n \cdot TS_d$ is written

$$X = x^m\left(ax\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right),\ a, b \in \mathbb{C}\{x, y\},$$

where $m = n \times n\left(E, D_i\right)$ where $n\left(E, D_i\right)$ is defined in Definition 5. Therefore, applying the basic operator leads to

$$\mathcal{B}(X) = x^m u^2 \frac{\partial a}{\partial y}\frac{\mathrm{d}x \wedge \mathrm{d}y}{x},$$

which is a local section of $\mathfrak{M}^n \cdot \Omega^2\left(2(F)^E - S_d^E + \overline{D}\right)$. Since the equation $\frac{\partial a}{\partial y} = h$ can be solved for any h, the operator $\mathcal{B}$ is onto locally around p. This property is true for any type of regular points for $\mathcal{F}\left[S_d\right]$.

The sheaf $\mathfrak{M}^n$ is generated by its global sections. Therefore, Proposition 18 ensures that

$$H^1\left(D, \mathfrak{M}^n \cdot \Omega^2\left(2(F)^E - S_d^E + \overline{D}\right)\right) = 0.$$

Finally, the long exact sequence in cohomology associated to (3.9) proves the end of Proposition 22. ■

3.4 Deformations of $\mathcal{F}\left[S_d\right]$

Proposition 22 can be expressed as follows: any infinitesimal deformation of S_d tangent to D at order n can be followed by an infinitesimal deformation of the foliation $\mathcal{F}\left[S_d\right]$ at the same level of tangency. Roughly speaking, the proof of Proposition 15 consists in an *non-commutative analog.* Actually, let us consider the following sheaves of non-abelian groups

Definition 23. For any *involutive* sub-sheaf $\mathfrak{I}$ of the sheaf of tangent vector fields to S_d^E that vanish along d and D, we consider

$$\mathfrak{G}(\mathfrak{I})$$

the sheaf of non-abelian groups generated by the flows of vector fields in $\mathfrak{I}$.

According to the Campbell–Hausdorff formula,

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}([X,[X,Y]]-[Y,[X,Y]])+\cdots} \tag{3.11}$$

any element of $\mathfrak{G}(\mathfrak{I})$ is a flow of an element of $\mathfrak{I}$.

The 1st step of the proof of Proposition 15 is the following:

Proposition 24. Extracted from the long exact sequence in cohomology induced by the embedding $\mathfrak{G}\left(\mathcal{B}_1\left(\mathcal{F}\left[S_d\right]\right)^{\circ}\right) \hookrightarrow \mathfrak{G}\left(\mathfrak{M}\cdot TS_d^{\circ}\right)$, the following sequence

$$H^1\left(D, \mathfrak{G}\left(\mathcal{B}_1\left(\mathcal{F}\left[S_d\right]\right)^{\circ}\right)\right) \to H^1\left(D, \mathfrak{G}\left(\mathfrak{M}\cdot TS_d^{\circ}\right)\right) \to 0$$

is exact.

Proof. Let us consider a 1−cocycle $\left\{\phi_{ij}\right\}_{ij} \in \mathcal{Z}^1\left(D, \mathfrak{G}\left(\mathfrak{M}\cdot TS_d^{\circ}\right)\right)$. By definition, this is a flow

$$\phi_{ij} = e^{X_{ij}}, \tag{3.12}$$

where $\left\{X_{ij}\right\}_{ij} \in \mathcal{Z}^1\left(D, \mathfrak{M}\cdot TS_d^{\circ}\right)$. By induction on n, we are going to prove that there exist $\left\{B_{ij}^n\right\}_{ij} \in \mathcal{Z}^1\left(D, \mathcal{B}_1\left(\mathcal{F}\left[S_d\right]\right)^{\circ}\right)$, $\left\{X_i^n\right\}_i \in \mathcal{Z}^0\left(D, \mathfrak{M}\cdot TS_d^{\circ}\right)$ and $\left\{X_{ij}^n\right\}_{ij} \in \mathcal{Z}^1\left(D, \mathfrak{M}^n\cdot TS_d^{\circ}\right)$ such that

$$e^{-X_i^n}\phi_{ij}e^{X_j^n} = e^{B_{ij}^n}e^{X_{ij}^n}. \tag{3.13}$$

For $n = 1$, this is the relation (3.12). Now, suppose this is true for n. According to Proposition 22, there exist $\left\{\tilde{B}_{ij}^n\right\}_{ij} \in \mathcal{Z}^1\left(D, \mathcal{B}_1\left(\mathcal{F}\left[S_d\right]\right)^{\circ}\right)$ and $\left\{Y_i^n\right\}_i \in \mathcal{Z}^0\left(D, \mathfrak{M}\cdot TS_d^{\circ}\right)$ such that

$$X_{ij}^n = Y_i^n + \tilde{B}_{ij}^n - Y_j^n.$$

Taking the flow at time 1 yields

$$\begin{aligned}
e^{-Y_i^n} e^{-X_i^n} \phi_{ij} e^{X_j^n} e^{Y_j^n} &= e^{-Y_i^n} e^{B_{ij}^n} e^{X_{ij}^n} e^{Y_j^n} \\
&= e^{B_{ij}^n} \left[e^{-B_{ij}^n}, e^{-Y_i^n} \right] e^{-Y_i^n} e^{X_{ij}^n} e^{Y_j^n} \\
&= e^{B_{ij}^n} \left[e^{-B_{ij}^n}, e^{-Y_i^n} \right] e^{\tilde{B}_{ij}^n} e^{Y_{ij}^{n+1}} \\
&= e^{B_{ij}^n} e^{\tilde{B}_{ij}^n} \underbrace{e^{-\tilde{B}_{ij}^n} \left[e^{-B_{ij}^n}, e^{-Y_i^n} \right] e^{\tilde{B}_{ij}^n} e^{Y_{ij}^{n+1}}}_{\in \mathfrak{G}\left(\mathfrak{M}^{n+1} \cdot TS_d^\circ\right)} \\
&= e^{B_{ij}^{n+1}} e^{X_{ij}^{n+1}},
\end{aligned}$$

where B_{ij}^{n+1} is given by the Campbell–Hausdorff (3.11) where $X = B_{ij}^n$ and $Y = \tilde{B_{ij}^n}$, which ensures the property by induction. Taking n as big as necessary, the proposition is a consequence of the stability property proved in [8]. ■

We can improve a bit the previous property taking advantage of the inductive structure of the desingularization of S_d.

Proposition 25. Let $E : (\mathcal{M}, D) \to (\mathbb{C}^2, 0)$ be the desingularization of $\mathcal{F}[S_d]$. Consider the sheaf $\mathfrak{I} \cdot TS_d$, where $\mathfrak{I}$ is the ideal of functions vanishing along D and $\mathcal{B}_0(\mathcal{F}[S_d]) = \ker\left(\mathcal{B}|_{\mathfrak{I}\cdot TS_d}\right)$. Then for every $\left\{\phi_{ij}\right\}_{ij} \in Z^1\left(D, \mathfrak{G}\left(\mathfrak{I} \cdot TS_d^\circ\right)\right)$ there exists a family $\left\{\psi_{ij}^k\right\}_{ij}$ $k = 0 \ldots l$ of 1-cocycles in $Z^1\left(D, \mathfrak{G}\left(\mathcal{B}_0\left(\mathcal{F}[S_d]\right)^\circ\right)\right)$ such that

$$\mathcal{M}\left[\phi_{ij}\right] \simeq \mathcal{M}\left[\psi_{ij}^0\right] \cdots \left[\psi_{ij}^l\right]. \tag{3.14}$$

In particular, $\mathcal{M}\left[\phi_{ij}\right]$ is the support of a foliation obtained by successive basic surgeries of $\mathcal{F}[S_d]$.

Proof. The proof is an induction on the length of the resolution of S_d. Let us consider a 1-cocyle $\left\{\phi_{ij}\right\}_{ij}$ in $\mathcal{Z}^1\left(\mathfrak{G}\left(\mathfrak{I} \cdot TS_d\right)^\circ\right)$. Let us consider $\left\{\overline{\phi_{ij}}\right\}_{ij}$ the restriction of the cocyle $\left\{\phi_{ij}\right\}_{ij}$ to D^2. We are going to apply inductively the property to S_{2,d_2} for some adapted direction d_2 of S_2 as defined in the proof of Proposition 18. Applying inductively

Proposition 25 to $\left\{\overline{\phi_{ij}}\right\}_{ij}$ yields the existence of 0−cocycles in $\mathfrak{G}\left(\mathfrak{I}\cdot\left(TS^2_{d_2}\right)^{\circ}\right)$ and of 1−cocycles $\mathfrak{G}\left(\mathcal{B}_0\left(\mathcal{F}\left[S^2_{d_2}\right]\right)^{\circ}\right)$ such that

$$\overline{\phi_{ij}} = \phi_i^1\psi_{ij}^1\phi_i^2\psi_{ij}^2\cdots\psi_{ij}^M\left(\phi_j^M\right)^{-1}\left(\phi_j^{M-1}\right)^{-1}\cdots\left(\phi_j^1\right)^{-1},$$

a relation that is equivalent to (3.14) for $\left\{\overline{\phi_{ij}}\right\}_{ij}$. Now, consider the following 1−cocyle

$$\tilde{\phi_{ij}} = \begin{cases}\phi_{12}\phi_j^1\phi_j^2\cdots\phi_j^M & \text{for } i=1 \text{ and } j=2.\\ \mathrm{Id} & \text{else.}\end{cases}$$

It belongs to $\mathcal{Z}^1\left(\mathfrak{G}\left(\mathfrak{I}\cdot TS_d\right)\right)$. Since $\mathfrak{M}$ and $\mathfrak{I}$ coincide along D_1, it belongs also to $\mathcal{Z}^1\left(\mathfrak{G}\left(\mathfrak{M}\cdot TS_d\right)^{\circ}\right)$. Therefore, Proposition 24 yields a 0−cocycle and 1-cocycle respectively in $\mathfrak{G}\left(\mathfrak{M}\cdot TS_d^{\circ}\right)$ and $\mathfrak{G}\left(\mathcal{B}_1\left(\mathcal{F}\left[S_d\right]\right)^{\circ}\right)$ such that

$$\tilde{\phi_{ij}} = \phi_i\psi_{ij}\phi_j^{-1}.$$

In particular, if $(i,j)\neq 2$, then $\phi_i^{-1}\phi_j = \psi_{ij}$. Therefore, for any $(i,j)\neq(1,2)$, one can write

$$\phi_{ij} = \phi_i^1\psi_{ij}^1\phi_i^2\psi_{ij}^2\cdots\psi_{ij}^M\phi_i\psi_{ij}\phi_j^{-1}\left(\phi_j^M\right)^{-1}\left(\phi_j^{M-1}\right)^{-1}\cdots\left(\phi_j^1\right)^{-1}$$

and

$$\phi_{12} = \phi_1\psi_{12}\phi_2^{-1}\left(\phi_2^M\right)^{-1}\left(\phi_2^{M-1}\right)^{-1}\cdots\left(\phi_2^1\right)^{-1},$$

which is equivalent to (3.14) for $\left\{\phi_{ij}\right\}_{ij}$. The proposition is proved. ■

Finally, we can prove Proposition 15. Let $E' : (\mathcal{M}', D') \to (\mathbb{C}^2, 0)$ be the desingularization of $\mathfrak{S}$. The curves $\mathfrak{S}$ and S_d are topologically equivalent. Since S is irreducible, the exceptional divisors D and D' are analytically equivalent. Following [8, Section 3.2], there exists a 1-cocycle $\left\{\phi_{ij}\right\}_{ij}$ in $\mathfrak{G}\left(\mathfrak{I}\cdot TS_d^{\circ}\right)$ such that

$$\mathcal{M}' \simeq \mathcal{M}\left[\phi_{ij}\right].$$

According to Proposition 25, $\mathcal{M}'$ is the support of a foliation obtained from a basic surgery of $\mathcal{F}\left[S_d\right]$ that leaves invariant the curve C, which completes the proof of Proposition 15.

As a corollary, we obtain Theorem 2, since under the hypothesis mentioned, p_1 cannot be equal to -1 and $\mathcal{F}\left[S_d\right]$ is not dicritical along the exceptional divisor of the 1st blowing-up.

4 Theorem 2$\Longrightarrow$Theorem 1

The proof consists in an argument by contradiction and four consecutive steps.

4.1 Step 1: construction of an equisingular family of curves $S(\epsilon)$ followed by an analytical family of forms $\omega(\epsilon) \in \Omega^1(S(\epsilon))$ reaching the minimal valuation in $\Omega^1(S(\epsilon))$.

Let S be an irreducible germ of curve in the generic component of its moduli space and let $E : (\mathcal{M}, D) \to \left(\mathbb{C}^2, 0\right)$ be its minimal desingularization. Let d be any direction for S. Let Σ be a mini-versal deformation of S_d for the topologically trivial deformations of S_d

$$\begin{array}{ccc} S_d & \hookrightarrow & \left(\mathbb{C}^{P+2}, \Sigma\right) \\ \downarrow & & \downarrow \pi \\ 0 & \hookrightarrow & \left(\mathbb{C}^P, 0\right) \end{array}.$$

Let H be an equation of Σ. The family $\epsilon \in \mathbb{C}^P \to \omega(\epsilon) = dH|_{\pi^{-1}(\epsilon)}$ is an analytic family of $1-$forms such that $\omega(\epsilon) \in \Omega^1\left(H^{-1}(0) \cap \pi^{-1}(\epsilon)\right)$ and $\nu(\omega(\epsilon)) = \nu(S) - 1$. Consider now, the smallest integer ν such that, there exists a Zariski open set such that for all $\epsilon \in \mathcal{U}$ there exists $\omega(\epsilon) \in \Omega^1\left(H^{-1}(0) \cap \pi^{-1}(\epsilon)\right)$ and $\nu(\omega(\epsilon)) = \nu$. Our main goal is to prove that

$$\nu \geq \left[\frac{\nu\left(S_d\right)}{2}\right]$$

Lemma 26. The family $\omega(\epsilon) : \epsilon \in \left(\mathbb{C}^P, 0\right) \to \Omega^1\left(S_d(\epsilon)\right)$ can be chosen analytic in ϵ.

Proof. Up to some change of coordinates $(x, y) \in \left(\mathbb{C}^2, 0\right)$, we can suppose that the direction d is a fixed curve equal to $\emptyset$, $\{x = 0\}$ or $\{xy = 0\}$ that does not depend on ϵ. In these three respective cases, any element in $\Omega^1\left(S_d(\epsilon)\right)$ can be written in coordinates

$$\omega(\epsilon)=\begin{cases} A_\epsilon \mathrm{d}x+B_\epsilon \mathrm{d}y, & d=\emptyset \\ A_\epsilon \mathrm{d}x+xB_\epsilon \mathrm{d}y, & d=\{x=0\} \\ \text{or} & \\ yA_\epsilon \mathrm{d}x+xB_\epsilon \mathrm{d}y, & d=\{xy=0\} \end{cases}$$

Let γ_ϵ be a Puiseux parametrization of $S(\epsilon)$ depending analytically on ϵ. The hypothesis ensures that for any $N\in\mathbb{N}$ and for any ϵ, the following system has a solution ω

$$\left(L_\epsilon\right):\begin{cases} \mathrm{Jet}^N_{t=0}\left(\gamma_\epsilon^*\omega\right)=0 & (1) \\ \mathrm{Jet}^{\nu-1}_{(x,y)}\omega(\epsilon)=0 & (2) \\ \mathrm{Jet}^{\nu}_{(x,y)}\omega(\epsilon)\neq 0 & (3) \end{cases}.$$

The family $\left(L_\epsilon\right)_\epsilon$ is an analytic family of linear systems with a finite number of unknown variables, say M, which are some coefficients of the Taylor expansion of A_ϵ and B_ϵ—(1) and (2)—and an open condition (3). Thus, the solutions can be viewed as a constructible set Z of $\mathcal{U}\times\mathbb{C}^M$ that projects onto $\mathcal{U}$ through the projection $p:\mathbb{C}^{P+M}\to\mathbb{C}^P$. Since the image of $p|_Z$ contains an open set, the set of points in Z where $p|_Z$ is a submersion cannot be empty. Thus, if $\left(z,\epsilon_0\right)$ is such a point, there exists a local analytic section σ of p defined in a neighborhood of ϵ_0 such that $\sigma\left(\epsilon_0\right)=\left(p,\epsilon_0\right)$. This provides a analytic family $\left(\omega_\epsilon\right)_{\epsilon\in\left(\mathbb{C}^P,\epsilon_0\right)}$ that is a solution of $\left(L_\epsilon\right)$ in a neighborhood of ϵ_0. Since the family γ_ϵ is topologically trivial, taking a bigger integer N if necessary, we can find a family of functions $f_k\in\mathbb{C}\{x,y\}$ with $\nu\left(df_k\right)>\nu$, $\nu\left(df_k\right)\xrightarrow[k\to\infty]{}+\infty$ such that for any $k\geq N$ and any ϵ, one has

$$\nu\left(\gamma_\epsilon^* df_k\right)=k.$$

Considering a form written

$$\Omega=\omega_\epsilon+\sum_{k\geq M}\alpha_k(\epsilon)\,\mathrm{d}f_k, \tag{4.1}$$

we can choose inductively $\alpha_k(\epsilon)$ such that (4.1) becomes a formal solution $\Omega \in \mathbb{C}\{\epsilon\}[[x,y]]$ of the system

$$\begin{cases} \gamma_\epsilon^*\Omega = 0 \\ \mathrm{Jet}^{\nu-1}_{(x,y)}\Omega = 0 \\ \text{and} \\ \mathrm{Jet}^{\nu}_{(x,y)}\Omega \neq 0 \end{cases}.$$

According to the Artin's approximation theorem [1], we can take Ω analytic as a whole, $\Omega \in \mathbb{C}\{\epsilon, x, y\}$. ■

For ϵ generic, we can also suppose that $\omega(\epsilon)$ is equireducible [22]. Finally, let

$$E(\epsilon) : (\mathcal{M}(\epsilon), D(\epsilon)) \to \left(\mathbb{C}^2, 0\right),\ \epsilon \in \left(\mathbb{C}^P, \epsilon_0\right)$$

be the equisingular family of minimal desingularizations of the foliations $\mathcal{F}(\epsilon)$ defined by $\omega(\epsilon)$. In particular, $E(\epsilon)$ is also an equisingular family of desingularizations of $S_d(\epsilon)$. For the sake of simplicity, we consider that $\epsilon_0 = 0$ and we still denote by $\mathcal{M}$, E, $\mathcal{F}$, ω and S_d respectively the manifold $\mathcal{M}(0)$, the desingularization $E(0)$, the foliation $\mathcal{F}(0)$, the $1-$form $\omega(0)$ and the curve $S_d(0)$.

4.2 Step 2: vanishing of some cohomology

Let $\left\{T_{ij}\right\}_{ij}$ be a $1-$cocycle in $\mathcal{Z}^1\left(\mathcal{M}, TS_d\right)$. Let us consider the deformation obtained by the gluing

$$\mathcal{M}\left[e^{(t)T_{ij}}\right].$$

Since the flow $e^{(t)T_{ij}}$ leaves globally invariant S_d, the manifold $\mathcal{M}\left[e^{(t)T_{ij}}\right]$ admits an invariant curve topologically equivalent to S_d^E. By versality, the so defined topologically trivial deformation is equivalent to a deformation $S_d(\epsilon(t))$ for some analytic factorization $\epsilon(t) : (\mathbb{C}, 0) \to \left(\mathbb{C}^P, 0\right)$. The deformation $S_d(\epsilon(t))$ is followed by the deformation of foliations $\mathcal{F}(\epsilon(t))$. Therefore, on the open set $\mathcal{M}(\epsilon)^*$ which is $\mathcal{M}(\epsilon)$ deprived of the singular locus of $E(\epsilon)^*\mathcal{F}(\epsilon)$, the cocycle $\left\{e^{(t)T_{ij}}\right\}_{ij}$ is equivalent to a cocycle of basic automorphisms. Thus, there exist a $0-$cocycle of automorphism $\left\{\phi_i(t)\right\}_i$ leaving globally invariant $S(\epsilon(t))_d^{E(\epsilon(t))}$ and $D(\epsilon(t))$ and a $1-$cocycle of basic automorphisms $\left\{B_{ij}(t)\right\}_{ij}$ for

$\mathcal{F}$, such that on $\mathcal{M}(\epsilon(t))^*$, one has

$$e^{(t)T_{ij}} = \phi_i(t) B_{ij}(t) \phi_j^{-1}(t).$$

Taking the derivative at $t = 0$ of the above expression yields to a cohomological relation on $\mathcal{M}(0) = \mathcal{M}$.

$$T_{ij} = T_i + b_{ij} - T_j, \tag{4.2}$$

where $\{T_i\}$ is a 0−cocycle in TS_d and $\left\{b_{ij}\right\}_{ij}$ is a 1−cocycle with values in the sub-sheaf of basic vector fields for $\mathcal{F}$ tangent to S_d, denoted simply by $\mathcal{B}(\mathcal{F})$.

Let us denote by Ω the image sheaf of TS_d by the basic operator (3.8) for $\mathcal{F}$ with a given balanced equation F.

The following diagram

$$\begin{tikzcd}
 & H^1(\mathcal{M}^*, \mathcal{B}(\mathcal{F})) \arrow[d, "i"] \\
H^1(\mathcal{M}, TS_d) \arrow[r, "\alpha"] \arrow[d, "\beta"] & H^1(\mathcal{M}^*, TS_d) \arrow[d, "\mathcal{B}"] \\
H^1(\mathcal{M}, \Omega) \arrow[r, "\gamma"] & H^1(\mathcal{M}^*, \Omega)
\end{tikzcd} \tag{4.3}$$

is commutative. Since for any 1-cocycle $\left\{T_{ij}\right\}_{ij} \in Z^1(\mathcal{M}, TS_d)$, a relation such as (4.2) exists, one has

$$\mathrm{Im}\alpha \subset \mathrm{Im}i.$$

Thus, the composed map $\mathcal{B} \circ \alpha$ is the zero map. The sheaf Ω on $\mathcal{M}^*$ can be described as follows:

$$\Omega = \Omega^2\left(2(F)^E - S_d^E + \sum n_i D_i\right), \tag{4.4}$$

where $D = \sum D_i$ and the n_i's are some integers depending on $\mathcal{F}$. This sheaf can be extended analytically on $\mathcal{M}$. The Mayer–Vietoris sequence applied to the covering $\{\mathcal{M}^*, \mathcal{U}\}$ of $\mathcal{M}$ where $\mathcal{U}$ is an union of some small open balls around each singularity is written

$$\cdots \to H^0\left(\mathcal{M}^*, \Omega\right) \bigoplus H^0\left(\mathcal{U}, \Omega\right) \xrightarrow{\Delta} H^0\left(\mathcal{M}^* \cap \mathcal{U}, \Omega\right)$$
$$\to H^1\left(\mathcal{M}, \Omega\right) \to H^1\left(\mathcal{M}^*, \Omega\right) \bigoplus H^1\left(\mathcal{U}, \Omega\right) \to \cdots$$

The Hartogs's extension result ensures that Δ is onto. Moreover, since $\mathcal{U}$ can be supposed to be Stein and Ω is coherent, we deduce that in the diagram (4.3) the map γ is injective.

The previous lemma and the properties of the diagram (4.3) ensure that the map

$$H^1\left(\mathcal{M}, TS_d\right) \xrightarrow{\beta} H^1\left(\mathcal{M}, \Omega\right)$$

is the zero map.

4.3 Step 3: the contradiction

We are going to prove that the fact that the above function β is the zero map leads to a contradiction with the inequality

$$\nu\left(\mathcal{F}\right) < \left[\frac{\nu\left(S_d\right)}{2}\right].$$

We recall that F being a balanced equation of $\mathcal{F}$ [9, Proposition 3.3], the next relation holds

$$\nu\left(\mathcal{F}\right) = \nu\left(F\right) - 1 + \tau\left(\mathcal{F}\right),$$

where $\tau\left(\mathcal{F}\right)$ is a positive integer called *the tangency excess of* $\mathcal{F}$ [9, Definition 3.2].

Fix a local system of coordinates $\left(x_1, y_1\right)$ along the 1st blowing-up such that $x_1 = 0$ is a local equation of D_1. Suppose first that $\mathcal{F}$ is not dicritical along the exceptional divisor of the blowing-up of its singularity. Then, one has the following inequality

$$\nu_D\left(E_1^*\omega\right) = \nu\left(\mathcal{F}\right) = \nu\left(F\right) - 1 + \tau\left(\mathcal{F}\right).$$

Let us suppose that in the coordinates $\left(x_1, y_1\right)$ the point $(0,0)$ is singular for $\frac{E_1^\star\omega}{x_1^{\nu_D\left(E_1^\star\omega\right)}}$. Then in small neighborhood of $(0,0)$, the 1$-$form $E_1^\star\frac{\omega}{F}$ can be written

$$x_1^{\tau(\mathcal{F})-1}\left((\cdots)\,\mathrm{d}x_1 + x_1\,(\cdots)\,\mathrm{d}y_1\right).$$

In particular, for any vector field $Y = x_1 (\cdots) \frac{\partial}{\partial x_1} + (\cdots) \frac{\partial}{\partial y_1}$ tangent to D_1 one has

$$L_Y E_1^\star \frac{\omega}{F} \wedge E_1^\star \frac{\omega}{F} = x_1^{2\tau(\mathcal{F})-1} (\cdots) .$$

Therefore, in the description (4.4), one has

$$n_1 = 1 - 2\tau (\mathcal{F}) .$$

Now, in neighborhood of $(0,0)$ deprived of $(0,0)$, $E_1^\star \frac{\omega}{F}$ can be more precisely written

$$x_1^{\tau(\mathcal{F})-1} \left(y_1^a (\cdots) \mathrm{d}x_1 + x_1 (\cdots) \mathrm{d}y_1\right)$$

for some integer a. Moreover if l stands for an equation of the divisor $2(F) - S_d$, then its strict transform can be locally written

$$l_1 = y_1^b (\cdots) + x_1 (\cdots) .$$

Finally, if one considers the meromorphic vector field $X = \frac{x_1}{y_1^{2a+b}} \frac{\partial}{\partial x_1}$ then

$$\beta (X) = L_X E_1^\star \frac{\omega}{F} \wedge E_1^\star \frac{\omega}{F} = -\frac{2a+b}{x_1^{1-2\tau(\mathcal{F})} l_1} \frac{1}{y_1} \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \cdots .$$

Since $\beta (X)$ vanishes in $H^1 \left(\mathcal{M}_1, \Omega^2 \left(2 (F)^{E_1} - S_d^{E_1} + n_1 D_1\right)\right)$, Lemma 19 shows that

$$\nu \left(2 (F) - S_d\right) \geq n_1,$$

which implies that

$$2\nu (\mathcal{F}) - \nu \left(S_d\right) \geq -1.$$

But $\nu (\mathcal{F}) \leq \left[\frac{\nu(S_d)}{2}\right] - 1$ gives us

$$2\nu (\mathcal{F}) - \nu \left(S_d\right) \leq 2 \left[\frac{\nu \left(S_d\right)}{2}\right] - \nu \left(S_d\right) - 2 < -1,$$

which is a contradiction.

Suppose now that $\mathcal{F}$ is dicritical along the exceptional divisor of the single blowing-up of its singularity. Then, one has the following inequality:

$$\nu_D\left(E_1^*\omega\right)=\nu\left(\mathcal{F}\right)+1=\nu\left(F\right)+\tau\left(\mathcal{F}\right).$$

Let us suppose that in the coordinates $\left(x_1,y_1\right)$ the point $(0,0)$ is regular for $\frac{E_1^\star\omega}{x_1^{\nu_D\left(E_1^\star\omega\right)}}$ and that l_1 has neither a zero nor a pole at $(0,0)$. Then in small neighborhood of $(0,0)$ deprived of $(0,0)$, the $1-$form $E_1^\star\frac{\omega}{F}$ can be written as

$$x_1^{\tau(\mathcal{F})}\mathrm{d}y_1.$$

In particular, for any vector field $Y=x_1\left(\cdots\right)\frac{\partial}{\partial x_1}+\left(\cdots\right)\frac{\partial}{\partial y_1}$ tangent to D_1 one has

$$L_YE_1^\star\frac{\omega}{F}\wedge E_1^\star\frac{\omega}{F}=x_1^{2\tau(\mathcal{F})}\left(\cdots\right).$$

Therefore, in the description (4.4), one has

$$n_1=-2\tau\left(\mathcal{F}\right).$$

If one considers the vector field $X=\frac{x_1}{y_1}\frac{\partial}{\partial y_1}$, then

$$\beta\left(X\right)=L_XE_1^\star\frac{\omega}{F}\wedge E_1^\star\frac{\omega}{F}=\frac{1}{l_1\left(0,0\right)x_1^{-2\tau(\mathcal{F})}}\frac{1}{y_1}\mathrm{d}x_1\wedge\mathrm{d}y_1+\cdots.$$

Since $\beta\left(X\right)$ vanishes, Lemma 19 shows again that

$$2\nu\left(F\right)-\nu\left(S_d\right)\geq-2\tau\left(\mathcal{F}\right)\Longleftrightarrow2\nu\left(\mathcal{F}\right)-\nu\left(S_d\right)\geq-2.$$

If $\nu\left(\mathcal{F}\right)\leq\left[\frac{\nu(S_d)}{2}\right]-2$ then we are led to a contradiction. Suppose that $\nu\left(\mathcal{F}\right)=\left[\frac{\nu(S_d)}{2}\right]-1$. If $\nu\left(S_d\right)$ is odd then

$$2\nu\left(\mathcal{F}\right)-\nu\left(S_d\right)=2\left(\frac{\nu\left(S_d\right)-1}{2}-1\right)-\nu\left(S_d\right)=-3,$$

which is still a contradiction. Suppose that $\nu\left(S_d\right)$ is even. Then, $\nu\left(\mathcal{F}\right)=\frac{\nu(S_d)}{2}-1$. The multiplicity $\nu\left(\mathcal{F}\right)$ being as small as possible in $\Omega^1\left(S_d\right)$, a basis of $\Omega^1\left(S_d\right)$ can be written

$\{\omega_1, \omega_2\}$ with

$$\frac{\nu(S_d)}{2} - 1 = \nu(\omega_1) \leq \nu(\omega_2) \text{ and} \nu(\omega_1) + \nu(\omega_2) \leq \nu(S_d).$$

Thus, there are only three possibilities for $\nu(\omega_2)$:

- if $\nu(\omega_2) = \frac{\nu(S_d)}{2}+1$, then any 1-form ω of multiplicity $\frac{\nu(S_d)}{2}$ in $\Omega^1(S_d)$ is written

$$\omega = a\omega_1 + b\omega_2,$$

  where a is a function of multiplicity 1 and b is any function. In particular, its jet of smallest order is written

$$(a)_1 \cdot (\omega_1)_{\frac{\nu(S_d)}{2}-1},$$

  where $(\star)_i$ stands for the jet of order i. Thus, as ω_1 is dicritical along the exceptional divisor of the single blowing-up of its singularity, ω is also. This would imply that any element of multiplicity $\frac{\nu(S_d)}{2}$ in the Saito module has this property. This is a contradiction with Theorem 2;
- if $\nu(\omega_2) = \frac{\nu(S_d)}{2}$ or $\nu(\omega_2) = \frac{\nu(S_d)}{2} - 1$ then using the criterion of Saito we have

$$(\omega_1)_{\nu(\omega_1)} \wedge (\omega_2)_{\nu(\omega_2)} = 0.$$

  Therefore, ω_2 is dicritical after one blowing-up. If $\nu(\omega_2) = \frac{\nu(S_d)}{2}$ then any 1−form of multiplicity $\frac{\nu(S_d)}{2}$ is dicritical, which is impossible. If $\nu(\omega_2) = \frac{\nu(S_d)}{2} - 1$, let us write

$$\begin{aligned} \omega_1 &= P_1\omega_r + \cdots \\ \omega_2 &= P_2\omega_r + \cdots, \end{aligned}$$

  where $\omega_r = x\mathrm{d}y - y\mathrm{d}x$. Consider ω in the module of Saito with multiplicity $\frac{\nu(S_d)}{2}$. It can be written

$$\omega = a\omega_1 + b\omega_2 = (aP_1 + bP_2)\,\omega_r + \cdots.$$

  If $\nu(a) = 0$ or $\nu(b) = 0$ then $\nu(\omega) = \frac{\nu(S_d)}{2} - 1$ unless there exists a non vanishing constant C such that $P_2 = CP_1$. But in that latter case $\{\omega_1, \omega_2 - C\omega_1\}$

is still a basis of the module of Saito with $\nu\left(\omega_2 - C\omega_1\right) > \frac{\nu(S_d)}{2} - 1$ which leads to a case already treated. Thus, $\nu(a) \geq 1$ and $\nu(b) \geq 1$ and necessarily, ω is dicritical along the exceptional divisor of one blowing-up. As before, any 1-form of multiplicity $\frac{\nu(S_d)}{2}$ would be dicritical along the exceptional divisor of the blowing-up of its singularity, which is impossible.

This completes the proof of the 1st part of Theorem 1, and thus for S generic, we prove that

$$\min_{\omega\in\Omega^1(S_d)} \nu(\omega) = \left[\frac{\nu\left(S_d\right)}{2}\right]. \tag{4.5}$$

4.4 Step 4: existence of a balanced basis

Let us prove now the existence of a balanced basis for $\Omega^1\left(S_d\right)$.

Let us suppose first that $\nu\left(S_d\right)$ is even. Consider a basis $\left\{\omega_1, \omega_2\right\}$ of $\Omega^1\left(S_d\right)$. According to (4.5) there are some 1-forms with multiplicity $\frac{\nu(S_d)}{2}$ in $\Omega^1\left(S_d\right)$. Hence, at least one of the forms in the basis, say ω_1, has a multiplicity equal to $\frac{\nu(S_d)}{2}$. The multiplicity of ω_2 is greater or equal to $\frac{\nu(S_d)}{2}$. If it is equal, then the basis is balanced. If not, $\left\{\omega_1, \omega_1 + \omega_2\right\}$ is still a basis and is balanced.

Suppose now that $\nu\left(S_d\right)$ is odd. If the direction of S_d is empty or contains one component, let us consider $\tilde{S} = S_d \cup L$ where L is a smooth curve transverse to the direction of S_d. Since the multiplicity of $\tilde{S}$ is even, according to the previous case, the module $\Omega^1\left(\tilde{S}\right)$ admits a balanced basis. Therefore, there exists a couple a $1-$forms $\left\{\omega_1, \omega_2\right\}$ of multiplicity $\frac{\nu(S_d)+1}{2}$ such that

$$\omega_1 \wedge \omega_2 = ulf\,\mathrm{d}x \wedge \mathrm{d}y, \qquad u(0) \neq 0,$$

where l is an irreducible equation of L and f a reduced equation of S_d. Now, according to (4.5), there exists $\overline{\omega}$ tangent to S_d such that $\nu(\overline{\omega}) = \frac{\nu(S_d)-1}{2}$. The 1-form $l\overline{\omega}$ is tangent to $\tilde{S}$. Hence, there exist two germs of functions a_1 and a_2 such that

$$l\overline{\omega} = a_1\omega_1 + a_2\omega_2.$$

The functions a_1 and a_2 cannot both vanish. Suppose by symmetry that a_1 does not vanish, then $\left\{l\overline{\omega}, \omega_2\right\}$ is a basis of $\Omega^1\left(\tilde{S}\right)$.Thus,

$$l\overline{\omega} \wedge \omega_2 = vlf\,\mathrm{d}x \wedge \mathrm{d}y, \qquad v(0) \neq 0.$$

Dividing by l the above expression leads to the criterion of Saito for the balanced basis $\{\overline{\omega}, \omega_2\}$ of $\Omega^1(S_d)$.

If the direction of S_d contains two components L_1 and L_2, then let us consider $\tilde{S} = S \cup L_1$. The module $\Omega^1\left(\tilde{S}\right)$ admits a balanced basis $\{\omega_1, \omega_2\}$ with $\nu(\omega_1) = \nu(\omega_2) = \left[\frac{\nu\left(\tilde{S}\right)}{2}\right] = \frac{\nu(S)+1}{2}$. Now, there exists $\overline{\omega}$ in $\Omega^1(S_d)$ with $\nu(\overline{\omega}) = \left[\frac{\nu(S_d)}{2}\right] = \frac{\nu(S)+1}{2}$. Since $\overline{\omega}$ is also tangent to $S \cup L_1$, there exist two functions a_1 and a_2 such that

$$\overline{\omega} = a_1\omega_1 + a_2\omega_2.$$

The functions a_1 and a_2 cannot both vanish so we can suppose that $a_1(0) \neq 0$. The family $\{\overline{\omega}, \omega_2\}$ is still a basis of $\Omega^1\left(\tilde{S}\right)$ that satisfies

$$\overline{\omega} \wedge \omega_2 = wfl_1 \mathrm{d}x \wedge \mathrm{d}y, \qquad w(0) \neq 0.$$

Thus, multiplying by l_2 leads to

$$\overline{\omega} \wedge l_2\omega_2 = wfl_1l_2 \mathrm{d}x \wedge \mathrm{d}y, \qquad w(0) \neq 0$$

and $\{\overline{\omega}, l_2\omega_2\}$ is a balanced basis of $\Omega^1(S_d)$.

This ends the proof of Theorem 1.

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