NUMBER OF MODULI OF A GERM OF COMPLEX PLANE CURVE.

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Abstract. In this article, we construct an algorithm to compute the number of the moduli of a germ of complex plane curve.

1. Introduction.

The number of moduli of a germ of curve in the complex plane. A germ of curve C in the complex plane is the zero locus of a germ of analytical reduced function $f(x,y) \in \mathbb{C}\{x,y\}$

$$C = \{f(x,y) = 0\}.$$

Such a curve can be classified up to continuous or analytic conjugacies of the ambient space $(\mathbb{C}^2, 0)$. The quotient of its equisingularity class $\mathrm{Top}(C)$ up to analytic conjugacies can be endowed with a structure of complex variety, yet not Hausdorff [4, 8] , and the dimension of this variety is precisely what we refer to as *the number of moduli of* C.

The problem of the determination of the number of moduli of a germ of complex plane curve goes back to the work of S. Ebey in 1965 [4] who computed the number of moduli for a particular equisingularity class of curve, namely, the one given by the equation $y^5 = x^9$. A few years after in 1973, O. Zariski in his seminal notes [29] focused on the case of a curve with only one irreducible component. The topological classification of an irreducible curve is well known and relies on a semi-group of integers extensively studied by Zariski himself in the 70s. Zariski proposed various approaches to obtain the number of moduli for irreducible curves beyond the case treated by Ebey. He introduced most of the concepts on which the forthcoming works relied. However, at this time, the analytical classification was a widely open question, even in the irreducible case. In 1978, C. Delorme [3] studied extensively the case of an irreducible curve with one Puiseux pair and established some formulas to compute the number of moduli. In 1979, M. Granger [12] and later, in 1988, J. Briançon, Granger and Ph. Maisonobe [1] produced an algorithm to compute the number of moduli for a non irreducible quasi-homogeneous curve defined by $x^m + y^n = 0$ first, for m and n relatively prime, and then in the general case. The common denominator of the two previous works is the algorithmic approach based upon arithmetic properties of the continuous fraction expansion associated to the pair (m,n). In 1988, O.A. Laudal, B. Martin and G. Pfister in [19], improved the work of Delorme and gave an explicit description of a universal family for curves with one Puiseux pair and a stratification of their moduli space. Finally, in 1998, R. Peraire exhibited an algorithm in [25] to compute the Tjurina number of a curve in its generic component, which is linked to the dimension of the number of moduli. Up to our knowledge, the initial question of the analytic classification can be today considered as mostly solved by a combination of the works of A. Hefez and M.E. Hernandes [14, 15, 16] in 2010 who adressed the irreducible case and very recently of M.E. Hernandes and M.E. Rodrigues Hernandes [17] for the general case : these works provide a normal form type result for a given equisingularity class of curve. Nevertheless, the extraction of the sole number of moduli form their very fine constructions can be quite involved, as it can be seen in the last example of [17]. From the algorithmic point of view, the approaches are based upon Gröbner basis like routine, which are known to be in general of high complexity.

In 2010 and 2011, in [9, 10], with E. Paul, we described the moduli space of a topologically quasi-homogeneous curve C as the spaces of leaves of an algebraic foliation defined on the moduli space of a foliation whose analytic invariant curve is precisely C. These works initiated an approach based upon the theory of foliations, which is at stake here. In 2022, in [6], we gave an explicit formula for the number of moduli for an irreducible curve : this formula involves only very elementary topological invariants of the curve, such as, the topological class of its desingularization. In [7], we studied the reducible case and constructed an algorithm to compute the number of moduli of a curve and proved that this algorithm yields the desired number under the assumption that this curve is a union of smooth curves.

The goal of the current article is to prove that the algorithm mentioned above provides the expected number of moduli in any case. The complexity of this algorithm is linear in the length of the desingularization of the curve.

The algorithm. Let C be a curve and $E : \left(\widetilde{\mathbb{C}^2}, D\right) \to (\mathbb{C}^2, 0)$ be its desingularization : it is a composition $E = E_1 \circ \cdots \circ E_N$ of elementary blowing-ups of points. According to [22], if C is any curve generic in its equisingularity class, its number of moduli, denoted by $\mathbb{M}^C$, is equal to the dimension of the cohomological space

$$\mathbb{M}^C = \dim_{\mathbb{C}} H^1(D, \Theta)$$

where Θ is the sheaf of germs of vector fields on $\widetilde{\mathbb{C}^2}$ tangent to the total transform $E^{-1}(C)$. Indeed, the first group of cohomology of the sheaf Θ can be identified as the tangent space to the space of parameters of any miniversal deformation of C. This dimension can be *inductively* computed along the desingularization of C following the result below.

Theorem ([6, 7])**.** *The number of moduli $\mathbb{M}^C$ is written*

$$\mathbb{M}^C = \dim_{\mathbb{C}} H^1\left(D_1, \Theta|_{D_1}\right) + \sum_k \mathbb{M}^{C_k \cup D_1}$$

where $\Theta|_{D_1}$ is the sheaf of germs of vector fields on the total space of E_1 tangent to $E_1^{-1}(C)$ and the C_k's are the connected components of $\overline{E_1^{-1}(C \setminus \{0\})}$.

The algorithm which provides the number of moduli is based upon the following remark : we can compute the dimension $\dim_{\mathbb{C}} H^1\left(D_1, \Theta|_{D_1}\right)$ using a study of the *Saito module of* C, denoted by $\mathrm{Der}(\log C)$, that is the set of vectors fields tangent to C. More precisely, this dimension can be expressed using mainly *the Saito number* $\mathfrak{s}_C$ of C defined by

$$\mathfrak{s}_C = \min_{X \in \mathrm{Der}(\log C)} \nu(X)$$

where $\nu(\cdot)$ is the standard valuation and *an optimal vector field X for C*, that is, a vector field reaching the minimum above. As an illustration, we quote the following result from [8, Proposition 3.15]

Proposition ([8]). *If C is generic in its equisingularity class and $E_1^{-1}(X)$ leaves generically invariant D_1 where X is an optimal vector field for C, then*

$$\dim_{\mathbb{C}} H^1\left(D_1, \Theta|_{D_1}\right) = \left\{ \begin{array}{cl} (\mathfrak{s}_C - 1)(\mathfrak{s}_C - 2) & \text{if } \nu(C) \text{ is even} \\ (\mathfrak{s}_C - 1)^2 & \text{if not.} \end{array} \right.$$

Saito vector field. As a matter of fact, for a given curve C there is a lot of optimal vector fields with various topologies. In order to *select* among them a better optimal vector field, we are going to require that not only X is optimal for C but also that for any k, $E_1^{-1}(X)$ is optimal for $C_k \cup D_1$ and finally, that this property of optimality propagates all along the desingularization process of C. Such a vector field will be said *Saito* for C. Not only is there a priori no reason for such a vector field to be indeed exceptional among the simply optimal vector fields, but there is no reason also that such a vector field exists.

The main goal of this article is to prove that a curve C, generic in its equisingularity class, always admits a Saito vector field, and that the topology of this vector field is somehow unique. More precisely, we will prove the following result

Theorem. *Let C be a curve generic in its equisingularity class. Then C admits a Saito vector field X. Moreover, let $\mathbb{A}$ be the dual tree of the desingularization process E of C. We number a vertex s of $\mathbb{A}$ by the number of tangency point between the irreducible component D_s of the exceptional divisor of E corresponding to s and the pull-back vector field $E^{-1}(X)$. Besides, we color a vertex s in white if D_s is invariant by $E^{-1}(X)$, otherwise we color it in black. Then, the colored numbered tree $\mathbb{A}$ does not depend on X.*

The article is divided in three sections. The first can be red independently : it focuses on a technical combinatorial result on trees. This result is a key element to describe the topology of the Saito vector field of C. More precisely, the result of the first section provides a guide to apply a *recipe* developped by A. Lins-Neto in [20] in order to build singular vector fields or foliations from local data in a blowing-up process. We control the topology of the resulting vector field and, in particular, its valuations along the initial desingularization of C, which ensures the Saito property. It remains to guarantee that the invariant curve of the obtained vector field is indeed generic in its equisingularity class : it is achieved by considering a deformation of the Saito vector field toward a vector field tangent to a curve that is actually generic in its equisingularity class. This last approach derives from technics established by X. Gomez-Mont in [11].

The final section focuses on examples : the Saito vector field of *the double cusp,* the number of moduli of a certain non irreducible curves and an application to the computation of the generic Tjurina number of a curve.

2. Saito dicriticity of an ordered numbered tree.

Let $\mathbb{A}$ be a tree. We can endow $\mathbb{A}$ with a partial order $\leq$ defined following an inductive description of $\mathbb{A}$: starting from a single vertex r, if $(\mathbb{A}, \leq)$ is defined, one can add a vertex to $\mathbb{A}$ following one of the two next rules

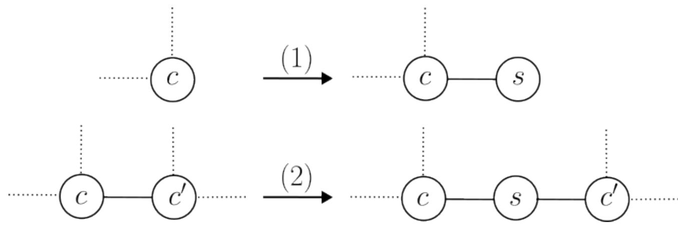


FIGURE 2.1. Inductive construction of the partial order $\leq$ on $\mathbb{A}$.

(1) a vertex s and an edge from s to a single vertex c are added to $\mathbb{A}$. The order $\leq$ is extended to $\mathbb{A} \cup \{s\}$ setting $s \geq c$.
(2) a vertex s is added to $\mathbb{A}$ deleting an egde from c to c' and adding two edges from s to c and from s to c'. The order is extended setting $s \geq c$ and $s \geq c'$.

The vertex r is the minimal element of $(\mathbb{A}, \leq)$ and is called the *root* of $\mathbb{A}$. In the sequel, in general, we will make no distinction between $\mathbb{A}$ and the set of vertices of $\mathbb{A}$.

We will denote by $n = (n_c)_{c\in\mathbb{A}}$ a numbering of the vertices of $\mathbb{A}$ by non negative integers. We will also consider an element $\Delta = (\Delta_c)_{c\in\mathbb{A}}$ in $\{0,1\}^{\mathbb{A}}$. The latter is called *a dicriticity for* $\mathbb{A}$. It induces a coloring of the tree $\mathbb{A}$: if $\Delta_c = 1$ the vertex c is colored in white, if not, it is colored in black.

Finally, the notation $\left\lfloor \begin{array}{c} a \\ b \end{array} \right._{c}$ stands for the the following :

$$\left\lfloor \begin{array}{c} a \\ b \end{array} \right._{c} = \begin{cases} a & \text{if } c \text{ is even} \\ b & \text{else} \end{cases}$$

Definition 1. In what follows, c stands for a vertex of $\mathbb{A}$.

(1) We denote by $\mathfrak{p}_c$ *the set of parents of* c that is the set of predecessors of c for the partial order $\leq$. Notice that $\mathfrak{p}_r = \emptyset$; in any other cases, $\mathfrak{p}_c$ contains one or two elements.

(1) Following [27], fixing a numbering $\{1, \ldots, N\}$ of the vertices such that $i \in \mathfrak{p}_j \Longrightarrow i \leq j$, we consider the proximity matrix $\mathbb{P}$ of $(\mathbb{A}, \leq)$ defined by

$$\begin{array}{ll} \mathbb{P}_{i,i} & = 1 \\ \mathbb{P}_{i,j} & = -1 \text{ if } i \in \mathfrak{p}_j \\ \mathbb{P}_{i,j} & = 0 \text{ else.} \end{array}.$$

It is an upper triangular invertible matrix.
(2) If n is numbering of $\mathbb{A}$, then $c \cdot n$ is the numbering defined by

$$(c \cdot n)_c = n_c + 1$$

and $(c \cdot n)_{c'} = n_{c'}$ if $c' \neq c$.
(3) In what follows, $\mathfrak{v}_\mathfrak{c}$ denotes *the set of neighbours of* c in $\mathbb{A}$, that is the set of vertices of $\mathbb{A}$ connected to c.

(4) We called *the multiplicity of* c in $\mathbb{A}$ the positive integer, denoted by ρ_c obtained inductively as follows : $\rho_r = 1$, and if $c \neq r$ then

$$\rho_c = \sum_{c' \in \mathfrak{p}_c} \rho_{c'}.$$

(5) We denote by δ_c the number of parents c' of c such that $\Delta_{c'} = 1$. In particular, this number depends not only on $\mathbb{A}$ but also on a dicriticity Δ.

(6) We called *the valuation of* c the number denoted by ν_c^n and defined by the matrix relation

$$(\nu_c^n)_{c\in\mathbb{A}} = \mathbb{P}^{-1} (n_c)_{c\in\mathbb{A}} .$$

In particular, from [27], it follows that

$$\nu_r^n = \sum_{c\in\mathbb{A}} \rho_c n_c$$

(7) The *square index* of c is defined by

$$\square_c = \frac{\delta_c}{2} - \left\lfloor \begin{matrix} \Delta_c \\ \frac{1}{2} \end{matrix} \right._{\nu_c^n - \delta_c} .$$

(8) If c and c' belong to $\mathbb{A}$, we defined *the acces tree* from c to c' the minimal subgraph of $\mathbb{A}$ that from c, leads to c' respecting the order $\leq$. It is denoted by

$$_{c}\mathbb{A}_{c'}.$$

If $c = r$ is the root of $\mathbb{A}$, then it is simply denoted by

$$\mathbb{A}_{c'}.$$

If $\rho_{c'} = 1$ then the access tree $\mathbb{A}_{c'}$ is a totally ordered linear chain of vertices whose multiplicities are equal to 1 as in Figure 2.2. The proximity matrix of this sub-graph is written

$$\begin{pmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & \ddots & -1 \\ & & & 1 \end{pmatrix}$$

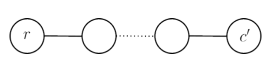


FIGURE 2.2. Acces tree from r to c' with $\rho_{c'} = 1$.

(9) We denote by $\epsilon = (\epsilon_c)_{c\in\mathbb{A}}$ the family of integers defined by the following matrix relation

$$(\epsilon_c)_{c\in\mathbb{A}} = \mathbb{P}\left(\frac{1}{2}(\nu_c^n)_{c\in\mathbb{A}} - (\square_c)_{c\in\mathbb{A}}\right) = \left(\frac{n_c}{2}\right)_{c\in\mathbb{A}} - \mathbb{P}\left((\square_c)_{c\in\mathbb{A}}\right).$$

This uple of integers is called *the configuration associated to the dicriticity* Δ.

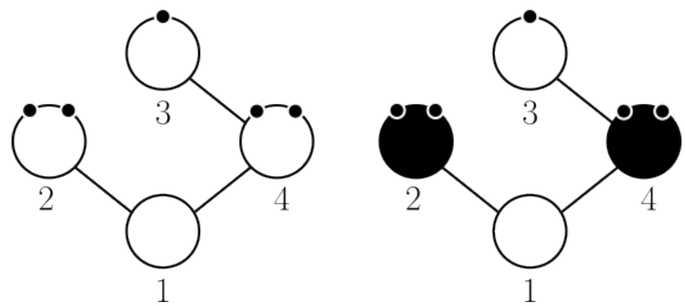


FIGURE 2.3. An ordered tree, its numbering and its dicriticity.

Example 2. Let us consider the tree represented in Figure (2.3).
In this example, the proximity matrix is written

$$\mathbb{P} = \begin{pmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The numbering is $n = (0, 2, 1, 2)$ and is represented in Figure (2.3) by dots attached to the vertices. We have

$$\begin{array}{c} \mathfrak{p}_1 = \emptyset,\ \mathfrak{p}_2 = \{1\},\ \mathfrak{p}_3 = \{1\},\ \mathfrak{p}_4 = \{1,3\} \\ \mathfrak{v}_1 = \{2,4\},\ \mathfrak{v}_2 = \{1\},\ \mathfrak{v}_4 = \{1,3\},\ \mathfrak{v}_3 = \{4\}. \end{array}$$

The partial order induced on $\mathbb{A}$ by the rules of construction is

$$1 \leq 2,\ 1 \leq 3 \leq 4.$$

The multiplicities are

$$\rho_1 = 1,\ \rho_2 = 1,\ \rho_3 = 1,\ \rho_4 = 2.$$

Given the numbering n, the valuations are

$$\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ \nu_4 \end{pmatrix} = \mathbb{P}^{-1} \begin{pmatrix} 0 \\ 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \\ 3 \\ 2 \end{pmatrix}.$$

Now, assuming the dicriticity is $\Delta = (1, 0, 1, 0)$ as in the figure, we obtain

$$\delta_1 = 0,\ \delta_2 = 1,\ \delta_3 = 1,\ \delta_4 = 2$$

and

$$\begin{array}{c} \square_1 = \frac{0}{2} - \left\lfloor \begin{matrix} 1 \\ \frac{1}{2} \end{matrix} \right._{7-0} = -\frac{1}{2},\ \square_2 = \frac{1}{2} - \left\lfloor \begin{matrix} 0 \\ \frac{1}{2} \end{matrix} \right._{2-1} = 0 \\ \square_3 = \frac{1}{2} - \left\lfloor \begin{matrix} 1 \\ \frac{1}{2} \end{matrix} \right._{3-1} = -\frac{1}{2},\ \square_4 = \frac{2}{2} - \left\lfloor \begin{matrix} 0 \\ \frac{1}{2} \end{matrix} \right._{2-2} = 1. \end{array}$$

Finally, the configuration ϵ is computed as follows

$$\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ 2 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} \\ 0 \\ -\frac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix}.$$

As it can be seen in the previous example, from any dicriticity Δ, one can compute a configuration ϵ following Definition 1.9. However, adding some constraints yields a unicity type result.

Theorem 3. *Consider a numbering n of $\mathbb{A}$. There exists a unique dicriticity, denoted by $\Delta^n = (\Delta^n_c)_{c\in\mathbb{A}}$ such that the associated configuration ϵ^n satisfies the following relations*

(1) if $\Delta^n_c = 0$, then $\epsilon^n_c \geq 2 - \sum_{c'\in\mathfrak{v}_c} \Delta^n_{c'}$
(2) if $\Delta^n_c = 1$, then $\epsilon^n_c \geq n_c$.

Such a dicriticity is said admissible *and is called* the Saito dicriticity of $\mathbb{A}$ numbered by n. *The exponent n appearing on any data in the sequel will mean that these datas are associated to the Saito dicriticity for a given numbering n.*

Moreover, the following properties hold

(A) *We define by ${}^n_{\mathbb{A}}\Theta^{01}_c$ the following invariant*

$${}^n_{\mathbb{A}}\Theta^{01}_c = \sum_{s\in\mathbb{A}_c} \square^n_s + \square^{c\cdot n}_s.$$

If $\rho_c = 1$ then we obtain

$$ {}^n_{\mathbb{A}}\Theta^{01}_c = -\Delta^n_{c_1} - \frac{|\mathbb{A}_{c_1}|}{2} \tag{2.1}$$

where $|\star|$ denotes the number of vertices in the subtree $\star$.

(B) *Let $c_0, c_1 \in \mathbb{A}$. We define by ${}^n_{\mathbb{A}}\Theta^{02}_{c_0,c_1}$ and ${}^n_{\mathbb{A}}\Theta^{11}_{c_0,c_1}$ the following invariants*

$${}^n_{\mathbb{A}}\Theta^{02}_{c_0,c_1} = \sum_{s\in\mathbb{A}_{c_1}} \square^n_s + \square^{c_1\cdot c_0\cdot n}_s$$
$${}^n_{\mathbb{A}}\Theta^{11}_{c_0,c_1} = \sum_{s\in\mathbb{A}_{c_1}} \square^{c_0\cdot n}_s + \square^{c_1\cdot n}_s.$$

If c_0 and c_1 satisfy both $\rho_{c_0} = \rho_{c_1} = 1$ then we have

$${}^n_{\mathbb{A}}\Theta^{02}_{c_0,c_1} = \pm\frac{1}{2} - \Delta^n_{c_1} - \frac{|\mathbb{A}_{c_1}|}{2} \tag{2.2}$$

$${}^n_{\mathbb{A}}\Theta^{11}_{c_0,c_1} = \pm\frac{1}{2} - \Delta^n_{c_1} - \frac{|\mathbb{A}_{c_1}|}{2} \tag{2.3}$$

(C) *Let c be a vertex of $\mathbb{A}$ such that $\rho_c = 1$. We say that the access tree $\mathbb{A}_c$* starts with a mixed branch *if there exists a vertex $m_c \in \mathbb{A}_c$ maximal for this property such that for any $s \in \mathbb{A}_{m_c}$, one has*

$$\Delta^n_s + \Delta^{c\cdot n}_s = 1.$$

It may happen that $m_c = c$. If not, let us denote by m_c^+ the vertex of $\mathbb{A}$ which succeeds m_c in $\mathbb{A}_c$. Depending on the type the mixed branch, Table (1) presents some properties of the valuations ν_r and $\nu_{m_c^+}$. In this table,

	ν_r	$\nu_{m_c^+}$
●···○—○··· ○···●—○···	odd	even
○···●—○··· ●···○—○···	even	odd
○···○—○··· ●···●—○···	even	even
●···●—○··· ○···○—○···	odd	odd

	ν_r	$\nu_{m_c^+}$
●···○—●··· ○···●—●···	odd	odd
○···●—●··· ●···○—●···	even	even
○···○—●··· ●···●—●···	even	odd
●···●—●··· ○···○—●···	odd	even

TABLE 1. The valuations ν_r and $\nu_{m_c^+}$ along a mixed branch.

$\lvert\mathbb{A}_c\rvert > 1$	ν_r
●···○ ○···●	odd
○···○ ●···●	even
○···● ●···○	impossible
●···● ○···○	impossible

$\lvert\mathbb{A}_c\rvert = 1$	ν_r
● ○	odd
○ ●	even

TABLE 2. The valuation ν_r along a *pure* mixed branch.

a picture such as ●···○—○···
○···●—○··· *represents the two Saito dicriticities along $\mathbb{A}_c$ respectively, above for the numbering c and below for the numbering $c \cdot n$. Besides, if $m_c = c$, Table (2) presents properties on the valuations ν_r depending also on the type of what is called in that case a* pure *mixed branch.*

(D) *Finally, for each connected component $\mathbb{K}$ of the sub-graph $\mathbb{A} \backslash \{c \in \mathbb{A} \mid \Delta_c^n = 0\}$, there exists $c \in \mathbb{K}$ with $\epsilon_c^n > 0$.*

The main statement of Theorem 3 has already been proved in [7] for the particular case of a tree $\mathbb{A}$ for which $\rho_c = 1$ for any $c \in \mathbb{A}$.

Example 4. Let us consider the *cusp tree,* that is the tree in Figure 2.4 numbered by $n_1 = n_2 = 0$ and $n_3 = 1$. The order is defined by the relations $1 \leq 2$ and $1 \leq 3$. The proximity matrix is

$$\mathbb{P} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Figure 2.4 presents also the associated Saito dicriticity represented by the coloring. Table 3 shows the various configurations obtained from the $8 = 2^3$ possible dicriticities on $\mathbb{A}$. For each configuration, we highlight by the notation $\langle \cdot \rangle$ a part of

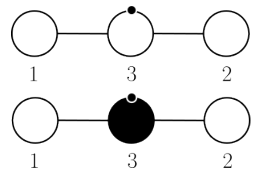


FIGURE 2.4. The *cusp* tree and its Saito dicriticity

$\Delta = (\Delta_1, \Delta_2, \Delta_3)$	$\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)$
$0, 0, 0$	$\langle -1 \rangle, 0, 1$
$0, 0, 1$	$\langle -1 \rangle, 0, 1$
$0, 1, 0$	$\langle 0 \rangle, 1, 1$
$0, 1, 1$	$\langle -1 \rangle, 0, 1$
$1, 0, 0$	$2, \langle 0 \rangle, 0$
$1, 0, 1$	$2, \langle 0 \rangle, 1$
$1, 1, 0$	$1, 1, 0$
$1, 1, 1$	$1, 1, \langle 0 \rangle$

TABLE 3. Dicriticities and configurations for the cusp tree numbered by $(0, 0, 1)$.

the configuration that violates one of the admissibility conditions. At the end, the unique and thus Saito dicriticity that satisfies all the three admissibility conditions is $(1, 1, 0)$ for which

$$\begin{aligned} \epsilon_1 = 1 \geq 0, \ \Delta_1 = 1 \\ \epsilon_2 = 1 \geq 0, \ \Delta_2 = 1 \\ \epsilon_3 = 0 \geq 2 - \Delta_1 - \Delta_2, \ \Delta_3 = 0. \end{aligned}$$

Notice that, 1 and 2 are two components of $\mathbb{A} \setminus \{3\}$ for which $\epsilon_1 > 0$ and $\epsilon_2 > 0$, as predicted by the property (D) of Theorem 3.

Example 5. Suppose that $\mathbb{A}$ is a tree reduced to two vertices. Its proximity matrix is

$$\mathbb{P} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Figure 2.5 presents the Saito dicriticity $\Delta = (\star, \star) \in \{0, 1\}^2$ depending on the values n_1 and n_2.

Let $c \in \mathbb{A}$. Along the row associated to c in the matrix $\mathbb{P}$, any occurrence of a coefficient -1 corresponds to a vertex that belongs to the access tree from r to v for some v in the neighborhood $\mathfrak{v}_r$. This remark leads to the following expression

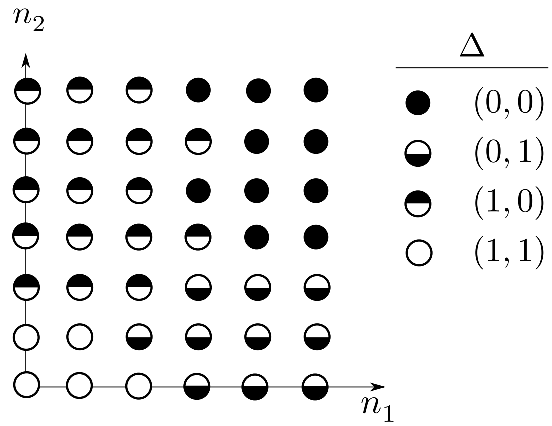


FIGURE 2.5. Unique admissible choice of $\Delta = (\Delta_1, \Delta_2)$.

of ϵ_c that is going to be used extensively in the sequel,

$$
\begin{aligned}
\epsilon_c &= \frac{n_c}{2} - \Box_c + \sum_{v\in\mathfrak{v}_c} \sum_{s\in_c\mathbb{A}_v\setminus\{c\}} \Box_s. \\
&= \frac{n_c}{2} - \frac{\delta_c}{2} + \Big\lfloor_{\nu_c^n-\delta_c} \begin{matrix}\Delta_c\\ \frac{1}{2}\end{matrix} + \sum_{v\in\mathfrak{v}_c} \sum_{s\in_c\mathbb{A}_v\setminus\{c\}} \frac{\delta_s}{2} - \Big\lfloor_{\nu_s^n-\delta_s} \begin{matrix}\Delta_s\\ \frac{1}{2}\end{matrix} .
\end{aligned} \tag{2.4}
$$

Proof of Theorem 3. The proof is, as a whole, an induction on the number of vertices in $\mathbb{A}$.

Suppose that $|\mathbb{A}| = 1$. The proximity matrix is $\mathbb{P} = (1)$ and the numbering $n = (n_r)$. In view of (2.4), we get

$$
\begin{aligned}
\epsilon_r &= \frac{n_r}{2} - \Box_r = \frac{n_r}{2} - \left(\frac{\delta_r}{2} - \Big\lfloor_{\nu_r^n-\delta_r} \begin{matrix}\Delta_r\\ \frac{1}{2}\end{matrix}\right) \\
&= \frac{n_r}{2} + \Big\lfloor_{n_r} \begin{matrix}\Delta_r\\ \frac{1}{2}\end{matrix}
\end{aligned}
$$

since $\delta_r = 0$ and $\nu_r^n = n_r$. Suppose $n_r = 0, 1$ or 2. Then it can be seen that $\Delta_r = 0$ is not admissible, since it would impose that $\epsilon_r \geq 2$, which is not true. However, if $\Delta_r = 1$, for $n_r = 0, 1$ or 2, we have respectively $\epsilon_r = 1,\ 1$ and 2 that always satisfies $\epsilon_r \geq n_r$. Thus for $n_r = 0, 1$ or 2, the Saito dicriticity is defined by $\Delta_r^{n_r} = 1$. To the contrary, if $n_r \geq 3$ then

$$
\epsilon_r \leq \frac{n_r}{2} + 1 < n_r
$$

thus $\Delta_r = 1$ is not an admissible dicriticity. However, $\Delta_r^{n_r} = 0$ is admissible since

$$
\epsilon_r = \frac{n_r}{2} + \Big\lfloor_{n_r} \begin{matrix}0\\ \frac{1}{2}\end{matrix} \geq 2,
$$

which concludes the proof of the main property of Theorem 3 for $|\mathbb{A}| = 1$.

Now consider property (A) and the invariant ${}^{n_r}_{\mathbb{A}}\Theta^{01}_r$. By specifying the definition, we obtain

$$\begin{aligned} {}^{n}_{\mathbb{A}}\Theta^{01}_r &= \square^n_r + \square^{r\cdot n}_r \\ &= \frac{\delta^n_r}{2} - \left\lfloor \begin{matrix} \Delta^n_r \\ \frac{1}{2} \end{matrix} \right._{\nu^n_r - \delta^n_r} + \frac{\delta^{r\cdot n}_r}{2} - \left\lfloor \begin{matrix} \Delta^{r\cdot n}_r \\ \frac{1}{2} \end{matrix} \right._{\nu^{r\cdot n}_r - \delta^n_r} \end{aligned}$$

Notice that $\delta^n_r = \delta^{r\cdot n}_r = 0$, $\nu^n_r = n_r$ and $\nu^{r\cdot n}_r = n_r + 1$. Consequently, this gives

$${}^{n}_{\mathbb{A}}\Theta^{01}_r = - \left\lfloor \begin{matrix} \Delta^n_r \\ \frac{1}{2} \end{matrix} \right._{n_r} - \left\lfloor \begin{matrix} \Delta^{r\cdot n}_r \\ \frac{1}{2} \end{matrix} \right._{n_r+1} = -\frac{1}{2} - \left\lfloor \begin{matrix} \Delta^n_r \\ \Delta^{r\cdot n}_r \end{matrix} \right._{n_r}$$

If $n_r = 0$ or 1 then $\Delta^n_r = \Delta^{r\cdot n}_r = 1$ and thus the invariant is written

$${}^{n}_{\mathbb{A}}\Theta^{01}_r = -\frac{3}{2} = -\frac{|\mathbb{A}_r|}{2} - \Delta^n_r.$$

If $n_r \geq 3$ then $\Delta^n_r = \Delta^{r\cdot n}_r = 0$ which induces

$${}^{n}_{\mathbb{A}}\Theta^{01}_r = -\frac{1}{2} = -\frac{|\mathbb{A}_r|}{2} - \Delta^n_r.$$

Finally, if $n_r = 2$ then $\Delta^n_r = 1$ and $\Delta^{r\cdot n}_r = 0$ and hence $\left\lfloor \begin{matrix} \Delta^n_r \\ \Delta^{r\cdot n}_r \end{matrix} \right._{n_r} = 1$. Therefore, it follows

$${}^{n}_{\mathbb{A}}\Theta^{01}_r = -\frac{3}{2} = -\frac{|\mathbb{A}_r|}{2} - \Delta^n_r.$$

Consequently, formula (2.1) is true for $|\mathbb{A}| = 1$. Now, the invariant ${}^{n}_{\mathbb{A}}\Theta^{02}_{r,r}$ is written

$${}^{n}_{\mathbb{A}}\Theta^{02}_{r,r} = \square^n_r + \square^{r\cdot r\cdot n}_r = - \left\lfloor \begin{matrix} \Delta^n_r \\ \frac{1}{2} \end{matrix} \right._{n_r} - \left\lfloor \begin{matrix} \Delta^{r\cdot r\cdot n}_r \\ \frac{1}{2} \end{matrix} \right._{n_r+2} = - \left\lfloor \begin{matrix} \Delta^n_r + \Delta^{r\cdot r\cdot n}_r \\ 1 \end{matrix} \right._{n_r} .$$

If $n_r = 0$ then it reduces to

$$- \left\lfloor \begin{matrix} \Delta^n_r + \Delta^{r\cdot r\cdot n}_r \\ 1 \end{matrix} \right._{n_r} = -2 = -\frac{1}{2} - \frac{|\mathbb{A}_r|}{2} - \Delta^n_r.$$

If $n_r = 1, 2$ then we obtain

$$- \left\lfloor \begin{matrix} \Delta^n_r + \Delta^{r\cdot r\cdot n}_r \\ 1 \end{matrix} \right._{n_r} = -1 = \frac{1}{2} - \frac{|\mathbb{A}_r|}{2} - \Delta^n_r.$$

Finally, if $n_r \geq 3$, then we get

$$- \left\lfloor \begin{matrix} \Delta^n_r + \Delta^{r\cdot r\cdot n}_r \\ 1 \end{matrix} \right._{n_r} = - \left\lfloor \begin{matrix} 0 \\ 1 \end{matrix} \right._{n_r} = \pm\frac{1}{2} - \frac{|\mathbb{A}_r|}{2} - \Delta^n_r,$$

thus formula (2.2) holds. We continue in this fashion obtaining the invariant ${}^{n}_{\mathbb{A}}\Theta^{11}_{r,r}$,

$${}^{n}_{\mathbb{A}}\Theta^{11}_{r,r} = \square^{r\cdot n}_r + \square^{r\cdot n}_r = - \left\lfloor \begin{matrix} 2\Delta^{r\cdot n}_r \\ 1 \end{matrix} \right._{n_r+1} .$$

If $n_r = 0, 1, 2$ then $\pm\frac{1}{2} - \frac{|\mathbb{A}_r|}{2} - \Delta_r^n = -2$ or -1. This implies (2.3). If $n_r \geq 3$, then we get

$$-\left\lfloor \begin{matrix} 2\Delta_r^{r\cdot n} \\ 1 \end{matrix} \right._{n_r+1} = -\left\lfloor \begin{matrix} 0 \\ 1 \end{matrix} \right._{n_r+1} = \pm\frac{1}{2} - \frac{|\mathbb{A}_r|}{2} - \Delta_r^n,$$

which completes the proof of formula (2.3). For $|\mathbb{A}| = 1$, there are only pure mixed branches of length one. It is enough to refer to the computations above to obtain the following correspondance : for $\mathbb{A} = \{r\}$, we get

$n = (n_r)$	0	1	2	≥ 3
Δ_r^n	1	1	1	0

which ensures the properties in Table (2). To conclude the case $|\mathbb{A}| = 1$,we observe that Property (D) follows from the computations of ϵ_r^n for $n_r = 0, 1$ and 2.

Now, we are going to prove inductively the main property of Theorem 3 from the theorem itself and Property (A). Let us consider the $|\mathfrak{v}_r|$ graphs obtained as the connected components of $\mathbb{A} \setminus \{r\}$. We index these graphs by $\mathfrak{v}_r$ itself by denoting each connected component $\mathbb{A}^c$ for $c \in \mathfrak{v}_r$. Each tree $\mathbb{A}^c$ inherits an order from the one of $\mathbb{A}$. Let us consider two different numberings of each component $\mathbb{A}^c$, $c \in \mathfrak{v}_r$. In the sequel, we refer to these two different numbered trees by the notation $\mathbb{A}^{\star,\nu}$ with $\star = 0$ or 1.

- $\star = 0$, $\mathbb{A}^{0,c} = \mathbb{A}^c$ numbered by the integer $n^0 = \left(n_s^0\right)_{s\in\mathbb{A}^{0,c}}$ with $n_s^0 = n_s$ for $s \in \mathbb{A}^{0,c}$.
- $\star = 1$, $\mathbb{A}^{1,c} = \mathbb{A}^c$ but numbered by the integer $n^1 = \left(n_s^1\right)_{s\in\mathbb{A}^{1,c}}$ with $n_s^1 = n_s$ for $c \neq s \in \mathbb{A}^{1,c}$, and $n_c^1 = n_c + 1$.

Note that by construction the tree $\mathbb{A}$ is obtained by gluing the family of trees $(\mathbb{A}^{\star,c})_{c\in\mathfrak{v}_r}$ with $\star = 0$ or 1 along the vertex r adding an edge between each vertex c and the root r. Each vertex s belongs exactly to one of the trees $\mathbb{A}^{\star,c}$. Applying the main property of Theorem 3 to each numbered tree $\mathbb{A}^{\star,c}$, we obtain a family of dicriticities $\Delta^{\star,c}$, that consists in the family of unique Saito dicriticities of the numbered trees $\mathbb{A}^{\star,c}$. As a result, we can define two new distinct dicriticities on the whole tree $\mathbb{A}$ induced by the $\Delta^{\star,c}$, $c \in \mathfrak{v}_r$ the following way :

- Δ^1, $\Delta_r^1 = 1$ and for any $s \neq r$ $\Delta_s^1 = \Delta_s^{1,c}$ if $s \in \mathbb{A}^{1,c}$.
- Δ^0, $\Delta_r^0 = 0$ and for any $s \neq r$ $\Delta_s^0 = \Delta_s^{0,c}$ if $s \in \mathbb{A}^{0,c}$.

We claim that both dicriticities Δ^0 or Δ^1 satisfy the admissibility conditions for the vertices $s \in \mathbb{A} \setminus \{r\}$. Indeed, let us denote $\star^{\star,c}$ the combinatorial datas resulting from Theorem 3 applied to each numbered tree $\mathbb{A}^{\star,c}$. We also denote simply by $\star^{0 \text{ or } 1}$ the combinatorial datas associated respectively to the dicriticities Δ^0 or Δ^1.

Let us focus first on the dicriticity Δ^1. For $c \in \mathfrak{v}_r$ and $s \in \mathbb{A}^{1,c}$, we get

$$\begin{array}{lll} \text{if } s \notin \mathbb{A}_c^{1,c} \setminus \{r\}, & \nu_s^{1,c} = \nu_s^1, \ \delta_s^{1,c} = \delta_s^1 \\ \text{if } s \in \mathbb{A}_c^{1,c} \setminus \{r\}, & \nu_s^{1,c} = \nu_s^1 + 1, \ \delta_s^{1,c} = \delta_s^1 - 1 \end{array} \tag{2.5}$$

Note that in any case, $\nu_s^{1,c} - \delta_s^{1,c}$ and $\nu_s^1 - \delta_s^1$ have the same parity. If $s \notin \mathbb{A}_c^{1,c} \setminus \{r\}$, relations (2.5) combined with the construction of Δ^1 ensures that $\epsilon_s^{1,c} = \epsilon_s^1$.

If $s \in \mathbb{A}^{1,c}_c \setminus \{r\}$ and $s \neq c$ then we obtain

$$\begin{aligned}
\epsilon^{1,c}_s &= \frac{n_s}{2} - \square^{1,c}_s + \sum_{v \in \mathfrak{v}_s} \sum_{u \in {}_s\mathbb{A}_v \setminus \{s\}} \square^{1,c}_u \\
&= \frac{n_s}{2} - \square^{1,c}_s + \square^{1,c}_c + \sum_{v \in \mathfrak{v}_s} \sum_{u \neq c \in {}_s\mathbb{A}_v \setminus \{s\}} \square^{1,c}_u \\
&= \frac{n_s}{2} - \frac{-1+\delta^1_s}{2} + \left\lfloor \begin{matrix} \Delta^1_s \\ \frac{1}{2} \end{matrix} \right._{\nu^{1,c}_s - \delta^{1,c}_s} + \frac{-1+\delta^1_c}{2} - \left\lfloor \begin{matrix} \Delta^1_c \\ \frac{1}{2} \end{matrix} \right._{\nu^{1,c}_c - \delta^{1,c}_c} + \sum_{v \in \mathfrak{v}_s} \sum_{u \neq c \in {}_s\mathbb{A}_v \setminus \{s\}} \square^{1,c}_u \\
&= \frac{n_s}{2} - \frac{\delta^1_s}{2} + \left\lfloor \begin{matrix} \Delta^1_s \\ \frac{1}{2} \end{matrix} \right._{\nu^1_s - \delta^1_s} + \frac{\delta^1_c}{2} - \left\lfloor \begin{matrix} \Delta^1_c \\ \frac{1}{2} \end{matrix} \right._{\nu^1_c - \delta^1_c} + \sum_{v \in \mathfrak{v}_s} \sum_{u \neq c \in {}_s\mathbb{A}_v \setminus \{s\}} \square^1_u \\
&= \epsilon^1_s.
\end{aligned}$$

If $s = c$, it follows from the numbering of $\mathbb{A}^{1,c}$ that

$$\begin{aligned}
\epsilon^{1,c}_c &= \frac{n_c+1}{2} - \square^{1,c}_c + \sum_{v \in \mathfrak{v}_c} \sum_{u \in {}_s\mathbb{A}_v \setminus \{s\}} \square^{1,c}_u \\
&= \frac{n_c+1}{2} - \frac{\delta^{1,c}_c}{2} + \left\lfloor \begin{matrix} \Delta^{1,c}_c \\ \frac{1}{2} \end{matrix} \right._{\nu^{1,c}_c - \delta^{1,c}_c} + \sum_{v \in \mathfrak{v}_c} \sum_{u \in {}_c\mathbb{A}_v \setminus \{c\}} \square^{1,c}_u \\
&= \frac{n_c+1}{2} - \frac{-1+\delta^1_c}{2} + \left\lfloor \begin{matrix} \Delta^1_c \\ \frac{1}{2} \end{matrix} \right._{\nu^1_c - \delta^1_c} + \sum_{v \in \mathfrak{v}_c} \sum_{u \in {}_c\mathbb{A}_v \setminus \{c\}} \square^1_u \\
&= \epsilon^1_c + 1.
\end{aligned}$$

Since the configuration $\left(\epsilon^{1,c}\right)_s$ is admissible for $\mathbb{A}^{1,c}$, ϵ^1_s satifies the admissibility conditions for $s \neq c$. For $s = c$, if $\Delta^1_c = 1$ then we get the following inequality

$$\epsilon^1_c = \epsilon^{1,c}_c - 1 \geq n^1_c + 1 - 1 \geq n^1_c,$$

and if $\Delta^1_c = 0$ then the relation becomes

$$\begin{aligned}
\epsilon^1_c = \epsilon^{1,c}_c - 1 &\geq \left(2 - \sum_{s \in \mathfrak{v}_c \setminus \{c\}} \Delta^{1,c}_s\right) - 1 \\
&\geq \left(2 - \sum_{s \in \mathfrak{v}_c \setminus \{c\}} \Delta^1_s\right) - \Delta^1_r \\
&\geq 2 - \sum_{s \in \mathfrak{v}_c} \Delta^1_s.
\end{aligned}$$

Thus, in any case, the configuration ϵ^1 is admissible for $s \neq r$. Using much the same computations, we can prove that ϵ^0 is also admissible for $s \neq r$.

However, we are going to prove that exactly one dicriticity among Δ^0 and Δ^1 satisfies the admissibility condition for $s = r$. Indeed, we have

$$\begin{aligned}
\epsilon_r^0 + \epsilon_r^1 &= \frac{n_r}{2} - \square_r^0 + \left(\sum_{v \in \mathfrak{v}_r} \sum_{s \in \mathbb{A}_v \setminus \{r\}} \square_s^0 \right) + \frac{n_r}{2} - \square_r^1 + \left(\sum_{v \in \mathfrak{v}_c} \sum_{s \in {}_c\mathbb{A}_v \setminus \{c\}} \square_s^1 \right) \\
&= n_r + \left\lfloor \begin{matrix} 0 \\ \frac{1}{2} \end{matrix} \right._{\nu_r} + \left\lfloor \begin{matrix} 1 \\ \frac{1}{2} \end{matrix} \right._{\nu_r} + \sum_{v \in \mathfrak{v}_r} \sum_{s \in \mathbb{A}_v \setminus \{r\}} \square_s^0 + \square_s^1 \\
&= n_r + 1 + \sum_{v \in \mathfrak{v}_r} \sum_{s \in \mathbb{A}_v \setminus \{r\}} \square_s^0 + \square_s^1.
\end{aligned}$$

Now, if $v \in \mathfrak{v}_r$ and $s \in {}_r\mathbb{A}_v \setminus \{r\}$ one has

$$\square_s^0 + \square_s^1 = \frac{\delta_s^0}{2} - \left\lfloor \begin{matrix} \Delta_s^0 \\ \frac{1}{2} \end{matrix} \right._{\nu_s^0 - \delta_s^0} + \frac{\delta_s^1}{2} - \left\lfloor \begin{matrix} \Delta_s^1 \\ \frac{1}{2} \end{matrix} \right._{\nu_s^1 - \delta_s^1}$$

The construction of $\Delta^\star$ and the relations (2.5) force

$$\square_s^0 + \square_s^1 = \frac{1}{2} + \square_s^{0,s} + \square_s^{1,s}$$

which leads to

$$\begin{aligned}
\epsilon_r^0 + \epsilon_r^1 &= n_r + 1 + \sum_{v \in \mathfrak{v}_r} \sum_{s \in \mathbb{A}_v \setminus \{r\}} \frac{1}{2} + \square_s^{0,s} + \square_s^{1,s} \\
&= n_r + 1 + \sum_{v \in \mathfrak{v}_r} \frac{|\mathbb{A}_v \setminus \{r\}|}{2} + {}_{\mathbb{A}^{0,v}}^{\;\;n}\Theta_v^{01}
\end{aligned}$$

Notice that in the tree $\mathbb{A}^{0,\nu}$ the vertex ν is of multiplicity 1. Property (A) gives the relation

$${}_{\mathbb{A}^{0,\nu}}^{\;\;n}\Theta_\nu^{01} = -\frac{|\mathbb{A}_v \setminus \{r\}|}{2} - \Delta_v^0.$$

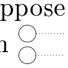
and the sum $\epsilon_r^0 + \epsilon_r^1$ reduces to

$$\epsilon_r^0 + \epsilon_r^1 = n_r + 1 - \sum_{v \in \mathfrak{v}_r} \Delta_v^0.$$

Finally, the above equality ensures that one of the two conditions

$$\epsilon_r^1 \geq n_r \qquad \text{or} \qquad \epsilon_r^0 \geq 2 - \sum_{v \in \mathfrak{v}_r} \Delta_v^0$$

holds but not both. As a consequence, either Δ^1 or Δ^0 is admissible for $s = r$, but not both. That concludes the proof of the main property of Theorem 3 for the tree $\mathbb{A}$.

Now, we are going to prove property (A) inductively from (A) and (C). Suppose first that the Saito dicricities respectively associated to n and $c \cdot n$ start with 

, then the invariant ${}^{n}_{\mathbb{A}}\Theta^{01}_{c}$ is written

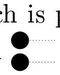
$$
\begin{aligned}
{}^{n}_{\mathbb{A}}\Theta^{01}_{c} &= \sum_{s\in\mathbb{A}_c} \square^n_s + \square^{c\cdot n}_s \\
&= \square^n_r + \square^{c\cdot n}_r + \sum_{s\in\mathbb{A}_c,\ s\neq r} \square^n_s + \square^{c\cdot n}_s \\
&= -\left\lfloor \begin{matrix} 1 \\ \frac{1}{2} \end{matrix}\right._{\nu^n_r} - \left\lfloor \begin{matrix} 1 \\ \frac{1}{2} \end{matrix}\right._{\nu^n_r+1} + \underbrace{\frac{\delta^n_{r^+}}{2} + \frac{\delta^{c\cdot n}_{r^+}}{2}}_{=1} - \left(\frac{\delta^n_{r^+}}{2} + \frac{\delta^{c\cdot n}_{r^+}}{2}\right) + \sum_{s\in\mathbb{A}_c,\ s\neq r} \square^n_s + \square^{c\cdot n}_s \\
&= -\frac{1}{2} - \left(\frac{\delta^n_{r^+}}{2} + \frac{\delta^{c\cdot n}_{r^+}}{2}\right) + \sum_{s\in\mathbb{A}_c,\ s\neq r} \square^n_s + \square^{c\cdot n}_s
\end{aligned}
$$

where r^+ is the successor of r in the branch $\mathbb{A}_c$. Consider the tree $\mathbb{A}^{r^+}$connected composent of r^+ in $\mathbb{A}\setminus\{r\}$. The inherited order of $\mathbb{A}^{r^+}$ makes of r^+ its root. Let s_0 be the vertex in the neighbobrhood $\mathfrak{v}_r$ of r in $\mathbb{A}$ such that $s_0 \geq r^+$. Note that from the unicity statement of Theorem 3 inductively applied to $\mathbb{A}^{r^+}$, the dicriticity of $\mathbb{A}^{r^+}$ inherited from the Saito dicriticity of $\mathbb{A}$ numbered by n is the Saito dicriticity of $\mathbb{A}^{r^+}$ numbered by the numbering inherited from the $s_0\cdot n$. Applying inductively property (A) to the tree $\mathbb{A}^{r^+}$, we get

$$
\begin{aligned}
-\left(\frac{\delta^n_{r^+}}{2} + \frac{\delta^{c\cdot n}_{r^+}}{2}\right) + \sum_{s\in\mathbb{A}_c,\ s\neq r} \square^n_s + \square^{c\cdot n}_s &= \sum_{s\in\mathbb{A}^{r^+}{}_c,} \square^{s_0\cdot n}_s + \square^{c\cdot s_0\cdot n}_s \\
&= {}^{s_0\cdot n}_{\mathbb{A}^{r^+}}\Theta^{01}_c \\
&= -\frac{|\mathbb{A}_c| - 1}{2} - \Delta^{s_0\cdot n}_c .
\end{aligned}
$$

Combining the two above relations, we are lead to

$$
{}^{n}_{\mathbb{A}}\Theta^{01}_{c} = -\frac{1}{2} - \frac{|\mathbb{A}_c| - 1}{2} - \Delta^{s_0\cdot n}_c = -\frac{|\mathbb{A}_c|}{2} - \Delta^n_c ,
$$

which is property (A). Now, if the Saito dicricities associated to n and $c\cdot n$ start with  , then the invariant ${}^{n}_{\mathbb{A}}\Theta^{01}_{c}$ becomes

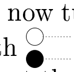
$$
\begin{aligned}
{}^{n}_{\mathbb{A}}\Theta^{01}_{c} &= \sum_{s\in\mathbb{A}_c} \square^n_s + \square^{c\cdot n}_s \\
&= \square^n_r + \square^{c\cdot n}_r + \sum_{s\in\mathbb{A}_c,\ s\neq r} \square^n_s + \square^{c\cdot n}_s \\
&= -\left\lfloor \begin{matrix} 0 \\ \frac{1}{2} \end{matrix}\right._{\nu^n_r} - \left\lfloor \begin{matrix} 0 \\ \frac{1}{2} \end{matrix}\right._{\nu^n_r+1} + \underbrace{\frac{\delta^n_{r^+}}{2} + \frac{\delta^{c\cdot n}_{r^+}}{2}}_{=0} - \left(\frac{\delta^n_{r^+}}{2} + \frac{\delta^{c\cdot n}_{r^+}}{2}\right) + \sum_{s\in\mathbb{A}_c,\ s\neq r} \square^n_s + \square^{c\cdot n}_s \\
&= -\frac{1}{2} - \left(\frac{\delta^n_{r^+}}{2} + \frac{\delta^{c\cdot n}_{r^+}}{2}\right) + \sum_{s\in\mathbb{A}_c,\ s\neq r} \square^n_s + \square^{c\cdot n}_s
\end{aligned}
$$

As above, applying inductively property (A) yields

$$\begin{aligned} -\left(\frac{\delta^n_{r^+}}{2}+\frac{\delta^{c\cdot n}_{r^+}}{2}\right)+\sum_{s\in\mathbb{A}_c,\ s\neq r}\Box^n_s+\Box^{c\cdot n}_s &= \sum_{s\in\mathbb{A}^{r^+}{}_c,}\Box^n_s+\Box^{c\cdot n}_s \\ &= {}^{\ n}_{\mathbb{A}^{r^+}}\Theta^{01}_c. \\ &= -\frac{|\mathbb{A}_c|-1}{2}-\Delta^n_c. \end{aligned}$$

As before, the two above relations lead to

$${}^{n}_{\mathbb{A}}\Theta^{01}_c=-\frac{1}{2}-\frac{|\mathbb{A}_c|-1}{2}-\Delta^n_c=-\Delta^n_c-\frac{|\mathbb{A}_c|}{2},$$

which is the desired property. We now turn to the case in which the Saito dicricities associated to n and $c\cdot n$ start with  . Hence, we are in the presence of a mixed or pure mixed branch. Suppose first that $|\mathbb{A}_c|=1$. In that case, the branch is pure and the invariant ${}^{n}_{\mathbb{A}}\Theta^{01}_c$ reduces to

$${}^{n}_{\mathbb{A}}\Theta^{01}_c=-\left\lfloor\begin{matrix}\Delta^n_r\\ \frac{1}{2}\end{matrix}\right._{\nu^n_r}-\left\lfloor\begin{matrix}\Delta^{r\cdot n}_r\\ \frac{1}{2}\end{matrix}\right._{\nu^{r\cdot n}_r}=-\left\lfloor\begin{matrix}1\\ \frac{1}{2}\end{matrix}\right._{\nu^n_r}-\left\lfloor\begin{matrix}0\\ \frac{1}{2}\end{matrix}\right._{\nu^n_r+1}.$$

From Table (C), we get

$${}^{n}_{\mathbb{A}}\Theta^{01}_c=-1-\frac{1}{2}=-\frac{|\mathbb{A}_c|}{2}-\Delta^n_c.$$

Suppose now that $|\mathbb{A}_c|\geq 2$. In that case, along the mixed branch, the nature of the square index allows us to simplify the expression of the invariant ${}^{n}_{\mathbb{A}}\Theta^{01}_c$. Suppose that s and s' are consecutive vertices $s\leq s'$ in $\mathbb{A}_c$ with

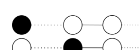
$$\Delta^n_\star+\Delta^{c\cdot n}_\star=1,\ \star=s,\ s'. \tag{2.6}$$

Then, evaluating the square indeces at s' yields

$$\Box^n_{s'}+\Box^{c\cdot n}_{s'}=\frac{\delta^n_{s'}}{2}+\frac{\delta^{c\cdot n}_{s'}}{2}-\left\lfloor\begin{matrix}\Delta^n_{s'}\\ \frac{1}{2}\end{matrix}\right._{\nu^n_{s'}-\delta^n_{s'}}-\left\lfloor\begin{matrix}\Delta^{c\cdot n}_{s'}\\ \frac{1}{2}\end{matrix}\right._{\nu^{c\cdot n}_{s'}-\delta^{c\cdot n}_{s'}}.$$

Now, according to the relations (2.6) one has $\delta^n_{s'}+\delta^{c\cdot n}_{s'}=1$, hence we obtain

$$\Box^n_{s'}+\Box^{c\cdot n}_{s'}=\frac{1}{2}-\left\lfloor\begin{matrix}\Delta^n_{s'}\\ \frac{1}{2}\end{matrix}\right._{\nu^n_{s'}-\delta^n_{s'}}-\left\lfloor\begin{matrix}1-\Delta^n_{s'}\\ \frac{1}{2}\end{matrix}\right._{\nu^n_{s'}+1-\delta^n_{s'}-1}=-\frac{1}{2}. \tag{2.7}$$

Let us focus nows on a mixed branch is of type  . Let m_c be the last vertex of the branch $\mathbb{A}_c$ where the mixing property (2.6) holds. Using the simplification (2.7), we obtain the following expression

$$\begin{aligned} {}^{n}_{\mathbb{A}}\Theta^{01}_c &= \sum_{s\in\mathbb{A}_c}\Box^n_s+\Box^{c\cdot n}_s \\ &= \Box^n_r+\Box^{c\cdot n}_r+\sum_{s\in\mathbb{A}_{m_c}\setminus\{r\}}\Box^n_s+\Box^{c\cdot n}_s+\Box^n_{m_c^+}+\Box^{c\cdot n}_{m_c^+}+\sum_{s>m_c^+,\ s\in\mathbb{A}_c}\Box^n_s+\Box^{c\cdot n}_s \\ &= -\left\lfloor\begin{matrix}0\\ \frac{1}{2}\end{matrix}\right._{\nu_r}-\left\lfloor\begin{matrix}1\\ \frac{1}{2}\end{matrix}\right._{\nu_r+1}-\frac{|\mathbb{A}_{m_c}|-1}{2}+\frac{1}{2}-\left\lfloor\begin{matrix}1\\ \frac{1}{2}\end{matrix}\right._{\nu_{m_c^+}-1}-\left\lfloor\begin{matrix}1\\ \frac{1}{2}\end{matrix}\right._{\nu_{m_c^+}+1}+1+{}^{s_0\cdot n}_{\mathbb{A}^{m_c^+}}\Theta^{01}_c \end{aligned}$$

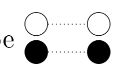
where $\mathbb{A}^{m_c^+}$ is the subtree of $\mathbb{A}$ whose root is m_c^+ and s_0 is the vertex in $\mathfrak{v}_r$ such that $s_0 \geq m_c^+$. Following Table 1, we are lead to

$$\begin{aligned}
{}^{n}_{\mathbb{A}}\Theta^{01}_{c} &= -\frac{1}{2} - 1 - \frac{|\mathbb{A}_{m_c}| - 1}{2} + \frac{1}{2} - 1 + 1 + {}^{s_0 \cdot n}_{\mathbb{A}^{m_c^+}}\Theta^{01}_{c} \\
&= -\frac{1}{2} - \frac{|\mathbb{A}_{m_c}|}{2} + {}^{s_0 \cdot n}_{\mathbb{A}^{m_c^+}}\Theta^{01}_{c}.
\end{aligned}$$

Applying inductively property (A), we obtain

$${}^{n}_{\mathbb{A}}\Theta^{01}_{c} = -\frac{1}{2} - \frac{|\mathbb{A}_{m_c}|}{2} - \frac{\left|{}^{m_c^+}\mathbb{A}_c\right|}{2} - \Delta^n_c = -\frac{|\mathbb{A}_c|}{2} - \Delta^n_c,$$

which is Property (A).

Suppose now the mixed branch is pure of type . Then the invariant ${}^{n}_{\mathbb{A}}\Theta^{01}_{c}$ is written

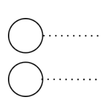
$$\begin{aligned}
{}^{n}_{\mathbb{A}}\Theta^{01}_{c} &= \sum_{s \in \mathbb{A}_c} \square^n_s + \square^{c \cdot n}_s = \square^n_r + \square^{c \cdot n}_r + \left(\sum_{s \in \mathbb{A}_{m_c} \setminus \{r\}} \square^n_s + \square^{c \cdot n}_s \right) \\
&= -\left\lfloor \begin{array}{c} 1 \\ \frac{1}{2} \end{array} \right._{\nu_r} - \left\lfloor \begin{array}{c} 0 \\ \frac{1}{2} \end{array} \right._{\nu_r + 1} - \frac{|\mathbb{A}_{m_c}| - 1}{2}.
\end{aligned}$$

According to Table (1), ν_r is even. Thus we obtain

$${}^{n}_{\mathbb{A}}\Theta^{01}_{c} = -1 - \frac{|\mathbb{A}_{m_c}|}{2} = -\Delta^n_c - \frac{|\mathbb{A}_{m_c}|}{2}$$

which is still Property (A). Any other type of mixed or pure mixed branch can be treated exactly the same way.

Now, we will prove inductively property (B) from properties (A) and (B). In the branch $\mathbb{A}_{c_1}$, we denote by r^+ the successor of r. Moreover, we denote by $d \in \mathfrak{v}_r$ such that $d \geq r^+$. Depending on how starts the Saito dicriticity of $\mathbb{A}$ numbered respectively by n and $c_0 \cdot c \cdot n$, we expand below the expression of the invariant ${}^{n}_{\mathbb{A}}\Theta^{02}_{c_0, c_1}$.



$$\begin{aligned}
{}^{n}_{\mathbb{A}}\Theta^{02}_{c_0, c_1} &= \square^n_r + \square^{c_0 \cdot c_1 \cdot n}_r + \sum_{s \in \mathbb{A}_{c_1} \setminus \{r\}} \square^n_s + \square^{c_0 \cdot c_1 \cdot n}_s \\
&= -\left\lfloor \begin{array}{c} 1 \\ \frac{1}{2} \end{array} \right._{\nu_r} - \left\lfloor \begin{array}{c} 1 \\ \frac{1}{2} \end{array} \right._{\nu_r + 2} + \sum_{s \in \mathbb{A}_{c_1} \setminus \{r\}} \square^{d \cdot n}_s + \square^{c_1 \cdot d \cdot n}_s \\
&= -\left\lfloor \begin{array}{c} 1 \\ \frac{1}{2} \end{array} \right._{\nu_r} - \left\lfloor \begin{array}{c} 1 \\ \frac{1}{2} \end{array} \right._{\nu_r + 2} + \frac{1}{2} + \frac{1}{2} + {}^{d \cdot n}_{\mathbb{A}^{r^+}}\Theta^{01}_{c_1} \\
&= \left\lfloor \begin{array}{c} -\frac{1}{2} \\ \frac{1}{2} \end{array} \right._{\nu_r} - \Delta^n_{c_1} - \frac{|\mathbb{A}_{c_1}|}{2}.
\end{aligned}$$

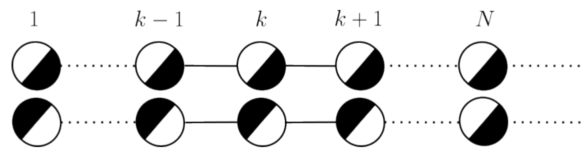
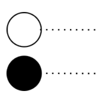
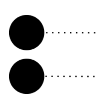


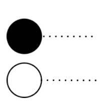
FIGURE 2.6. Mixed branch stopping from being mixed at the N^{th} vertex.



$$\begin{aligned}
{}^{n}_{\mathbb{A}}\Theta^{02}_{c_0,c_1} &= \square^{n}_{r} + \square^{c_0\cdot c_1\cdot n}_{r} + \sum_{s\in\mathbb{A}_{c_1}\setminus\{r\}} \square^{n}_{s} + \square^{c_0\cdot c_1\cdot n}_{s}\\
&= -\Big\lfloor\begin{smallmatrix}1\\ \frac{1}{2}\end{smallmatrix}_{\nu_r} - \Big\lfloor\begin{smallmatrix}0\\ \frac{1}{2}\end{smallmatrix}_{\nu_r+2} + \sum_{s\in\mathbb{A}_{c_1}\setminus\{r\}} \square^{d\cdot n}_{s} + \square^{c_1\cdot n}_{s}\\
&= -\Big\lfloor\begin{smallmatrix}1\\ \frac{1}{2}\end{smallmatrix}_{\nu_r} - \Big\lfloor\begin{smallmatrix}0\\ \frac{1}{2}\end{smallmatrix}_{\nu_r+2} + \frac{1}{2} + {}^{n}_{\mathbb{A}_r^+}\Theta^{11}_{d,c_1}\\
&= \pm\frac{1}{2} - \Delta^{n}_{c} - \frac{|\mathbb{A}_c|}{2}
\end{aligned}$$



$$\begin{aligned}
{}^{n}_{\mathbb{A}}\Theta^{02}_{c_0,c_1} &= \square^{n}_{r} + \square^{c_0\cdot c_1\cdot n}_{r} + \sum_{s\in\mathbb{A}_{c_1}\setminus\{r\}} \square^{n}_{s} + \square^{c_0\cdot c_1\cdot n}_{s}\\
&= -\Big\lfloor\begin{smallmatrix}0\\ \frac{1}{2}\end{smallmatrix}_{\nu_r} - \Big\lfloor\begin{smallmatrix}0\\ \frac{1}{2}\end{smallmatrix}_{\nu_r+2} + \sum_{s\in\mathbb{A}_{c_1}\setminus\{r\}} \square^{n}_{s} + \square^{c_1\cdot n}_{s}\\
&= -\Big\lfloor\begin{smallmatrix}0\\ \frac{1}{2}\end{smallmatrix}_{\nu_r} - \Big\lfloor\begin{smallmatrix}0\\ \frac{1}{2}\end{smallmatrix}_{\nu_r+2} + {}^{n}_{\mathbb{A}_r^+}\Theta^{01}_{c_1}\\
&= \Big\lfloor\begin{smallmatrix}\frac{1}{2}\\ -\frac{1}{2}\end{smallmatrix}_{\nu_r} - \Delta^{n}_{c_1} - \frac{|\mathbb{A}_{c_1}|}{2}.
\end{aligned}$$



$$\begin{aligned}
{}^{n}_{\mathbb{A}}\Theta^{02}_{c_0,c_1} &= \square^{n}_{r} + \square^{c_0\cdot c_1\cdot n}_{r} + \sum_{s\in\mathbb{A}_{c_1}\setminus\{r\}} \square^{n}_{s} + \square^{c_0\cdot c_1\cdot n}_{s}\\
&= -\Big\lfloor\begin{smallmatrix}0\\ \frac{1}{2}\end{smallmatrix}_{\nu_r} - \Big\lfloor\begin{smallmatrix}1\\ \frac{1}{2}\end{smallmatrix}_{\nu_r+2} + \sum_{s\in\mathbb{A}_{c_1}\setminus\{r\}} \square^{n}_{s} + \square^{c_1\cdot d\cdot n}_{s}\\
&= -\Big\lfloor\begin{smallmatrix}0\\ \frac{1}{2}\end{smallmatrix}_{\nu_r} - \Big\lfloor\begin{smallmatrix}1\\ \frac{1}{2}\end{smallmatrix}_{\nu_r+2} + \frac{1}{2} + {}^{n}_{\mathbb{A}_r^+}\Theta^{02}_{d,c_1}\\
&= \pm\frac{1}{2} - \Delta^{n}_{c_1} - \frac{|\mathbb{A}_{c_1}|}{2}.
\end{aligned}$$

That concludes the proof of Property (B) for the invariant ${}^{n}_{\mathbb{A}}\Theta^{02}_{c_0,c_1}$. The case of the invariant ${}^{n}_{\mathbb{A}}\Theta^{11}_{c_0,c_1}$ is obtained much the same way.

To prove inductively property (C) as a consequence of all previous properties, we consider a mixed branch of any type as in Figure 2.6. In the sequel, the vertex is designated by its position k in the branch $k = 1, \cdots$. The N^{th} is the first for which the mixing property does not hold.

Let us denote by $\epsilon^{\star}_{k}$, $\star = n, c\cdot n$ the configuration of the k^{th} vertices of the branch. Since the configuration is supposed to be admissible, summing the two inequalities associated to the admissibility conditions of Theorem 3, we get

$$\epsilon_k^n + \epsilon_k^{c\cdot n} \geq n_k + 2 - \sum_{s\in\mathfrak{v}_k} \Delta_s^{\sigma_{k,n}}$$

where $\sigma_{k,n} = n$ if $\Delta_k^n = 0$, else $\sigma_{k,n} = c\cdot n$. Therefore,

$$\sum_{k=1}^{N-1} \epsilon_k^n + \epsilon_k^{c\cdot n} \geq 2\,(N-1) + \sum_{k=1}^{N-1} n_k - \sum_{k=1}^{N-1}\sum_{s\in\mathfrak{v}_k} \Delta_s^{\sigma_{k,n}}. \tag{2.8}$$

Now, we want to estimate the expression in the left of the above inequality. Suppose first that $k = 2,\cdots, N-2$. Notice that in this situation $\delta_k^\star = \Delta_{k-1}^\star$, hence

$$\begin{aligned}
\epsilon_k^n + \epsilon_k^{c\cdot n} &= \frac{n_k}{2} - \frac{\Delta_{k-1}^n}{2} + \left\lfloor \begin{matrix}\Delta_k^n \\ \frac{1}{2}\end{matrix}\right._{\nu_k^n-\Delta_{k-1}^n} + \sum_{s\in\mathfrak{v}_k}\ \sum_{u\in_k\mathbb{A}_s\setminus\{k\}} \Box_u^n \\
&\quad + \frac{n_k}{2} - \frac{\Delta_{k-1}^{c\cdot n}}{2} + \left\lfloor \begin{matrix}\Delta_k^{c\cdot n} \\ \frac{1}{2}\end{matrix}\right._{\nu_k^{c\cdot n}-\Delta_{k-1}^{c\cdot n}} + \sum_{s\in\mathfrak{v}_k}\ \sum_{u\in_k\mathbb{A}_s\setminus\{k\}} \Box_u^{c\cdot n} \\
&= n_k + \frac{1}{2} + \sum_{s\in\mathfrak{v}_k}\ \sum_{u\in_k\mathbb{A}_s\setminus\{k\}} \Box_u^n + \Box_u^{c\cdot n}
\end{aligned}$$

Let us denote by k^- and k^+ the vertices in $\mathfrak{v}_k$ such that $k^- \geq (k-1)$, $k^- \neq k-1$ and $k^+ \geq (k+1)$ for the order $\leq$ on the tree. Notice that k^- may not exists and k^+ may be equal to $k+1$.

From the previous expressions we obtain,

$$\begin{aligned}
\epsilon_k^n + \epsilon_k^{c\cdot n} &= n_k + \frac{1}{2} + \sum_{s\in\mathfrak{v}_k\setminus\{k^-,k^+\}}\ \sum_{u\in_k\mathbb{A}_s\setminus\{k\}} \Box_u^n + \Box_u^{c\cdot n} \\
&\quad + \sum_{u\in_k\mathbb{A}_{k^-}\setminus\{k\}} \Box_u^n + \Box_u^{c\cdot n} + \sum_{u\in_k\mathbb{A}_{k^+}\setminus\{k\}} \Box_u^n + \Box_u^{c\cdot n}.
\end{aligned}$$

For $s\in\mathfrak{v}_k\setminus\{k^-,k^+\}$, we are lead to

$$\begin{aligned}
\sum_{u\in_k\mathbb{A}_s\setminus\{k\}} \Box_u^n + \Box_u^{c\cdot n} &= \frac{|{}_k\mathbb{A}_s\setminus\{k\}|}{2} + \sum_{u\in_k\mathbb{A}_s\setminus\{k\}} \Box_u^n + \Box_u^{s\cdot n} \\
&= \frac{|{}_k\mathbb{A}_s\setminus\{k\}|}{2} + {}^{\,n}_{\mathbb{A}^k}\Theta_s^{01}.
\end{aligned}$$

Hence, according to Property (A), it

$$\sum_{u\in_k\mathbb{A}_s\setminus\{k\}} \Box_u^n + \Box_u^{c\cdot n} = -\Delta_s^{\sigma_{k,n}}.$$

In the same way, we find

$$\begin{aligned}
\sum_{u\in_k\mathbb{A}_{k^-}\setminus\{k\}} \Box_u^n + \Box_u^{c\cdot n} &= \frac{1}{2} + \frac{\left|{}_k\mathbb{A}_{k^-}\setminus\{k\}\right|}{2} + \sum_{u\in_k\mathbb{A}_{k^-}\setminus\{k\}} \Box_u^a + \Box_u^b \\
&= \frac{1}{2} + \frac{\left|{}_k\mathbb{A}_{k^-}\setminus\{k\}\right|}{2} + \left\{\begin{array}{l} {}^{\,n}_{\mathbb{A}^k}\Theta_{k,k^-}^{02} \text{ or} \\ {}^{\,n}_{\mathbb{A}^k}\Theta_{k,k^-}^{11} \end{array}\right.
\end{aligned}$$

where

$$\{a,b\} = \left\{ \begin{array}{l} \{n, k\cdot k^{-}\cdot n\} \\ \{k\cdot n, k^{-}\cdot n\} \end{array} \right. \text{ if } \left(\Delta_{k-1}^{n}, \Delta_{k-1}^{n}\right) = \left\{ \begin{array}{l} (0,0) \text{ or } (1,1) \\ (0,1) \text{ or } (1,0) \end{array} \right.$$

Thus, Property (A) ensures that

$$\sum_{u\in {}_k\mathbb{A}_{k^-}\setminus\{k\}} \square_u^n + \square_u^{c\cdot n} = \frac{1}{2} \pm \frac{1}{2} - \Delta_{k^-}^{\sigma_{k,n}}.$$

In the same way, one can prove that

$$\sum_{u\in {}_k\mathbb{A}_{k^+}\setminus\{k\}} \square_u^n + \square_u^{c\cdot n} = \pm\frac{1}{2} - \Delta_{k^+}^{\sigma_{k,n}}.$$

Finally, for $k\in\{2,\cdots,N-2\}$

$$\epsilon_k^n + \epsilon_k^{c\cdot n} = n_k + \frac{1}{2} \pm \frac{1}{2} + \left\{ \begin{array}{ll} \frac{1}{2}\pm\frac{1}{2} & \text{if } k^{-} \text{ exists} \\ 0 & \text{else} \end{array} \right. - \sum_{s\in\mathfrak{v}_k} \Delta_s^{\sigma_{k,n}} \tag{2.9}$$

For $k=1$ the computation is slightly different bu we obtain

$$\epsilon_1^n + \epsilon_1^{c\cdot n} = n_1 + \frac{1}{2} + \left\lfloor \begin{array}{c} \Delta_1^n \\ \Delta_1^{c\cdot n} \\ \nu_1 \end{array} \right. \pm \frac{1}{2} - \sum_{s\in\mathfrak{v}_1} \Delta_s^{\sigma_{1,n}}. \tag{2.10}$$

Finally for $k=N-1$, we can write

$$\begin{aligned} \epsilon_{N-1}^n + \epsilon_{N-1}^{c\cdot n} &= \frac{n_{N-1}}{2} - \frac{\Delta_{N-2}^n}{2} + \left\lfloor \begin{array}{c} \Delta_{N-1}^n \\ \frac{1}{2} \\ \nu_N^n - \Delta_{N-2}^n \end{array} \right. + \frac{n_{N-1}}{2} - \frac{\Delta_{N-2}^{c\cdot n}}{2} + \left\lfloor \begin{array}{c} \Delta_{N-1}^{c\cdot n} \\ \frac{1}{2} \\ \nu_N^{c\cdot n} - \Delta_{N-2}^{c\cdot n} \end{array} \right. \\ &\quad + \sum_{s\in\mathfrak{v}_{N-1}} \sum_{u\in {}_{N-1}\mathbb{A}_s\setminus\{N-1\}} \square_u^n + \square_u^{c\cdot n} \\ &= n_{N-1} + \frac{1}{2} + \sum_{u\in {}_{(N-1)}\mathbb{A}_{(N-1)^+}\setminus\{(N-1)\}} \square_u^n + \square_u^{c\cdot n} \\ &\quad + \sum_{s\in\mathfrak{v}_{N-1}, s\neq(N-1)^+} \sum_{u\in {}_{N-1}\mathbb{A}_s\setminus\{N-1\}} \square_u^n + \square_u^{c\cdot n} \end{aligned}$$

Now, we focus on a term of the above sum : we have

$$\begin{aligned} \sum_{u\in {}_{N-1}\mathbb{A}_{(N-1)^+}\setminus\{N-1\}} \square_u^n + \square_u^{c\cdot n} &= \frac{\Delta_{N-1}^n}{2} + \frac{\Delta_{N-1}^{c\cdot n}}{2} - \left\lfloor \begin{array}{c} \Delta_N^n \\ \frac{1}{2} \\ \nu_N^n - \Delta_{N-1}^n \end{array} \right. - \left\lfloor \begin{array}{c} \Delta_N^{c\cdot n} \\ \frac{1}{2} \\ \nu_N^{c\cdot n} - \Delta_{N-1}^{c\cdot n} \end{array} \right. \\ &\quad + \frac{\Delta_N^n}{2} + \frac{\Delta_N^{c\cdot n}}{2} \\ &\quad - \left(\frac{\Delta_N^n}{2} + \frac{\Delta_N^{c\cdot n}}{2}\right) + \sum_{\substack{u\in {}_{(N-1)}\mathbb{A}_{(N-1)^+} \\ u\neq N, N-1}} \square_u^n + \square_u^{c\cdot n}. \end{aligned}$$

Since $\Delta_N^n = \Delta_N^{c\cdot n}$, $\nu_N^n = \nu_N^{c\cdot n} - 1$ and $\Delta_{N-1}^n + \Delta_{N-1}^{c\cdot n} = 1$, we are lead to the expression

$$\sum_{u \in {}_{N-1}\mathbb{A}_{(N-1)^+} \setminus \{N-1\}} \square_u^n + \square_u^{c\cdot n} = \frac{1}{2} - \left\lfloor \begin{matrix} \Delta_N^n \\ 1-\Delta_N^n \end{matrix} \right._{\nu_N - \Delta_{N-1}^n} - \Delta_{(N-1)^+}^{\sigma(N-1),n}$$

Finally, we find

$$\epsilon_{N-1}^n + \epsilon_{N-1}^{c\cdot n} = n_{N-1} + 1 - \left\lfloor \begin{matrix} \Delta_N^n \\ 1-\Delta_N^n \end{matrix} \right._{\nu_N - \Delta_{N-1}^n} + \begin{cases} \frac{1}{2} \pm \frac{1}{2} & \text{if } (N-1)^- \text{exists} \\ 0 & \text{else} \end{cases} - \sum_{s \in \mathfrak{v}_{N-1}} \Delta_s^{\sigma_{N-1},n} \tag{2.11}$$

Summing the equations (2.9), (2.10) and (2.11) yields

$$\sum_{k=1}^{N-1} \epsilon_k^n + \epsilon_k^{c\cdot n} = \left\lfloor \begin{matrix} \Delta_1^n \\ \Delta_1^{c\cdot n} \end{matrix} \right._{\nu_1} - \left\lfloor \begin{matrix} \Delta_N^n \\ 1-\Delta_N^n \end{matrix} \right._{\nu_N - \Delta_{N-1}^n} + \underbrace{(\cdots)}_{\text{at most } 2N-3} + \sum_{k=1}^{N-1} n_k - \sum_{k=1}^{N-1} \sum_{s \in \mathfrak{v}_k} \Delta_s^{\sigma_{k,n}}.$$

Combining with the inequality (2.8), we obtain

$$\left\lfloor \begin{matrix} \Delta_1^n \\ \Delta_1^{c\cdot n} \end{matrix} \right._{\nu_1} - \left\lfloor \begin{matrix} \Delta_N^n \\ 1-\Delta_N^n \end{matrix} \right._{\nu_N - \Delta_{N-1}^n} \geq 1.$$

This inequality induces all the properties presented in Table 1.

To prove the statements in Table 2, we add one white component at the end of each pure mixed branches, providing thus standard mixed branches. Numbering the vertices of these branches $1, \cdots, N, N+1$, the $N+1^{\text{th}}$ being the added one and setting $n_{N+1} = 0$, we obtain two mixed branches numbered respectively by n and $(N+1)\cdot n$ whose dicricities are Saito. Thus, the computations performed for mixed branches yield

$$\left\lfloor \begin{matrix} \Delta_1^n \\ \Delta_1^{c\cdot n} \end{matrix} \right._{\nu_1} - \left\lfloor \begin{matrix} 1 \\ 0 \end{matrix} \right._{0-\Delta_N^n} \geq 1.$$

Thus if $\Delta_N^n = 0$ the above inequality is impossible ; that excludes the two last cases of Table 2. If $\Delta_N^n = 1$, then the inequality reduces to $\left\lfloor \begin{matrix} \Delta_1^n \\ \Delta_1^{c\cdot n} \end{matrix} \right._{\nu_1} \geq 1$, which implies the two first cases of Table 2. Finally, suppose that the mixed branch is reduced to a single couple of vertices and starts with $\overset{\bullet}{\circ}$. Assume that ν_r^n is even. We can write,

$$\epsilon_r^n = \frac{n_r}{2} + \left\lfloor \begin{matrix} 0 \\ \frac{1}{2} \end{matrix} \right._{\nu_r^n} + \sum_{v \in \mathfrak{v}_c} \square_v.$$

Hence, we deduce that

$$\epsilon_r^n + 1 = \frac{n_r + 1}{2} + \left\lfloor \begin{matrix} 0 \\ \frac{1}{2} \end{matrix} \right._{\nu_r^n + 1} + \sum_{v \in \mathfrak{v}_c} \square_v = \frac{n_r + 1}{2} + \left\lfloor \begin{matrix} 0 \\ \frac{1}{2} \end{matrix} \right._{\nu_r^{r\cdot n}} + \sum_{v \in \mathfrak{v}_c} \square_v$$

Since, ϵ^n is the configuration of the Saito dicriticity for the numbering n, we get

$$\epsilon_r^n \geq 2 - \sum_{s \in \mathfrak{v}_r} \Delta_s^n$$

and consequently, $\epsilon_r^n + 1 \geq 2 - \sum_{s\in\mathfrak{v}_r} \Delta_s^n$. Therefore, the dicriticity Δ^n keeps on being Saito for the tree $\mathbb{A}$ numbered by $r \cdot n$. Since, this dicriticity is unique, we get $\Delta_r^{r\cdot n} = 1$ which contradicts the hypothesis of a mixed branch. Finally, ν_r^n has to be odd. In the same way, suppose the mixed branch is reduced to $\overset{\circ}{\bullet}$ and ν_r^n is odd. The arguments are the same as above and from the following computations

$$\epsilon_r^n = \frac{n_r}{2} + \left\lfloor \begin{matrix} 1 \\ \frac{1}{2} \end{matrix} \right._{\nu_r^n} + \sum_{v\in\mathfrak{v}_c} \square_v \geq n_r$$

$$\epsilon_r^n + 1 = \frac{n_r + 1}{2} + \left\lfloor \begin{matrix} 1 \\ \frac{1}{2} \end{matrix} \right._{\nu_r^{r\cdot n}} + \sum_{v\in\mathfrak{v}_c} \square_v \geq n_r + 1$$

we get a contradiction. Hence, ν_r^n is even. This concludes the proof of Property (C).

It remains to prove Property (D). Let $\mathbb{K}$ be the connected component of r in the sub-graph $\mathbb{A} \setminus \{c \in \mathbb{A} | \, \Delta_c^n = 0\}$. If $\mathbb{K} = \emptyset$, the property is proved by induction on $|\mathbb{A}|$. If not, suppose that there exists $s \in \mathbb{K}$ such that $n_s > 0$. Then, since $\Delta_s^n = 1$, the admissibility condition ensures that $\epsilon_s^n \geq n_s > 0$ which is the property. Finally, we suppose that for any $s \in \mathbb{K}$, $n_s = 0$. Assume also that for any $s \in \mathbb{K}$, $\epsilon_s^n = 0$. For any $s \in \mathfrak{v}_r$, we consider $k_s \in \mathbb{A}_s$ the minimal vertex such that ${}_{k_s}\mathbb{A}_s$ is in $\mathbb{K}$. Now, one can write

$$0 = \epsilon_r^n = -\square_r + \sum_{v\in\mathfrak{v}_r} \sum_{s\in\mathbb{A}_v\setminus\{r\}} \square_s^n. \tag{2.12}$$

$$= \left\lfloor \begin{matrix} 1 \\ \frac{1}{2} \end{matrix} \right._{\nu_r^n} + \sum_{v\in\mathfrak{v}_r} \sum_{\mathbb{A}_{k_v^{-1}}\setminus\{r\}} \square_s^n + \sum_{v\in\mathfrak{v}_r} \sum_{{}_{k_v}\mathbb{A}_v\setminus\{r\}} \square_s^n$$

Now, extracting the intermediary sum in the expression above yields

$$\begin{aligned}
\sum_{\mathbb{A}_{k_v^{-1}}\setminus\{r\}} \square_s^n &= \sum_{\mathbb{A}_{k_v^{-1}}\setminus\{r\}} \frac{\delta_s^n}{2} - \left\lfloor \begin{matrix} \Delta_s^n \\ \frac{1}{2} \end{matrix} \right._{\star} \\
&= \sum_{\mathbb{A}_{k_v^{-1}}\setminus\{r\}} \frac{\delta_s^n}{2} - \sum_{\mathbb{A}_{k_v^{-1}}\setminus\{r\}} \left\lfloor \begin{matrix} \Delta_s^n \\ \frac{1}{2} \end{matrix} \right._{\star} \\
&= \frac{\delta_{r^+}^n}{2} + \sum_{\mathbb{A}_{k_v^{-1}}\setminus\{r,r^+\}} \frac{\delta_s^n}{2} - \sum_{\mathbb{A}_{k_v^{-1}}\setminus\{r\}} \left\lfloor \begin{matrix} \Delta_s^n \\ \frac{1}{2} \end{matrix} \right._{\star} \\
&= \frac{\delta_{r^+}^n}{2} + \sum_{\mathbb{A}_{k_v^{-2}}\setminus\{r\}} \frac{\delta_{s^+}^n}{2} - \left(\sum_{\mathbb{A}_{k_v^{-2}}\setminus\{r\}} \left\lfloor \begin{matrix} \Delta_s^n \\ \frac{1}{2} \end{matrix} \right._{\star} \right) - \left\lfloor \begin{matrix} \Delta_{k_v^{-1}}^n \\ \frac{1}{2} \end{matrix} \right._{\star} .
\end{aligned}$$

where r^+ is the successor of r in $\mathbb{A}_v$, k_v^{-i} the predecessor of k_v^{-i+1}. Since $\Delta_r^n = 1$, $\delta_{r^+}^n = 1$ and $\delta_{s^+}^n = 1 + \Delta_s^n$. Hence, we obtain

$$\sum_{{}_r\mathbb{A}_{k_v^{-1}} \setminus \{r\}} \square_s^n = \underbrace{\frac{1}{2} - \left\lfloor \begin{matrix} \Delta^n_{k_v^{-1}} \\ \frac{1}{2} \\ \star \end{matrix} \right.}_{A} + \underbrace{\sum_{{}_r\mathbb{A}_{k_v^{-2}} \setminus \{r\}} \left\lfloor \begin{matrix} \frac{1-\Delta_s^n}{2} \\ \frac{\Delta_s^n}{2} \\ \star \end{matrix} \right.}_{\geq 0} . \tag{2.13}$$

Now, since $k_v \in \mathbb{K}$, we get $0 = \epsilon^n_{k_v}$ and thus,

$$\square^n_{k_v} = \sum_{v \in \mathfrak{v}_{k_v}} \sum_{s \in {}_{k_v}\mathbb{A}_v \setminus \{k_v\}} \square_s^n .$$

If $\Delta^n_{k_v^{-1}} = 0$ then, in expression (2.13) $A = \frac{1}{2} - \left\lfloor \begin{matrix} \Delta^n_{k_v^{-1}} \\ \frac{1}{2} \\ \star \end{matrix} \right. \geq 0$. If $\Delta^n_{k_v^{-1}} = 1$, then by construction of k_v, $k_v^{-1} \notin \mathfrak{v}_{k_v}$. In the latter case, there exists $s \in \mathfrak{v}_{k_v}$ and $c \in {}_{k_v}\mathbb{A}_s$ with $\mathfrak{p}_c = \left\{k_v, k_v^{-1}\right\}$. Thus

$$\delta_c = \frac{1}{2} + \frac{1}{2}$$

which comes to compensate the fact that in relation (2.13), A might be equal to $\frac{-1}{2}$. Finally, if $s \in {}_{k_v}\mathbb{A}_v \setminus \{r\}$, $s \neq k_v$ then as before,

$$\square_s^n = \sum_{v \in \mathfrak{v}_s} \sum_{u \in {}_s\mathbb{A}_v \setminus \{s\}} \square_u^n .$$

Doing so step by step, from (2.12), we are lead to an expression of the form

$$0 = \left\lfloor \begin{matrix} 1 \\ \frac{1}{2} \\ \nu_r^n \end{matrix} \right. + \underbrace{(\cdots)}_{\geq 0}$$

which is impossible. That concludes the proof of Property (D) and, at the same time, the proof of Theorem 3. $\square$

3. Saito foliations of a germ of curve and its deformation.

The $\mathcal{O}_{(\mathbb{C}^2,0)}-$module $\mathrm{Der}\,(\log C)$ of germs of vector fields tangent to a germ of curve $C \subset \left(\mathbb{C}^2, 0\right)$ has been introduced as a particular case of a far more general object by K. Saito in [26]. We are interesting in the valuations of the vector fields in this module, for the standard valuation ν defined by

$$\nu\left(a\partial_x + b\partial_y\right) = \min\left(\nu\left(a\right), \nu\left(b\right)\right), \ a, b \in \mathbb{C}\left\{x, y\right\},$$

where

$$\nu\left(\sum_{i,j} a_{ij} x^i y^j\right) = \min_{a_{ij} \neq 0} \left\{i + j\right\}.$$

In particular, we define *the number of Saito of C* by

$$\mathfrak{s}_C = \min_{X \in \mathrm{Der}(\log C)} \nu\left(X\right)$$

A vector field tangent to C is said *optimal* if its valuation is equal to the Saito number of C.

Let π be the blowing-up of the singular point of C. At any singular point s of the total transform $\pi^{-1}(C)$, the strict transform X^{π} of X leaves invariant the strict transfotm C^{π} and maybe the exceptional divisor of π. When the latter occurs, X is said *non dicritical*. Otherwise, it is said *dicritical*. The vector field X^{π} may not be optimal for the germ of C^{π} at s although X is optimal for C. When the optimality property is preserved along the desingularization process of C, we said that X is *Saito* for C. More precisely, we consider the following inductive definition :

Definition 6. X is *Saito* for C when X is optimal for C and when X^{π} is Saito for each germ of $d \cup C^{\pi}$ at any of its singular points where

- $d = \pi^{-1}(0)$ if X is not dicritical,
- $d = \emptyset$, otherwise.

To initiate the definition, we require that if C is regular, then $\nu(X) = 0$ and if C is the union of two transversal regular curves, then $\nu(X) = 1$.

The goal of the current section is to prove the existence of a curve C' equisingular to C that admits a Saito foliation. To do so, we are going to construct a foliation using gluing techniques of [20]. The elementary pieces of this gluing are semi-local models for Saito foliations introduced just below. The results of the first section will provide a global data prescribing the gluing. The obtained foliation will be studied from the point of view of deformations and the curve C' will be found as an invariant curve of a generic deformation of the constructed foliation.

3.1. **Semi-local models for Saito foliations.** First, let us describe the two families of semi-local models for Saito foliations. These models are said to be semi-local because they are defined in the neighborhood of a compact divisor embedeed in a surface.

Let $\mathcal{M}_p$ be the germ of neighborhood of the divisor, given locally by $x_1 = 0$ and $y_2 = 0$, in the $2-$dimensional manifold defined by the disjont union of two charts

$$\left(\mathbb{C}^2, (x_1, y_1)\right) \coprod \left(\mathbb{C}^2, (x_2, y_2)\right)$$

with the identification $y_2 = y_1^p x_1 \quad x_2 = \frac{1}{y_1}$, $p \geq 0$. The divisor $\{x_1 = y_2 = 0\}$ is a regular rational compact curve embedded in $\mathcal{M}_p$ with negative self-intersection equal to $-p$.

3.1.1. *The dicritical model.* The manifold $\mathcal{M}_p$ can be foliated by the foliation $\mathcal{R}_{p,N}$ given in coordinates (x_1, y_1) by the $1-$form

$$\mathrm{d}x_1 + \prod_{i=1}^{N} (y_1 - i)\,\mathrm{d}y_1. \tag{3.1}$$

This foliation is transverse to the compact divisor except at the points given in coordinates (x_1, y_1) by $(0, i)$, $i = 1, \ldots, N$ where it is tangent at order 1. Using the changes of coordinates form (x_1, y_1) to (x_2, y_2), we can see that the foliation is regular and transverse to the compact divisor at $+\infty$. Note that we have

$$\sum_{p \in \{x_1 = 0\}} \mathrm{Tan}\left(\mathcal{R}_{p,N}, \{x_1 = 0\}, p\right) = N$$

where Tan is an index introduced in particular in [18].

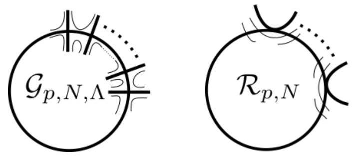


FIGURE 3.1. Local models for Saito foliations.

3.1.2. *The non dicritical model.* The manifold $\mathcal{M}_p$ can also be foliated by a foliation given by the $1-$form $\mathcal{G}_{p,N,\Lambda}$, $\Lambda = (\lambda_1, \ldots, \lambda_N)$ written in the coordinates (x_1, y_1)

$$\frac{dx_1}{x_1} + \sum_{i=1}^{N} \lambda_i \frac{\mathrm{d}y_1}{y_1 - i}. \tag{3.2}$$

with the following condition, known as the Camacho-Sad relation [2],

$$\sum_{i=1}^{N} \lambda_i = p. \tag{3.3}$$

This foliation leaves invariant the divisor $x_1 = 0$ and the relation above ensure that it is regular at $+\infty$. By construction, for any i, we get

$$\mathrm{CS}\left(\mathcal{G}, \{x_1 = 0\}, i\right) = \lambda_i.$$

where CS is the so-called Camacho-Sad index [2]. Moreover, it follows that

$$\sum_{p \in \{x_1 = 0\}} \mathrm{Ind}\left(\mathcal{R}_{p,N}, \{x_1 = 0\}, p\right) = N$$

where Ind is the second index introduced in [18].

Figure 3.1 presents the topology of $\mathcal{R}$ and $\mathcal{G}$.

3.2. **Gluing local models.** Let E be the process of desingularization of C. Let $\mathbb{A}$ be the dual tree of the exceptional divisor $E^{-1}(0)$. The map E is a composition of elementary blowing-ups that we denote

$$E = \bigcirc_{s \in \mathbb{A}} E_s.$$

Here E_s is the elementary blowing-up whose exceptional divisor is the component s. For any c, the notation $\star_c^E$ will refer to the germ at the point leading to the component c of the strict transform of $\star$ by the sub-process $\bigcirc_{s \in \mathbb{A}_c \setminus \{c\}} E_s$ where $\mathbb{A}_c$ is the access tree from r to c, as defined in the previous section.

For a germ of vector field X (or its associated germ of foliation $\mathcal{F}$) and $s \in \mathbb{A}$, we will set $\Delta_s^{X \text{ (or } \mathcal{F})} = 1$ if X_s^E is non dicritical, otherwise, $\Delta_s^X = 0$. It defines a dicriticity on $\mathbb{A}$.

Proposition 7. *There exists C' equisingular to C such that there exists $X \in$ Der $(\log C')$ satisfying the following : for any $s \in \mathbb{A}$*

$$\nu\left(X_s^E\right) = \frac{\nu\left(C_s^E\right) + \delta_s^X}{2} - \left\lfloor \begin{array}{c} 1 - \Delta_s^X \\ \frac{1}{2} \\ \nu(C_s^E) + \delta_s^X \end{array} \right.$$

Proof. The process E of desingularization of C induces an numbered ordered tree $\mathbb{A}$ as defined in the previous section. The tree $\mathbb{A}$ is the dual tree of the exceptional divisor of E ; the order is the one induced by the process it-self ; the numbering n is setting as follows : n_s is equal to the number of component of the strict transform C^E attached to s. Consider the associated Saito dicriticity Δ^n and configuration ϵ^n given by Theorem 3.

Using a result of A.-L. Neto [20] of construction of singular foliations in dimension 2 from elementary elements, we are going to construct a foliation from the data of Δ^n and ε^n by gluing semi-local models. The matrix $\mathbb{P}$ being the proximity matrix of $\mathbb{A}$ it is known that $\mathbb{P}^t\mathbb{P}$ is the intersection matrix I of $E^{-1}(0)$ embedded in its neighborhood [27].

To $s \in \mathbb{A}$ with $\Delta^n_s = 1$, we associate the semi-local model $\mathcal{G}_{I_{s,s},\epsilon^n_s+|\{c\in\mathfrak{v}_s,\Delta^n_c=1\}|,\Lambda_s}$ where

$$\Lambda_s = \left(\lambda_1, \cdots, \lambda_{\epsilon^n_s}, \lambda_{s,c_1}, \cdots, \lambda_{s,c_{|\{c\in\mathfrak{v}_s,\Delta^n_c=1\}|}}\right)$$

The only obstruction for such a semi-local construction is the Camacho-Sad relation

$$\sum_{i=1}^{\epsilon^n_s} \lambda_i + \sum_{i\in\{c\in\mathfrak{v}_s,\Delta^n_c=1\}} \lambda_{s,i} = -I_{s,s} \tag{3.4}$$

To $s \in \mathbb{A}$ with $\Delta^n_s = 0$, we associate the semi-local model

$$\mathcal{R}_{I_{s,s},\epsilon^n_s-2+\sum_{c\in\mathfrak{v}_s}\Delta^n_c}.$$

Since Δ^n is the Saito dicriticity, the admissibility condition yields the inequality

$$\epsilon^n_s - 2 + \sum_{c\in\mathfrak{v}_s} \Delta^n_c \geq 0,$$

so that, the definition of the model does make sense.

From [20], all these semi-local models can be glued together by gluing maps following the edges of $\mathbb{A}$ provided that at any intersection point of two components s and s' with $\Delta^n_s = \Delta^n_{s'} = 1$, the following relation is satisfied

$$\lambda_{s,s'} \cdot \lambda_{s',s} = 1. \tag{3.5}$$

Property ((D)) of Theorem 3 ensures that, along any connected component $\mathbb{K}$ of $\mathbb{A} \setminus \{s \in \mathbb{A} |\, \Delta^n_s = 1\}$, no incompatiblity will occur between the relations (3.4) and (3.5). Indeed, along $\mathbb{K}$ the number of induced relations is

$$\sharp\text{vertices}(\mathbb{K}) + \sharp\text{edges}(\mathbb{K}).$$

However, the number of variables involved in the mentioned relations is

$$\sum_{s\in\mathbb{K}} \epsilon^n_s + |\{c \in \mathfrak{v}_s, \Delta^n_c = 1\}|.$$

Following Property ((D)) the above number of variables satisfies

$$\begin{aligned}\sum_{s\in\mathbb{K}} \epsilon^n_s + |\{c \in \mathfrak{v}_s, \Delta^n_c = 1\}| \geq 1 + \sum_{s\in\mathbb{K}} |\{c \in \mathfrak{v}_s, \Delta^n_c = 1\}| &= 2\sharp\text{vertices}(\mathbb{K}) - 1 \\ &\geq \sharp\text{vertices}(\mathbb{K}) + \sharp\text{edges}(\mathbb{K})\end{aligned}$$

since $\sharp\text{vertices}(\mathbb{K}) = \sharp\text{edges}(\mathbb{K}) + 1$. Therefore the system of equations, union of (3.4) and (3.5), has always a solution. Note that these solutions can be chosen to be rational numbers.

The gluing leads to a foliation defined in a neighborhood of a compact divisor $\mathcal{D}$, union of $|\mathbb{A}|$ regular rational curves, with same intersection matrix as the one of the exceptional divisor of E. According to a classical result of H. Grauert [13], the neighborhood of $\mathcal{D}$ is analytically equivalent to the neighborhood of the exceptional divisor of some blowing-up process E' with same dual graph as E. The latter neighborhood is foliated by a foliation $\mathcal{F}'$ that can be contracted by E' in a foliation $\mathcal{F}$.

For any component $s \in \mathbb{A}$, either $\Delta_s^n = 0$ and $\mathcal{F}'$ is generically transverse to s. Then, we choose arbitraly n_s regular and transverse invariant curves attached to s. Or $\Delta_s^n = 1$ and $\mathcal{F}'$ locally given by (3.2) leaves invariant at least n_s regular and transverse curves attached to s: indeed, Δ^n being Saito, we have $\epsilon_s^n \geq n_s$. The union of all these curves yields a curve C' whose desingularization process has for associated numbered dual tree the tree $\mathbb{A}$ itself. Thus, C' and C are equisingular [28]. In the sequel, for the sake of simplicity, we denote C' simple by C. According to [18, Theorem 3], we have

$$\nu(\mathcal{F}) + 1 = \sum_{s\in\mathbb{A}} \rho_s \times \begin{cases} -\left|\{c \in \mathfrak{v}_s, \Delta_c^n = 1\}\right| + \sum_{p\in s} \operatorname{Ind}\left(\mathcal{F}^E, s, p\right) & \text{if } \Delta_s^n = 1 \\ 2 - \left|\{c \in \mathfrak{v}_s, \Delta_c^n = 1\}\right| + \sum_{p\in s} \operatorname{Tan}\left(\mathcal{F}^E, s, p\right) & \text{if } \Delta_s^n = 0 \end{cases}$$

In our construction, the definition of the semi-local models induces the relations

$$\begin{aligned} \sum_{p\in s} \operatorname{Ind}\left(\mathcal{F}^E, s, p\right) &= \epsilon_s^n + \left|\{c \in \mathfrak{v}_s, \Delta_c^n = 1\}\right| \\ \sum_{p\in s} \operatorname{Tan}\left(\mathcal{F}^E, s, p\right) &= \epsilon_s^n - 2 + \left|\{c \in \mathfrak{v}_s, \Delta_c^n = 1\}\right|. \end{aligned} \tag{3.6}$$

Moreover, by construction for any $s \in \mathbb{A}$, we find

$$\Delta_s^n = \Delta_s^{\mathcal{F}}, \ \delta_s^n = \delta_s^{\mathcal{F}}.$$

Thus, since the configuration ϵ^n satisfies the system (9) of Theorem 3, the valuation of $\mathcal{F}$ can be expressed as follows

$$\nu(\mathcal{F}) = \sum_{s\in\mathbb{A}} \rho_s \epsilon_s^n = \frac{\nu_1^n}{2} - \left\lfloor \begin{matrix} 1-\Delta_1^n \\ \frac{1}{2} \\ \nu_1^n \end{matrix} \right. = \frac{\nu(C)}{2} - \left\lfloor \begin{matrix} 1-\Delta_1^{\mathcal{F}} \\ \frac{1}{2} \\ \nu(C) \end{matrix} \right. .$$

Doing the same remark along the whole process of blowing-ups of C, we obtain, for any $s \in \mathbb{A}$,

$$\nu\left(\mathcal{F}_s^E\right) = \frac{\nu\left(C_s^E\right) + \delta_s^{\mathcal{F}}}{2} - \left\lfloor \begin{matrix} 1-\Delta_s^{\mathcal{F}} \\ \frac{1}{2} \\ \nu(C_s^E)+\delta_s^{\mathcal{F}} \end{matrix} \right. \tag{3.7}$$

□

If a foliation $\mathcal{F}$ leaves invariant C and satisfies the relations (3.7), then the proof above highlights that the dicriticity $\Delta^{\mathcal{F}}$ together with the configuration ϵ defined by the relations (3.6) provide an admissible configuration for the numbered tree associated to C. Since, Theorem 3 ensures the unicity of this dicriticity and thus of its configuration, the last statement of the introduction is proved.

3.3. **Deformation of $\mathcal{F}$.** In the previous section, we obtained a foliation $\mathcal{F}$ leaving invariant a curve C whose valuations satisfy the relations described in Proposition 7. However, we cannot still claim that a vector field X defining $\mathcal{F}$ is neither optimal nor Saito, since the curve C could be *special* in its equisingularity and could admit a tangent vector field with small valuations. In order to overcome this difficulty, we are going to prove that $\mathcal{F}$ can be put in a *weakly equisingular* deformation that follows a deformation of toward generic elements of the equisingularity class of C, for which lower bound for Saito numbers is known. To implement this strategy, we will gather material from [6, 8, 5, 11].

Theorem 8. *There exists C' equisingular to C such that C' admits a Saito vector field X further satisfying for any $s \in \mathbb{A}$*

$$\nu\left(X_s^E\right) = \frac{\nu\left(C_s^E\right) + \delta_s^X}{2} - \left\lfloor \begin{matrix} 1 - \Delta_s^X \\ \frac{1}{2} \end{matrix} \right._{\nu(C_s^E)+\delta_s^X}$$

Proof. Let E be the desingularization process of C. Denote by Ω the volume form

$$\Omega = E^\star\left(\mathrm{d}x \wedge \mathrm{d}y\right).$$

Let $\mathfrak{X}$ be the global vector field $\mathfrak{X} = E^\star\left(\frac{X}{f}\right)$ where X is a vector field defining $\mathcal{F}$ and f is *a balanced equation of the separatricies of* X, as introduced in [5, Definition 1.2]. Following [6, Proposition 18], we associate to $\mathfrak{X}$ the following divisor

$$D_{\mathfrak{X}} = 2\left((f=0)^E - (f=\infty)^E\right) - C^E + \overline{D} \tag{3.8}$$

defined in the total space of E. Here, $\overline{D}$ is the union of components of D invariant by $\mathfrak{X}$. Let us consider $\mathbb{F}$ the sheaf based upon D of $\mathcal{O}-$modules of vector fields tangent to the foliation given by $\mathfrak{X}$ and Θ the sheaf based on D of vector fields tangent to $E^{-1}(C)$. In [11, Theorem 1.6], Gomez-Mont exhibits the existence of an exact sequence in cohomology written

$$\mathbb{H}^1(D,\mathbb{F}) \to H^1(D,\Theta) \to H^1\left(D, \operatorname{Hom}\left(\mathbb{F}, \frac{\Theta}{\mathbb{F}}\right)\right). \tag{3.9}$$

The space $\mathbb{H}^1(D,\mathbb{F})$ is identified with the space of infinitemisal deformations of $\mathcal{F}$; the space $H^1(D,\Theta)$ is identified with the space of infinitesimal deformations of C. Now, the sheaf $\mathbb{F}$ is locally free of rank 1. Thus, a section α of $\operatorname{Hom}\left(\mathbb{F}, \frac{\Theta}{\mathbb{F}}\right)$ is completely determined by the image of $E^\star X$ or, equivalently by the image of $\mathfrak{X}$. By contruction, $\mathcal{F}$ is of *second kind* as defined in [5]. The relations established in [5, Lemme 2.1] are written in our context

$$\nu_s\left(i_{E^\star X}\Omega\right) = \nu_s\left(E^\star f\right) + \begin{cases} 1 & \text{if } s \text{ is invariant by } E^\star X \\ 0 & \text{if not} \end{cases}$$

where i stands for the inner product. It can be seen that, as a consequence, the morphism of sheaves defined by

$$\operatorname{Hom}\left(\mathbb{F}, \frac{\Theta}{\mathbb{F}}\right) \to \Omega^2\left(D_{\mathfrak{X}}\right), \ \alpha \mapsto i_{\alpha(\mathfrak{X})}\Omega \wedge i_{\mathfrak{X}}\Omega$$

is an isomorphism of sheaves : here, $\Omega^2(D_{\mathfrak{X}})$ is the sheaf over D of $2-$forms η for which the divisor $(\eta) = (\eta = 0) - (\eta = \infty)$ satisfies

$$(\eta) \geq -D_{\mathfrak{X}}.$$

Moreover, in [6, Proposition 18], it is proved that, provided that the relations (3.7) are satisfied, we have

$$\mathrm{H}^1\left(D, \Omega^2(D_{\mathfrak{X}})\right) = 0.$$

Thus, the exact sequence (3.9) reduces to

$$\mathbb{H}^1(D, \mathcal{F}) \to \mathrm{H}^1(D, \Theta) \to 0. \tag{3.10}$$

Now, let $(C_t)_{t\in(\mathbb{C}^N,0)}$ be a versal deformation of C. In $\left(\mathbb{C}^N, 0\right)$ the generic component in the sense of [8, Theorem 2.8] is the complement of an analytical subset Σ. Therefore, we can set a direction t_i such that $\left.\frac{\partial C_t}{\partial t_i}\right|_{t=0} \in \mathrm{H}^1(D, \Theta)$ is transverse to Σ. According to 3.10 and [11, Theorem 3.3] there exists a deformation $(\mathcal{F}_t)_{t\in(\mathbb{C},0)}$ of $\mathcal{F}$ such the image of $\left.\frac{\partial \mathcal{F}_t}{\partial t}\right|_{t=0} \in \mathbb{H}^1(D, \mathcal{F})$ in 3.10 is $\left.\frac{\partial C_t}{\partial t_i}\right|_{t=0}$. The deformation $(\mathcal{F}_t)_{t\in(\mathbb{C},0)}$ being locally equisingular, it leaves invariant a curve C_t' equisingular to C that does not belong to Σ for $t \neq 0$. Moreover, the valuations are invariant, and we get

$$\forall t \in (\mathbb{C}, 0),\ \nu\left((\mathcal{F}_t)_s^E\right) = \nu\left(\mathcal{F}_s^E\right) = \frac{\nu\left(C_s^E\right) + \delta_s^{\mathcal{F}}}{2} - \left\lfloor 1 - \Delta_s^{\mathcal{F}} \right\rfloor_{\nu(C_s^E)+\delta_s^{\mathcal{F}}}^{\frac{1}{2}}.$$

Now, if c is generic in its equisingularity class then, according to [8, Theorem 4], for any X in $\mathrm{Der}(\log c)$, the following lower bound holds

$$\forall s \in \mathbb{A},\ \nu\left(X_s^E\right) \geq \frac{\nu\left(c_s^E\right) + \delta_s^X}{2} - \left\lfloor 1 - \Delta_s^X \right\rfloor_{\nu(c_s^E)+\delta_s^X}^{\frac{1}{2}}.$$

Thus, for any $t \neq 0$, the foliation $(\mathcal{F}_t)_{t\in\mathbb{C}}$ - or a vector field X_t defining $\mathcal{F}_t$ - leaves invariant a curve C_t' equisingular to C, is optimal for C_t' and keeps on being optimal along the desingularization process of C_t', that means precisely, is Saito for C_t'. $\square$

3.4. **Number of moduli of C.** According to [22, Theorem 4.2], the number of moduli $\mathbb{M}^C$ of C is equal to $\dim \mathrm{H}^1(D, \Theta)$ when C is chosen generic in its equisingularity class. The results of this section and these of [8] ensure that this dimension can be computed from the topological datas associated to a Saito foliation. In [7], a precise description of an algorithm is given to compute this topological datas when C is an union of regular curve. This article implies that the exact same algorithm, presented briefly here, still provides this topological datas in the general case, and as a product the number of moduli of the initial curve.

We implemented, among other procedures this algorithm on Sage 9.* - or Python 3 -. See the routine *Courbes.Planes* following the link

https://perso.math.univ-toulouse.fr/genzmer/

Algorithm 1 Algorithm to compute the number of moduli of C.

INPUT : C a curve

Compute the numbered tree $(\mathbb{A}, n)$ of C.
Compute the Saito dicriticity of $(\mathbb{A}, n)$.
Using the associated configuration, compute $\sigma(C) = \dim \mathrm{H}^1\left(D_1, \Theta|_{D_1}\right)$.
For C_k connected component of $\overline{E_1^{-1}(C \setminus \{0\})}$, compute inductively $\mathbb{M}^{C_k \cup D_1}$.

RETURN : $\mathbb{M}^C = \sigma(C) + \sum_k \mathbb{M}^{C_k \cup D_1}$.

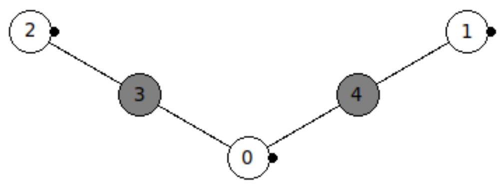


FIGURE 4.1. Saito dicriticities of the double cusp.

4. EXAMPLES

Example 9 (*The Saito foliation of the double cusp*)**.** The double cusp is the curve C defined by

$$\left(y^2 + x^3\right)\left(y^2 - x^3\right) = 0.$$

It is a curve with no moduli and its Saito number is 2. Its desingularization E consists in five elementary blowing-ups

$$E = \bigcirc_{i=0}^{4} E_i.$$

The Saito dicriticity of C is given in Figure 4.1. The number on each vertex allows us to identify the order on the tree defined by

$$0 \leq 1,\ 0 \leq 2,\ 1 \leq 4,\ 2 \leq 3.$$

The dots in Figure 4.1 encode the configuration. Here, the configuration associated to the Saito dicriticity is

$$\epsilon_0 = \epsilon_1 = \epsilon_2 = 1,\ \epsilon_3 = \epsilon_4 = 0.$$

It can be seen, by computing its desingularization, that the vector field X defined by

$$\begin{aligned} X = & \left(\frac{9}{5}x^3y - x^2y^2 + y^3 - \frac{4}{5}x^2 + xy\right)\partial_x \\ & + \left(\frac{6}{5}x^2y^2 - \frac{3}{2}xy^3 - \frac{5}{6}x^3 - \frac{6}{5}xy + \frac{2}{3}y^2\right)\partial_y \end{aligned}$$

is Saito for the double cusp. Indeed, it is tangent to C and non dicritical. Its valuation satisfies

$$2 = \frac{\nu(C) = 4}{2} - \left\lfloor \begin{array}{c} 1-1 \\ \frac{1}{2} \\ 4 \end{array} \right. .$$

After one blowing-up, it has three singularities along the exceptional divisor given in the coordinates $(y = y_1, x = y_1x_1)$ by

$$s_1 = (0,0)\,,\ s_3 = \left(0, -\frac{6}{5}\right) \text{ and } s_2 = (0,\infty)\,.$$

The singularity s_3 is reduced : the quotient of the eigenvalues of $E_0^\star X$ at s_2 is actually equal to 5. At s_1 and s_3, $E_0^\star X$ is of valuation 1 which satisfies

$$1 = \frac{\nu\left(C_{s_1}^E\right)+1}{2} - \Big\lfloor \begin{matrix} 1-1 \\ \frac{1}{2} \\ 2 \end{matrix} = \frac{\nu\left(C_{s_3}^E\right)+1}{2} - \Big\lfloor \begin{matrix} 1-1 \\ \frac{1}{2} \\ 2 \end{matrix}\ .$$

After blowing-up s_1, the vector field $(E_0 \circ E_1)^\star X$ has two singularities along the new exceptional divisor. One is reduced with positive and rational quotient of the eigenvalues. The other is radial, that is, its linear part is locally in coordinates written $x\partial_x + y\partial_y$. The same occurs at s_3. At the radial singularities s_5 and s_6 which are dicritical, one has

$$1 = \frac{\nu\left(C_{s_5 \text{ or } s_6}^E\right)+2}{2} - \Big\lfloor \begin{matrix} 1-0 \\ \frac{1}{2} \\ 3 \end{matrix}\ .$$

As a consequence, X is indeed Saito for C.

Example 10 (*Number of moduli of the union of r cusps equisingular to $y^2+x^3=0$*)**.** In [17], the authors give a formula for the number of moduli of the curve

$$\mathcal{C}_r = \left\{\prod_{i=1}^{r}\left(y^2+a_ix^3\right)=0\right\}$$

where $a_i \neq a_j \neq 0$ for $i \neq j$. When r is even, this dimension happens to be equal to

$$\frac{(r-1)(3r-5)+1}{2}. \tag{4.1}$$

Let us illustrate how our algorithm works in this situation. The proximity matrix of the $\mathcal{C}_r$ is

$$\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

and the numbering of $\mathbb{A}$ is $(0,0,r)$. The Saito dicriticity is equal to $(1,1,0)$ and the associated configuration is $\left(2,1,\frac{r}{2}\right)$. After one blowing-up, according to [8, Proposition 4], we get

$$\dim H^1\left(D_1,\Theta\right) = \frac{(r-1)(r-2)}{2} + \frac{(r-1)(r-2)}{2} = (r-1)(r-2)\,.$$

Now, after one blowing-up the curve $D_1 \cup \mathcal{C}_r^{E_1}$ is given in local coordinates $y = y_1x_1$ by

$$x_1\prod_{i=1}^{r}\left(y_1^2+a_ix_1\right)=0.$$

The proximity matrix of the desingularization of the latter curve is now

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

and the numbering $(0, r+1)$. The Saito dicriticity is equal to $(1,0)$ and the associated configuration is $\left(1, \frac{r}{2}\right)$. Thus, we obtain

$$\dim H^1\left(D_2, \Theta\right) = \frac{\left(\frac{r}{2}-1\right)\left(\frac{r}{2}-2\right)}{2} + \frac{\frac{r}{2}\left(\frac{r}{2}-1\right)}{2}.$$

Finally, after one more blowing-up the curve $D_1 \cup D_2 \cup \mathcal{C}_r^{E_2}$ is given by

$$x_2 y_2 \prod_{i=1}^{r}\left(y_2 + a_i x_2\right).$$

The proximity matrix reduces to (1) and the numbering to $(r+2)$. The Saito dicriticity is just (0) and the configuration $\left(\frac{r}{2}+1\right)$. Thus, still following [8, Proposition 4], one has

$$\dim H^1\left(D_3, \Theta\right) = \frac{\left(\frac{r}{2}-1\right)\left(\frac{r}{2}-2\right)}{2} + \frac{\frac{r}{2}\left(\frac{r}{2}-1\right)}{2} + r - 1.$$

Adding the above dimensions leads to

$$\begin{aligned}\dim H^1\left(D, \Theta\right) &= (r-1)(r-2) + \frac{\left(\frac{r}{2}-1\right)\left(\frac{r}{2}-2\right)}{2} + \frac{\frac{r}{2}\left(\frac{r}{2}-1\right)}{2} \\ &\quad + \frac{\left(\frac{r}{2}-1\right)\left(\frac{r}{2}-2\right)}{2} + \frac{\frac{r}{2}\left(\frac{r}{2}-1\right)}{2} + r - 1 \\ &= \frac{(r-1)(3r-5)+1}{2}.\end{aligned}$$

Example 11 (*Number of moduli of a union of r cusps equisingular to $y^n + x^{n+1} = 0$*)**.** Consider the curve $C_{r,n}$ defined by

$$\left(y^n + a_1 x^{n+1}\right)\left(y^n + a_2 x^{n+1}\right) \cdots \left(y^n + a_r x^{n+1}\right) = 0,\ a_i \neq a_j \neq 0.$$

The Saito vector field of $C_{r,n}$ is non dicritical of valuation $\left[\frac{rn}{2}\right]$. Therefore, we get

$$\dim H^1\left(D_1, \Theta\right) = \begin{cases} \frac{(rn-2)(rn-4)}{4} & \text{if } n \text{ or } r \text{ is even} \\ \frac{(rn-3)^2}{4} & \text{else} \end{cases}.$$

After the first blowing-up, the curve $C_{r,n}^{E_1} \cup D_1$ is a union of $r+1$ regular curves tangent at order n. Its Saito vector field is non dicritical of valuation $\left[\frac{r+1}{2}\right]$ and thus

$$\dim H^1\left(D_2, \Theta\right) = \begin{cases} \frac{(r-1)(r-3)}{4} & \text{if } r \text{ is odd} \\ \frac{(r-2)^2}{4} & \text{else} \end{cases}.$$

The next $n-2$ blowing-ups produce curves which, at each step, are a union of $r+1$ regular curves tangent as a whole and a transverse curve. Its Saito vector field is non dicritical of valuation $\left[\frac{r}{2}\right]+1$. Therefore, we find

$$\dim H^1\left(D_i, \Theta\right) = \begin{cases} \frac{r(r-2)}{4} & \text{if } r \text{ is even} \\ \frac{(r-1)^2}{4} & \text{else} \end{cases},\ i = 2, \ldots, n-1.$$

Finally, the n^{th} blowing-up yields a curve union of $r+2$ transverse curves. Its Saito vector field is dicritical and

$$\dim H^1\left(D_{n-1}, \Theta\right) = \begin{cases} \frac{r^2}{4} & \text{if } r \text{ is even} \\ \frac{r^2-1}{4} & \text{else} \end{cases}.$$

Finally, adding the contributions above, we find

$$\mathbb{M}^{C_{r,n}}=\begin{cases}\frac{n^2r^2+nr^2-8rn}{4}+3 & \text{if } r \text{ is even}\\ \frac{n^2r^2+nr^2-8rn+n}{4}+2 & \text{if } r \text{ is odd and } n \text{ is even}\\ \frac{n^2r^2+nr^2-8rn+n+9}{4} & \text{if } r \text{ is odd and } n \text{ is odd.}\end{cases}$$

Example 12 (*Generic Tjurina number of a curve*)**.** The algorithm defined above allows us to provide immediately a computation of the generic Tjurina number, that is, the dimension of the quotient of $\mathbb{C}\{x,y\}$ by the Tjurina ideal of C, i.e $(f,\partial_x f,\partial_y f)$ where f is an equation of C. Let E be the desingularization process of C. On the exceptional divisor D of E, we consider the sheaves T_{df} and Θ of vector fields tangent respectively to the foliation $E^\star df$ and $E^{-1}\left(f^{-1}(0)\right)$. The following sequence of sheaves

$$0\to T_{df}\to\Theta\xrightarrow{E^\star df(\cdot)}(f\circ E)\,\mathcal{O}_D\to 0$$

is exact [23]. The associated long exact sequence in cohomology is written

$$\begin{aligned}0\to H^0(D,T_{df})\to H^0(D,\Theta)\to H^0(D,(f\circ E)\,\mathcal{O}_D)\\ \to H^1(D,T_{df})\to H^1(D,\Theta)\to 0\end{aligned}$$

since $H^1(D,(f\circ E)\,\mathcal{O}_D)=0$. Now, we can identify the global sections of the above sheaves :

$$\begin{aligned}H^0(D,(f\circ E)\,\mathcal{O}_D)&=(f)\\ H^0(D,\Theta)&=\{X\text{ vector field}|\,X\cdot f\in(f)\}\end{aligned}$$

Therefore, the following sequence is exact

$$0\to\frac{(f)}{\{X\cdot f|\,X\text{ tangent to }f=0\}}\to H^1(D,T_{df})\to H^1(D,\Theta)\to 0.$$

Now, it can be seen that

$$\frac{(f)}{\{X\cdot f|\,X\text{ tangent to }f=0\}}\simeq\frac{(f,\mathrm{Jac}f)}{\mathrm{Jac}f}.$$

The previous short exact sequence ensures that

$$\dim_{\mathbb{C}}H^1(D,\Theta)-\dim_{\mathbb{C}}H^1(D,T_{df})+\dim_{\mathbb{C}}\frac{(f,\mathrm{Jac}f)}{\mathrm{Jac}f}=0$$

which can be also written

$$\tau(C)=\mu(C)-\delta(C)+\dim_{\mathbb{C}}H^1(D,\Theta)$$

where $\mu(C)$ is the Milnor number of C and $\delta(C)$ its modularity [21]. Now, if C is chosen generic, we obtain

$$\tau_{\text{gen}}(C)=\mu(C)-\delta(C)+\mathbb{M}^C. \tag{4.2}$$

Since the Milnor number and the modularity can be computed from the numbered tree of C, the formula above yields an agorithm to compute the generic Tjurina number of C - which happens to be also the minimal Tjurina number.

As an example, the curve given by the following parametrization $C=\left(t^9,t^{12}+t^{17}\right)$ has been studied by Peraire [24] and she found

$$\tau_{\text{gen}}(C)=80.$$

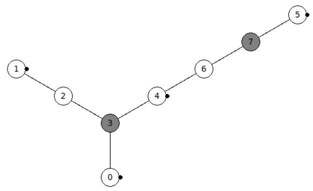
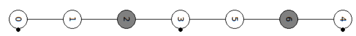
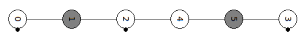
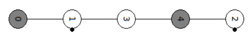
Saito vector field of the successive blown-up curves	$\nu(C)$	$\mathfrak{s}_C$	Contribution to the moduli
	9	4	9
	4	2	0
	5	2	1
	5	2	1

TABLE 4. Algorithm for Peraire's example.

It can be seen that

$$\mu(C) = 98 \text{ and } \delta(C) = 29.$$

Table 4 presents the four first steps of the inductive algorithm : beyond, no new contribution in the number of moduli appears. Thus, it provides the number moduli of C and we find

$$\mathbb{M}^C = 9 + 0 + 1 + 1 = 11,$$

which confirms the result of Peraire in view of (4.2).

Declarations. Conflict of interest The author states that there is no conflict of interest.

Data availability Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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