LENGTH DERIVATIVE OF THE GENERATING SERIES OF WALKS CONFINED IN THE QUARTER PLANE

THOMAS DREYFUS AND CHARLOTTE HARDOUIN

Abstract. In the present paper, we use difference Galois theory to study the nature of the generating series counting walks in the quarter plane. These series are trivariate formal power series $Q(x, y, t)$ that count the number of walks confined in the first quadrant of the plane with a fixed set of admissible steps, called the model of the walk. While the variables x and y are associated to the ending point of the path, the variable t encodes its length. In this paper, we prove that if $Q(x, y, t)$ does not satisfy any algebraic differential relations with respect to x or y, it does not satisfy any algebraic differential relations with respect to the parameter t. Combined with [BBMR16, DHRS18, DHRS17], we are able to characterize the t-differential transcendence of the generating series for any unweighted model of walk.

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Introduction

Classifying lattice walks in restricted domains is an important problem in enumerative combinatorics. Recently much progress has been made in the study of walks with small steps in the quarter plane. A small steps model in the plane is composed of a set of admissible cardinal directions. For a given model, one defines $q_{i,j,k}$ to be the number of walks confined to the first quadrant of the plane that begin at $(0, 0)$ and end at (i, j) in k admissible steps. The algebraic nature of the associated complete generating series

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$Q(x,y,t) = \sum_{i,j,k=0}^{\infty} q_{i,j,k} x^i y^j t^k$ captures many important combinatorial properties of the model: symmetries, asymptotic information, and recursive relations of the coefficients.

Among the 256 models in the first quadrant of the plane, Bousquet-Mélou and Mishna proved in [BMM10] that, after accounting for symmetries and eliminating the trivial and one dimensional cases, only 79 cases remained. It is worth mentioning that the generating series is algebraic in all the trivial cases. Figure 1 classifies the models into four groups depending on the algebraic nature of the series.

- *Algebraic cases:* the series $Q(x,y,t)$ satisfies a non-trivial polynomial relation with coefficients in $\mathbb{Q}(x,y,t)$.
- *Holonomic cases:* the series $Q(x,y,t)$ is transcendental and holonomic, i.e. it satisfies a non-trivial linear differential equation with coefficients in $\mathbb{Q}(x,y,t)$ with respect to each of the three derivations ([BMM10, BvHK10, FR10]).
- *Differentially algebraic cases:* the series $Q(x,y,t)$ is non-holonomic and differentially algebraic, i.e. it satisfies a non-trivial polynomial differential equation with coefficients in $\mathbb{Q}$ with respect to each of the three derivations ([KR12, MM14, BBMR16]).
- *Differentially transcendental cases:* the series is not differentially algebraic with respect to the derivation $\frac{d}{dx}$, nor with respect to the derivation $\frac{d}{dy}$ ([DHRS18, DHRS17]).

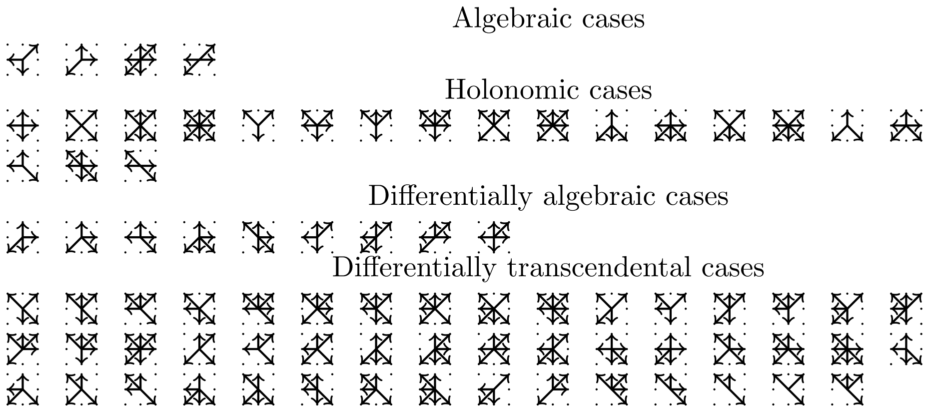


FIGURE 1. Classification of the 79 models. The algebraic and holonomic cases correspond to walks with a finite group.

These classification results come from many approaches: probabilistic methods, combinatorial classification, computer algebra and “ Guess and Prove ”, analysis and boundary value problems, and more recently difference Galois theory and algebraic geometry. The analytic approach consists in studying the asymptotic growth of the generating series, or else showing that it has an infinite number of singularities, in order to prove its non-holonomicity. Thus, this approach also allows for the study of some important specializations of the complete generating series as for instance $Q(1,1,t)$ the generating series for the number of nearest neighbor walks in the quarter plane (see [MM14, MR09]). Though very powerful, these analytic techniques are unable to detect the differentially algebraic generating functions among the non-holonomic ones. For instance, the generating function $\prod_{k=1}^{\infty} \frac{1}{(1-x^k)}$ counting the number of partitions has an infinite number of singularities, and yet is differentially algebraic. In order to detect these more subtle kinds of functional dependance it is necessary, to use new arguments that focus on the functional equation satisfied by the complete generating series. Indeed, the combinatorial decomposition of a walk into a shorter walk followed by an admissible step translates into a functional equation for the generating series. Following the ideas of Fayolle, Iasnogorodski and Malyshev [FIM99], one specializes this functional equation to the so-called *kernel curve* to find a linear discrete equation. Difference Galois theory allows then to characterize the differentially transcendental complete generating series ([DHRS18]) whereas the clever use of Tutte invariants produces explicit differential algebraic relations for the 9 non-holomic but differentially algebraic cases ([BBMR16]).

The aim of this paper is to complete the picture by solving the problem of the algebraic relations satisfied by the complete generating series and its derivatives with respect to the length variable t. Until now only partial results were known. Any complete generating function that is differentially algebraic with respect to x and y was known to be also differentially algebraic with respect to t. This comes for instance from an explicit description of the series via elliptic functions (see [BBMR16] for the 9 differentially algebraic cases). We prove that the converse holds. More precisely,

Theorem 1. *For any unweighted model, the following facts are equivalent:*

(1) *the complete generating series is* $\frac{d}{dx}$*-differentially algebraic over* $\mathbb{Q}$*;*
(2) *the complete generating series is* $\frac{d}{dy}$*-differentially algebraic over* $\mathbb{Q}$*;*
(3) *the complete generating series is* $\frac{d}{dt}$*-differentially algebraic over* $\mathbb{Q}$*.*

Thus the classification in Figure 1 remains valid after adding the t-derivation. In the holonomic cases, Bousquet-Mélou and Mishna showed that differential behavior was the same with respect to the three variables x, y and t. The same property was expected to be true for the differentially algebraic cases but was far from being obvious to prove. Indeed, in general, there is a priori no relation between the $\frac{d}{dx}$ and $\frac{d}{dt}$ differential algebraic properties of a function in these two variables. For instance, the function $t\Gamma(x)$ is holonomic with respect to t but not differentially algebraic with respect to x, thanks to Hölder's result. It happens that, for the walks in the first quadrant, the differential algebraic behavior is governed by the geometry of the kernel curve. This curve is the generic fiber of a rational or elliptic fibration over the projective line in t. Then, the connection between the t and x derivatives of the complete generating series is related to the fact that this fibration is not trivial, i.e. not a direct product*. Unfortunately, Theorem 1 gives only a partial answer for specializations of the generating series such as for instance the series $Q(1,1,t)$ counting nearest neighbor walks in the quarter plane. If the complete generating series is differential algebraic, the Tutte invariants of [BBMR16] produce some explicit algebraic differential equations that one can try to specialize. However, when the complete series is differentially transcendental, it seems very difficult to know whether its specialization $Q(1,1,t)$ is differentially transcendental or not. Methods based on the kernel curve will fail because most of the time the point $(1,1)$ is not on the

*The function $t\Gamma(x)$ satisfies the finite difference equation $F(x+1,t) = xF(x,t)$ over the trivial fibration $\mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1, (x,t) \mapsto t$

kernel curve and analytic methods will only prove non-holomy. It seems that, the study of the differential transcendence of this specialization will require drastically new ideas.

Our work relies on a non-archimedean uniformization of the Kernel curve. The advantage of this framework is that it unifies the so called singular and non-singular models but also allows us to address the question of the t-derivation. Indeed, in earlier works, former uniformizations of the kernel curve were constructed for a fixed value of t and resulted in the study of a $\mathbf{q}$-difference equation for singular models ([DHRS17]) and finite difference elliptic equations for non-singular ones ([KR12]). We use here the formalism of Tate curves over $\mathbb{Q}(t)$ as in [Roq70] to show that for any singular or non-singular model, the differential algebraic properties of the complete generating series are encoded by the differential algebraic properties of a solution of a rank one non-homogeneous linear $\mathbf{q}$-difference equation. Then, we generalize some Galoisian criteria for $\mathbf{q}$-difference equations of [HS08] to prove Theorem 1. Our result holds in the more general context of weighted models.That is, given any (unweighted) model, one can add weights associated to each admissible step and ask what happens to the algebraic nature of the series. In the weighted situation, we are able to prove only one direction of the equivalence.

Theorem 2. *For any non-degenerate*[†] *walk with infinite group of the walk, if the complete generating series series is* $\frac{d}{dt}$*-differentially algebraic over* $\mathbb{Q}$ *then it is* $\frac{d}{dx}$*-differentially algebraic over* $\mathbb{Q}$*. A similar result holds for the derivation* $\frac{d}{dy}$*.*

The paper is organized as follows. In Section 1 we present some reminders and notations for walks in the quarter plane. In Section 2 we consider walks with genus zero kernel curve, while Section 3 deals with the genus one case. Since this paper combines many different fields, non-archemedian uniformization, combinatorics, and Galois theory, we choose to postpone many technical intermediate results to the appendices. This should allow the reader to understand the articulation of our proof of Sections 2 and 3 in three steps without being lost in too many technicalities. These three steps are the uniformization of the kernel and the construction of a linear $\mathbf{q}$-difference equation, the Galoisian criteria, and finally, the resolution of telescoping problems. Appendix A is devoted to the non-archimedean estimates that we used in the uniformization procedure. Appendix B contains some reminders on special functions on Tate curves and their normal forms. Appendix C proves the Galoisian criteria mentioned above. Finally, Appendix D studies the transcendence properties of special functions on Tate curves that will be used for the descent of our telescoping equations.

1. The walks in the quadrant

The goal of this section is to introduce some basic properties of walks in the quarter plane. In §1.1, we introduce the generating series $Q(x, y, t)$ of a walk confined in the quarter plane. In §1.2, we attach to any walk a *kernel curve*, which is an algebraic curve defined over $\mathbb{Q}[t]$. This curve has been intensively studied as an algebraic curve over $\mathbb{C}$ by fixing a morphism from $\mathbb{Q}[t]$ to $\mathbb{C}$. For instance, [FIM99] is concerned with $t = 1$ whereas the papers [DHRS18] and [DR19] focus respectively on $t \in \mathbb{C}$ transcendental over $\mathbb{Q}$ and $t \in]0, 1[$. Unfortunately, specializing t even generically does not allow to

[†]See Definition 1.4

study the t-dependencies of the generating series. In this paper, we do not work with a specialization of t. This forces us to move away from the archimedean framework of the field of complex numbers and to consider the kernel curve over a suitable valued field extension of $\mathbb{Q}(t)$ endowed with the valuation at 0.

1.1. **The walks.** The cardinal directions of the plane $\{\leftarrow,\nwarrow,\uparrow,\nearrow,\rightarrow,\searrow,\downarrow,\swarrow\}$ are identified with pairs of integers $(i,j)\in\{0,\pm 1\}^2\backslash\{(0,0)\}$. A walk $\mathcal{W}$ in the quarter plane $\mathbb{Z}^2_{\geq 0}$ is a sequence of points $(M_n)_{n\in\mathbb{Z}_{\geq 0}}$ such that

- it starts at $(0,0)$, that is, $M_0=(0,0)$;
- for all $n\in\mathbb{Z}_{\geq 0}$, the point M_n belong to the quadrant $\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}$;
- for all $n\in\mathbb{Z}_{\geq 0}$, the vector $M_{n+1}-M_n$ belongs to a given subset $\mathcal{D}$ of the set of cardinal directions.

The set $\mathcal{D}$ is called *the set of steps* or *the model* of the walk. Fixing a family of elements $(d_{i,j})_{(i,j)\in\{0,\pm1\}^2}$ of $\mathbb{Q}\cap[0,1]$ such that $\sum_{i,j} d_{i,j}=1$ and $d_{i,j}\neq 0$ inf and only if $(i,j)\in\mathcal{D}$, one can choose to weight the model of the walk in order to add a probabilistic flavor to our study. In that case, the $d_{i,j}$ are called the weights and the model is called a weighted model. Note that the set of steps of the walk $\mathcal{W}$ is the set of cardinal directions with non-zero weight, that is,

$$\mathcal{D}=\{(i,j)\in\{0,\pm1\}^2\backslash\{(0,0)\}|d_{i,j}\neq 0\}.$$

A model is *unweighted* if $d_{0,0}=0$ and if the non-zero $d_{i,j}$'s all have the same value.

For any $(i,j)\in\mathbb{Z}^2_{\geq 0}$ and any $k\in\mathbb{Z}_{\geq 0}$, we let $q_{i,j,k}\in[0,1]$ be the probability for the walk confined in the quadrant $\mathbb{Z}^2_{\geq 0}$ to reach the position (i,j) from the initial position $(0,0)$ after k steps. We introduce the corresponding trivariate generating series

$$Q(x,y,t):=\sum_{i,j,k\geq 0} q_{i,j,k}x^i y^j t^k.$$

Note that the generating series is not exactly the same as the one that we defined in the introduction. To recover the latter, we should take $d_{i,j}\in\{0,1\}$ and $d_{i,j}=1$ if and only if the corresponding direction belongs to $\mathcal{D}$. Fortunately, the assumption $\sum_{i,j} d_{i,j}=1$ can be relaxed by rescaling the t-variable, and the results of the present paper stay valid for the generating series of the introduction.

Remark 1.1. For simplicity, we assume that the weights $d_{i,j}\in\mathbb{Q}$ and that $t\in\mathbb{R}$ is transcendental over $\mathbb{Q}$. However, we would like to mention that any of the arguments and statements below will hold with arbitrary real weights in $[0,1]$. One just needs to replace the field $\mathbb{Q}$ with the field $\mathbb{Q}(d_{i,j})$.

The *kernel* polynomial of a weighted model $(d_{i,j})_{i,j\in\{0,\pm1\}^2}$ is defined by

$$K(x,y,t):=xy(1-tS(x,y)) \tag{1.1}$$

where

$$\begin{aligned} S(x,y) &= \textstyle\sum_{(i,j)\in\{0,\pm1\}^2} d_{i,j}x^i y^j \\ &= A_{-1}(x)\frac{1}{y}+A_0(x)+A_1(x)y \\ &= B_{-1}(y)\frac{1}{x}+B_0(y)+B_1(y)x, \end{aligned} \tag{1.2}$$

and $A_i(x)\in x^{-1}\mathbb{Q}[x]$, $B_i(y)\in y^{-1}\mathbb{Q}[y]$.

By [DHRS17, Lemma 1.1], see also [BMM10, Lemma 4], the generating series $Q(x,y,t)$ satisfies the following functional equation:

$$K(x,y,t)Q(x,y,t) = xy + F^1(x,t) + F^2(y,t) + td_{-1,-1}Q(0,0,t), \tag{1.3}$$

where

$$F^1(x,t) := K(x,0,t)Q(x,0,t), \quad \text{and} \quad F^2(y,t) := K(0,y,t)Q(0,y,t).$$

Remark 1.2. We shall often use the following symmetry argument between x and y. Exchanging x and y in the kernel polynomial amounts to consider the kernel polynomial of a weighted model $\mathcal{D}' := \{(i,j) \text{ such that } (j,i) \in \mathcal{D}\}$ with weights $d'_{i,j} := d_{j,i}$.

We conclude with a remark concerning the field of definition of any polynomial relations between the derivatives of the series.

Remark 1.3. Let K be a field generated over $\mathbb{Q}$ by elements that are $(\frac{d}{dx}, \frac{d}{dt})$ (resp. $(\frac{d}{dy}, \frac{d}{dt})$) differentially algebraic over $\mathbb{Q}$. By [Kol73, Proposition 8, Page 101], the series $Q(x,y,t)$ is $(\frac{d}{dx}, \frac{d}{dt})$ (resp. $(\frac{d}{dy}, \frac{d}{dt})$)-differentially transcendental over $\mathbb{Q}$ if and only if it is $(\frac{d}{dx}, \frac{d}{dt})$ (resp. $(\frac{d}{dy}, \frac{d}{dt})$)-differentially transcendental over K. Similar statements hold for $Q(x,0,t)$ and $Q(0,y,t)$.

1.2. **The kernel curve.** The *kernel* polynomial has coefficients in the valued field $\mathbb{Q}(t)$ endowed with the valuation at zero. The latter field is neither algebraically closed nor complete. In order to use the theory of Tate curves, one needs to consider a complete algebraically closed field extension of $\mathbb{Q}(t)$. We consider the field C of Hahn series or Mal'cev-Neumann series with coefficients in $\overline{\mathbb{Q}}$, an algebraic closure of $\mathbb{Q}$, and monomials from $\mathbb{Q}$. We recall that a Hahn series f is a formal power series $\sum_{\gamma\in\mathbb{Q}} c_\gamma t^\gamma$ with coefficients c_γ in $\overline{\mathbb{Q}}$ and such that the subset $\{\gamma | c_\gamma \neq 0\}$ is a well ordered subset of $\mathbb{Q}$. Its valuation $v_0(f)$ is the smallest element of the subset $\{\gamma | c_\gamma \neq 0\}$. The field C is algebraically closed and complete with respect to the valuation at zero, see [AvdDvdH17, Ex. 3.2.23 and p. 151]. One can endow C with a derivation ∂_t as follows

$$\partial_t \left(\sum_{\gamma\in\mathbb{Q}} c_\gamma t^\gamma \right) = \sum_{\gamma\in\mathbb{Q}} c_\gamma \gamma t^\gamma.$$

Then, ∂_t extends the derivation $t\frac{d}{dt}$ of $\mathbb{Q}(t)$, see [AvdDvdH17, Ex.(2), §4.4].

Let us fix once for all $\alpha \in \mathbb{R}$ such that $0 < \alpha < 1$. For any $f \in C$, we define the norm of f as $|f| = \alpha^{v_0(f)}$. Note for any Hahn series f such that $|f| < 1$, we have $|\partial_t(f)| < 1$. This is not true when ∂_t is replaced by $\frac{d}{dt}$.

We need to discard some degenerate cases. Following [FIM99], we have the following definition.

Definition 1.4. A weighted model is called *degenerate* if one of the following holds:

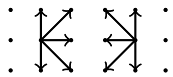
- $K(x,y,t)$ is reducible as an element of the polynomial ring $C[x,y]$,
- $K(x,y,t)$ has x-degree less than or equal to 1,
- $K(x,y,t)$ has y-degree less than or equal to 1.

Remark 1.5. In [DHRS17], the authors specialize the variable t as a transcendental complex number. Then, they study the kernel curve as a complex algebraic curve in $\mathbf{P}^1(\mathbb{C}) \times \mathbf{P}^1(\mathbb{C})$. In this work, we shall use any algebraic geometric result of [DHRS17] by appealing to Lefschetz Principle: every true statement about an algebraic variety defined over $\mathbb{C}$ remains true when $\mathbb{C}$ is replaced by an algebraically closed field of characteristic zero.

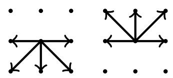
The following proposition gives very simple conditions on $\mathcal{D}$ to decide whether a weighted model is degenerate or not.

Proposition 1.6 (Lemma 2.3.2 in [FIM99])**.** *A weighted model is degenerate if and only if at least one of the following holds:*

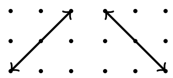
(1) *There exists $i \in \{-1, 1\}$ such that $d_{i,-1} = d_{i,0} = d_{i,1} = 0$. This corresponds to walks with steps supported in one of the following configurations*



(2) *There exists $j \in \{-1, 1\}$ such that $d_{-1,j} = d_{0,j} = d_{1,j} = 0$. This corresponds to walks with steps supported in one of the following configurations*



(3) *All the weights are zero except maybe $\{d_{1,1}, d_{0,0}, d_{-1,-1}\}$ or $\{d_{-1,1}, d_{0,0}, d_{1,-1}\}$. This corresponds to walks with steps supported in one of the following configurations*



Note that we only discard one dimensional problems as explained in [BMM10]. For all the degenerate cases, the generating series $Q(x, y, t)$ is algebraic.

From now on, we shall always assume that the weighted model under consideration is non-degenerate.

To any weighted model $\mathcal{D}$, we attach a curve E, called the *kernel curve*, that is defined as the zero set in $\mathbf{P}^1(C) \times \mathbf{P}^1(C)$ of the following homogeneous polynomial

$$\widetilde{K}(x_0, x_1, y_0, y_1, t) = x_0 x_1 y_0 y_1 - t \sum_{i,j=0}^{2} d_{i-1,j-1} x_0^i x_1^{2-i} y_0^j y_1^{2-j} = x_1^2 y_1^2 K\left(\frac{x_0}{x_1}, \frac{y_0}{y_1}, t\right).$$

Let us write $\widetilde{K}(x_0, x_1, y_0, y_1, t) = \sum_{i,j=0}^{2} A_{i,j} x_0^i x_1^{2-i} y_0^j y_1^{2-j}$ where $A_{i,j} = -t d_{i-1,j-1}$ if $(i, j) \neq (1, 1)$ and $A_{1,1} = 1 - t d_{0,0}$. The partial discriminants of $\widetilde{K}(x_0, x_1, y_0, y_1, t)$ are defined as the discriminants of the second degree homogeneous polynomials $y \mapsto \widetilde{K}(x_0, x_1, y, 1, t)$ and $x \mapsto \widetilde{K}(x, 1, y_0, y_1, t)$, respectively, i.e.

$$\Delta_x(x_0, x_1) = \left(\sum_{i=0}^{2} x_0^i x_1^{2-i} A_{i,1}\right)^2 - 4\left(\sum_{i=0}^{2} x_0^i x_1^{2-i} A_{i,0}\right) \times \left(\sum_{i=0}^{2} x_0^i x_1^{2-i} A_{i,2}\right)$$

and

$$\Delta_y(y_0,y_1)=\left(\sum_{j=0}^{2} y_0^j y_1^{2-j} A_{1,j}\right)^2 - 4\left(\sum_{j=0}^{2} y_0^j y_1^{2-j} A_{0,j}\right)\times\left(\sum_{j=0}^{2} y_0^j y_1^{2-j} A_{2,j}\right).$$

Introduce

$$\mathfrak{D}(x):=\Delta_x(x,1)=\sum_{j=0}^{4}\alpha_j x^j \quad \text{and} \quad \mathfrak{E}(y):=\Delta_y(y,1)=\sum_{j=0}^{4}\beta_j y^j, \tag{1.4}$$

where
(1.5)

$$\begin{array}{lcl}
\alpha_4 &=& \big(d_{1,0}^2-4d_{1,1}d_{1,-1}\big)t^2\\
\alpha_3 &=& 2t^2 d_{1,0}d_{0,0}-2td_{1,0}-4t^2(d_{0,1}d_{1,-1}+d_{1,1}d_{0,-1})\\
\alpha_2 &=& 1+t^2d_{0,0}^2+2t^2d_{-1,0}d_{1,0}-4t^2(d_{-1,1}d_{1,-1}+d_{0,1}d_{0,-1}+d_{1,1}d_{-1,-1})-2td_{0,0}\\
\alpha_1 &=& 2t^2d_{-1,0}d_{0,0}-2td_{-1,0}-4t^2(d_{-1,1}d_{0,-1}+d_{0,1}d_{-1,-1})\\
\alpha_0 &=& \big(d_{-1,0}^2-4d_{-1,1}d_{-1,-1}\big)t^2\\
\\
\beta_4 &=& \big(d_{0,1}^2-4d_{1,1}d_{-1,1}\big)t^2\\
\beta_3 &=& 2t^2d_{0,1}d_{0,0}-2td_{0,1}-4t^2(d_{1,0}d_{-1,1}+d_{1,1}d_{-1,0})\\
\beta_2 &=& 1+t^2d_{0,0}^2+2t^2d_{0,-1}d_{0,1}-4t^2(d_{1,-1}d_{-1,1}+d_{1,0}d_{-1,0}+d_{1,1}d_{-1,-1})-2td_{0,0}\\
\beta_1 &=& 2t^2d_{0,-1}d_{0,0}-2td_{0,-1}-4t^2(d_{1,-1}d_{-1,0}+d_{1,0}d_{-1,-1})\\
\beta_0 &=& \big(d_{0,-1}^2-4d_{1,-1}d_{-1,-1}\big)t^2.
\end{array}$$

The discriminants $\Delta_x(x_0,x_1),\Delta_y(y_0,y_1)$ are homogeneous polynomials of degree 4. The Eisenstein invariants are defined as follows (see [Dui10, §2.3.5]):

Definition 1.7. For any homogeneous polynomial of the form

$$f(x_0,x_1)=a_0x_1^4+4a_1x_0x_1^3+6a_2x_0^2x_1^2+4a_3x_0^3x_1+a_4x_0^4\in C[x_0,x_1],$$

we define the Eisenstein invariants of $f(x_0,x_1)$ as

- $D(f)=a_0a_4+3a_2^2-4a_1a_3$
- $E(f)=a_0a_3^2+a_1^2a_4-a_0a_2a_4-2a_1a_2a_3+a_2^3$
- $F(f)=27E(f)^2-D(f)^3$.

Since C is algebraically closed of characteristic zero, we can apply [Dui10, §2.4] to the kernel curve. The following proposition characterizes the smoothness of the kernel curve in terms of the invariants $F(\Delta_x)$, $F(\Delta_y)$.

Proposition 1.8 (Proposition 2.4.3 in [Dui10] and Lemma 4.4 in [DHRS17])**.** *The following statements are equivalent*

- *The kernel curve E is smooth, i.e. it has no singular point;*
- $F(\Delta_x)\neq 0$*;*
- $F(\Delta_y)\neq 0$.

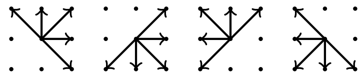
Furthermore, if E is smooth then it is an elliptic curve with J-invariant given by

$$J(E)=12^3\frac{D(\Delta_y)^3}{-F(\Delta_y)}\in C.$$

Otherwise, if E is non-degenerate and singular, E has a unique singular point and is a genus zero curve.

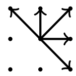
We define the genus of a weighted model as the genus of the associated kernel curve E. We recall the results obtained in [FIM99, Theorem 6.1.1] that classify all the weighted models attached to a genus zero kernel.

Theorem 1.9. *Any non-degenerate weighted model of genus zero has steps included in one of the following 4 sets of steps:*



Otherwise, for any other non-degenerate weighted model, the kernel curve E is an elliptic curve.

Remark 1.10. The walks corresponding to the fourth configuration never enter the quarter-plane. As described in [BMM10, Section 2.1], if we consider walks corresponding to the second and third configurations we are in the situation where one of the quarter plane constraints implies the other. In the last three configurations, the generating series is algebraic. So the only interesting genus zero weighted models have steps included in



Note that due to Proposition 1.6, the anti-diagonal steps have non-zero attached weights. See (G0) for the enumeration of the possible set of steps of the interesting genus zero weighted models.

Moreover, by Theorem 1.9, combined with Proposition 1.6, the non-degenerate weighted models of genus one are the walks where there are no three consecutive cardinal directions with weight zero. Or equivalently, this corresponds to the situation where the set of steps is not included in any half plane.

Thanks to Theorem 1.9, one can reduce our study to two cases depending on the genus of the kernel curve attached to a non-degenerate weighted model. The following lemma proves that when the kernel curve is of genus one, its J-invariant has modulus strictly greater than 1. This property allows us to use the theory of Tate curves in order to analytically uniformize the kernel curve.

Lemma 1.11. *When E is smooth, the invariant $J(E) \in \mathbb{Q}(t)$ is such that $|J(E)| > 1$, where $|\;|$ denotes the norm of $(C, |\;|)$.*

Proof. At $t = 0$, $\Delta_y(y_0, y_1)$ reduces to $y_0^2 y_1^2$. This proves that the reduction of $D(\Delta_y)$ (resp. $E(\Delta_y)$) at $t = 0$ is $\frac{1}{12}$ (resp. $\frac{1}{6^3}$). One concludes that $F(\Delta_y)$ vanishes for $t = 0$. By Proposition 1.8, $J(E) \in \mathbb{Q}(t)$ has a strictly negative valuation at $t = 0$. Thus, $|J(E)| > 1$. □

1.3. **The automorphism of the walk.** Following [BMM10, Section 3] or [KY15, Section 3], we introduce the involutive birational transformations of $\mathbf{P}^1(C) \times \mathbf{P}^1(C)$ given by

$$i_1(x, y) = \left(x, \frac{A_{-1}(x)}{A_1(x)y}\right) \text{ and } i_2(x, y) = \left(\frac{B_{-1}(y)}{B_1(y)x}, y\right),$$

(see §1.1 for the significance of the A_i, B_i's).

They induce two involutive automorphisms $\iota_1, \iota_2 : E \dashrightarrow E$ given by

$$\begin{array}{rcl} \iota_1([x_0 : x_1],[y_0 : y_1]) & = & \left([x_0 : x_1], \left[\dfrac{A_{-1}(\frac{x_0}{x_1})}{A_1(\frac{x_0}{x_1})\frac{y_0}{y_1}} : 1\right]\right), \\ \text{and} \quad \iota_2([x_0 : x_1],[y_0 : y_1]) & = & \left(\left[\dfrac{B_{-1}(\frac{y_0}{y_1})}{B_1(\frac{y_0}{y_1})\frac{x_0}{x_1}} : 1\right], [y_0 : y_1]\right). \end{array}$$

Note that ι_1 and ι_2 are nothing but the vertical and horizontal switches of E, see Figure 2. That is, for any $P = (x, y) \in E$, we have

$$\{P, \iota_1(P)\} = E \cap (\{x\} \times \mathbf{P}^1(C)) \text{ and } \{P, \iota_2(P)\} = E \cap (\mathbf{P}^1(C) \times \{y\}).$$

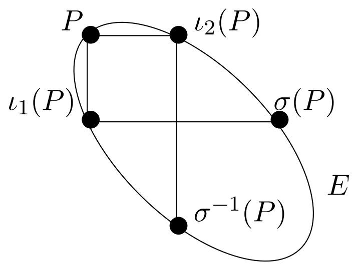


FIGURE 2. The maps ι_1, ι_2 restricted to the kernel curve E

The automorphism of the walk σ is defined by

$$\sigma = \iota_2 \circ \iota_1.$$

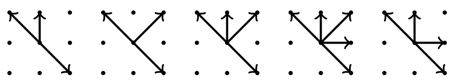
The following holds.

Lemma 1.12 (Lemma 4.14 in [DHRS17])**.** *Let $P \in E$. The following statements are equivalent:*

- *P is fixed by σ;*
- *P is fixed by ι_1 and ι_2;*
- *P is the only singular point of E that is of genus zero.*

2. Generating functions for walks, genus zero case

In this section, we fix a non-degenerate weighted model of genus zero. Following Remark 1.10, after eliminating duplications of trivial cases and the interchange of x and y, we should focus on walks $\mathcal{W}$ arising from the following 5 sets of steps:

(G0) 

In this section, we prove the following theorem:

Theorem 2.1. *For any weighted model of Table* (G0)*, the generating series* $Q(x,0,t)$ *is* $(\frac{d}{dx},\frac{d}{dt})$*-differentially transcendental over* $\mathbb{Q}$*.*

For any weighted model of Table (G0)*, the generating series* $Q(0,y,t)$ *is* $(\frac{d}{dy},\frac{d}{dt}))$*-differentially transcendental over* $\mathbb{Q}$*.*

Theorem 2.1 implies the differential transcendence of the complete generating series.

Corollary 2.2. *For any weighted model of Table* (G0)*, the generating series* $Q(x,y,t)$ *is* $(\frac{d}{dx},\frac{d}{dt})$ *and* $(\frac{d}{dy},\frac{d}{dt})$*-differentially transcendental over* $\mathbb{Q}$*.*

Proof of Corollary 2.2. Suppose to the contrary that $Q(x,y,t)$ is $(\frac{d}{dx},\frac{d}{dt})$-algebraic over $\mathbb{Q}$. Let $P(X,X_x,X_t,X_{xt},\dots)$ be a non-zero polynomial with coefficients in $\mathbb{Q}$ such that $P(Q(x,y,t),\frac{d}{dx}Q(x,y,t),\frac{d}{dt}Q(x,y,t),\dots)=0$. Specializing at $y=0$ this relation and noting that $\frac{d^i}{dx^i}\frac{d^j}{dt^j}(Q(x,0,t)$ is the specialization of $\frac{d^i}{dx^i}\frac{d^j}{dt^j}(Q(x,y,t)$, one finds a non-trivial differential algebraic relations for $Q(x,0,t)$ in the derivatives $\frac{d}{dx}$ and $\frac{d}{dt}$. This contradicts Theorem 2.1. The proof for the $(\frac{d}{dy},\frac{d}{dt})$-differential transcendence is similar. $\square$

As detailed in the introduction, our proof has three major steps:

Step 1: we attach the incomplete generating series $Q(x,0,t)$ and $Q(0,y,t)$ to some auxiliary functions that share the same differential behavior but satisfy simple **q**-difference equations. This is done via the uniformization of the kernel (see §2.1 and §2.2).

Step 2: we apply difference Galois theory to the **q**-difference equations satisfied by the auxiliary functions in order to relate the differential algebraicity of the incomplete generating series to the existence of *telescoping relations.* These telescoping relations are of the form (2.7).

Step 3: we prove that there is no such telescoping relation. This allows us to conclude that the incomplete generating series are differentially transcendental (see §2.3).

2.1. **Uniformization of the kernel curve.** With the notation of §1, especially (1.5), any weighted model of Table (G0) satisfies $\alpha_0=\alpha_1=\beta_0=\beta_1=0$. Moreover, since the weighted model is non-degenerate, one finds that the product $d_{1,-1}d_{-1,1}$ is non-zero. Furthermore,

$$-1+d_{0,0}t\pm\sqrt{(1-d_{0,0}t)^2-4d_{1,-1}d_{-1,1}t^2}\neq 0.$$

The uniformization of the kernel curve of a weighted model of Table (G0) is given by the following proposition.

Proposition 2.3 (Propositions 1.5 in [DHRS17])**.** *Let us consider a weighted model of Table* (G0) *and let* E *be its kernel curve. There exist* $\lambda\in C^*$ *and a parameterization* $\phi:\mathbf{P}^1(C)\to E$ *with*

$$\phi(s)=(x(s),y(s))=\left(\frac{4\alpha_2}{\sqrt{\alpha_3^2-4\alpha_2\alpha_4}(s+\frac{1}{s})-2\alpha_3},\frac{4\beta_2}{\sqrt{\beta_3^2-4\beta_2\beta_4}(\frac{s}{\lambda}+\frac{\lambda}{s})-2\beta_3}\right),$$

such that

- $\phi:\mathbf{P}^1(C)\setminus\{0,\infty\}\to E\setminus\{(0,0)\}$ *is a bijection and* $\phi^{-1}((0,0))=\{0,\infty\}$*;*

- *The automorphisms ι_1, ι_2, σ of E induce automorphisms $\tilde{\iota}_1, \tilde{\iota}_2, \sigma_{\mathbf{q}}$ of $\mathbf{P}^1(C)$ via ϕ that satisfy $\tilde{\iota}_1(s) = \frac{1}{s}$, $\tilde{\iota}_2(s) = \frac{\mathbf{q}}{s}$, $\sigma_{\mathbf{q}}(s) = \mathbf{q}s$, with $\lambda^2 = \mathbf{q} \in \{\widetilde{\mathbf{q}}, \widetilde{\mathbf{q}}^{-1}\}$ and*
$$\widetilde{\mathbf{q}} = \frac{-1 + d_{0,0}t - \sqrt{(1-d_{0,0}t)^2 - 4d_{1,-1}d_{-1,1}t^2}}{-1 + d_{0,0}t + \sqrt{(1-d_{0,0}t)^2 - 4d_{1,-1}d_{-1,1}t^2}} \in C^*.$$
Thus, we have the commutative diagrams
$$\begin{tikzcd} E \arrow[r, "\iota_k"] & E \\ \mathbb{P}^1(C) \arrow[u, "\phi"] \arrow[r, "\tilde{\iota}_k"'] & \mathbb{P}^1(C) \arrow[u, "\phi"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} E \arrow[r, "\sigma"] & E \\ \mathbb{P}^1(C) \arrow[u, "\phi"] \arrow[r, "\sigma_{\mathbf{q}}"'] & \mathbb{P}^1(C) \arrow[u, "\phi"'] \end{tikzcd}$$

The following estimate on the norm of $\widetilde{\mathbf{q}}$ holds:

Lemma 2.4. *We have $|\widetilde{\mathbf{q}}| > 1$.*

Proof. We consider the expansion as a Puiseux series of $\widetilde{\mathbf{q}}$. It is then easily seen that its valuation is negative,which gives $|\widetilde{\mathbf{q}}| > 1$. □

2.2. **Meromorphic continuation of the generating series.** In this paragraph, we combine the functional equation (1.3) with the uniformization of the kernel obtained above to meromorphically continue the generating series.

We define the norm of an element $b = [b_0 : b_1] \in \mathbf{P}^1(C)$ as follows: if $b_1 \neq 0$, we set $|b| = |\frac{b_0}{b_1}|$ and $|[1:0]| = \infty$ by convention. Since $|t| < 1$, the generating series $Q(x,y,t)$ as well as $F^1(x,t), F^2(y,t)$ converge for any $(x,y) \in \mathbf{P}^1(C) \times \mathbf{P}^1(C)$ such that $|x|$ and $|y|$ are smaller than or equal to 1. On that domain, they satisfy

$$K(x,y,t)Q(x,y,t) = xy + F^1(x,t) + F^2(y,t) + td_{-1,-1}Q(0,0,t). \tag{2.1}$$

We claim that there exist two positive real numbers c_0, c_∞ such that ϕ maps the disks $U_0 = \{s \in \mathbf{P}^1(C) | |s| < c_0\}$ and $U_\infty = \{s \in \mathbf{P}^1(C) | |s| > c_\infty\}$ into the domain $\mathcal{U}$ defined by $\{(x,y) \in E$ such that $|x| \leq 1$ and $|y| \leq 1\}$. Indeed, the α_i and β_i are of norm smaller or equal to 1 and $|\alpha_2| = 1$ (see (1.5)). Thus, if $|s| < min(1, |\sqrt{\alpha_3^2 - 4\alpha_2\alpha_4}|)$, then
$$|x(s)| = \left| \frac{4\alpha_2 s}{\sqrt{\alpha_3^2 - 4\alpha_2\alpha_4}(s^2+1) - 2\alpha_3 s} \right| = \frac{|4\alpha_2 s|}{|\sqrt{\alpha_3^2 - 4\alpha_2\alpha_4}|} < 1.$$
An analogous reasoning for $y(s)$ shows that when $|s|$ is sufficiently small, we find $|x(s)|, |y(s)| \leq 1$. Similarly, one can prove that, when $|s|$ is sufficiently big, one has $|x(s)|, |y(s)| \leq 1$. This proves our claim.

We set $\breve{F}^1(s) = F^1(x(s),t)$ and $\breve{F}^2(s) = F^2(y(s),t)$. Based on the above, these functions are well defined on $U_0 \cup U_\infty$. Evaluating (2.1) for $(x,y) = (x(s), y(s))$, one finds

$$0 = x(s)y(s) + \breve{F}^1(s) + \breve{F}^2(s) + td_{-1,-1}Q(0,0,t). \tag{2.2}$$

The following lemma shows that one can use the above equation to meromorphically continue the functions $\breve{F}^i(s)$ so that they satisfy a $\mathbf{q}$-difference equation.

Lemma 2.5. *For $i = 1, 2$, the restriction of the function $\breve{F}^i(s)$ to U_0 can be continued to a meromorphic function $\widetilde{F}^i(s)$ on C such that*

$$\widetilde{F}^1(\mathbf{q}s) - \widetilde{F}^1(s) = b_1 = (x(\mathbf{q}s) - x(s))y(\mathbf{q}s)$$

and

$$\widetilde{F}^2(\mathbf{q}s) - \widetilde{F}^2(s) = b_2 = (y(\mathbf{q}s) - y(s))x(s).$$

Proof. We just give a sketch of a proof since the arguments are the exact analogue in our ultrametric context of those employed in [DHRS17, §2.1]. Since $\tilde{\iota}_1(s) = \frac{1}{s}$ and $\tilde{\iota}_2(s) = \frac{\mathbf{q}}{s}$, we can assume without loss of generality that $\tilde{\iota}_1(U_0) \subset U_\infty$ and $\tilde{\iota}_2(U_\infty) \subset U_0$. Then one can evaluate (2.2) at any $s \in U_0$. We obtain

$$0 = x(s)y(s) + \breve{F}^1(s) + \breve{F}^2(s) + td_{-1,-1}Q(0, 0, t).$$

Evaluating (2.2) at $\tilde{\iota}_1(s) \in U_\infty$, we find

$$0 = x(\tilde{\iota}_1(s))y(\tilde{\iota}_1(s)) + \breve{F}^1(\tilde{\iota}_1(s)) + \breve{F}^2(\tilde{\iota}_1(s)) + td_{-1,-1}Q(0, 0, t).$$

Using the invariance of $x(s)$ (resp. $y(s)$) with respect to $\tilde{\iota}_1$ (resp. $\tilde{\iota}_2$), the second equation is

$$0 = x(s)y(\mathbf{q}s) + \breve{F}^1(s) + \breve{F}^2(\mathbf{q}s) + td_{-1,-1}Q(0, 0, t).$$

Subtracting this last equation to the first, we find that, for any $s \in U_0$, we have

$$\breve{F}^2(\mathbf{q}s) - \breve{F}^2(s) = (y(\mathbf{q}s) - y(s))x(s). \tag{2.3}$$

By Lemma 2.4, the norm of $\widetilde{\mathbf{q}}$ is strictly greater than one and therefore the norm of $|\mathbf{q}|$ is distinct from 1. This allows us to use (2.3) to meromorphically continue $\breve{F}^2$ to C so that it satisfies (2.3) everywhere. The proof for $\breve{F}^1$ is similar. $\square$

Note that, for $i = 1, 2$, the function $\widetilde{F}^i(s)$ does not coincide a priori with $\breve{F}^i(s)$ in the neighborhood of infinity.

2.3. **Differential transcendence in the genus zero case.** We recall that any holomorphic function f on C^* can be represented as an everywhere convergent Laurent series with coefficients in C. Moreover any non-zero meromorphic function on C^* can be written as the quotient of two holomorphic functions on C^* with no common zeros. We denote by $\mathcal{M}er(C^*)$ the field of meromorphic functions over C^* and by $\sigma_{\mathbf{q}}$ the $\mathbf{q}$-difference operator that maps a meromorphic function $g(s)$ onto $g(\mathbf{q}s)$. Finally, let $C_{\mathbf{q}}$ be the the field formed by the meromorphic functions over C^* fixed by $\sigma_{\mathbf{q}}$.

We now define the $\mathbf{q}$-logarithm. If $|\mathbf{q}| > 1$, the Jacobi Theta function is the meromorphic function defined by $\theta_{\mathbf{q}}(s) = \sum_{n\in\mathbb{Z}} \mathbf{q}^{-n(n+1)/2}s^n \in \mathcal{M}er(C^*)$. It satisfies the the $\mathbf{q}$-difference equation

$$\theta_{\mathbf{q}}(\mathbf{q}s) = s\theta_{\mathbf{q}}(s).$$

Its logarithmic derivative $\ell_{\mathbf{q}}(s) = \frac{\partial_s(\theta_{\mathbf{q}})}{\theta_{\mathbf{q}}} \in \mathcal{M}er(C^*)$ satisfies $\ell_{\mathbf{q}}(\mathbf{q}s) = \ell_{\mathbf{q}}(s) + 1$. If $|\mathbf{q}| < 1$ then the meromorphic function $-\ell_{1/\mathbf{q}}$ is solution of $\sigma_{\mathbf{q}}(-\ell_{1/\mathbf{q}}) = -\ell_{1/\mathbf{q}} + 1$. Abusing the notation, we still denote by $\ell_{\mathbf{q}}$ the function $-\ell_{1/\mathbf{q}}$ when $|\mathbf{q}| < 1$.

Since we want to use the $\mathbf{q}$-difference equations of Lemma 2.5 as a constraint for the form of the differential algebraic relations satisfied by the $\widetilde{F}^i(s)$'s, we need to consider derivations that are compatible with $\sigma_{\mathbf{q}}$ in the sense that they commute with each other. This is not the case for the derivation $\partial_t = t\frac{d}{dt}$. By Lemma D.2, the derivations $\partial_s = s\frac{d}{ds}$ and $\Delta_{t,\mathbf{q}} = \partial_t(\mathbf{q})\ell_{\mathbf{q}}(s)\partial_s + \partial_t$ commute with $\sigma_{\mathbf{q}}$. The following lemma relates the differential transcendence of the incomplete generating series $Q(x,0,t)$ and $Q(0,y,t)$ to the differential transcendence of the auxiliary functions $\widetilde{F}^i(s)$.

Lemma 2.6. *If the generating series is $Q(x,0,t)$ is $\left(\frac{d}{dx},\frac{d}{dt}\right)$-differentially algebraic over $\mathbb{Q}$, then $\widetilde{F}^1(s)$ is $(\partial_s,\Delta_{t,\mathbf{q}})$-differentially algebraic over $\widetilde{K} = C_{\mathbf{q}}(s,\ell_{\mathbf{q}}(s))$.*

If the generating series is $Q(0,y,t)$ is $\left(\frac{d}{dy},\frac{d}{dt}\right)$-differentially algebraic over $\mathbb{Q}$, then $\widetilde{F}^2(s)$ is $(\partial_s,\Delta_{t,\mathbf{q}})$-differentially algebraic over $\widetilde{K} = C_{\mathbf{q}}(s,\ell_{\mathbf{q}}(s))$.

Proof. The statement being symmetrical in x and y, we prove it only for $Q(x,0,t)$. Assume that the generating series is $Q(x,0,t)$ is $\left(\frac{d}{dx},\frac{d}{dt}\right)$-differentially algebraic over $\mathbb{Q}$. Since $F^1(x,t)$ is the product of $Q(x,0,t)$ by the polynomial $K(x,0,t)\in\mathbb{Q}[x,t]$, the function $F^1(x,t)$ is $\left(\frac{d}{dx},\frac{d}{dt}\right)$-differentially algebraic over $\mathbb{Q}$. It is therefore $\left(\frac{d}{dx},\partial_t\right)$-differentially algebraic over $\mathbb{Q}(t)$, and finally $\left(\frac{d}{dx},\partial_t\right)$-differentially algebraic over $\mathbb{Q}$, since t is ∂_t-differentially algebraic over $\mathbb{Q}$. Remember that $\widetilde{F}^1(s)$ coincides with $F^1(x(s),t)$ for $s\in U_0$ where $x(s)$ is defined thanks to Proposition 2.3. Therefore, we need to understand the relations between the x and t derivatives of $F^1(x,t)$ and the derivatives of $F^1(x(s),t)$ with respect to ∂_s and $\Delta_{t,\mathbf{q}}$.

Let us study these relations for an arbitrary bivariate function $G(x,t)$ which converges on $|x|,|y|\leq 1$. Denote by δ_x the derivation $\frac{d}{dx}$ and by $\widetilde{G}(s) = G(x(s),t)$. From the equality $(\partial_s\widetilde{G}(s)) = \partial_s(x(s))(\delta_x G)(x(s),t)$, we conclude that

$$\partial_t(\widetilde{G}(s)) = (\partial_t G)(x(s),t) + \partial_t(x(s))(\delta_x G)(x(s),t) = (\partial_t G)(x(s),t) + c\partial_s(\widetilde{G}(s)),$$

where $c = \frac{\partial_t(x(s))}{\partial_s(x(s))}$. The element c belongs to $\widetilde{K}$ because $x(s)\in\widetilde{K}$ and $\widetilde{K}$ is stable by $\partial_s,\Delta_{t,\mathbf{q}}$ and thereby by $\partial_t = \Delta_{t,\mathbf{q}} - \partial_t(\mathbf{q})\ell_{\mathbf{q}}(s)\partial_s$ (see Lemma D.5). An easy induction shows that

$$(\partial_t^n G)(x(s),t) = \partial_t^n(\widetilde{G}(s)) + \sum_{i\leq n, j<n} b_{i,j}\partial_t^j\partial_s^i(\widetilde{G}(s)), \tag{2.4}$$

where the $b_{i,j}$'s belong to $\widetilde{K}$. By Lemma D.2, we have $\partial_s\Delta_{t,\mathbf{q}} - \Delta_{t,\mathbf{q}}\partial_s = f\partial_s$, where $f = \partial_t(\mathbf{q})\partial_s(\ell_{\mathbf{q}})\in\widetilde{K}$. Combining (2.4) with $\partial_t = \Delta_{t,\mathbf{q}} - \partial_t(\mathbf{q})\ell_{\mathbf{q}}(s)\partial_s$, we find that

$$(\partial_t^n G)(x(s),t) = \Delta_{t,\mathbf{q}}^n(\widetilde{G}(s)) + \sum_{i\leq 2n, j<n} d_{i,j}\Delta_{t,\mathbf{q}}^j\partial_s^i(\widetilde{G}(s)), \tag{2.5}$$

for some $d_{i,j}$'s in $\widetilde{K}$. Moreover, an easy induction shows that, for any $m\in\mathbb{N}^*$, we have

$$(\delta_x^m G)(x(s),t) = \frac{1}{\partial_s(x(s))^m}\partial_s^m(\widetilde{G}(s)) + \sum_{i=1}^{m-1} a_i\partial_s^i(\widetilde{G}(s)), \tag{2.6}$$

where $a_i\in\widetilde{K}$. Applying (2.5) with G replaced by $\delta_x^m G$, we find that for every $m,n\in\mathbb{N}$,

$$(\partial_t^n\delta_x^m G)(x(s),t) = \Delta_{t,\mathbf{q}}^n((\delta_x^m G)(x(s),t)) + \sum_{i\leq 2n, j<n} d_{i,j}\Delta_{t,\mathbf{q}}^j\partial_s^i((\delta_x^m G)(x(s),t)).$$

Combining this equation with (2.6), we conclude that

$$(\partial_t^n \delta_x^m G)(x(s),t) = \frac{1}{\partial_s(x(s))^m}\Delta_{t,\mathbf{q}}^n \partial_s^m(\widetilde{G}(s)) + \sum_{i\leq 2n+m, j<n} r_{i,j}\Delta_{t,\mathbf{q}}^j \partial_s^i(\widetilde{G}(s)),$$

where the $r_{i,j}$'s are elements of $\widetilde{K}$.

Applying the computations above to $G = F^1(x,t)$, we find that any non-trivial polynomial equation in the derivatives $\delta_x^m \partial_t^n F^1(x,t)$ over $\mathbb{Q}$ yields to a non-trivial polynomial equation over $\widetilde{K}$ between the derivatives $\Delta_{t,\mathbf{q}}^j \partial_s^i(\widetilde{F}^1(s))$. □

Thus, we have reduced the proof of Theorem 2.1 to the following proposition:

Proposition 2.7. *The functions $\widetilde{F}^1(s)$ and $\widetilde{F}^2(s)$ are $(\partial_s, \Delta_{t,\mathbf{q}})$-differentially transcendental over $\widetilde{K}$.*

Proof. Suppose to the contrary that $\widetilde{F}^1(s)$ is $(\partial_s, \Delta_{t,\mathbf{q}})$-differentially algebraic over $\widetilde{K}$. By Lemma 2.5, the meromorphic function $\widetilde{F}^1(s)$ satisfies $\widetilde{F}^1(\mathbf{q}s) - \widetilde{F}^1(s) = b_1 = (x(\mathbf{q}s) - x(s))y(\mathbf{q}s)$ with $b_1 \in C(s) \subset C_{\mathbf{q}}(s)$. We now apply difference Galois theory to this $\mathbf{q}$-difference equation. More precisely, by Proposition D.6 and Corollary D.14 with $K = C_{\mathbf{q}}(s)$, there exist $m \in \mathbb{N}$, $d_0, \ldots, d_m \in C_{\mathbf{q}}$ not all zero and $h \in C_{\mathbf{q}}(s)$ such that

$$d_0 b_1 + d_1\partial_s(b_1) + \cdots + d_m\partial_s^m(b_1) = \sigma_{\mathbf{q}}(h) - h. \tag{2.7}$$

Let $(e_\beta)_{\beta\in B}$ be a C-basis of $C(s)$. Then, $(e_\beta)_{\beta\in B}$ is a $C_{\mathbf{q}}$-basis of $C_{\mathbf{q}}(s)$ by [Wib10, Lemma 1.1.6]. Now, decompose the d_k's and h over $(e_\beta)_{\beta\in B}$. Since $b_1 \in C(s)$, it is easily seen that (2.7) amounts into a collection of polynomial equations with coefficients in C that should satisfy the coefficients of the d_k's and h with respect to the basis $(e_\beta)_{\beta\in B}$. Since this collection of polynomial equations has a non-zero solution in $C_{\mathbf{q}}$, we can conclude that it has a non-zero solution in C because C is algebraically closed. Therefore, there exists $c_k \in C$ not all zero and $g \in C(s)$ such that

$$\sum_k c_k \partial_s^k(b_1) = \sigma_{\mathbf{q}}(g) - g.$$

By [HS08, Lemma 6.4] there exist $f \in C(s)$ and $c \in C$, such that

$$\widetilde{F}^1(\mathbf{q}s) - \widetilde{F}^1(s) = b_1 = \sigma_{\mathbf{q}}(f) - f + c.$$

Since $\widetilde{F}^1$ is meromorphic at $s = 0$, we conclude that c must be equal to zero. Finally, we have shown that there exist $f \in C(s)$ such that

$$b_1 = \sigma_{\mathbf{q}}(f) - f. \tag{2.8}$$

By duality, the morphism $\phi : \mathbf{P}^1 \to E$ gives rise to a field isomorphism ϕ^* from the field $C(x,y)$[‡] of rational functions on E and the field $C(s)$ of rational functions on $\mathbf{P}^1$. Moreover, one has $\sigma_{\mathbf{q}}\phi^* = \phi^*\sigma^*$, where σ^* is the action induced by the automorphism of the walk on $C(E)$. Then, it is easily seen that the equation (2.8) is equivalent to

$$(\sigma(x) - x)\sigma(y) = \sigma(\tilde{f}) - \tilde{f}, \tag{2.9}$$

[‡]Here x and y denote the coordinate functions on the curve E.

where $\tilde{f} \in C(x,y)$ is the rational function corresponding to f via ϕ^*. The coefficients of $\tilde{f}$ as a rational function over E belong to a finitely generated extension F of $\mathbb{Q}(t)$.

There exists a $\mathbb{Q}$-embedding ψ of F into $\mathbb{C}$ that maps t onto a transcendental complex number. Since σ and E are defined over $\mathbb{Q}(t)$, we apply ψ to (2.9) and we find

$$(\overline{\sigma}(x) - x)\overline{\sigma}(y) = \overline{\sigma}(\overline{f}) - \overline{f},$$

where $\overline{f}$ belongs to $\mathbb{C}(\overline{E})$ the field of rational functions on the complex algebraic curve $\overline{E}$ defined by the kernel polynomial $K(x,y,\psi(t))$ and where $\overline{\sigma}$ is the automorphism of $\mathbb{C}(\overline{E})$ induced by the automorphism of the walk corresponding to $\overline{E}$. In [DHRS17, §3.2], the authors proved that there is no such equation. This concludes the proof by contradiction. □

3. Generating functions of walks, genus one case

In this section we consider the situation where the kernel curve E is an elliptic curve. By Remark 1.10, this corresponds to the case where the set of steps is not included in an half plane. Moreover, we work under the assumption that the group of the walk is infinite. In [DR19], the authors study the finite group case and prove that the uniformization of the generating series is an elliptic function over an elliptic curve isogeneous to the kernel curve. This allows them to prove that for any genus one kernel curve and finite group of the walk, the generating series is holonomic with respect to any of the two variables x, y. Unfortunately, the t-dependency is not known for general weighted models of genus one with finite group.

Our strategy follows the basic lines of the one employed in the genus zero situation. However, our uniformization procedure in the genus one case is more delicate and differs from previous works such as [FIM99, KR12, DR19] that relied on the uniformization of elliptic curves over $\mathbb{C}$ by a fundamental parallelogram of periods. Over a non-archimedean field C, there might be a lack of non-trivial lattices. One has to consider multiplicative analogues, that is, discrete subgroups of C^* of the form $q^{\mathbb{Z}}$. Then, rigid analytic geometry gives a geometric meaning to the quotient $C^*/q^{\mathbb{Z}}$. This geometric quotient is called a Tate curve (see [Roq70] for more details). For simplicity of exposition, we won't give here many details on this non-archimedean geometry The multiplicative uniformization of the kernel curve allows us as in §2.2 to attach to the incomplete generating series $Q(x,0,t)$ and $Q(0,y,t)$ some meromorphic functions $\widetilde{F}^i(s)$ satisfying

$$\widetilde{F}^i(\mathbf{q}s) - \widetilde{F}^i(s) = b_i(s),$$

for some $\mathbf{q} \in C^*$ and $b_i(s) \in C_q$, the field of q-periodic meromorphic functions over C^*. This process detailed in §3.1, 3.2 and 3.3 has many advantages. Though technical, it is much more simple than the uniformization by a fundamental parallelogram of periods since we only have to deal with one generator of the fundamental group of the elliptic curve, precisely the loop around the origin in C^*. Moreover, it gives a unified framework to study the genus zero and one case, namely, the Galois theory of $\mathbf{q}$-difference equations. This is the content of §3.4 where we apply the Galoisian criteria of Appendix C to translate the differential algebraicity of the generating series in terms of the existence of a telescoper. Finally, we show how one can apply the results of [DHRS18] to our context in order to conclude that there is no such telescoper for all but 9 of the non-degenerate unweighted models of genus one with infinite group.

3.1. **Uniformization of the kernel curve.** Let us fix a weighted model of genus one. By Lemma 1.11, the norm of the J-invariant $J(E)$ of the kernel curve is such that $|J(E)| > 1$. By Proposition B.2, there exists $q \in C$ such that $0 < |q| < 1$ and $J(E) = J(E_q) = \frac{1}{|q|}$, where E_q is the elliptic curve attached to the Tate curve $C^*/q^{\mathbb{Z}}$ (see Proposition 3.1, Lemmas B.5, and B.7). The curve E_q can be analytically uniformized by C^* thanks to special functions, which have their origins in the theory of Jacobi q-theta functions (see Proposition 3.1 below). Finally, since E and E_q have the same J-invariant, there exists an algebraic isomorphism between these two elliptic curves. In order to describe the uniformization of the kernel curve E, one needs to explicit this algebraic isomorphism. This is not completely obvious since E_q is given by its Tate normal form in $\mathbf{P}^2$, i.e. by an equation of the form

$$Y^2 + XY = X^3 + BX + \widetilde{C}.$$

Therefore, many intermediate results are quite technical and we choose to postpone these results to the appendix B. The following proposition describes the multiplicative uniformization of an elliptic curve given by a Tate normal form.

Following [Roq70, Page 28], we set $s_k = \sum_{n>0} \frac{n^k q^n}{1-q^n} \in C$ for $k \geq 1$.

Proposition 3.1. *The series*

- $X(s) = \sum_{n\in\mathbb{Z}} \frac{q^n s}{(1-q^n s)^2} - 2s_1$*;*
- $Y(s) = \sum_{n\in\mathbb{Z}} \frac{(q^n s)^2}{(1-q^n s)^3} + s_1$*;*

are q-periodic meromorphic functions over C^*. Furthermore $X(s) = X(1/s)$, and $X(s)$ has a pole of order 2 at any element of the form $q^{\mathbb{Z}}$. Moreover, the analytic map

$$\begin{array}{rccc} \pi: & C^* & \rightarrow & \mathbf{P}^2(C), \\ & s & \mapsto & [X(s) : Y(s) : 1] \end{array}$$

is onto and his image is E_q, the elliptic curve defined by the following Tate normal form

$$Y^2 + XY = X^3 + BX + \widetilde{C}, \tag{3.1}$$

where $B = -5s_3$ and $\widetilde{C} = -\frac{1}{12}(5s_3 + 7s_5)$. Moreover, $\pi(s_1) = \pi(s_2)$ if and only if $s_1 \in s_2 q^{\mathbb{Z}}$.

Proof. This is [FvdP04, Theorem 5.1.4, Corollary 5.1.5, and Theorem 5.1.10]. □

In the notation of Section 1.2, set $\mathfrak{D}(x) := \Delta_x(x, 1)$. Let us write the kernel polynomial

$$K(x, y, t) = \widetilde{A}_0(x) + \widetilde{A}_1(x)y + \widetilde{A}_2(x)y^2 = \widetilde{B}_0(y) + \widetilde{B}_1(y)x + \widetilde{B}_2(y)x^2$$

with $\widetilde{A}_i(x) \in C[x]$ and $\widetilde{B}_i(y) \in C[y]$. For $i \geq 1$, let $\mathfrak{D}^{(i)}(x)$ denote the i-th derivative with respect to x of $\mathfrak{D}(x)$. The analytic uniformization of the kernel curve is given by the following proposition.

Theorem 3.2. *There exists a root a of $\mathfrak{D}(x)$ in C such that $|a|, |\mathfrak{D}^{(2)}(a)-2|, |\mathfrak{D}^{(i)}(a)| < 1$ for $i = 3, 4$, $|q|^{1/2} < |\mathfrak{D}^{(1)}(a)| < 1$. For any such a, there exists $u \in C^*$ with $|u| = 1$ such that the map ϕ given by*

$$\begin{array}{llll} \phi: & C^* & \rightarrow & E, \\ & s & \mapsto & (\overline{x}(s), \overline{y}(s)), \end{array}$$

is surjective where

$$\begin{aligned} \overline{x}(s) &= a + \frac{\mathfrak{D}^{(1)}(a)}{u^2 X(s) + \frac{u^2}{12} - \frac{\mathfrak{D}^{(2)}(a)}{6}} \\ \overline{y}(s) &= \frac{\frac{\mathfrak{D}^{(1)}(a)\left(2u^3Y(s)+u^3X(s)\right)}{2\left(u^2X(s)+\frac{u^2}{12}-\frac{\mathfrak{D}^{(1)}(a)}{6}\right)^2} - \widetilde{A}_1\left(a + \frac{\mathfrak{D}^{(1)}(a)}{u^2X(s)+\frac{u^2}{12}-\frac{\mathfrak{D}^{(2)}(a)}{6}}\right)}{2\widetilde{A}_2\left(a + \frac{\mathfrak{D}^{(1)}(a)}{u^2X(s)+\frac{u^2}{12}-\frac{\mathfrak{D}^{(2)}(a)}{6}}\right)}. \end{aligned} \tag{3.2}$$

Proof. Lemma A.1 and Lemma B.7 guaranty the existence of a. The element a allows us to write down the isomorphism between the kernel curve E and one of its Weierstrass normal form E_1. More precisely, by Proposition B.4, the application w_E

$$\begin{array}{lll} E_1 & \rightarrow & E \subset \mathbf{P}^1(C) \times \mathbf{P}^1(C) \\ [x_1 : y_1 : 1] & \mapsto & (\overline{x}, \overline{y}) \end{array}$$

where

$$\overline{x} = a + \frac{\mathfrak{D}^{(1)}(a)}{x_1 - \frac{\mathfrak{D}^{(2)}(a)}{6}} \text{ and } \overline{y} = \frac{\frac{\mathfrak{D}^{(1)}(a)y_1}{2(x_1-\frac{\mathfrak{D}^{(1)}(a)}{6})^2} - \widetilde{A}_1\left(a + \frac{\mathfrak{D}^{(1)}(a)}{x_1-\frac{\mathfrak{D}^{(2)}(a)}{6}}\right)}{2\widetilde{A}_2\left(a + \frac{\mathfrak{D}^{(1)}(a)}{x_1-\frac{\mathfrak{D}^{(2)}(a)}{6}}\right)},$$

is an isomorphism between the elliptic curve $E_1 \subset \mathbf{P}^2(C)$ given by the equation $y_1^2 = 4x_1^3 - g_2x_1 - g_3$ and the kernel curve E. Now, it remains to explicit the isomorphism between E_q and one of its Weierstrass normal form $\widetilde{E}_1$. By Lemma B.5, the application $\begin{array}{llll} w_T: & E_q & \rightarrow & \widetilde{E}_1, \\ & [X : Y : 1] & \mapsto & [X + \frac{1}{2} : 2Y + X : 1] \end{array}$ induces an isomorphism between E_q and the curve $\widetilde{E}_1$ given by $y^2 = 4x^3 - h_2x - h_3$. Since E and E_q have the same J-invariants and are therefore isomorphic, the same holds for their Weierstrass normal forms. Thus, there exists $u \in C^*$ such that $\begin{array}{llll} \psi: & \widetilde{E}_1 & \rightarrow & E_1, \\ & [x : y : 1] & \mapsto & [u^2x : u^3y : 1] \end{array}$ induces an isomorphism of elliptic curves (see Lemma B.6). To conclude, we set $\phi = w_E \circ \psi \circ w_T \circ \pi$ where π is the uniformization of E_q by C^* given in Proposition 3.1. The norm estimate on u is Lemma B.7. □

Remark 3.3.

- Note that by construction $\phi(s_1) = \phi(s_2)$ if and only if if $s_1 \in s_2 q^{\mathbb{Z}}$ (see Proposition 3.1).
- Via ϕ, the field of rational functions over E can be identified with field of q-periodic meromorphic functions over C.
- The conditions on a are crucial to guaranty the meromorphic continuation of the generating series (see the proof of Lemma 3.7).

- The symmetry arguments between x and y of Remark 1.2 can be pushed further and one can construct another uniformization of E as follows. Denoting by $\mathfrak{E}(y)$ the polynomial $\Delta_y(y,1)$. One can prove that there exist a root $b \in C^*$ of $\mathfrak{E}$ such that $|b|, |\mathfrak{E}^{(2)}(b)-2|, |\mathfrak{E}^{(i)}(b)| < 1$ for $i = 3,4$ and $|q|^{1/2} < |\mathfrak{E}^{(1)}(b)| < 1$ and $v \in C^*$ with $|v| = 1$ such that the analytic map ψ given by

$$\begin{array}{llll} \psi: & C^* & \rightarrow & E, \\ & s & \mapsto & (\overline{x}(s), \overline{y}(s)), \end{array}$$

is surjective with $\overline{y}(s) = b + \frac{\mathfrak{E}^{(1)}(b)}{v^2 X(s) + \frac{v^2}{12} - \frac{\mathfrak{E}^{(2)}(b)}{6}}$ (see [DR19, (2.16)] for similar arguments).

3.2. **The group of the walk.** The following proposition gives an explicit form for the automorphisms of C^* induced via ϕ by the automorphisms σ, ι_1, ι_2 of E.

Proposition 3.4. *There exists $\mathbf{q}$ in C^* such that the automorphism of C^* defined by $\sigma_{\mathbf{q}} : s \mapsto \mathbf{q}s$ induces via ϕ the automorphism σ, that is $\sigma \circ \phi = \phi \circ \sigma_{\mathbf{q}}$. Similarly, the involutions $\tilde{\iota}_1, \tilde{\iota}_2$ of C^*, that are defined by $\tilde{\iota}_1(s) = 1/s$ and $\tilde{\iota}_2(s) = \mathbf{q}/s$, induce via ϕ the automorphisms ι_1, ι_2.*

Proof. The automorphism σ corresponds to the addition by a prescribed point Ω of E (see [Dui10, Prop. 2.5.2]). Let $\pi : C^* \rightarrow E_q$ be the surjective map defined in Proposition 3.1. By [FvdP04, Exercise 5.1.9], the map π is a group isomorphism between the multiplicative group $(C^*, *)$ and the Mordell-Weil group of E_q [§]. Moreover, since E_q and E are elliptic curves, any isomorphism between E_q and E is a group morphism between their respective Mordell-Weil groups. This proves that ϕ is a group morphism. Then, there exists $\mathbf{q} \in C^*$ such that $\sigma \circ \phi = \phi \circ \sigma_{\mathbf{q}}$. Since ϕ is q-invariant, the element $\mathbf{q}$ is determined modulo $q^{\mathbb{Z}}$ (see Remark 3.3). This proves the first statement.
Let us denote by $\tilde{\iota}_1, \tilde{\iota}_2$ some automorphisms of C^*, obtained by pulling back to C^* via ϕ the automorphisms ι_1, ι_2 of E. The automorphisms $\tilde{\iota}_1, \tilde{\iota}_2$ are uniquely determined up to multiplication by some power of q. The automorphisms of C^* are of the form $s \mapsto ls^{\pm 1}$ with $l \in C^*$. Note that $\overline{x}(q^{\mathbb{Z}}) = a$, and $(a, \frac{-B(a)}{2A(a)}) \in E$ is fixed by ι_1. Indeed, by construction $\mathfrak{D}(a) = 0$. This proves that $\tilde{\iota}_1(1)$ belongs to $q^{\mathbb{Z}}$. Since ι_1 is not the identity, we can modify $\tilde{\iota}_1$ by a suitable power of q to get $\tilde{\iota}_1(s) = 1/s$. The expression of $\tilde{\iota}_2$ follows with $\sigma = \iota_2 \circ \iota_1$. □

Remark 3.5.
- The choice of the element $\mathbf{q}$ is unique up to multiplication by $q^{\mathbb{Z}}$. Since $|q| \neq 1$, we can choose $\mathbf{q}$ such that $|q|^{1/2} \leq |\mathbf{q}| < |q|^{-1/2}$.
- Pursuing the symmetry arguments of Remark 3.3, we easily note that Proposition 3.4 has a straightforward analogue when one replaces ϕ by ψ and one exchanges $\tilde{\iota}_1$ and $\tilde{\iota}_2$.

The proof of the following lemma is straightforward.

[§] This is the group whose underlying set is the set of points of E and whose group law is given by the addition on the elliptic curve E.

Lemma 3.6. *The automorphism σ has infinite order if and only if $\mathbf{q}$ and q are* multiplicatively independent[¶], *that is, there is no $(r,l) \in \mathbb{Z}^2 \setminus (0,0)$ such that $q^r = \mathbf{q}^l$.*

3.3. **Meromorphic continuation.** In this section, we prove that the functions

$$F^1(x,t) := K(x,0,t)Q(x,0,t), \text{ and } \quad F^2(y,t) := K(0,y,t)Q(0,y,t),$$

can be meromorphically continued to C^*. We follow the ideas initiated in [FIM99]. We note that, since $|t| < 1$, the series $F^1(x,t)$ and $F^2(y,t)$ converge on the affinoid subset $U = \{(x,y) \in E \subset \mathbf{P}^1(C) \times \mathbf{P}^1(C) | |x| \leq 1, |y| \leq 1\}$ of E. With Lemma A.3, U is not empty. For $(x,y) \in U$, we have

$$0 = xy + F^1(x,t) + F^2(y,t) + td_{-1,-1}Q(0,0,t).$$

Set $U_x = \{(x,y) \in E \subset \mathbf{P}^1(C) \times \mathbf{P}^1(C) | |x| \leq 1\}$. Note that $F^1(x,t)$ is analytic on U_x. We continue $F^2(y,t)$ on U_x by setting

$$F^2(y,t) = -xy - F^1(x,t) - td_{-1,-1}Q(0,0,t).$$

Composing $F^i(x,t)$ with the surjective map

$$\begin{array}{rccc} \phi : & C^* & \to & E \\ & s & \mapsto & (\overline{x}(s), \overline{y}(s)), \end{array}$$

we define the functions $\breve{F}^1(s) = F^1(\overline{x}(s),t)$ and $\breve{F}^2(s) = F^2(\overline{y}(s),t)$ for any s in the set

$$\mathcal{U}_x := \phi^{-1}(U_x) \cap \{s \in C^* | |s| \in [|q|^{1/2}, |q|^{-1/2}[\}.$$

The goal of the following lemma is to prove that $\mathcal{U}_x$ is an annulus whose size is large enough in order to continue the functions $\breve{F}^1, \breve{F}^2$, to the whole C^* (see Figure 3).

Lemma 3.7. *Let $|s| \in [|q|^{1/2}, |q|^{-1/2}[$. The following statements hold:*

- *if $|s| \in]|\mathfrak{D}^{(1)}(a)|, |\mathfrak{D}^{(1)}(a)|^{-1}[$, then $|\overline{x}(s)| < 1$;*
- *if $|s| = |\mathfrak{D}^{(1)}(a)|^{\pm 1}$, then $|\overline{x}(s)| = 1$;*
- *otherwise $|\overline{x}(s)| > 1$.*

In conclusion, $\mathcal{U}_x = [|\mathfrak{D}^{(1)}(a)|, |\mathfrak{D}^{(1)}(a)|^{-1}]$.

Proof. From the definition of $X(s)$, we have $X(s) = X(1/s)$ so that $\overline{x}(s) = \overline{x}(1/s)$. Using this symmetry, we just have to prove Lemma 3.7 for $|s| \in [|q|^{1/2}, 1]$. We have

$$|\overline{x}(s)| = \left| a + \frac{\mathfrak{D}^{(1)}(a)}{u^2 X(s) + \frac{u^2}{12} - \frac{\mathfrak{D}^{(2)}(a)}{6}} \right| \leq \max\left(|a|, \left| \frac{\mathfrak{D}^{(1)}(a)}{u^2 X(s) + \frac{u^2}{12} - \frac{\mathfrak{D}^{(2)}(a)}{6}} \right| \right), \tag{3.3}$$

with equality if $|a| \neq \left| \frac{\mathfrak{D}^{(1)}(a)}{u^2 X(s) + \frac{u^2}{12} - \frac{\mathfrak{D}^{(2)}(a)}{6}} \right|$. Remember that $|u| = 1$, $|a| < 1$, and $|q|^{1/2} < |\mathfrak{D}^{(1)}(a)| < 1$, see Theorem 3.2. Let us first assume that $|s| \in [|\mathfrak{D}^{(1)}(a)|, 1[$. By Lemma B.3, $|u^2 X(s)| = |s|$ and by Lemma B.8, $|\frac{u^2}{12} - \frac{\mathfrak{D}^{(2)}(a)}{6}| < |\mathfrak{D}^{(1)}(a)|$. Therefore

$$\left| \frac{\mathfrak{D}^{(1)}(a)}{u^2 X(s) + \frac{u^2}{12} - \frac{\mathfrak{D}^{(2)}(a)}{6}} \right| = \left| \frac{\mathfrak{D}^{(1)}(a)}{s} \right|.$$

[¶]Note that multiplicatively independent is sometimes replaced in the literature by non-commensurable (see [Roq70, §6]).

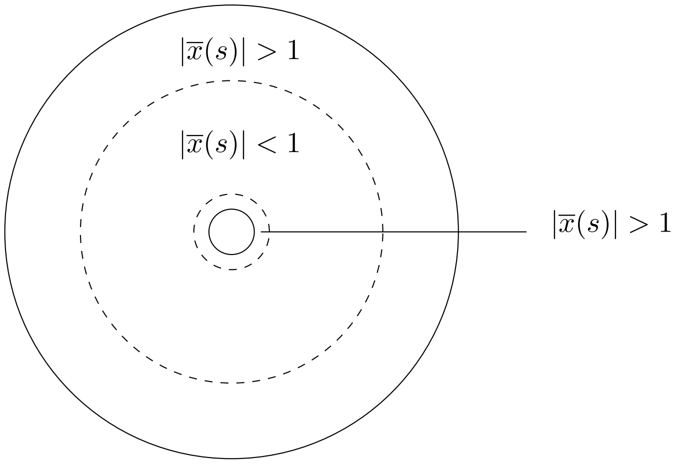


FIGURE 3. The plain circles correspond to $|s| = |q|^{\pm 1/2}$. The dashed circles correspond to $|\overline{x}(s)| = 1$.

Combining this equality with (3.3) and $|a| < 1$, we find that $|\overline{x}(s)| < 1$ if $|s| \in]|\mathfrak{D}^{(1)}(a)|, 1[$, and $|\overline{x}(s)| = 1$ if $|s| = |\mathfrak{D}^{(1)}(a)|$.

Assume now that $|s| = 1$. By construction, $|\overline{x}(1)| = |a| < 1$. So let us assume that $s \neq 1$. Since $|\frac{u^2}{12} - \frac{\mathfrak{D}^{(2)}(a)}{6}| < |\mathfrak{D}^{(1)}(a)| < 1$ and $|u^2 X(s)| \geq 1$ by Lemma B.3, we find

$$\left| \frac{\mathfrak{D}^{(1)}(a)}{u^2 X(s) + \frac{u^2}{12} - \frac{\mathfrak{D}^{(2)}(a)}{6}} \right| = \left| \frac{\mathfrak{D}^{(1)}(a)}{u^2 X(s)} \right| \leq |\mathfrak{D}^{(1)}(a)| < 1.$$

This concludes the proof of the first two points.

Assume that $|s| \in]|q|^{1/2}, |\mathfrak{D}^{(1)}(a)|[$. By Lemma B.3, $|u^2 X(s)| = |X(s)| = |s|$. Since

$$\left| \frac{u^2}{12} - \frac{\mathfrak{D}^{(2)}(a)}{6} \right| < |\mathfrak{D}^{(1)}(a)| < 1,$$

we find that $|u^2 X(s) + \frac{u^2}{12} - \frac{\mathfrak{D}^{(2)}}{6}| < |\mathfrak{D}^{(1)}(a)|$ and therefore, $|\overline{x}(s)| > 1$. If we have $|s| = |q|^{1/2} < |\mathfrak{D}^{(1)}(a)|$ then Lemma B.3 implies that $|u^2 X(s)| = |X(s)| \leq |s| < |\mathfrak{D}^{(1)}(a)|$. Since $|\frac{u^2}{12} - \frac{\mathfrak{D}^{(2)}(a)}{6}| < |\mathfrak{D}^{(1)}(a)|$, we deduce that $|u^2 X(s) + \frac{u^2}{12} - \frac{\mathfrak{D}^{(2)}(a)}{6}| < |\mathfrak{D}^{(1)}(a)|$ and therefore, $|\overline{x}(s)| > 1$. This concludes the proof. $\square$

Remark 3.8. By symmetry between x and y, one could have define $U_y = \{(x, y) \in E \subset \mathbf{P}^1(C) \times \mathbf{P}^1(C) | |y| \leq 1\}$ and continue $F^1(x, t)$ on U_y by setting

$$F^1(x, t) = -xy - F^2(y, t) - t d_{-1,-1} Q(0, 0, t).$$

Then, the composition of the F^i with the surjective map ψ defined in Remark 3.3 yields to functions $\breve{F}^i$ that are defined on $\mathcal{U}_y := \psi^{-1}(U_y) \cap \{s \in C^* | |s| \in [|q|^{1/2}, |q|^{-1/2}[\}$. The analogue of Lemma 3.7 is as follows. For $|s| \in [|q|^{1/2}, |q|^{-1/2}[$, the following statements hold:

- if $|s| \in]|\mathfrak{E}^{(1)}(b)|, |\mathfrak{E}^{(1)}(b)|^{-1}[$, then $|\overline{y}(s)| < 1$;

- if $|s| = |\mathfrak{E}^{(1)}(b)|^{\pm 1}$ then $|\overline{y}(s)| = 1$;
- otherwise $|\overline{y}(s)| > 1$.

By Proposition 3.4, the automorphism of the walk corresponds to the $\mathbf{q}$-dilatation on C^*. The following lemma shows that one can cover C^* either with the $\mathbf{q}$-orbit of the set $\mathcal{U}_x$ or with the $\mathbf{q}$-orbit of $\mathcal{U}_y$.

Lemma 3.9. *The following statement hold:*

- $|\mathbf{q}| \neq 1$;
- *moreover, up to replace $\mathbf{q}$ by some convenient $q^{\mathbb{Z}}$-multiple, the following hold:*
  - *if either $d_{-1,1} = 0$ or $d_{1,-1} \neq 0$, then,*
$$\bigcup_{\ell \in \mathbb{Z}} \sigma_{\mathbf{q}}^{\ell}(\mathcal{U}_x) = C^*;$$
  - *if either $d_{-1,1} \neq 0$ or $d_{1,-1} = 0$ then,*
$$\bigcup_{\ell \in \mathbb{Z}} \sigma_{\mathbf{q}}^{\ell}(\mathcal{U}_y) = C^*.$$

Proof. Let us first prove that $|\mathbf{q}| \neq 1$. By Remark 3.5, one can choose $\mathbf{q}$ so that we have $|q|^{1/2} \leq |\mathbf{q}| < |q|^{-1/2}$. By construction, $\overline{x}(1) = a$. Let $b \in \mathbf{P}^1(C)$ such that $(a,b) \in E$. Since $\iota_1(a,b) = (a,b)$ we have $\iota_2(a,b) \neq (a,b)$ by Lemma 1.12. So let $a' \in \mathbf{P}^1(C)$ distinct from a such that $\sigma(a,b) = (a',b)$. Then, $\overline{x}(\mathbf{q}) = a'$. By Lemma 3.7, $|\overline{x}(s)| < 1$ for $|s| = 1$. Thus, it suffices to prove that $|\overline{x}(\mathbf{q})| = |a'| \geq 1$ to conclude that $|\mathbf{q}| \neq 1$.

Remember that $K(x,y,t) = \widetilde{A}_{-1}(x) + \widetilde{A}_0(x)y + \widetilde{A}_1(x)y^2 = \widetilde{B}_{-1}(y) + \widetilde{B}_0(y)x + \widetilde{B}_1(y)x^2$ with $\widetilde{A}_i(x) \in C[x]$ and $\widetilde{B}_i(y) \in C[y]$. With $\iota_1(a,b) = (a,b)$ and the formulas in §1.3, one finds that
$$b^2 = \frac{A_{-1}(a)}{A_1(a)} = \frac{\widetilde{A}_{-1}(a)}{\widetilde{A}_1(a)}.$$

Let ν be the valuation at $X = 0$ of $\frac{\widetilde{A}_{-1}(X)}{\widetilde{A}_1(X)}$. Lemma A.2 with $|a| < 1$ gives $|b|^2 = |a|^{\nu}$. Note that $\widetilde{A}_1$ and $\widetilde{A}_{-1}$ are polynomial of degree at most two in X, so the integer ν belongs to $\{-2,-1,0,1,2\}$. We have

$$a' = \frac{\widetilde{B}_{-1}(b)}{\widetilde{B}_1(b)a}. \tag{3.4}$$

We will prove that $|a'| \geq 1$ with a case by case study of the values of ν.

Remember that

$$\begin{array}{lcl} \widetilde{A}_{-1} & = & d_{-1,-1} + d_{0,-1}x + d_{1,-1}x^2 \\ \widetilde{A}_1 & = & d_{-1,1} + d_{0,1}x + d_{1,1}x^2 \\ \widetilde{B}_{-1} & = & d_{-1,-1} + d_{-1,0}y + d_{-1,1}y^2 \\ \widetilde{B}_1 & = & d_{1,-1} + d_{1,0}y + d_{1,1}y^2. \end{array} \tag{3.5}$$

Case $\nu \geq 1$. Then, $|b| = |a|^{\nu/2} < 1$. Combining (3.4) and Lemma A.2, we find $|a||a'| = |b|^l$ where l is the valuation at $X = 0$ of $\frac{\widetilde{B}_{-1}(X)}{\widetilde{B}_1(X)}$. This gives $|a'| = |a|^{l\nu/2-1}$. Since l belongs to $\{-2,\dots,2\}$ and ν is in $\{1,2\}$, we get $-3 \leq l\nu/2 - 1 \leq 1$. If $l\nu/2 - 1$

equals 1 then ν must be equal to 2 and by (3.5), we must have $d_{-1,-1} = d_{0,-1} = 0$ and $d_{-1,1} \neq 0$. By Remark 1.10, we must have $d_{-1,0}d_{1,-1} \neq 0$ so that $l = 1$ and $l\nu/2 - 1 = 0$. A contradiction. Then, $l\nu/2 - 1 \leq 0$ and $|a'| \geq 1$.

Case $\nu = 0$. Then, $|b| = 1$. With Lemma A.3 and $|a| < 1$, we obtain $|a'| > 1$.

Case $\nu \leq -1$. Then $|b| = |a|^{\nu/2} > 1$. Combining (3.4) and Lemma A.2, we find $|a'| = |a|^{l\nu/2-1}$ where $l \in \{-2, \dots, 2\}$ is the degree in X of $\frac{\widetilde{B}_{-1}(X)}{\widetilde{B}_1(X)}$. Since l belongs to $\{-2, \dots, 2\}$ and ν is in $\{-1, -2\}$, we get $1 \geq l\nu/2 - 1 \geq -3$. If $l\nu/2 - 1 = 1$ then $\nu = -2$ and by (3.5), we must have $d_{-1,1} = d_{0,1} = 0$ and $d_{-1,-1} \neq 0$. By Remark 1.10, we must have $d_{-1,0}d_{1,1} \neq 0$ so that $l = -1$ and $l\nu/2 - 1 = 0$. A contradiction. Then, $l\nu/2 - 1 \leq 0$ and $|a'| \geq 1$.

Assume that either $d_{-1,1} = 0$ or $d_{1,-1} \neq 0$ and let us prove that

$$\bigcup_{\ell \in \mathbb{Z}} \sigma_{\mathbf{q}}^{\ell}(\mathcal{U}_x) = C^*.$$

By Lemma A.4, there exists $(a_0, b_0) \in E$ such that $|a_0| = 1$ and $\sigma(a_0, b_0) = (a_1, b_1)$ with $|a_1| \leq 1$. By Lemma 3.7, there exists $s_0 \in C^*$ with $|s_0| = |\mathfrak{D}^{(1)}(a)|^{\pm 1}$ such that $\overline{x}(s_0) = a_0$. Since $|q|^{1/2} \leq |\mathbf{q}| < |q|^{-1/2}$ and $|q|^{1/2} < |\mathfrak{D}^{(1)}(a)| < 1$, we find that $|q| < |\mathbf{q}s_0| < |q|^{-1/2}$. Since $|\overline{x}(\mathbf{q}s_0)| = |a_1| \leq 1$, we conclude using Lemma 3.7 that

- either $|\mathbf{q}s_0| \in \mathcal{U}_x$. This proves that
$$\mathcal{U}_x \cap \sigma_{\mathbf{q}}(\mathcal{U}_x) = [|\mathfrak{D}^{(1)}(a)|, |\mathfrak{D}^{(1)}(a)|^{-1}] \cap \sigma_{\mathbf{q}}([|\mathfrak{D}^{(1)}(a)|, |\mathfrak{D}^{(1)}(a)|^{-1}]) \neq \varnothing.$$
Since $|\mathbf{q}| \neq 1$, we deduce that
$$\bigcup_{\ell \in \mathbb{Z}} \sigma_{\mathbf{q}}^{\ell}(\mathcal{U}_x) = C^*.$$
- or $|\mathbf{q}s_0| \in [|q||\mathfrak{D}^{(1)}(a)|, |q||\mathfrak{D}^{(1)}(a)|^{-1}]$. Replacing $\mathbf{q}$ by $\mathbf{q}/q$ allows to conclude.
- or $|\mathbf{q}s_0| \in [|q|^{-1}|\mathfrak{D}^{(1)}(a)|, |q|^{-1}|\mathfrak{D}^{(1)}(a)|^{-1}]$. Replacing $\mathbf{q}$ by $q\mathbf{q}$ allows to conclude.

The proof for $\mathcal{U}_y$ is obtained by a symmetry argument using Lemma A.4 and Remark 3.8.

□

According to Lemma 3.9, we define some auxiliary functions as follows

- if $d_{-1,1} = 0$, we define, for $i = 1, 2$, the function $\widetilde{F}^i(s)$ on $\mathcal{U}_x$ as $F^i(\phi(s), t)$;
- if $d_{-1,1} \neq 0$, the function $\widetilde{F}^i(s)$ is defined on $\mathcal{U}_y$ as $F^i(\psi(s), t)$.

A priori the auxiliary functions $\widetilde{F}^1(s), \widetilde{F}^2(s)$ are defined on $\mathcal{U}_x$ if $d_{-1,1} = 0$ and on $\mathcal{U}_y$ otherwise. Theorem 3.10 below shows that one can meromorphically continue the functions $\widetilde{F}^i(s)$ on C^* so that they satisfy some non-homogeneous rank 1 linear $\mathbf{q}$-difference equations.

Theorem 3.10. *The auxiliary functions $\widetilde{F}^1(s), \widetilde{F}^2(s)$ can be continued meromorphically on C^* so that they satisfy*

$$\widetilde{F}^1(\mathbf{q}s) - \widetilde{F}^1(s) = b_1$$

and

$$\widetilde{F}^2(\mathbf{q}s) - \widetilde{F}^2(s) = b_2,$$

where $b_1 = (x(\mathbf{q}s) - x(s))y(\mathbf{q}s)$ *and* $b_2 = (y(\mathbf{q}s) - y(s))x(s)$ *are two q-periodic meromorphic functions over* C^**.*

Proof. The proof is completely similar to the proof of Lemma 2.5 and relies on the fact that either the $\mathbf{q}$-orbit of $\mathcal{U}_x$ or the $\mathbf{q}$-orbit of $\mathcal{U}_y$ covers C^*. □

Note that by Remark 3.3, the coefficients b_1, b_2 of the $\mathbf{q}$-difference can be identified with rational functions on the algebraic curve E.

3.4. **Differential transcendence.** The strategy to study the differential transcendence of generating functions of non-degenerate weighted models of genus one with infinite group is similar to the one employed in §2. One first relate the differential behavior of the incomplete generating series to the differential algebraic properties of their associated auxiliary functions. Then, one applies to these auxiliary functions the Galois theory of $\mathbf{q}$-difference equations. However, since the coefficients of the $\mathbf{q}$-difference equations satisfied by the auxiliary functions are no longer rational but elliptic, the Galoisian criteria as well as the descent method to obtain some "simple telescopers" are quite technical and postponed to Appendix C. Then, one obtains a first criteria to guaranty the differential transcendence of the incomplete generating series.

Theorem 3.11. *Assume that the weighted model is non-degenerate, of genus one, and that the group of the walk is infinite. If* $Q(x,0,t)$ *is* $\left(\frac{d}{dx}, \frac{d}{dt}\right)$*-differentially algebraic over* $\mathbb{Q}$ *then there exist* $c_0, \dots, c_n \in C$ *not all zero and* $h \in C_q$ *such that*

$$c_0 b_1 + c_1 \partial_s(b_1) + \cdots + c_n \partial_s^n(b_1) = \sigma_{\mathbf{q}}(h) - h. \tag{3.6}$$

A symmetrical result holds for $Q(0,y,t)$ replacing b_1 by b_2.

Proof. Since the group of the walk is of infinite order, the automorphism σ is of infinite order. Therefore by Lemma 3.6 the elements $\mathbf{q}$ and q defined in Proposition 3.4 are multiplicatively independent. Assume that $Q(x,0,t)$ is $\left(\frac{d}{dx}, \frac{d}{dt}\right)$-differentially algebraic over $\mathbb{Q}$. Let $\widetilde{F}^1(s)$ be the auxiliary function defined above.

We denote by $C_{\mathbf{q}}.C_q$ the compositum of the fields C_q and $C_{\mathbf{q}}$ inside the field of meromorphic functions over C^*. We claim that $\widetilde{F}^1(s)$ is $(\partial_s, \Delta_{t,\mathbf{q}})$-differentially algebraic over $C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}}, \ell_q)$. Let us prove this claim when $d_{-1,1} = 0$, the proof when $d_{-1,1} \neq 0$ being similar. Reasoning as in Lemma 2.6, one can show that, for $n, m \in \mathbb{N}$, one has

$$(\partial_t^n \partial_x^m F^1)(\overline{x}(s), t) = \frac{1}{\partial_s(\overline{x}(s))^m} \Delta_{t,q}^n \partial_s^m (\widetilde{F}^1(s)) + \sum_{i \leq 2n+m, j<n} r_{i,j} \Delta_{t,q}^j \partial_s^i (\widetilde{F}^1(s)),$$

where $r_{i,j} \in C_{\mathbf{q}}(\ell_{\mathbf{q}})(\overline{x}(s), \partial_s^l \partial_t^k(\overline{x}(s)), \dots)$. By construction, $\overline{x}(s)$ is in C_q so that Lemma D.5 implies that $\partial_s^l \partial_t^k(\overline{x}(s)) \in C_q(\ell_q)$ for any positive integers k, l. Then, the field $C_{\mathbf{q}}(\ell_{\mathbf{q}})(\overline{x}(s), \partial_s^l \partial_t^k(\overline{x}(s)), \dots)$ generated by $\overline{x}$ and its derivatives with respect to ∂_s and ∂_t is contained in $C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}}, \ell_q)$. Thus, any non-trivial polynomial relation between the x-t-derivatives of $Q(x,0,t)$ yields to a non-trivial polynomial relation between the derivatives of $\widetilde{F}^1(s)$ with respect to ∂_s and $\Delta_{t,\mathbf{q}}$ over $C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}}, \ell_q)$. This proves the claim.

By Theorem 3.10, the function $\widetilde{F}^1(s)$ satisfies $\widetilde{F}^1(\mathbf{q}s) - \widetilde{F}^1(s) = b_1(s)$ with $b_1(s) \in C_q \subset C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}}, \ell_q)$. Since $\widetilde{F}^1(s)$ is $(\partial_s, \Delta_{t,\mathbf{q}})$-differentially algebraic over $C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}}, \ell_q)$, Proposition D.6 and Corollary D.14 imply that there exist $m \in \mathbb{N}$ and $d_0, \dots, d_m \in C_{\mathbf{q}}$ not all zero and $g \in C_{\mathbf{q}}.C_q(\ell_q)$ such that

$$d_0 b_1 + d_1 \partial_s(b_1) + \cdots + d_m \partial_s^m(b_1) = \sigma_{\mathbf{q}}(g) - g.$$

Since b_1 is in C_q, Lemma D.13 allows to perform a descent on the coefficients of the telescoping relation above. Thus, there exist $c_0, \dots, c_n \in C$ not all zero and $h \in C_q$ such that

$$c_0 b_1 + c_1 \partial_s(b_1) + \cdots + c_n \partial_s^n(b_1) = \sigma_{\mathbf{q}}(h) - h.$$

This concludes the proof. The symmetry argument between x and y gives the proof for $Q(0, y, t)$. □

Theorem 3.11 has an easy corollary concerning the differential transcendence of the complete generating series for weighted models of genus one with infinite group.

Theorem 3.12. *For any non-degenerate weighted model of genus one with infinite group, the following statements are equivalent:*

(1) *the series $Q(x, 0, t)$ is $\left(\frac{d}{dx}, \frac{d}{dt}\right)$-differentially algebraic over $\mathbb{Q}$;*
(2) *there exist $c_0, \dots, c_n \in C$ not all zero and $h \in C_q$ such that*

$$c_0 b_1 + c_1 \partial_s(b_1) + \cdots + c_n \partial_s^n(b_1) = \sigma_{\mathbf{q}}(h) - h;$$

(3) *the series $Q(x, 0, t)$ is $\frac{d}{dx}$-differentially algebraic over $\mathbb{C}$.*

Furthermore, if none of the above conditions hold, then $Q(x, y, t)$ is $\left(\frac{d}{dx}, \frac{d}{dt}\right)$-differentially transcendental over $\mathbb{Q}$.

Remark 3.13. Similarly, we may prove that the following statements are equivalent:

(1) The series $Q(0, y, t)$ is $\left(\frac{d}{dy}, \frac{d}{dt}\right)$-differentially algebraic over $\mathbb{Q}$.
(2) There exist $c_0, \dots, c_n \in C$ not all zero and $h \in C_q$ such that

$$c_0 b_2 + c_1 \partial_s(b_2) + \cdots + c_n \partial_s^n(b_2) = \sigma_{\mathbf{q}}(h) - h.$$

(3) The series $Q(0, y, t)$ is $\frac{d}{dy}$-differentially algebraic over $\mathbb{C}$.

Furthermore, if none of the above conditions holds then $Q(x, y, t)$ is $\left(\frac{d}{dy}, \frac{d}{dt}\right)$-differentially transcendental over $\mathbb{Q}$.

Proof. Since the group is infinite, the automorphism σ is of infinite order. Therefore by Lemma 3.6 the elements $\mathbf{q}$ and q defined in Proposition 3.4 are multiplicatively independent.

Theorem 3.11 gives (1) $\Rightarrow$ (2). Assume that (2) holds. There exist $c_0, \dots, c_n \in C$ not all zero and $h \in C_q$ such that

$$c_0 b_1 + c_1 \partial_s(b_1) + \cdots + c_n \partial_s^n(b_1) = \sigma_{\mathbf{q}}(h) - h. \tag{3.7}$$

Combining (3.7) with the functional equation satisfied by $\widetilde{F}^1(s)$ and using the commutativity of $\sigma_{\mathbf{q}}$ and ∂_s, one finds that

$$\sigma_{\mathbf{q}}\left[c_0 \widetilde{F}^1(s) + \cdots + c_n \partial_s^n(\widetilde{F}^1(s)) - h\right] = c_0 \widetilde{F}^1(s) + \cdots + c_n \partial_s^n(\widetilde{F}^1(s)) - h. \tag{3.8}$$

Since $\widetilde{F}^1$ and h are meromorphic over C^*, there exists $g \in C_{\mathbf{q}}$ such that

$$c_0\widetilde{F}^1(s) + \cdots + c_n\partial_s^n(\widetilde{F}^1(s)) - h = g.$$

Therefore, $\widetilde{F}^1(s)$ is ∂_s-differentially algebraic over C_q. Reasoning as in Lemma 2.6, one finds a non-trivial algebraic relation with coefficients in C_q between the first n-th derivatives of F^1 with respect to ∂_x evaluated in $(\overline{x}(s), t)$. Any element of $C_q = C(\overline{x}(s), \overline{y}(s))$ is algebraic over $C(\overline{x}(s))$. Therefore, the first n-th derivatives of F^1 with respect to ∂_x evaluated in $(\overline{x}(s), t)$ are still algebraically dependent over $C(\overline{x}(s))$. We conclude that $F^1(x,t) = K(x,0,t)Q(x,0,t)$ is $\frac{d}{dx}$-differentially algebraic over $C(x)$ and therefore over $\mathbb{Q}$ by Remark 1.3. This proves that (2) $\Rightarrow$ (3). Statement (3) implies obviously (1). $\square$

As a corollary, one finds criteria ensuring the differential transcendence of the incomplete generating series.

Corollary 3.14. *For all but 9 of the non-degenerate unweighted models of genus one, with an infinite group, the generating series $Q(x,0,t)$ and $Q(0,y,t)$ are respectively $\left(\frac{d}{dx}, \frac{d}{dt}\right)$ and $\left(\frac{d}{dy}, \frac{d}{dt}\right)$-transcendental over $\mathbb{Q}$ (see Figure 1).*
If the weighted model is non-degenerate, of genus one, with an infinite group, and at least one of the following situation holds:

- $d_{1,0}^2 - 4d_{1,1}d_{1,-1}$ *is not a square in* $\mathbb{Q}$*;*
- $d_{0,1}^2 - 4d_{1,1}d_{-1,1}$ *is not a square in* $\mathbb{Q}$*;*
- $d_{1,1} = 0$, $d_{1,0}d_{0,1} \neq 0$ *and there are no* $\mathbb{Q}$ *points of* E *fixed by* ι_1 *or* ι_2*;*
- $d_{1,1} = d_{1,0} = 0$, $d_{0,1} \neq 0$*;*
- $d_{1,1} = d_{0,1} = 0$, $d_{1,0} \neq 0$*;*

then, the generating series $Q(x,0,t)$ and $Q(0,y,t)$ are respectively $\left(\frac{d}{dx}, \frac{d}{dt}\right)$ and $\left(\frac{d}{dy}, \frac{d}{dt}\right)$-transcendental over $\mathbb{Q}$. In all the above cases, the complete generating series is $\left(\frac{d}{dx}, \frac{d}{dt}\right)$ and $\left(\frac{d}{dy}, \frac{d}{dt}\right)$-transcendental over $\mathbb{Q}$.

Proof of Corollary 3.14. By Theorem 3.12 and Remark 3.13, it is sufficient to prove that the the generating series $Q(x,0,t)$ and $Q(0,y,t)$ are respectively $\frac{d}{dx}$ and $\frac{d}{dy}$-transcendental over $\mathbb{Q}$. This is

- the main result of [DHRS18, Section 5] for all but 9 of the unweighted non-degenerate models of genus one with infinite group;
- of [DR19, Section 3.2] for the weighted models above.

$\square$

Appendix A. Non-archemedean estimates

In this section, we give some non-archimedean estimates, which will be crucial to uniformize the kernel curve.

A.1. **Discriminants of the kernel equation.** Lemma A.1 relates the genus of the kernel curve to the simplicity of the roots of the discriminant of the kernel polynomial. It also ensures the existence of a root with convenient norm estimates. Let us remind, see (1.4), that we have defined $\mathfrak{D}(x) := \Delta_x(x,1)$, where $\Delta_x(x_0,x_1)$ is the discriminants of the second degree homogeneous polynomials $y \mapsto \widetilde{K}(x_0,x_1,y,1,t)$.

Lemma A.1. *For any non-degenerate weighted model of genus one, the following holds:*

- *all the roots of $\Delta_x(x_0,x_1)$ in $\mathbf{P}^1(C)$ are simple;*
- *the discriminant $\mathfrak{D}(x) := \Delta_x(x,1)$ has a root $a \in C$ such that $|a| < 1$, $|\mathfrak{D}^{(2)}(a) - 2| < 1$, and $|\mathfrak{D}^{(1)}(a)|, |\mathfrak{D}^{(3)}(a)|, |\mathfrak{D}^{(4)}(a)| < 1$ where $\mathfrak{D}^{(i)}$ denote the i-th derivative with respect to x of $\mathfrak{D}(x)$.*

A symmetric statement holds for $\Delta_y(y_0,y_1)$ by replacing $\mathfrak{D}$ by $\mathfrak{E}$.

Proof. The first assertion is [DHRS17, Lemma 4.4]. First, let us prove the existence of a root $a \in C$ of $\mathfrak{D}(x)$ such that $|a| < 1$. Suppose to the contrary that all the roots of $\mathfrak{D}(x)$ have a norm greater than or equal to 1. If α_0 is zero then zero is a root: a contradiction. Thus, we can assume that α_0 is non-zero.

Let us first assume that $\alpha_4 \neq 0$. The product of the roots of $\mathfrak{D}(x)$ equals

$$\frac{\alpha_0}{\alpha_4} = \frac{t^2(d_{-1,0}^2 - 4d_{-1,-1}d_{-1,1})}{t^2(d_{1,0}^2 - 4d_{1,-1}d_{1,1})}.$$

Then we conclude that $|\frac{\alpha_0}{\alpha_4}| = 1$ so that each of the roots must have norm 1. Then, considering the symmetric functions of the roots of $\mathfrak{D}(x)$, we conclude that, for any $i = 0, \ldots, 3$, the element $\frac{\alpha_i}{\alpha_4}$ should have norm smaller than or equal to 1. Since

$$\frac{\alpha_2}{\alpha_4} = \frac{-4d_{-1,-1}d_{1,1}t^2 - 4d_{0,-1}d_{0,1}t^2 - 4d_{1,-1}d_{-1,1}t^2 + 2d_{-1,0}d_{1,0}t^2 + d_{0,0}^2t^2 - 2td_{0,0} + 1}{t^2(d_{1,0}^2 - 4d_{1,-1}d_{1,1})},$$

has norm strictly greater than 1, we find a contradiction.

Assume now that $\alpha_4 = 0$. Since the roots of $\Delta_x(x_0,x_1)$ in $\mathbf{P}^1(C)$ are simple, the coefficient α_3 is non-zero. The product of the roots of $\mathfrak{D}(x)$ equals

$$-\frac{\alpha_0}{\alpha_3} = \frac{-t^2(d_{-1,0}^2 - 4d_{-1,-1}d_{-1,1})}{2t^2d_{1,0}d_{0,0} - 2td_{1,0} - 4t^2(d_{0,1}d_{1,-1} + d_{1,1}d_{0,-1})}.$$

Then, it is clear that $|\frac{\alpha_0}{\alpha_3}| \leq 1$ and that each of the roots has norm 1. Thus, the symmetric function $\frac{\alpha_2}{\alpha_3}$ should also have norm smaller than or equal to 1. But

$$-\frac{\alpha_2}{\alpha_3} = \frac{-4d_{-1,-1}d_{1,1}t^2 - 4d_{0,-1}d_{0,1}t^2 - 4d_{1,-1}d_{-1,1}t^2 + 2d_{-1,0}d_{1,0}t^2 + d_{0,0}^2t^2 - 2td_{0,0} + 1}{2t^2d_{1,0}d_{0,0} - 2td_{1,0} - 4t^2(d_{0,1}d_{1,-1} + d_{1,1}d_{0,-1})},$$

has norm strictly bigger than 1. We find a contradiction again.

Let a be a root of $\mathfrak{D}(x)$ in C with $|a| < 1$. Since $a, \alpha_1, \alpha_3, \alpha_4$ have norm smaller than 1, $|\alpha_2 - 1| < 1$, and

- $\mathfrak{D}^{(1)}(a) = \alpha_1 + 2\alpha_2 a + 3\alpha_3 a^2 + 4\alpha_4 a^3$;
- $\mathfrak{D}^{(2)}(a) = 2\alpha_2 + 6\alpha_3 a + 12\alpha_4 a^2$;
- $\mathfrak{D}^{(3)}(a) = 6\alpha_3 + 24\alpha_4 a$;
- $\mathfrak{D}^{(4)}(a) = 24\alpha_4$,

we have $|\mathfrak{D}^{(2)}(a)-2|<1$, and $|\mathfrak{D}^{(1)}(a)|, |\mathfrak{D}^{(3)}(a)|, |\mathfrak{D}^{(4)}(a)|<1$. The statement for $\Delta_y(y_0,y_1)$ is symmetrical and we omit its proof. □

A.2. **Automorphisms of the walk on the domain of convergence.** In this section, we study the action of the group of the walk on the product of the unit disks in $\mathbf{P}^1(C)\times\mathbf{P}^1(C)$. This product is the fundamental domain of convergence of the generating series.

We need a preliminary lemma that explains how one can compute the norm of the values of a rational function.

Lemma A.2. *Let $f\in C(X)$ be a non-zero rational function and let $a\in\mathbf{P}^1(C)$. Let ν (resp. d) be the valuation at $X=0$ (resp. ∞) of f with the convention that $\nu=+\infty$, $d=-\infty$ if $f=0$. The following statements hold:*

- *if $|a|<1$, then $|f(a)|=|a|^{\nu}$;*
- *if $|a|>1$, then $|f(a)|=|a|^{d}$.*

Proof. Let us prove the first case, the second being completely symmetrical. Let us write $f(X)$ as $\frac{\sum_{i=\nu_1}^{r_1} c_i X^i}{\sum_{j=\nu_2}^{r_2} d_j X^j}$ with $c_{\nu_1}d_{\nu_2}\neq 0$. If $k>l$, we note that $|a^k|<|a^l|$. Then

$$|f(a)|=\frac{|\sum_{i=\nu_1}^{r_1} c_i a^i|}{|\sum_{j=\nu_2}^{r_2} d_j a^j|}=|a|^{\nu_1-\nu_2}=|a|^{\nu}.$$

□

The following lemma explains how the fundamental involutions permute the interior and the exterior of the fundamental domain of convergence.

Lemma A.3. *For any non-degenerate weighted model, the following statements hold:*

1. *for any $a\in C$ with $|a|=1$, there exist $b_{\pm}\in\mathbf{P}^1(C)$ with $|b_-|<1$, and $|b_+|>1$, such that $K(a,b_{\pm},t)=0$;*
2. *for any $b\in C$ with $|b|=1$, there exist $a_{\pm}\in\mathbf{P}^1(C)$ with $|a_-|<1$, and $|a_+|>1$, such that $K(a_{\pm},b,t)=0$.*

Proof. See [DR19, Section 1.3] for a similar result in the situation where C is replaced by $\mathbb{C}$.

The statements are symmetrical, so we only prove the first one. Since C is algebraically closed and the model is non-degenerate, Proposition 1.6 implies that $K(x,y,t)$ is of degree 2 in y. Then, for any $a\in C$, there are two elements $b_{\pm}\in\mathbf{P}^1(C)$ such that $K(a,b_{\pm},t)=0$. let $a\in C$ with $|a|=1$. We write

$$K(a,y,t)=t\alpha+\beta y+t\gamma y^2 \tag{A.1}$$

where

- $\alpha=-\sum_{i=-1}^{1} d_{i,-1}a^{i+1}$;
- $\beta=a-t\sum_{i=-1}^{1} d_{i,0}a^{i+1}$;
- $\gamma=-\sum_{i=-1}^{1} d_{i,1}a^{i+1}$.

Since $|a|=1$, we find $|\beta|=1$, $|\alpha|,|\gamma|\leq 1$. First let us prove that there is no point $(a_0,b_0)\in E$ such that $|a_0|=|b_0|=1$. Indeed, suppose to the contrary that $|a_0|=|b_0|=1$ and $K(a_0,b_0,t)=0$. Then, $|\beta|=|a_0|=1$ and $|\gamma|,|\alpha|\leq 1$ so that the equality $|\beta b_0| = |t(\alpha+\gamma b_0^2)|$ implies $|b_0|<1$. We find a contradiction. From the equation $K(a,b,t)=0$, we deduce that

$$\text{if } |b|<1, \text{ then } |t\alpha| = |\beta b + t\gamma b^2| = |\beta b| \text{ which gives } |b| = |t\alpha|; \tag{A.2}$$

$$\text{if } |b|>1, \text{ then } |\frac{1}{b}|<1 \text{ and we find } |t\gamma| = \left|\frac{t\alpha}{b^2}+\frac{\beta}{b}\right| = \left|\frac{\beta}{b}\right| = \left|\frac{1}{b}\right|. \tag{A.3}$$

Using $K(a,b_\pm,t)=0$, we find

$$b_- b_+ = \frac{\alpha}{\gamma}, \tag{A.4}$$

with the convention that b_+ is $[1:0]$ if $\gamma=0$. If $\gamma=0$ then $b_- = \frac{-t\alpha}{\beta}$ has norm smaller than 1, which concludes the proof in that case. Assume now that $\gamma\neq 0$. Since $|b_+|$ and $|b_-|$ cannot have norm 1, we just need to discard the cases "$|b_+|<1$ and $|b_-|<1$" or "$|b_+|>1$ and $|b_-|>1$". If $\alpha=0$, then one of the root is zero, say $b_-=0$, and $|b_+| = \frac{|\beta|}{|t\gamma|}>1$, which concludes the proof in that case. If $\alpha\neq 0$ then one can suppose to the contrary that $|b_+|<1$ and $|b_-|<1$. From (A.2), we obtain $|b_+|=|b_-|=|t\alpha|$, which gives

$$|b_+b_-| = |t\alpha|^2 = \frac{|\alpha|}{|\gamma|}.$$

Then, $|t^2\alpha| = \frac{1}{|\gamma|}\geq 1$, which contradicts $|t^2\alpha|<1$. Suppose to the contrary that $|b_+|>1$ and $|b_-|>1$. By (A.3), $|b_+|=|b_-|=\frac{1}{|t\gamma|}$ which gives

$$|b_+b_-| = \frac{1}{|t\gamma|^2} = \frac{|\alpha|}{|\gamma|}.$$

Thus, $|t^2\alpha| = \frac{1}{|\gamma|}\geq 1$, and once again, we find a contradiction. □

Lemma A.4 explains how the the intersection of the fundamental domain of convergence of the generating series and its image by σ is non-empty. This result is therefore crucial in order to continue the generating series to the whole C^*.

Lemma A.4. *For any non-degenerate weighted model, the following statements hold:*

- *if $d_{-1,1}=0$ or $d_{1,-1}\neq 0$ there exists $(a,b)\in E$ with $|a|=1$ such that $\sigma(a,b)=(a',b')$ with $|a'|\leq 1$;*
- *if $d_{-1,1}\neq 0$ or $d_{1,-1}=0$ there exists $(a,b)\in E$ with $|b|=1$ such that $\sigma(a,b)=(a',b')$ with $|b'|\leq 1$.*

Proof. Using the symmetry between x and y mentioned in Remark 1.2, we only prove the first statement of Lemma A.4.

Let $a\in \mathbf{P}^1(C)$ such that $|a|=1$. By Lemma A.3, there exist $b_+\in\mathbf{P}^1(C)$ with $|b_+|>1$ and $b_-\in C$ with $|b_-|<1$ such that $(a,b_\pm)\in E$. Let B_i as in (1.2) and let ν (resp d) be the valuation at 0 (resp ∞) of the rational fraction $\frac{B_{-1}(y)}{B_1(y)} = \frac{\sum_{j=-1}^{1} d_{-1,j}y^j}{\sum_{j=-1}^{1} d_{1,j}y^j}\in C(y)$ (note that B_1 is not identically zero by Proposition 1.6). We claim that either $\nu\geq 0$ or $d\leq 0$. If $d_{1,-1}\neq 0$ then $\nu\geq 0$. If $d_{-1,1}=0$ then either $d\leq 0$ or $d=1$. In the latter situation, we must have $d_{1,1}=d_{1,0}=0$ and $d_{-1,0}\neq 0$. Since the model is non-degenerate, we must have $d_{1,-1}\neq 0$ by Proposition 1.6. In that case, $\nu\geq 0$. This proves the claim.

Let $a_+, a_- \in \mathbf{P}^1(C)$ such that $\iota_2(a, b_+) = (a_+, b_+)$ and $\iota_2(a, b_-) = (a_-, b_-)$. This gives

$$a_+ = \frac{B_{-1}(b_+)}{B_1(b_+)a} \text{ and } a_- = \frac{B_{-1}(b_-)}{B_1(b_-)a}. \tag{A.5}$$

Since $\sigma(a, b_-) = (a_+, b_+)$ (resp $\sigma(a, b_+) = (a_-, b_-)$), it is enough to prove that either a_+ or a_- has norm smaller or equal to 1. If $d \leq 0$, we combine (A.5), Lemma A.2 and $|b+| > 1$ to find $|aa_+| = |a_+| = |b_+|^d \leq 1$. If $\nu \geq 0$, we combine (A.5), Lemma A.2 and $|b_-| < 1$ to find $|aa_-| = |a_-| = |b_-|^\nu \leq 1$. This ends the proof. □

Appendix B. Tate curves and their normal forms

Let $(C, |\ |)$ be a complete non-archimedean algebraically closed valued field of zero characteristic and let $q \in C$ such that $0 < |q| < 1$. In this section, we recall some of the basic properties of elliptic curves over non-archimedean fields. The period lattice is here replaced by a discrete multiplicative group of the form $q^{\mathbb{Z}}$. Then, the quotient of $\mathbb{C}$ by a period lattice is replace by the so called *Tate curve*, which corresponds to the naive quotient of the multiplicative group C^* by $q^{\mathbb{Z}}$. However, in the non-archimedean context, only elliptic curves with J-invariant of norm greater than equal to one can be analytically uniformized by Tate curves (see Proposition B.2). The analytic geometry behind is the rigid analytic geometry as developed in [FvdP04]. We will not introduce this theory here but we just recall briefly the algebraic geometrical and special functions aspects of Tate curves.

B.1. **Special functions on a Tate curve.** We recall that any holomorphic function f on C^* can be represented by an everywhere convergent Laurent series $\sum_{n\in\mathbb{Z}} a_n s^n$ with $a_n \in C$. Moreover any non-zero meromorphic function on C^* can be written as $\frac{g}{h}$ such that the holomorphic functions g and h have no common zeros. We shall denote by $\mathcal{M}er(C^*)$ the field of meromorphic functions over C^*.

Remark B.1. If k is a complete non-archimedian sub-valued field of C and q belongs to k, every result quoted above still holds over k.

The analytification of the elliptic curve E_q is isomorphic to the Tate curve, which is the rigid analytic space corresponding to the naive quotient of $C^*/q^{\mathbb{Z}}$. The curve E_q is therefore a“canonical” elliptic curve. A natural question is ”Given an elliptic curve E defined over C, is there a q such that E is isomorphic to E_q?” The answer is positive under certain assumption on the J-invariant $J(E)$ of E.

Proposition B.2 (Theorem 5.1.18 in [FvdP04])**.** *Let E be an elliptic curve over C such that $|J(E)| > 1$. Then, there exists $q \in C$ such that $0 < |q| < 1$ and E is isomorphic to the elliptic curve E_q.*

Remind that we have defined $s_k = \sum_{n>0} \frac{n^k q^n}{1-q^n} \in C$ for $k \geq 1$, and

$$X(s) = \sum_{n\in\mathbb{Z}} \frac{q^n s}{(1-q^n s)^2} - 2s_1, \quad Y(s) = \sum_{n\in\mathbb{Z}} \frac{(q^n s)^2}{(1-q^n s)^3} + s_1.$$

They are q-periodic meromorphic functions over C^*. By Proposition 3.1, the field C_q of q-periodic meromorphic functions over C^* coincides with the field generated over C by $X(s)$ and $Y(s)$.

Since we need to understand what is the pullback of the fundamental domain of convergence of the generating series via this uniformization, we prove some basic properties on the norm of $X(s)$. Remind that $X(s) = X(1/s)$ and $X(qs) = X(s)$. Thus it suffices to study $|X(s)|$ for $|q|^{1/2} \leq |s| \leq 1$. The following study follows the arguments of [Sil94, §V.4].

Lemma B.3. *Let $s \in C^*$. The following holds:*

- *If $|q|^{1/2} < |s| < 1$, then $|X(s)| = |s|$;*
- *If $|s| = 1$, then $|X(s)| \geq 1$;*
- *If $|s| = |q|^{1/2}$, then $|X(s)| \leq |s|$.*

Proof. Since $X(s)$ has a pole in $s = 1$ we may further assume that $s \neq 1$. Let us rewrite $X(s)$:

$$X(s) = \frac{s}{(1-s)^2} + \sum_{n>0} \frac{q^n s}{(1-q^n s)^2} + \frac{q^n s^{-1}}{(1-q^n s^{-1})^2} - 2\frac{q^n}{1-q^n}.$$

This means that we have

$$|X(s)| \leq \max\left(\left|\frac{s}{(1-s)^2}\right|, \left|\sum_{n>0} \frac{q^n s}{(1-q^n s)^2} + \frac{q^n s^{-1}}{(1-q^n s^{-1})^2} - 2\frac{q^n}{1-q^n}\right| \right), \tag{B.1}$$

with equality when $|\frac{s}{(1-s)^2}| \neq |\sum_{n>0} \frac{q^n s}{(1-q^n s)^2} + \frac{q^n s^{-1}}{(1-q^n s^{-1})^2} - 2\frac{q^n}{1-q^n}|$. Let us consider $s \in C^* \setminus \{1\}$ with $|q|^{1/2} \leq |s| \leq 1$. Using $|q| < 1$ we find that $|q^n s| \leq |qs| < 1$ for every $n \geq 1$. This shows that the norm of $q^n s$ is strictly smaller than 1. Then, $\left|\frac{q^n s}{(1-q^n s)^2}\right| = |q^n s| < |s|$. On the other hand, $|q^n| \leq |q| < |s|$ and $|\frac{q^n}{1-q^n}| < |s|$. Finally, when $|q|^{1/2} < |s|$, we have $|q^n s^{-1}| \leq |qs^{-1}| < |qq^{-1/2}| < |s|$ and therefore $\left|\frac{q^n s^{-1}}{(1-q^n s^{-1})^2}\right| = |q^n s^{-1}| < |s|$. This proves that, for any $s \in \mathbf{P}^1(C)$ such that $|q|^{1/2} < |s| \leq 1$, we have

$$\left|\sum_{n>0} \frac{q^n s}{(1-q^n s)^2} + \frac{q^n s^{-1}}{(1-q^n s^{-1})^2} - 2\frac{q^n}{1-q^n}\right| < |s|. \tag{B.2}$$

When, $|q|^{1/2} = |s|$ and $n \geq 2$, we have $|q^n s^{-1}| \leq |q^2 s^{-1}| = |q^2 q^{-1/2}| < |s|$, and therefore $\left|\frac{q^n s^{-1}}{(1-q^n s^{-1})^2}\right| = |q^n s^{-1}| < |s|$. Moreover, if $|q|^{1/2} = |s|$ then $|qs^{-1}| = |qq^{-1/2}| = |s|$. Therefore $\left|\frac{qs^{-1}}{(1-qs^{-1})^2}\right| = |qs^{-1}| = |s|$. We conclude that

$$\left|\sum_{n>0} \frac{q^n s}{(1-q^n s)^2} + \frac{q^n s^{-1}}{(1-q^n s^{-1})^2} - 2\frac{q^n}{1-q^n}\right| = |s|. \tag{B.3}$$

It remains to consider the term $\frac{s}{(1-s)^2}$. If $|s| < 1$ then we have $\left|\frac{s}{(1-s)^2}\right| = |s|$. Combining with (B.1), (B.2) and (B.3) respectively, we obtain the result when $|q|^{1/2} < |s| < 1$ and

$|q|^{1/2}=|s|<1$ respectively. If $|s|=1$ and $s\neq 1$ then $|1-s|\leq 1$. Thus, $\left|\frac{s}{(1-s)^2}\right|\geq |s|=1$, which, combined with (B.1) and (B.2) concludes the proof. □

B.2. **Tate and Weierstrass normal forms.** In [DR19], the authors generalize the results of [KR12] and attach a Weierstrass normal form to the kernel curve. The following proposition proves that, with some care, their result passes to a non-archimedean framework.

Let us consider a non-degenerate weighted model of genus one and let us write its kernel polynomial as follows: $K(x,y,t)=\widetilde{A}_0(x)+\widetilde{A}_1(x)y+\widetilde{A}_2(x)y^2=\widetilde{B}_0(y)+\widetilde{B}_1(y)x+\widetilde{B}_2(y)x^2$ with $\widetilde{A}_i(x)\in C[x]$ and $\widetilde{B}_i(y)\in C[y]$. The following proposition gives a Weierstrass normal form for the kernel curve.

Proposition B.4. *Let $a\in C$ be as in Lemma A.1. Let E_1 be the elliptic curve defined by the Weierstrass equation*

$$y_1^2=4x_1^3-g_2x_1-g_3, \tag{B.4}$$

with

$$\begin{aligned} g_2 &= \frac{\mathfrak{D}^{(2)}(a)^2}{3}-2\frac{\mathfrak{D}^{(1)}(a)\mathfrak{D}^{(3)}(a)}{3}\\ g_3 &= -\frac{\mathfrak{D}^{(2)}(a)^3}{27}+\frac{\mathfrak{D}^{(1)}(a)\mathfrak{D}^{(2)}(a)\mathfrak{D}^{(3)}(a)}{9}-\frac{\mathfrak{D}^{(1)}(a)^2\mathfrak{D}^{(4)}(a)}{6}. \end{aligned} \tag{B.5}$$

Then, the rational map

$$\begin{array}{ccc} E_1 & \rightarrow & E\subset \mathbf{P}^1(C)\times\mathbf{P}^1(C)\\ [x_1:y_1:1] & \mapsto & (\overline{x},\overline{y}) \end{array}$$

where

$$\overline{x}=a+\frac{\mathfrak{D}^{(1)}(a)}{x_1-\frac{\mathfrak{D}^{(2)}(a)}{6}} \text{ and } \overline{y}=\frac{\frac{\mathfrak{D}^{(1)}(a)y_1}{2(x_1-\frac{\mathfrak{D}^{(1)}(a)}{6})^2}-\widetilde{A}_1\left(a+\frac{\mathfrak{D}^{(1)}(a)}{x_1-\frac{\mathfrak{D}^{(2)}(a)}{6}}\right)}{2\widetilde{A}_2\left(a+\frac{\mathfrak{D}^{(1)}(a)}{x_1-\frac{\mathfrak{D}^{(2)}(a)}{6}}\right)},$$

is an isomorphism of elliptic curves that sends the point $\mathcal{O}=[1:0:0]$ in E_1 to the point $\left(a,\dfrac{-\widetilde{A}_1(a)}{2\widetilde{A}_2(a)}\right)\in E$.

Proof. This is the same proof as in [DR19, Proposition 18]. Note that there is only one configuration here since we have chosen a root of the discriminant $|a|<1$ which can not be infinity. □

We recall that the J-invariant $J(E_1)$ of the elliptic curve E_1 given in a Weierstrass form $y_1^2=4x_1^3-g_2x_1-g_3$ equals to $J(E_1)=12^3\frac{g_2^3}{g_2^3-27g_3^2}$. For a weighted model of genus one, the J-invariant $J(E)$ of the kernel curve has modulus strictly greater than 1 by Lemma 1.11. Since $J(E)=J(E_1)$, Proposition B.2 shows that there exists $q\in C^*$ such that $0<|q|<1$ and E_1 is isomorphic to E_q. In order to explicit this isomorphism, we need to understand how one passes from to a Tate normal form to a Weierstrass normal form. This is the content of the following lemmas.

Lemma B.5. *[§6, Page 29 in [Roq70]] In the notation of Proposition 3.1, the change of variable $X = x - \frac{1}{12}$ and $Y = \frac{1}{2}(y - x + \frac{1}{12})$ maps the Tate equation*

$$Y^2 + XY = X^3 + BX + \widetilde{C}$$

onto the Weierstrass equation

$$y^2 = 4x^3 - h_2 x - h_3,$$

where $h_2 = \frac{1}{12} + 20s_3$ and $h_3 = \frac{-1}{6^3} + \frac{7}{3}s_5$.

As detailed above, the elliptic curves E_1 and E_q are isomorphic. The following lemma gives the form of an explicit isomorphism between theses two curves.

Lemma B.6. *Let $y^2 = 4x^3 - h_2x - h_3$ be the Weierstrass normal form (resp. $Y^2 + XY = X^3 + BX + \widetilde{C}$ its Tate normal form) of E_q as in Lemma B.5 and let $y_1^2 = 4x_1^3 - g_2x_1 - g_3$ be the Weierstrass normal form of E_1 as in Proposition B.4.*

There exists $u \in C^*$ such that the following map

$$\begin{array}{rcl} E_q & \to & E_1, \\ (X, Y) & \mapsto & (u^2(X + \frac{1}{12}), u^3(2Y + X)) \end{array}$$

is an isomorphism of elliptic curves. Moreover, the following holds

- *$h_2 = \frac{g_2}{u^4}$ and $h_3 = \frac{g_3}{u^6}$;*
- *$\Delta_q = \frac{\Delta_1}{u^{12}}$ where Δ_1 and Δ_q denote the discriminants of the Weierstrass equations of E_1 and E_q respectively.*

Proof. From [Sil09, Proposition 3.1, Chapter III], we deduce that any isomorphism between the elliptic curves E_1 and E_q is given by $x_1 = u^2x + \alpha$ and $y_1 = u^3y + \beta u^2 x + \gamma$ with $u \in C^*$, $\alpha, \beta, \gamma \in C$. Since both equations are in Weierstrass normal form, we necessarily have $\alpha = \beta = \gamma = 0$. This proves the first point. From $y_1^2 = 4x_1^3 - g_2x_1 - g_3$, we substitute x_1, y_1 by x, y to find

$$u^6y^2 = 4u^6x^3 - g_2u^2x - g_3.$$

Dividing the both sides by u^6 we find $h_2 = \frac{g_2}{u^4}$ and $h_3 = \frac{g_3}{u^6}$. The assertion on the discriminants follows from $\Delta_q = h_2^3 - 27h_3^2$ and $\Delta_1 = g_2^3 - 27g_3^2$. $\square$

The lemma below gives some precise estimate for the norms of $\Delta_q = h_2^3 - 27h_3^2$ and $\Delta_1 = g_2^3 - 27g_3^2$, the discriminants of the elliptic curves E_q, E_1, and the element u defined in Lemma B.6.

Lemma B.7. *The following statement hold:*

- *$|\Delta_q| = |q|$, with $|h_2 - \frac{1}{12}| = |q|$ and $|h_3 - (-\frac{1}{6^3})| = |q|$;*
- *$|\Delta_1| = |q|$ with $|g_2 - \frac{4}{3}| < 1$, $|g_3 - (-\frac{8}{27})| < 1$;*
- *$|u| = 1$;*
- *$|\mathfrak{D}^{(1)}(a)| \in]\,|q|^{1/2}, 1[$.*

Proof. Following [Roq70, Pages 29-30], we find $|\Delta_q| = |q|, |s_3| = |q| = |s_5|$. Combining the latter norm estimates with Lemma B.5, we find $|h_2 - \frac{1}{12}| = |q|$ and $|h_3 - (-\frac{1}{6^3})| = |q|$. Let us prove the second point. It follows from (1.5) that $|1 - \alpha_2| < 1$ and $|\alpha_i| < 1$ for $i = 0, 1, 3, 4$. By Lemma A.1, $|\mathfrak{D}^{(1)}(a)|, |\mathfrak{D}^{(3)}(a)|, |\mathfrak{D}^{(4)}(a)| < 1, |\mathfrak{D}^{(2)}(a) - 2| < 1.$

Combining these norm estimates with (B.5), we find $|g_2-\frac{4}{3}|<1$, $|g_3-(-\frac{8}{27})|<1$. Since $|J(E_1)|=|J(E_q)|=|\frac{12^3 g_2}{\Delta_1}|=|\frac{12^3 h_2}{\Delta_q}|$ and $|g_2|=|h_2|=1$, we find $|\Delta_q|=|\Delta_1|=|q|$. By Lemma B.6, $\Delta_q=\frac{\Delta_1}{u^{12}}$, and then $|u|=1$.
Let us prove the last point. Let us expand $\Delta_1=g_2^3-27g_3^2$ with the expression of g_2,g_3 given in (B.5):

$$\begin{array}{rcl}\Delta_1 & = & \left(\frac{\mathfrak{D}^{(2)}(a)^2}{3}-2\frac{\mathfrak{D}^{(1)}(a)\mathfrak{D}^{(3)}(a)}{3}\right)^3-27\left(\frac{-\mathfrak{D}^{(2)}(a)^3}{27}+\frac{\mathfrak{D}^{(1)}(a)\mathfrak{D}^{(2)}(a)\mathfrak{D}^{(3)}(a)}{9}-\frac{\mathfrak{D}^{(1)}(a)^2\mathfrak{D}^{(4)}(a)}{6}\right)^2\\
& = & \frac{\mathfrak{D}^{(2)}(a)^6}{27}-\frac{2\mathfrak{D}^{(1)}(a)\mathfrak{D}^{(2)}(a)^4\mathfrak{D}^{(3)}(a)}{9}+\frac{4\mathfrak{D}^{(1)}(a)^2\mathfrak{D}^{(2)}(a)^2\mathfrak{D}^{(3)}(a)^2}{9}-\frac{8\mathfrak{D}^{(1)}(a)^3\mathfrak{D}^{(3)}(a)^3}{27}\\
& & -\frac{\mathfrak{D}^{(2)}(a)^6}{27}-\frac{\mathfrak{D}^{(1)}(a)^2\mathfrak{D}^{(2)}(a)^2\mathfrak{D}^{(3)}(a)^2}{3}-\frac{3\mathfrak{D}^{(1)}(a)^4\mathfrak{D}^{(4)}(a)^2}{4}+\frac{2\mathfrak{D}^{(1)}(a)\mathfrak{D}^{(2)}(a)^4\mathfrak{D}^{(3)}(a)}{9}\\
& & -\frac{\mathfrak{D}^{(1)}(a)^2\mathfrak{D}^{(2)}(a)^3\mathfrak{D}^{(4)}(a)}{3}+\mathfrak{D}^{(1)}(a)^3\mathfrak{D}^{(2)}(a)\mathfrak{D}^{(3)}(a)\mathfrak{D}^{(4)}(a)\\
& = & \frac{\mathfrak{D}^{(1)}(a)^2\mathfrak{D}^{(2)}(a)^2\mathfrak{D}^{(3)}(a)^2}{9}-\frac{8\mathfrak{D}^{(1)}(a)^3\mathfrak{D}^{(3)}(a)^3}{27}-\frac{3\mathfrak{D}^{(1)}(a)^4\mathfrak{D}^{(4)}(a)^2}{4}\\
& & -\frac{\mathfrak{D}^{(1)}(a)^2\mathfrak{D}^{(2)}(a)^3\mathfrak{D}^{(4)}(a)}{3}+\mathfrak{D}^{(1)}(a)^3\mathfrak{D}^{(2)}(a)\mathfrak{D}^{(3)}(a)\mathfrak{D}^{(4)}(a).\end{array}$$

Since $|\mathfrak{D}^{(1)}(a)|,|\mathfrak{D}^{(3)}(a)|,|\mathfrak{D}^{(4)}(a)|<1,|\mathfrak{D}^{(2)}-2|<1$, the previous expression is a sum of terms that are all strictly smaller in norm than $|\mathfrak{D}^{(1)}(a)|^2$. This proves that $|\Delta_1|=|q|<|\mathfrak{D}^{(1)}(a)|^2$.

□

The following estimate will be required to uniformize the generating series.

Lemma B.8. *In the notation of Theorem 3.2, we have* $|\frac{u^2}{12}-\frac{\mathfrak{D}^{(2)}(a)}{6}|<|\mathfrak{D}^{(1)}(a)|$.

Proof. Using (B.5) and the norm estimate on the $\mathfrak{D}^{(i)}(a)$'s, we get

$$g_2=\frac{\mathfrak{D}^{(2)}(a)^2}{3}+\mathfrak{D}^{(1)}(a)\omega,\quad g_3=\frac{-\mathfrak{D}^{(2)}(a)^3}{27}+\mathfrak{D}^{(1)}(a)\omega',\tag{B.6}$$

where $|\omega|,|\omega'|<1$. This proves that

$$\frac{g_3}{g_2}=\frac{-\mathfrak{D}^{(2)}(a)}{9}+\mathfrak{D}^{(1)}(a)\omega''$$

with $|\omega''|<1$. Then, we find

$$\left|\frac{u^2}{12}-\frac{\mathfrak{D}^{(2)}(a)}{6}\right|=\left|\frac{u^2}{12}+\frac{3g_3}{2g_2}-\frac{3g_3}{2g_2}-\frac{\mathfrak{D}^{(2)}(a)}{6}\right|\leq\max\left(\left|\frac{u^2}{12}+\frac{3g_3}{2g_2}\right|,\left|\frac{3}{2}\mathfrak{D}^{(1)}(a)\omega''\right|\right).$$

Finally, with the norm estimate of Lemma B.7, it is sufficient to show that $|\frac{u^2}{12}+\frac{3g_3}{2g_2}|\leq|q|$. By Lemma B.6, we have $\frac{u^2}{12}=\frac{g_3h_2}{12g_2h_3}$. By Lemma B.7, $|h_2-\frac{1}{12}|=|q|$ and $|h_3-(-\frac{1}{6^3})|=|q|$. Then, by Lemma B.7 again, we find

$$\left|\frac{u^2}{12}+\frac{3g_3}{2g_2}\right|=\left|\frac{g_3h_2}{12g_2h_3}+\frac{3g_3}{2g_2}\right|=\left|\frac{g_3}{g_2}\right|\left|\frac{h_2}{12h_3}+\frac{3}{2}\right|=\left|\frac{h_2+18h_3}{12h_3}\right|$$
$$=|h_2+18h_3|=\left|\left(h_2-\frac{1}{12}\right)+18\left(h_3-\left(-\frac{1}{6^3}\right)\right)\right|\leq\max\left(\left|h_2-\frac{1}{12}\right|,\left|h_3-\left(-\frac{1}{6^3}\right)\right|\right)\leq|q|.$$

□

Appendix C. Difference Galois theory

In this section, we establish some criteria to guaranty the transcendence of functions satisfying a difference equation of order 1. This criteria is based on the Galois theory of difference fields as developed in [vdPS97] but generalizes some of the existing results in the literature, for instance the assumption that the field of constants is algebraically closed (see for instance Theorem C.8).

The algebraic framework of this section is difference algebra and more precisely the notion of difference fields. A difference field is a pair (K, σ) where K is a field and σ is an automorphism of K. The field σ-constants K^σ of (K, σ) is formed by the elements $f \in K$ such that $\sigma(f) = f$. An extension $(K, \sigma_K) \subset (L, \sigma_L)$ of difference fields is a field extension $K \subset L$ such that σ_L coincides with σ_K on K. If there is no confusion, we shall denote by σ the automorphism σ_K and σ_L. For a complete introduction on difference algebra, we shall refer to [Coh65].

C.1. **Rank one difference equations.**

In this section, we focus on rank one difference equations.

Lemma C.1. *Let $(K, \sigma) \subset (L, \sigma)$ be an extension of difference fields such that $L^\sigma = K^\sigma$. Let $x \in L$. The following statements are equivalent*

(1) *x is algebraic over K^σ;*
(2) *there exists $r \in \mathbb{N}^*$ such that $\sigma^r(x) = x$.*

Proof. Assume that x is algebraic over K^σ. Then, σ induces a permutation on the set of roots of the minimal polynomial of x over K^σ. Thus, there exists $r \in \mathbb{N}^*$ such that $\sigma^r(x) = x$. Conversely, if there exists $r \in \mathbb{N}^*$ such that $\sigma^r(x) = x$, the polynomial $P(X) = \prod_{i=0}^{r-1}(X - \sigma^i(x)) \in L[X]$ is fixed by σ and thereby $P(X) \in L^\sigma[X] = K^\sigma[X]$. Since $P(x) = 0$, we have proved that x is algebraic over K^σ. □

Lemma C.2. *Let $(K, \sigma) \subset (L, \sigma)$ be an extension of difference fields such that $L^\sigma = K^\sigma$. Let $f \in L$ and $0 \neq c \in K$, such that $\sigma(f) = f + c$. The following statements are equivalent*

(1) *$f \in K$;*
(2) *f is algebraic over K;*
(3) *There exists $\alpha \in K$ such that $\sigma(\alpha) = \alpha + c$.*

Moreover, let $\overline{K}$ be the algebraic closure of K endowed with a structure of σ-field extension of K. For all $\alpha \in \overline{K}, i \in \mathbb{Z}$ we denote by α_i the element of $\overline{K}$ such that $\sigma^i(f - \alpha) = f - \alpha_i$. If f is transcendental over K then for $i, j \in \mathbb{Z}$ such that $i \neq j$, the elements α_j and α_i are distinct.

Proof. Let us prove the first part of the proposition. The first statement implies trivially the second one. Assume that f is algebraic over K and let $P(X) = X^n + a_{n-1}X^{n-1} + \dots a_0 \in K[X]$ be its minimal polynomial over K. Note that $n \neq 0$. Using $\sigma(f) - f = c$ and $P(f) = 0$, we find that $\sigma(P(f)) - P(f) = 0 = (nc + \sigma(a_{n-1}) - a_{n-1})f^{n-1} + b_{n-2}f^{n-2} + \dots + b_0$ with $b_i \in K$ for $i = 0, \dots, n-2$. By minimality of $P(X)$, we find that $\sigma(a_{n-1}) - a_{n-1} = -nc$ with $a_{n-1} \in K$. Then, $\sigma(\alpha) - \alpha = c$ with $\alpha = \frac{a_{n-1}}{-n} \in K$. We have shown that the second statement implies the third. Finally, assume that there exists $\alpha \in K$ such that $\sigma(\alpha) = \alpha + c$. With $\sigma(f) - f = c$, we find that $\sigma(\alpha - f) = \alpha - f$. This gives that $\alpha - f \in L^\sigma = K^\sigma$ and the element f belongs to K.

Now, let us assume that f is transcendental over K. Suppose to the contrary that there exist $\alpha \in \overline{K}$ and $i > j \in \mathbb{Z}$ such that

$$\alpha_i = (\sigma^i(\alpha) - c - \sigma(c) - \cdots - \sigma^{i-1}(c)) = \alpha_j = (\sigma^j(\alpha) - c - \sigma(c) - \cdots - \sigma^{j-1}(c)).$$

The latter equality gives $\sigma^r(\beta) - \beta = \gamma$ where $r = i - j > 0$, $\beta = \sigma^j(\alpha)$ and $\gamma = \sigma^{i-1}(c) + \cdots + \sigma^j(c)$. Since α is algebraic over K, the same holds for β. Let $P(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0 \in K[X] \setminus K$ be the minimal polynomial of β over K. Using the fact that $\sigma^r(\beta) - \beta = \gamma$ and the minimality of P, we conclude, as above, that $\sigma^r(a_{n-1}) - a_{n-1} = -n\gamma$, that is $\sigma^r(\tilde{\beta}) - \tilde{\beta} = \gamma$ where $\tilde{\beta} = \frac{a_{n-1}}{-n} \in K$. Combining this equality with $\sigma^r(\sigma^j(f)) - \sigma^j(f) = \gamma$, we find that $\tilde{\beta} - \sigma^j(f) \in L$ is fixed by σ^r. By Lemma C.1, this means that $\tilde{\beta} - \sigma^j(f)$ is algebraic over K^σ, which yields to f algebraic over K. We find a contradiction. □

Lemma C.3. *Let $(K,\sigma) \subset (L,\sigma)$ be an extension of difference fields such that $L^\sigma = K^\sigma$. Let $f \in L$ and $0 \neq c \in K$, such that $\sigma(f) = f + c$. Assume that f is transcendental over K. If there exists $g \in K(f)$ such that $\sigma(g) - g \in K[f]$, then $g \in K[f]$.*

Proof. Let $\overline{K}$ be an algebraic closure of K, endowed with a structure of σ-field extension of K. Since f is transcendental over K, we can write a partial fraction decomposition of $g \in \overline{K}(f)$. Let R be the largest integer such that there exists $\alpha \in \overline{K}$ so that the element $\frac{1}{(f-\alpha)^R}$ appears in the partial fraction decomposition of g. Suppose to the contrary that $R > 0$ and let $\alpha \in \overline{K}$ such that $\frac{1}{(f-\alpha)^R}$ appears in the partial fraction decomposition of g. We deduce from Lemma C.2 applied to K and f, that the elements $\{\alpha_i, i \in \mathbb{Z}\}$ are all distinct. Then, there exists N, the largest integer such that $\sigma^N(\frac{1}{(f-\alpha)^R})$ appears in the partial fraction decomposition of g. The element $\sigma^{N+1}(\frac{1}{(f-\alpha)^R})$ appears in the partial fraction decomposition of $\sigma(g)$. This proves that $\sigma^{N+1}(\frac{1}{(f-\alpha)^R})$ appears in the partial fraction decomposition of $\sigma(g) - g$. A contradiction with $\sigma(g) - g \in K[f]$. This proves that $g \in K[f]$. □

C.2. **Differential transcendence criteria.** In this section, a $(\sigma, \partial, \Delta)$-field K is a difference field (K, σ) endowed with two derivations ∂, Δ commuting with σ such that $\partial\Delta - \Delta\partial = c_K\partial$ with $c_K \in K^\sigma$. We assume that ∂ is non-trivial on K, that is, it is not the zero derivation. The element c_K has to be considered as part of the data of the notion of $(\sigma, \partial, \Delta)$-field. An extension of $(\sigma, \partial, \Delta)$-fields is an inclusion of two $(\sigma, \partial, \Delta)$-fields $(K, \sigma_K, \partial_K, \Delta_K) \subset (L, \sigma_L, \partial_L, \Delta_L)$ such that

- $K \subset L$ is a field extension;
- $\sigma_K, \partial_K, \Delta_K$ are the restrictions of $\sigma_L, \partial_L, \Delta_L$ to K;
- $c_K = c_L$.

If there is no confusion, we shall omit the subscripts $_K$, $_L$. If σ is the identity, we shall speak of (∂, Δ)-fields, (∂, Δ)-fields extension for short.

Example C.4. As proved in §D, the following fields are $(\sigma, \partial, \Delta)$-fields, that correspond respectively to the framework of the genus zero and genus one kernel curve. Remind that $\sigma_{\mathbf{q}}$ denote the automorphism of $\mathcal{M}er(C^*)$ defined by $f(s) \mapsto f(\mathbf{q}s)$ and $C_{\mathbf{q}}$ denote the field of meromorphic functions fixed by $\sigma_{\mathbf{q}}$. In the two examples, we have $\Delta_{\mathbf{q},t} =$

$\partial_t(\mathbf{q})\ell_{\mathbf{q}}(s)\partial_s + \partial_t$ where $\ell_{\mathbf{q}}$ is the so called $\mathbf{q}$-logarithm. That is, an element of $\mathcal{M}er(C^*)$ satisfying $\sigma_{\mathbf{q}}(\ell_{\mathbf{q}}) = \ell_{\mathbf{q}} + 1$, and $c_K = \partial_t(\mathbf{q})\partial_s(\ell_{\mathbf{q}}) \in C_{\mathbf{q}}$.

- Let $\mathbf{q} \in C^*$ with $|\mathbf{q}| \neq 1$. Then, the inclusion
$$(C_{\mathbf{q}}(s, \ell_{\mathbf{q}}), \sigma_{\mathbf{q}}, \partial_s, \Delta_{t,\mathbf{q}}) \subset (\mathcal{M}er(C^*), \sigma_{\mathbf{q}}, \partial_s, \Delta_{t,\mathbf{q}})$$
is an extension of $(\sigma, \partial, \Delta)$-fields.
- Let $\mathbf{q}$ and q two elements of C^* such that $|q|, |\mathbf{q}| \neq 1$, that are *multiplicatively independent*, that is, there are no $r, l \in \mathbb{Z}^2 \setminus (0,0)$ such that $q^r = \mathbf{q}^l$. Since $C_{\mathbf{q}} \subset \mathcal{M}er(C^*)$ and $C_q \subset \mathcal{M}er(C^*)$, we consider $C_{\mathbf{q}}.C_q \subset \mathcal{M}er(C^*)$, the field compositum of $C_{\mathbf{q}}$ and C_q inside $\mathcal{M}er(C^*)$. Then, the inclusion
$$(C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}}, \ell_q), \sigma_{\mathbf{q}}, \partial_s, \Delta_{t,\mathbf{q}}) \subset (\mathcal{M}er(C^*), \sigma_{\mathbf{q}}, \partial_s, \Delta_{t,\mathbf{q}})$$
is an extension of $(\sigma, \partial, \Delta)$-fields.

Definition C.5. Let $(K, \partial, \Delta) \subset (L, \partial, \Delta)$. An element $f \in L$ is said to be (∂, Δ)-differentially algebraic over K if there exists $N \in \mathbb{N}$, such that the elements

- $\partial^i(f)$ for $i \leq N$ are algebraically dependent over K if Δ is a K-multiple of ∂;
- $\partial^i \Delta^j(f)$ for $i, j \leq N$ are algebraically dependent over K otherwise.

Otherwise, we will say that f is (∂, Δ)-transcendental over K.

Remark C.6. Note that since $\partial\Delta - \Delta\partial = c\partial$ with $c \in K^\sigma \subset K$, the (∂, Δ)-field extension of K generated by some element $f \in L$ coincides with the field extension of K generated by the set $\{\partial^i \Delta^j(f), \text{ for } i, j \in \mathbb{N}\}$.

The following lemma will be crucial in many arguments:

Lemma C.7. *If $K \subset M$ is a σ-field extension such that $M^\sigma = K$ and $K \subset L$ is a σ-field extension with $L^\sigma = L$. Then M and L are linearly disjoint over K.*

Proof. Let $c_1, \ldots, c_r \in L$ be K-linearly independent elements, that become dependent over M. Up to a permutation of the c_i's, a minimal linear relation among these elements over M has the following form

$$c_1 + \sum_{i=2}^{r} \lambda_i c_i = 0, \tag{C.1}$$

with $\lambda_i \in M$ for $i = 2, \ldots, r$. Computing $\sigma((\text{C.1})) - (\text{C.1})$, we find

$$\sum_{i=2}^{r} (\sigma(\lambda_i) - \lambda_i) c_i = 0.$$

By minimality, $\sigma(\lambda_i) = \lambda_i$ and $\lambda_i \in M^\sigma = K$. By K-linear independence of the c_i, we find that $\lambda_i = 0$ for $i = 2, \ldots, r$ and then $c_1 = 0$. A contradiction. □

The following statement, whose proof is due to Michael Singer, is a version of an old theorem of Ostrowski [Ost46, Kol68] and its proof follows the lines of the proof of [DHRS18, Proposition 3.6]. In this last paper, it was assumed that K^σ is algebraically closed, which is not the case in this article. One could use the powerful scheme-theoretic tools developed in [OW15] to prove the result in our more general setting. Instead we will argue in a more elementary way to reduce Theorem C.8 to the case where K^σ is algebraically closed.

Theorem C.8. *Let $(K,\sigma,\partial,\Delta)$ be a (σ,∂,Δ)-field such that K^σ is relatively algebraically closed in K, that is there are no proper algebraic extension of K^σ inside K. Let $(L,\sigma,\partial,\Delta)$ be a (σ,∂,Δ)-ring extension of $(K,\sigma,\partial,\Delta)$. Let $f\in L$ and $b\in K$ such that $\sigma(f)=f+b$. If f is (∂,Δ)-differentially algebraic over K then there exist $\ell_1,\ell_2\in\mathbb{N}$, $c_{i,j}\in K^\sigma$ not all zero and $g\in K$ such that*

$$\sum_{\substack{0\leq i\leq \ell_1,\\ 0\leq j\leq \ell_2}} c_{i,j}\partial^i\Delta^j(b)=\sigma(g)-g. \tag{C.2}$$

Furthermore, we may take $\ell_2=0$ in the case where ∂ and Δ are K-linearly dependent. We call (C.2) a telescoping relation for b.

The proof of this result depends on results from the Galois theory of linear difference equations and we will refer to [DHRS18, Appendix A] and the references given there for relevant facts from this theory. Let (K,σ) be a difference field and consider the system of difference equations

$$\sigma(y_0)-y_0=b_0,\ldots,\sigma(y_n)-y_n=b_n, \quad \text{with } b_0,\ldots,b_n\in K. \tag{C.3}$$

Let us see (C.3) as a system $\sigma(Y)=AY$, where $A\in \mathrm{GL}_{2(n+1)}(K)$ is a diagonal bloc matrix $A=\mathrm{Diag}(A_0,\ldots,A_n)$ with $A_i=\begin{pmatrix}1 & b_i\\ 0 & 1\end{pmatrix}$ which correspond to the equation $\sigma(y_i)-y_i=b_i$. A Picard-Vessiot extension for $\sigma(Y)=AY$ is a difference ring extension (R,σ) of (K,σ) such that:

- there exists $U\in \mathrm{GL}_{2(n+1)}(R)$ such that $\sigma(U)=AU$;
- R is generated as a K-algebra by the entries of U and $\det(U)^{-1}$;
- R is a simple difference ring, that is, the σ-ideals of R are $\{0\}$ and R.

We will need the following result.

Lemma C.9 (Proposition A.9 in [DHRS18]). *Assume that (K,σ) is a difference field with K^σ algebraically closed. Let R be a Picard-Vessiot extension for the system (C.3) and $z_0,\ldots,z_n\in R$ be solutions of this system. If $z_0,\ldots,z_n$ are algebraically dependent over K, then there exist $c_i\in K^\sigma$, not all zero, and $g\in K$ such that*

$$c_0b_0+\ldots+c_nb_n=\sigma(g)-g.$$

Before proving Theorem C.8, we give a slight generalization of Lemma C.9.

Lemma C.10. *Let (K,σ) be a difference field with K^σ relatively algebraically closed in K and let $b_0,\ldots,b_n$ be some elements in K. Let (L,σ) be a σ-ring extension of (K,σ). Let $z_0,\ldots,z_n\in L$ be solutions of $\sigma(z_i)-z_i=b_i$. If $z_0,\ldots,z_n$ are algebraically dependent over K, then there exist $c_i\in K^\sigma$, not all zero, and $g\in K$ such that*

$$c_0b_0+\ldots+c_nb_n=\sigma(g)-g.$$

Proof. Let $\mathbf{k}$ be the algebraic closure of K^σ. We extend σ to be the identity on $\mathbf{k}$[‖]. Under the assumption that K^σ is relatively algebraically closed, the ring $\widetilde{K}=K\otimes_{K^\sigma}\mathbf{k}$ is an integral domain and in fact is a field. We have $\widetilde{K}^\sigma=\mathbf{k}$. Let $\widetilde{L}=L\otimes_{K^\sigma}\mathbf{k}$. We then

‖On the other hand, there is no unique procedure to extend a field automorphism of K^σ to the algebraic closure $\mathbf{k}$. Indeed, these extensions are controlled by the Galois group of the field $\mathbf{k}$ over K^σ.

have a natural inclusion of $\widetilde{K} \subset \widetilde{L}$. Let $S = \widetilde{K}[z_0, \ldots, z_n] \subset \widetilde{L}$. It is easily seen that S is a σ-ring extension of $\widetilde{K}$. Let I be a maximal difference ideal in S and let $R = S/I$. For each $r = 0, \ldots, n$, let u_r be the image of z_r in R. Since $\widetilde{K}^\sigma = \mathbf{k}$ is algebraically closed and R is a simple difference ring, we have that R is a Picard-Vessiot ring for the system associated to $\sigma(y_r) - y_r = b_r$, $r = 0, \ldots, n$, over $\widetilde{K}$. The elements $u_0, \ldots, u_n$ are algebraically dependent over K and solutions of $\sigma(y_r) - y_r = b_r, r = 0, \ldots, n$. Lemma C.9 proves that there exist $c_i \in \mathbf{k}$, not all zero, and $g \in \widetilde{K}$ such that

$$\sum_{0 \leq i \leq n} c_i b_i = \sigma(g) - g.$$

Let $\{d_r\} \subset \mathbf{k}$ be a K^σ-basis of $\mathbf{k}$. By Lemma C.7, it is also a K-basis of $\widetilde{K}$. We may write each c_i and g as

$$c_i = \sum_r c_{i,r} d_r \text{ and } g = \sum_r g_r d_r$$

for some $c_{i,r} \in K^\sigma$ and $g_r \in K$. Since not all the c_i are zero, there exists r such that $c_{i,r}$ are not all zero. For this r, we have

$$\sum_{i \leq n} c_{i,r} b_i = \sigma(g_r) - g_r.$$

This yields the conclusion of the proof. □

Proof of Theorem C.8. Assuming that f is (∂, Δ)-differentially algebraic over K, there is some finite set $\{\partial^{i_0}\Delta^{j_0}(f), \ldots, \partial^{i_n}\Delta^{j_n}(f)\} \subset L$ of elements that are algebraically dependent over K. Note that $j_k = 0$ for all k if Δ is K-linearly dependent from ∂. Since σ commutes with Δ and ∂, we have for all $r = 0, \ldots, n$,

$$\sigma(\partial^{i_r}\Delta^{j_r}(f)) - \partial^{i_r}\Delta^{j_r}(f) = \partial^{i_r}\Delta^{j_r}(b).$$

To conclude it remains to apply Lemma C.10 with $z_r = \partial^{i_r}\Delta^{j_r}(f)$ and $b_r = \partial^{i_r}\Delta^{j_r}(b)$ for $r = 0, \ldots, n$. □

Appendix D. Meromorphic functions on a Tate curve and their derivations

In this section we translate the galoisian criteria of Theorem C.8 in the context of elliptic functions field. We start by defining the derivations. Studying the transcendence properties of the $\mathbf{q}$-logarithm, we then perform a descent on the field of coefficients and on the number of derivations involved in the telescoping relation.

D.1. **Derivation on non-archemedean elliptic functions field.** Let $\mathbf{q} \in C^*$ such that $|\mathbf{q}| \neq 1$ and let $\sigma_{\mathbf{q}}$ denote the automorphism of $\mathcal{M}er(C^*)$ defined by $\sigma_{\mathbf{q}}(f(s)) = f(\mathbf{q}s)$. We denote by $C_{\mathbf{q}}$ the field of meromorphic functions fixed by $\sigma_{\mathbf{q}}$. By Proposition 3.1, it is the field of rational functions on the Tate curve $E_{\mathbf{q}}$ or $E_{1/\mathbf{q}}$, depending whether $|\mathbf{q}| < 1$ or $|\mathbf{q}| > 1$. In this section, we construct, as in [DVH12, §2] a derivation of these functions that encode their t-depencies and commute with $\sigma_{\mathbf{q}}$.

The fact that $\partial_s = s\frac{d}{ds}$ acts on $\mathcal{M}er(C^*)$, and its commutation with $\sigma_{\mathbf{q}}$ is straightforward. Unfortunately, the t-derivative of $\mathbf{q}$ may be non-trivial, implying a more complicated commutation rule between $\partial_t = t\frac{d}{dt}$ and $\sigma_{\mathbf{q}}$. More precisely, we have

$$\begin{array}{l} \partial_s \circ \sigma_{\mathbf{q}} = \sigma_{\mathbf{q}} \circ \partial_s; \\ \partial_t \circ \sigma_{\mathbf{q}} = \partial_t(\mathbf{q})\sigma_{\mathbf{q}} \circ \partial_s + \sigma_{\mathbf{q}} \circ \partial_t. \end{array}$$

The following statement holds.

Lemma D.1. *The ∂_s-constants $\mathcal{M}er(C^*)^{\partial_s} = \{f \in \mathcal{M}er(C^*) | \partial_s(f) = 0\}$ of $\mathcal{M}er(C^*)$ are precisely the constant functions C.*

Next Lemma introduces a twisted t-derivation that commutes with $\sigma_{\mathbf{q}}$. Remind that the q-logarithm $\ell_{\mathbf{q}}$ has been defined in §2.3.

Lemma D.2 (Lemma 2.1 in [DVH12])**.** *The following derivations of $\mathcal{M}er(C^*)$*

$$\left\{ \begin{array}{l} \partial_s \\ \Delta_{t,\mathbf{q}} = \partial_t(\mathbf{q})\ell_{\mathbf{q}}(s)\partial_s + \partial_t, \end{array} \right.$$

commute with $\sigma_{\mathbf{q}}$. Moreover, we have

$$\partial_s \Delta_{t,q} - \Delta_{t,q}\partial_s = \partial_t(\mathbf{q})\partial_s(\ell_{\mathbf{q}})\partial_s,$$

where $\partial_t(\mathbf{q})\partial_s(\ell_{\mathbf{q}}) \in C_{\mathbf{q}}$.

Remark D.3. Note that since $\partial_s, \Delta_{t,\mathbf{q}}$ commute with $\sigma_{\mathbf{q}}$, we can derive the equation $\sigma_{\mathbf{q}}(\ell_{\mathbf{q}}) = \ell_{\mathbf{q}} + 1$ to find $\sigma_{\mathbf{q}}(\partial_s(\ell_{\mathbf{q}})) = \partial_s(\ell_{\mathbf{q}})$ and $\sigma_{\mathbf{q}}(\Delta_{t,\mathbf{q}}(\ell_{\mathbf{q}})) = \Delta_{t,\mathbf{q}}(\ell_{\mathbf{q}})$. We then conclude that $\partial_s(\ell_{\mathbf{q}}), \Delta_{t,\mathbf{q}}(\ell_{\mathbf{q}})$ belong to $C_{\mathbf{q}}$.

The link with the iterates of $\Delta_{t,\mathbf{q}}$ and the derivatives ∂_s, ∂_t is now made in the following lemma.

Lemma D.4. *For any $i \in \mathbb{N}$, there exist $c_{j,k,l} \in C_{\mathbf{q}}$ such that*

$$\Delta_{t,\mathbf{q}}^i = (\partial_t(\mathbf{q})\ell_{\mathbf{q}})^i \partial_s^i + \sum_{k=0}^{i-1}\sum_{j=0}^{k}\sum_{l=0}^{i} c_{j,k,l}\ell_{\mathbf{q}}^j \partial_s^k \partial_t^l.$$

Proof. Let us prove the result by induction on i. For $i = 1$, this comes from the fact that $\Delta_{t,\mathbf{q}} = \partial_t(\mathbf{q})\ell_{\mathbf{q}}\partial_s + \partial_t$. Let us fix $i \in \mathbb{N}$ and assume that the result holds for i. We find

$$\Delta_{t,\mathbf{q}}^{i+1} = (\partial_t(\mathbf{q})\ell_{\mathbf{q}}\partial_s + \partial_t)\left((\partial_t(\mathbf{q})\ell_{\mathbf{q}})^i \partial_s^i + \sum_{k=0}^{i-1}\sum_{j=0}^{k}\sum_{l=0}^{i} c_{j,k,l}\ell_{\mathbf{q}}^j \partial_s^k \partial_t^l\right),$$

that is

$$\begin{aligned} &\Delta_{t,\mathbf{q}}^{i+1} = (\partial_t(\mathbf{q})\ell_{\mathbf{q}})^{i+1}\partial_s^{i+1} + \Delta_{t,\mathbf{q}}((\partial_t(\mathbf{q})\ell_{\mathbf{q}})^i)\partial_s^i + (\partial_t(\mathbf{q})\ell_{\mathbf{q}})^i\partial_t\partial_s^i + \\ &\sum_{k=0}^{i-1}\sum_{j=0}^{k}\sum_{l=0}^{i} \Delta_{t,\mathbf{q}}(c_{j,k,l}\ell_{\mathbf{q}}^j)\partial_s^k\partial_t^l + \sum_{k=0}^{i-1}\sum_{j=0}^{k}\sum_{l=0}^{i} c_{j,k,l}\partial_t(\mathbf{q})\ell_{\mathbf{q}}^{j+1}\partial_s^{k+1}\partial_t^l + \sum_{k=0}^{i-1}\sum_{j=0}^{k}\sum_{l=0}^{i} c_{j,k,l}\ell_{\mathbf{q}}^j\partial_s^k\partial_t^{l+1}. \end{aligned}$$

Note that the commutation of $\sigma_{\mathbf{q}}$ with $\Delta_{t,\mathbf{q}}$ implies that $C_{\mathbf{q}}$ is stabilized by $\Delta_{t,\mathbf{q}}$. Since by Remark D.3, $\Delta_{t,\mathbf{q}}(\ell_{\mathbf{q}})$ belongs to $C_{\mathbf{q}}$, we get that, for any integer j, any $\tilde{c} \in C_{\mathbf{q}}$, we have $\Delta_{t,\mathbf{q}}(\tilde{c}(\ell_{\mathbf{q}})^j) = \Delta_{t,\mathbf{q}}(\tilde{c})(\ell_{\mathbf{q}})^j + \tilde{c}c(\ell_{\mathbf{q}})^{j-1}$ where $c = j\Delta_{t,\mathbf{q}}(\ell_{\mathbf{q}}) \in C_{\mathbf{q}}$. Therefore, with

$\Delta_{t,\mathbf{q}}(\tilde{c}) \in C_{\mathbf{q}}$, we find that $\Delta_{t,\mathbf{q}}(\tilde{c}(\ell_{\mathbf{q}})^j) \in C_{\mathbf{q}}[\ell_{\mathbf{q}}]$ is of degree at most j in $\ell_{\mathbf{q}}$. With $\partial_t(\mathbf{q}), c_{j,k,l} \in C_{\mathbf{q}}$, this ends the proof. $\square$

From now on, let us fix $q \in C^*$ with $|q| \neq 1$, that is multiplicatively independent to $\mathbf{q}$, that is there are no $r, l \in \mathbb{Z}^2 \setminus (0,0)$ such that $q^r = \mathbf{q}^l$. Remind that $C_{\mathbf{q}}.C_q \subset \mathcal{M}er(C^*)$ is the compositum of fields and $\ell_{\mathbf{q}} \in \mathcal{M}er(C^*)$ is a solution of $\sigma_{\mathbf{q}}(\ell_{\mathbf{q}}) = \ell_{\mathbf{q}} + 1$. We now give examples of difference differential fields for $\sigma_{\mathbf{q}}, \partial_s$ and $\Delta_{t,\mathbf{q}}$.

Lemma D.5. *The following statement hold.*

1. *The field $C_{\mathbf{q}}(s, \ell_{\mathbf{q}})$ is stabilized by $\sigma_{\mathbf{q}}$, ∂_s and $\Delta_{t,\mathbf{q}}$. The field $C_{\mathbf{q}}(s)$ is stabilized by $\sigma_{\mathbf{q}}$, and ∂_s. The field $C(s)$ is stabilized by ∂_s, ∂_t.*
2. *The field $C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}}, \ell_q)$ is stabilized by $\sigma_{\mathbf{q}}$, ∂_s and $\Delta_{t,\mathbf{q}}$. The field $C_{\mathbf{q}}.C_q(\ell_q)$ is stabilized by $\sigma_{\mathbf{q}}$, and ∂_s. The field $C_q(\ell_q)$ is stabilized by ∂_s, ∂_t.*

Proof. (1) Since $\sigma_{\mathbf{q}}(\ell_{\mathbf{q}}) = \ell_{\mathbf{q}} + 1$, we easily see that $C_{\mathbf{q}}(s, \ell_{\mathbf{q}}), C_{\mathbf{q}}(s)$ are stabilized by $\sigma_{\mathbf{q}}$. Since $\sigma_{\mathbf{q}}$ commutes with ∂_s and $\Delta_{t,\mathbf{q}}$, the field $C_{\mathbf{q}}$ is stabilized by ∂_s and $\Delta_{t,\mathbf{q}}$. It is now clear that $C_{\mathbf{q}}(s)$ is stabilized by ∂_s and $\Delta_{t,\mathbf{q}}(C_{\mathbf{q}}(s)) \subset C_{\mathbf{q}}(s, \ell_{\mathbf{q}})$. By Remark D.3, $\Delta_{t,\mathbf{q}}(\ell_{\mathbf{q}}), \partial_s(\ell_{\mathbf{q}}) \in C_{\mathbf{q}}$. Combining the lasts assertions, we obtain the result for $C_{\mathbf{q}}(s, \ell_{\mathbf{q}})$. Finally, the field $C(s)$ is stable by ∂_s, ∂_t, since C is stable by ∂_s, ∂_t, and $\partial_s(s) = s$, $\partial_t(s) = 0$.
(2) Let us prove that $C_q(\ell_q)$ is stabilized by $\sigma_{\mathbf{q}}$. Using $\sigma_q(\ell_q) = \ell_q + 1$ and the commutation between $\sigma_{\mathbf{q}}$ and σ_q, we find that $\sigma_{\mathbf{q}}(\ell_q) - \ell_q \in C_q$. Similarly, $\sigma_{\mathbf{q}}(C_q) \subset C_q$, proving that $C_q(\ell_q)$ is stabilized by $\sigma_{\mathbf{q}}$. Using $\partial_s(C_q) \subset C_{\mathbf{q}}$ and $\partial_s(\ell_q) \in C_q$, we find that the field $C_{\mathbf{q}}.C_q(\ell_q)$ is stabilized by $\sigma_{\mathbf{q}}$ and ∂_s.

Let us now consider the field $C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}}, \ell_q)$. The field $C_{\mathbf{q}}(\ell_{\mathbf{q}})$ is clearly stable by $\sigma_{\mathbf{q}}$. From what preceede, $C_q(\ell_q)$ is stable by $\sigma_{\mathbf{q}}$, and therefore, $C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}}, \ell_q)$ is stable by $\sigma_{\mathbf{q}}$. The same arguments than those used in (1), prove that $\Delta_{t,\mathbf{q}}(C_{\mathbf{q}}(\ell_{\mathbf{q}})) \subset C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}})$ and $\partial_s(C_{\mathbf{q}}(\ell_{\mathbf{q}})) \subset C_{\mathbf{q}}(\ell_{\mathbf{q}})$. It remains to prove that $\Delta_{t,\mathbf{q}}(C_q(\ell_q)) \subset C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}}, \ell_q)$. We note that $\partial_t(\mathbf{q})\ell_{\mathbf{q}}\partial_s + \partial_t = \Delta_{t,\mathbf{q}} = \Delta_{t,q} + (\partial_t(\mathbf{q})\ell_{\mathbf{q}} - \partial_t(q)\ell_q)\partial_s$. Since C_q is stabilized by $\Delta_{t,q}$ and ∂_s, we find that $\Delta_{t,\mathbf{q}}(C_q) \subset C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}}, \ell_q)$. Moreover, since $\partial_s(\ell_q), \Delta_{t,q}(\ell_q)$ belong to C_q, see Remark D.3, we find that $\Delta_{t,\mathbf{q}}(\ell_q) \in C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}}, \ell_q)$. We have shown the inclusion $\Delta_{t,\mathbf{q}}(C_q(\ell_q)) \subset C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}}, \ell_q)$. This concludes the proof for $C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}}, \ell_q)$.

Let us now consider $C_q(\ell_q)$. By Remark D.3 and $\partial_t = \Delta_{t,q} - \partial_t(q)\ell_q\partial_s$, we find that the inclusion holds $\partial_s(\ell_q), \partial_t(\ell_q) \in C_q(\ell_q)$. Since $\partial_s, \Delta_{t,q}$ commute with σ_q, C_q is stable by $\partial_s, \Delta_{t,q}$. With $\partial_t = \Delta_{t,q} - \partial_t(q)\ell_q\partial_s$, it follows that $\partial_t(C_q) \subset C_q(\ell_q)$. Finally, we obtain that the field $C_q(\ell_q)$ is stable by ∂_s, ∂_t.

$\square$

D.2. **Difference Galois theory for elliptic function fields.** In this section, we apply the results of §C to the specific cases of elliptic function fields introduced in Lemma D.5. We recall that the following fields extensions are $(\sigma, \partial, \Delta)$-fields extensions.

- Let $\mathbf{q} \in C^*$ with $|\mathbf{q}| \neq 1$. Then, let us consider
$$(C_{\mathbf{q}}(s, \ell_{\mathbf{q}}), \sigma_{\mathbf{q}}, \partial_s, \Delta_{t,\mathbf{q}}) \subset (\mathcal{M}er(C^*), \sigma_{\mathbf{q}}, \partial_s, \Delta_{t,\mathbf{q}}).$$
- Let $\mathbf{q}$ and q two elements of C^* such that $|q|, |\mathbf{q}| \neq 1$, that are *multiplicatively independent*. Let us consider
$$(C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}}, \ell_q), \sigma_{\mathbf{q}}, \partial_s, \Delta_{t,\mathbf{q}}) \subset (\mathcal{M}er(C^*), \sigma_{\mathbf{q}}, \partial_s, \Delta_{t,\mathbf{q}}).$$

In that framework, the criteria obtained in §C to guaranty the $(\partial_s, \Delta_{t,\mathbf{q}})$-differential transcendence of a solution of a rank one $\mathbf{q}$-difference equation can be simplified and some descent arguments prove that the existence of a telescoping relation involving the two derivatives implies the existence of a telescoping relations involving only the derivation ∂_s. More precisely, we find the following proposition:

Proposition D.6. *Let $K \subset \mathcal{M}er(C^*)$ be a $(\sigma_{\mathbf{q}}, \partial_s)$-field and let us assume that*

(H1) $L = K(\ell_{\mathbf{q}})$ *is a $(\sigma_{\mathbf{q}}, \partial_s, \Delta_{t,\mathbf{q}})$-field;*
(H2) $K^{\sigma_{\mathbf{q}}} = L^{\sigma_{\mathbf{q}}} = C_{\mathbf{q}}$ *is relatively algebraically closed in L;*
(H3) $\ell_{\mathbf{q}}$ *is transcendental over K.*

Let $f \in \mathcal{M}er(C^*)$, that satisfies $\sigma_{\mathbf{q}}(f) = f + b$, for some b that belongs to a subfield of K stable by ∂_s, ∂_t.

If f is $(\partial_s, \Delta_{t,\mathbf{q}})$-differentially algebraic over L then, there exist $m \in \mathbb{N}$, $d_0, \dots, d_m \in C_{\mathbf{q}}$ not all zero, and $h \in K$ such that

$$d_0 b_1 + d_1 \partial_s(b) + \cdots + d_m \partial_s^m(b) = \sigma_{\mathbf{q}}(h) - h.$$

Proof. Since f is $(\partial_s, \Delta_{t,\mathbf{q}})$-differentially algebraic over L and $K^{\sigma_{\mathbf{q}}}$ is relatively algebraically closed, Theorem C.8 yields that there exist $M \in \mathbb{N}$, $c_{i,j} \in L^{\sigma_{\mathbf{q}}}$ not all zero, and $g \in L$ such that

$$\sum_{i,j \leq M} c_{i,j} \partial_s^i \Delta_{t,\mathbf{q}}^j(b) = \sigma_{\mathbf{q}}(g) - g. \tag{D.1}$$

By Lemma D.4, for all $i \in \mathbb{N}$, there exist $c_{j,k,l} \in C_{\mathbf{q}}$ such that

$$\Delta_{t,\mathbf{q}}^i = (\partial_t(\mathbf{q})\ell_{\mathbf{q}})^i \partial_s^i + \sum_{k=0}^{i-1} \sum_{j=0}^{k} \sum_{l=0}^{i} c_{j,k,l} \ell_{\mathbf{q}}^j \partial_s^k \partial_t^l. \tag{D.2}$$

The left hand side of (D.1) is a polynomial in $\ell_{\mathbf{q}}$ with coefficients in K. By Lemma C.3 with **(H2)** and **(H3)**, we find that $g \in K[\ell_{\mathbf{q}}]$ as well.

Thus, let us write $g = \sum_{k=0}^{R} \alpha_k \ell_{\mathbf{q}}^k$ with $\alpha_k \in K$ and $\alpha_R \neq 0$. Let

$$N = \max\{j \in \mathbb{N} | \exists i \text{ such that } c_{i,j} \neq 0\}.$$

By (D.2), the coefficient of highest degree in $\ell_{\mathbf{q}}$ of the left hand side of (D.1) is

$$\left(\sum_{i \leq M} c_{i,N} (\partial_t(\mathbf{q}))^N \partial_s^{N+i}(b) \right) \ell_{\mathbf{q}}^N. \tag{D.3}$$

On the other hand, we have

$$\sigma_{\mathbf{q}}(g) - g = \ell_{\mathbf{q}}^R(\sigma_{\mathbf{q}}(\alpha_R) - \alpha_R)) + \ell_{\mathbf{q}}^{R-1}(\sigma_{\mathbf{q}}(\alpha_{R-1}) - \alpha_{R-1} + R\sigma_{\mathbf{q}}(\alpha_R)) + P(\ell_{\mathbf{q}}), \tag{D.4}$$

where $P(X) \in K[X]$ is a polynomial of degree strictly smaller than $R-1$. Then, comparing (D.3) and (D.4), we find that

- either $R < N$ so that

$$\sum_{i \leq M} c_{i,N} (\partial_t(\mathbf{q}))^N \partial_s^{N+i}(b) = 0, \tag{D.5}$$

- either $R=N$ so that

$$\sum_{i\leq M} c_{i,N}(\partial_t(\mathbf{q}))^N \partial_s^{N+i}(b) = \sigma_{\mathbf{q}}(\alpha_N) - \alpha_N, \tag{D.6}$$

- or $R>N$ so that $R>0$, $0\neq \alpha_R \in L^{\sigma_{\mathbf{q}}}$. We claim that $R=N-1$. Indeed, $R>N-1$ implies $\sigma_{\mathbf{q}}(\alpha_R)=\sigma_{\mathbf{q}}(\alpha_R)$, $\sigma_{\mathbf{q}}(\alpha_{R-1})-\alpha_{R-1}+R\alpha_R=0$ and then $\sigma_{\mathbf{q}}(\frac{\alpha_{R-1}}{\alpha_R})-\frac{\alpha_{R-1}}{\alpha_R}+R=0$ with $\frac{\alpha_{R-1}}{\alpha_R}\in K$ in contradiction with Lemma C.2 applied to $f=\ell_{\mathbf{q}}$. Thus, we get $R=N-1$ and

$$\sum_{i\leq M} \frac{c_{i,N}}{\alpha_R}(\partial_t(\mathbf{q}))^N \partial_s^{N+i}(b) = \sigma_{\mathbf{q}}\left(\frac{\alpha_{R-1}}{\alpha_R}\right) - \frac{\alpha_{R-1}}{\alpha_R} + R. \tag{D.7}$$

For all these cases, note that there exists i_0 such that $c_{i_0,N}\neq 0$ by definition of N. Since ∂_s commutes with $\sigma_{\mathbf{q}}$, we can derive (D.7) with respect to ∂_s and obtain that in any case, there exists $d_k \in L^{\sigma_{\mathbf{q}}} = C_{\mathbf{q}}$ not all zero and $h\in K$ such that

$$\sum_{k\leq M+1} d_k \partial_s^k(b) = \sigma_{\mathbf{q}}(h) - h. \tag{D.8}$$

□

D.3. **Transcendence properties.** The goal of this subsection is to prove some transcendence properties of the $\mathbf{q}$-logarithm in order to perform some descent procedure on telescopers. More precisely, we need to prove that the assumptions **(H1)** to **(H3)** of Proposition D.6 are satisfied for the fields $C_{\mathbf{q}}(s)$ and $C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}},\ell_q)$ for $\mathbf{q}$ and q two multiplicatively independent elements of C^* with $|q|\neq 1$, $|\mathbf{q}|\neq 1$. We recall that q and $\mathbf{q}$ are multiplicatively independent if there are no $(r,l)\in\mathbb{Z}^2\setminus(0,0)$ such that $q^r=\mathbf{q}^l$. Remind that $C_{\mathbf{q}}.C_q \subset \mathcal{M}er(C^*)$ is the compositum of fields and $\ell_{\mathbf{q}}\in\mathcal{M}er(C^*)$ is a solution of $\sigma_{\mathbf{q}}(y)=y+1$. With Lemma D.5, **(H1)** of Proposition D.6 is satisfied for $K=C_{\mathbf{q}}(s)$ and $K=C_{\mathbf{q}}.C_q(\ell_q)$.

Lemma D.7. *Any element in a $\sigma_{\mathbf{q}}$-extension of C_q[**] that is algebraic over C_q and invariant by $\sigma_{\mathbf{q}}$ is in C. Any element in a σ_q-extension of $C_{\mathbf{q}}$ that is algebraic over $C_{\mathbf{q}}$ and invariant by σ_q is in C.*

Proof. The two statements are symmetrical, so let us only prove the first one. First let us prove that $C_q\cap C_{\mathbf{q}}=C$. Let f be an element of C_q that is $\sigma_{\mathbf{q}}$-invariant. Suppose to the contrary that f is non-constant. Then f has a non-zero pole c. Since $\sigma_{\mathbf{q}}(f)=f$, the multiplication by $\mathbf{q}$ induces a permutation of the poles of f modulo q. Since the set of poles modulo q is a finite set, there exists $m\in\mathbb{N}$ such that $\mathbf{q}^m c = q^d c$ for some $d\in\mathbb{Z}$. A contradiction with the fact that q and $\mathbf{q}$ are multiplicatively independent. Now, let f be in a $\sigma_{\mathbf{q}}$-extension of C_q algebraic over C_q and invariant by $\sigma_{\mathbf{q}}$. Let $\mu(X)\in C_q[X]$ be the monic minimal polynomial of f above C_q. Since $\sigma_{\mathbf{q}}(f)=f$, we easily see that the coefficients of μ must be fixed by $\sigma_{\mathbf{q}}$. Then, these coefficients belong to $C_q\cap C_{\mathbf{q}}$, which is equal to C. Then, f is algebraic over C. The latter field being algebraically closed, we conclude that $f\in C$. □

Lemma D.8. *The following statements hold:*

[**]We recall that since $\sigma_{\mathbf{q}}$ and σ_q commute, the field C_q is a $\sigma_{\mathbf{q}}$-field.

(1) *the fields* $C_{\mathbf{q}}$ *and* C_q *are linearly disjoint over* C*;*
(2) *for all* $\alpha \in C_{\mathbf{q}}.C_q$, $\sigma_q(\alpha) \neq \alpha + 1$ *and* $\sigma_{\mathbf{q}}(\alpha) \neq \alpha + 1$*;*
(3) *for all* $\alpha \in C_{\mathbf{q}}(s)$, $\sigma_{\mathbf{q}}(\alpha) \neq \alpha + 1$.

Proof. (1) This is Lemmas D.7 and C.7 with $K = C$, $M = C_q$ and $L = C_{\mathbf{q}}$, $\sigma = \sigma_{\mathbf{q}}$.
(2) Suppose to the contrary that there exists $\alpha \in C_{\mathbf{q}}.C_q$, such that $\sigma_q(\alpha) = \alpha + 1$. Since C_q is by Proposition 3.1, the field of meromorphic functions over a Tate curve, there exist $x, y \in C_q$ such that x is transcendental over C, y algebraic of degree 2 over $C(x)$ and $C_q = C(x, y)$. Since $C_{\mathbf{q}}$ is linearly disjoint from C_q over C, the field $C_{\mathbf{q}}.C_q$ equals $C_{\mathbf{q}}(x, y)$ and there are $P(X), Q(X) \in C_{\mathbf{q}}(X)$ such that $\alpha = P(x)y + Q(x)$. Since x, y are fixed by σ_q and y is of degree 2 over $C_{\mathbf{q}}(x)$, we deduce from $\sigma_q(\alpha) = \alpha + 1$ that $P^{\sigma_q}(x) = P(x)$ and $Q^{\sigma_q}(x) - Q(x) = 1$ where $P^{\sigma_q}(X)$ (resp. $Q^{\sigma_q}(X)$) denotes the fraction obtained from $P(X)$ (resp. $Q(X)$) by applying σ_q to the coefficients. Let $\overline{C_{\mathbf{q}}}$ be some algebraic closure of $C_{\mathbf{q}}$. We endow $\overline{C_{\mathbf{q}}}$ with a structure of σ_q-field extension of $C_{\mathbf{q}}$. Let us write $Q(X) = \frac{c_r}{X^r} + \cdots + \frac{c_1}{X} + R(X)$ with $R \in \overline{C_{\mathbf{q}}}(X)$ with no pole at $X = 0$. Then, since x is transcendental over $\overline{C_{\mathbf{q}}}$ and fixed by σ_q

$$Q^{\sigma_q}(x) - Q(x) = 1 = \frac{\sigma_q(c_r) - c_r}{x^r} + \cdots + \frac{\sigma_q(c_1) - c_1}{x} + R^{\sigma_q}(x) - R(x).$$

Using the transcendence of x over $\overline{C_{\mathbf{q}}}$, we find that $1 = \sigma_q(\tilde{\beta}) - \tilde{\beta}$ for $\tilde{\beta} = R(0) \in \overline{C_{\mathbf{q}}}$. There exists a unique derivation extending ∂_s to $\overline{C_{\mathbf{q}}}$ and this derivation commutes with σ_q. Denoting this derivation by ∂_s and deriving $1 = \sigma_q(\tilde{\beta}) - \tilde{\beta}$, we conclude that $\partial_s(\tilde{\beta}) \in C_q \cap C_{\mathbf{q}^r}$. Note that q and $\mathbf{q}^r$ are multiplicatively independent. By Lemma D.7, we find that $\partial_s(\tilde{\beta}) \in C$ which leads to $\tilde{\beta} = cs + d$ for some $c, d \in C$. A contradiction with $1 = \sigma_q(\tilde{\beta}) - \tilde{\beta}$. The proof for $\mathbf{q}$ is similar.
(3) Let $\alpha \in C_{\mathbf{q}}(s)$. Using the partial fraction decomposition of α in $\overline{C_{\mathbf{q}}}(s)$, the fact that $\sigma_{\mathbf{q}}(s) = \mathbf{q}s$ and the transcendence of s over $C_{\mathbf{q}}$, one can easily see that $\sigma_{\mathbf{q}}(\alpha) - \alpha \neq 1$. □

Lemma D.9. *The following statements hold:*

(1) *the function* $\ell_{\mathbf{q}}$ *(resp.* ℓ_q*) is transcendental over* $C_{\mathbf{q}}.C_q$*;*
(2) *the function* $\ell_{\mathbf{q}}$ *is transcendental over* $C_{\mathbf{q}}(s)$*. In particular,* ***(H3)*** *of Proposition D.6 is satisfied for* $K = C_{\mathbf{q}}(s)$*.*

Proof. (1) Since $\sigma_{\mathbf{q}}(\ell_{\mathbf{q}}) = \ell_{\mathbf{q}} + 1$ and $C_{\mathbf{q}} \subset (C_{\mathbf{q}}.C_q)^{\sigma_{\mathbf{q}}} \subset \mathcal{M}er(C^*)^{\sigma_{\mathbf{q}}} = C_{\mathbf{q}}$, we can apply Lemma C.2 and find that $\ell_{\mathbf{q}}$ is algebraic over $C_{\mathbf{q}}.C_q$ if and only if there exists $\alpha \in C_{\mathbf{q}}.C_q$ such that $\sigma_{\mathbf{q}}(\alpha) = \alpha + 1$. We conclude by Lemma D.8. The proof for ℓ_q is symmetrical.
(2) Since $\sigma_{\mathbf{q}}(\ell_{\mathbf{q}}) = \ell_{\mathbf{q}} + 1$ and $C_{\mathbf{q}} \subset (C_{\mathbf{q}}(s))^{\sigma_{\mathbf{q}}} \subset \mathcal{M}er(C^*)^{\sigma_{\mathbf{q}}} = C_{\mathbf{q}}$, we can apply Lemma C.2 and find that $\ell_{\mathbf{q}}$ is algebraic over $C_{\mathbf{q}}(s)$ if and only if there exists $\alpha \in C_{\mathbf{q}}(s)$ such that $\sigma_{\mathbf{q}}(\alpha) = \alpha + 1$. We again conclude by Lemma D.8. □

Lemma D.10. *The following statement hold:*

(1) *let* $f \in C_q$*. If there exists* $\alpha \in C_{\mathbf{q}}.C_q$ *satisfying* $\sigma_{\mathbf{q}}(\alpha) - \alpha = f$*, then there exists* $\beta \in C_q$ *such that* $\sigma_{\mathbf{q}}(\beta) - \beta = f$*;*
(2) *let* $f \in C_{\mathbf{q}}.C_q$*. If there exists* $\alpha \in C_{\mathbf{q}}.C_q(\ell_q)$ *satisfying* $\sigma_{\mathbf{q}}(\alpha) - \alpha = f$*, then, there exist* $\tilde{a} \in C_{\mathbf{q}}, \tilde{b} \in C_{\mathbf{q}}.C_q$ *such that* $\sigma_{\mathbf{q}}(\tilde{a}\ell_q + \tilde{b}) - (\tilde{a}\ell_q + \tilde{b}) = f$*.*

Proof. (1) Analogously to the proof of Lemma D.8, let us write $\alpha = P(x)y + Q(x)$ for $P(X), Q(X) \in C_q(X)$ and $C_{\mathbf{q}} = C(x,y)$. Reasoning as in the proof of Lemma D.8, we find that $Q^{\sigma_{\mathbf{q}}}(x) - Q(x) = f$. Since x is transcendental over C_q, we conclude as in Lemma D.8 that there is $\tilde{\beta} \in \overline{C_q}$, for some $\overline{C_q}$ algebraic closure of C_q such that $\sigma_{\mathbf{q}}(\tilde{\beta}) - \tilde{\beta} = f$. Since by Lemma D.7, $\overline{C_q}^{\sigma_{\mathbf{q}}} = C_q^{\sigma_{\mathbf{q}}} = C$, Lemma C.2 implies that there exists $\beta \in C_q$ such that $\sigma_{\mathbf{q}}(\beta) - \beta = f$.
(2) First of all, let us note that since $\sigma_{\mathbf{q}}$ and σ_q commute, there exists $d \in C_q$ such that

$$\sigma_{\mathbf{q}}(\ell_q) = \ell_q + d. \tag{D.9}$$

By Lemma D.9, the function ℓ_q is transcendental over $C_{\mathbf{q}}.C_q$. This implies that $\ell_q \notin C_{\mathbf{q}}$ and then $d \neq 0$. Since $C_{\mathbf{q}}.C_q(\ell_q)^{\sigma_{\mathbf{q}}} = C_{\mathbf{q}} = \mathcal{M}er(C^*)^{\sigma_{\mathbf{q}}} = C_{\mathbf{q}}.C_q^{\sigma_{\mathbf{q}}} = C_{\mathbf{q}}$, Lemma C.3, applied to $\sigma_{\mathbf{q}}(\ell_q) = \ell_q + d$, implies that there exists $P \in C_{\mathbf{q}}.C_q[X]$ such that

$$f = \sigma_{\mathbf{q}}(P(\ell_q)) - P(\ell_q).$$

Now, let us write $P(X) = \sum_{k=0}^{N} a_k X^k$ with $a_k \in C_{\mathbf{q}}.C_q$, and N minimal. We find

$$f = (\sigma_{\mathbf{q}}(a_N) - a_N)\ell_q^N + (\sigma_{\mathbf{q}}(a_{N-1}) - a_{N-1} + Nd\sigma_{\mathbf{q}}(a_N))\ell_q^{N-1} + \text{terms of order less than } N-1. \tag{D.10}$$

We conclude in view of (D.10) that if $N=0$ we are done by setting $\tilde{a} = 0$ and $\tilde{b} = a_N$. Let us now assume that $N > 0$. Then, by minimality of N, $\sigma_{\mathbf{q}}(a_N) = a_N$. We claim that $\sigma_{\mathbf{q}}(a_{N-1}) - a_{N-1} + Nd\sigma_{\mathbf{q}}(a_N) = \sigma_{\mathbf{q}}(a_{N-1}) - a_{N-1} + Nda_N \neq 0$. To the contrary, $\sigma_{\mathbf{q}}(a_{N-1}) = a_{N-1} - Nda_N$ implies $\sigma_{\mathbf{q}}(\frac{a_{N-1}}{a_N} + N\ell_q) = \frac{a_{N-1}}{a_N} + N\ell_q$ and $\frac{a_{N-1}}{a_N} + N\ell_q \in C_{\mathbf{q}}$, contradicting the transcendence of ℓ_q over $C_{\mathbf{q}}.C_q$, see Lemma D.9. This proves the claim. If $N > 1$, then (D.10) with $\sigma_{\mathbf{q}}(a_N) = a_N$ and $\sigma_{\mathbf{q}}(a_{N-1}) - a_{N-1} + Nda_N \neq 0$, would give an equation of order $N-1$ which would contradicts the transcendence of ℓ_q over $C_{\mathbf{q}}.C_q$. This proves that $N=1$ and $f = \sigma_{\mathbf{q}}(a_1\ell_q + a_0) - (a_1\ell_q + a_0)$ for some $a_1 \in C_{\mathbf{q}}, a_0 \in C_{\mathbf{q}}.C_q$. □

Lemma D.11. *The function $\ell_{\mathbf{q}}$ is transcendental over $C_{\mathbf{q}}.C_q(\ell_q)$. In particular, the assumption **(H3)** of Proposition D.6 holds for $K = C_{\mathbf{q}}.C_q(\ell_q)$.*

Proof. By Lemma C.2, the function $\ell_{\mathbf{q}}$ is algebraic over $C_{\mathbf{q}}.C_q(\ell_q)$ if and only if we have $\ell_{\mathbf{q}} \in C_{\mathbf{q}}.C_q(\ell_q)$. Suppose to the contrary that $\ell_{\mathbf{q}} \in C_{\mathbf{q}}.C_q(\ell_q)$. Since $1 = \sigma_{\mathbf{q}}(\ell_{\mathbf{q}}) - \ell_{\mathbf{q}}$, we conclude by Lemma D.10 that there exist $\tilde{a} \in C_{\mathbf{q}}, \tilde{b} \in C_{\mathbf{q}}.C_q$ such that $1 = \sigma_{\mathbf{q}}(\tilde{a}\ell_q + \tilde{b}) - (\tilde{a}\ell_q + \tilde{b})$. Combining this equation with $\sigma_{\mathbf{q}}(\ell_{\mathbf{q}}) - \ell_{\mathbf{q}} = 1$, we find that $\sigma_{\mathbf{q}}(\ell_{\mathbf{q}}) - \ell_{\mathbf{q}} = \sigma_{\mathbf{q}}(\tilde{a}\ell_q + \tilde{b}) - (\tilde{a}\ell_q + \tilde{b})$, proving that $\sigma_{\mathbf{q}}(\tilde{a}\ell_q + \tilde{b} - \ell_{\mathbf{q}}) = \tilde{a}\ell_q + \tilde{b} - \ell_{\mathbf{q}} \in C_{\mathbf{q}}$. Then, there exists $\widetilde{b_1} \in C_{\mathbf{q}}.C_q$ such that

$$\ell_{\mathbf{q}} = \widetilde{a}\ell_q + \widetilde{b_1}. \tag{D.11}$$

Deriving (D.11) with respect to ∂_s, we find

$$\partial_s(\ell_{\mathbf{q}}) = \partial_s(\widetilde{a})\ell_q + \widetilde{a}\partial_s(\ell_q) + \partial_s(\widetilde{b_1}).$$

By Remark D.3, $\partial_s(\ell_{\mathbf{q}}), \partial_s(\ell_q) \in C_{\mathbf{q}}.C_q$. In virtue of the commutation between ∂_s and $\sigma_{\mathbf{q}}, \sigma_q$, the fields $C_q, C_{\mathbf{q}}$ are stabilized by ∂_s, which implies $\partial_s(\widetilde{a}), \partial_s(\widetilde{b_1}) \in C_{\mathbf{q}}.C_q$. By Lemma D.9, the function ℓ_q is transcendental over the latter field, we conclude that $\partial_s(\widetilde{a}) = 0$ and therefore $\widetilde{a} \in C$. In particular it belongs to $C_{\mathbf{q}}$ and C_q. Using $1 = \sigma_{\mathbf{q}}(\tilde{a}\ell_q + \tilde{b}) - (\tilde{a}\ell_q + \tilde{b})$, we find

$$1 - \widetilde{a}d = \sigma_{\mathbf{q}}(\widetilde{b}) - \widetilde{b},$$

where $d = \sigma_{\mathbf{q}}(\ell_q) - \ell_q \in C_q$, see (D.9). Since $1 - \widetilde{a}d \in C_q$, we conclude by Lemma D.10, that there exists $\widetilde{b_2} \in C_q$ such that $1 - \widetilde{a}d = \sigma_{\mathbf{q}}(\widetilde{b_2}) - \widetilde{b_2}$. Replacing the left hand side gives

$$\sigma_{\mathbf{q}}(\ell_{\mathbf{q}}) - \ell_{\mathbf{q}} - \sigma_{\mathbf{q}}(\widetilde{a}\ell_q) + \widetilde{a}\ell_q = \sigma_{\mathbf{q}}(\widetilde{b_2}) - \widetilde{b_2}.$$

This shows that $\ell_{\mathbf{q}} - \widetilde{a}\ell_q - \widetilde{b_2} \in C_{\mathbf{q}}$ and then, there exists $c \in C_{\mathbf{q}}$ such that $\ell_{\mathbf{q}} + c = \widetilde{a}\ell_q + \widetilde{b_2}$. Deriving this equation with respect to ∂_s, we find (we use $\partial_s(\widetilde{a}) = 0$)

$$\partial_s(\ell_{\mathbf{q}}) + \partial_s(c) = \widetilde{a}\partial_s(\ell_q) + \partial_s(\widetilde{b_2}).$$

By Remark D.3, the left hand side of the equation belongs to $C_{\mathbf{q}}$ whereas the right hand side is in C_q. By Lemma D.7, we conclude that $\partial_s(\ell_{\mathbf{q}} + c) \in C$. This means that there exist $a_0, b_0 \in C$ such that $\ell_{\mathbf{q}} = a_0 s + b_0 - c$ in contradiction with $\ell_{\mathbf{q}}$ transcendental over $C_{\mathbf{q}}(s)$, see Lemma D.9. □

We can now prove that our fields satisfy the assumption **(H2)** of Proposition D.6.

Lemma D.12. *The following holds:*

(1) *$C_{\mathbf{q}}$ is relatively algebraically closed in $C_{\mathbf{q}}(s, \ell_{\mathbf{q}})$;*
(2) *$C_{\mathbf{q}}$ is relatively algebraically closed in $C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}}, \ell_q)$.*

*In particular, **(H2)** of Proposition D.6 holds for $K = C_{\mathbf{q}}(s)$ and $K = C_{\mathbf{q}}.C_q(\ell_q)$.*

Proof. (1) The first point is a consequence of transcendence of s over $C_{\mathbf{q}}$, and the transcendence of $\ell_{\mathbf{q}}$ over $C_{\mathbf{q}}(s)$, see Lemma D.9.
(2) Let us prove the second point. Let us start by proving that $C_{\mathbf{q}}$ is relatively algebraically closed in $C_{\mathbf{q}}.C_q$. As in the proof of Lemma D.8, we have $C_{\mathbf{q}} = C(x, y)$ and $C_{\mathbf{q}}.C_q = C_q(x, y)$ where y is of degree 2 over both $C(x)$ and $C_q(x)$. Let $f \in C_q(x, y)$. Then $f = P(x)y + Q(x)$ with $P(x), Q(x) \in C_q(x)$. If f is algebraic over $C_{\mathbf{q}}$ then Lemma C.1 implies that $\sigma_{\mathbf{q}}^r(f) = f$ for some $r \in \mathbb{Z}^*$ and therefore $\sigma_{\mathbf{q}}^r(P(x)) = P(x)$ and $\sigma_{\mathbf{q}}^r(Q(x)) = Q(x)$. We claim that $P(x)$ and $Q(x)$ are in $C(x)$, and therefore that $f \in C_{\mathbf{q}}$. Let $P(x) = P_1(x)/P_2(x)$ where $P_1(x), P_2(x) \in C_q[x]$ are relatively prime and $P_1(x)$ is monic. We then have that $\sigma_{\mathbf{q}}^r(P_1(x))P_2(x) = \sigma_{\mathbf{q}}^r(P_2(x))P_1(x)$ and consequently $P_1(x)$ divides $\sigma_{\mathbf{q}}^r(P_1(x))$ (resp. $\sigma_{\mathbf{q}}^r(P_1(x))$ divides $P_1(x)$). Since $P_1(x)$ is monic, $P_1(x) = \sigma_{\mathbf{q}}^r(P_1(x))$ and $P_2(x) = \sigma_{\mathbf{q}}^r(P_2(x))$. This implies that the coefficients of $P_1(x)$ and $P_2(x)$ are left fixed by $\sigma_{\mathbf{q}}^r$. Note that by assumption, q and $\mathbf{q}^r$ are multiplicatively independent. Therefore, by Lemma D.7, applied with $\mathbf{q}$ replaced by $\mathbf{q}^r$, $P_1, P_2 \in C[X]$. The proof for Q is similar. This proves our claim and show that $f \in C_{\mathbf{q}}$. Then $C_{\mathbf{q}}$ is relatively algebraically closed in $C_{\mathbf{q}}.C_q$.

Note that Lemma D.9 implies that $\ell_{\mathbf{q}}$ is transcendental over $C_{\mathbf{q}}.C_q$ and Lemma D.11 implies that ℓ_q is transcendental over $C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}})$. Therefore $C_{\mathbf{q}}$ is relatively algebraically closed in $C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}}, \ell_q)$.

□

Finally, we prove a lemma that will allows us to descend some *telescoping relations* on smaller base fields.

Lemma D.13. *Let $b \in C_q$ such that there exist $N \in \mathbb{N}$, $c_i \in C_{\mathbf{q}}$ with $c_N \neq 0$, and $g \in C_{\mathbf{q}}.C_q(\ell_{\mathbf{q}}, \ell_q)$ that satisfy*

$$\sum_{i=0}^{N} c_i \partial_s^i(b) = \sigma_{\mathbf{q}}(g) - g. \tag{D.12}$$

Then, there exist $m \in \mathbb{N}$, $d_0, \dots, d_m \in C$ not all zero and $h \in C_q$ such that

$$d_0 b_2 + d_1 \partial_s(b_2) + \cdots + d_m \partial_s^m(b_2) = \sigma_{\mathbf{q}}(h) - h.$$

Proof. First of all note that the left hand side of (D.12) belongs to $C_{\mathbf{q}}.C_q$. By Lemma D.11, the function $\ell_{\mathbf{q}}$ is transcendental over $C_{\mathbf{q}}.C_q(\ell_q)$. By Lemma C.3, $g \in C_{\mathbf{q}}.C_q(\ell_q)[\ell_{\mathbf{q}}]$. So let us write $g = \sum_{k=0}^{R} \alpha_k \ell_{\mathbf{q}}^k$ with $\alpha_k \in C_{\mathbf{q}}.C_q(\ell_q)$, $\alpha_R \neq 0$.

Claim. There exist $m \in \mathbb{N}$, $c'_k \in C_{\mathbf{q}}$, $c'_m \neq 0$, and $\alpha \in C_{\mathbf{q}}.C_q(\ell_q)$ such that

$$\sum_{k=0}^{m} c'_k \partial_s^k(b) = \sigma_{\mathbf{q}}(\alpha) - \alpha. \tag{D.13}$$

If $R = 0$ the claim is proved. Assume that $R > 0$. Then, we have

$$\sigma_{\mathbf{q}}(g) - g = \ell_{\mathbf{q}}^R(\sigma_{\mathbf{q}}(\alpha_R) - \alpha_R)) + \ell_{\mathbf{q}}^{R-1}(\sigma_{\mathbf{q}}(\alpha_{R-1}) - \alpha_{R-1} + R\alpha_R) + P(\ell_{\mathbf{q}}), \tag{D.14}$$

where $P(X) \in C_{\mathbf{q}}.C_q(\ell_q)[X]$ is a polynomial of degree smaller than $R-1$. Then, comparing (D.14) and (D.12), we find, by transcendence of $\ell_{\mathbf{q}}$ over $C_{\mathbf{q}}.C_q(\ell_q)$, see Lemma D.11, that $\sigma_{\mathbf{q}}(\alpha_R) = \alpha_R$. Let us prove that $\sigma_{\mathbf{q}}(\alpha_{R-1}) - \alpha_{R-1} + R\alpha_R \neq 0$. Indeed if $\sigma_{\mathbf{q}}(\alpha_{R-1}) - \alpha_{R-1} + R\alpha_R = 0$ then $\sigma_{\mathbf{q}}(\frac{\alpha_{R-1}}{\alpha_R}) - \frac{\alpha_{R-1}}{\alpha_R} + R = 0$ with $\frac{\alpha_{R-1}}{\alpha_R} \in C_{\mathbf{q}}.C_q$ in contradiction with Lemma D.9 and Lemma C.2. We then obtain that $R = 1$ since otherwise we would deduce from (D.14) an algebraic relation for $\ell_{\mathbf{q}}$ over $C_{\mathbf{q}}.C_q(\ell_q)$, contradicting Lemma D.11. Thus,

$$\sum_{i=0}^{N} \frac{c_i}{\alpha_1} \partial_s^i(b) = \sigma_{\mathbf{q}}\left(\frac{\alpha_0}{\alpha_1}\right) - \frac{\alpha_0}{\alpha_1} + 1. \tag{D.15}$$

Remind that $\alpha_1 \in C_{\mathbf{q}}$ and the latter field is stable by ∂_s due to the commutation between ∂_s and $\sigma_{\mathbf{q}}$. By Lemma D.5, the field $C_{\mathbf{q}}.C_q(\ell_q)$ is stabilized by ∂_s. We can derive (D.15) with respect to ∂_s and using the commutation between $\sigma_{\mathbf{q}}$ and ∂_s, we obtain our claim.

Claim. There exist $M \in \mathbb{N}$, $d_k \in C_{\mathbf{q}}$, $d_M \neq 0$ and $\beta \in C_{\mathbf{q}}.C_q$ such that

$$\sum_{k=0}^{M} d_k \partial_s^k(b) = \sigma_{\mathbf{q}}(\beta) - \beta.$$

Indeed, by Lemma D.10, we can find $a \in C_{\mathbf{q}}, b \in C_{\mathbf{q}}.C_q$ such that

$$\sum_{k=0}^{m} c'_k \partial_s^k(b) = \sigma_{\mathbf{q}}(a\ell_q + b) - (a\ell_q + b). \tag{D.16}$$

Either $a=0$ and $\sum_k c'_k \partial_s^k(b) = \sigma_{\mathbf{q}}(b) - (b)$ for some $b \in C_{\mathbf{q}}.C_q$. Or $a \neq 0$ and dividing (D.16) by a and deriving with respect to ∂_s, we find

$$\sum_{k=0}^{m+1} d_k \partial_s^k(b) = \sigma_{\mathbf{q}}(\partial_s(\ell_q) + \partial_s(b/a)) - (\partial_s(\ell_q) + \partial_s(b/a)),$$

where the d_k are in $C_{\mathbf{q}}$, $d_{m+1} = \frac{c'_m}{a} \neq 0$. Furthermore, by Remark D.3 and the fact that $C_{\mathbf{q}}$, C_q, are stable by ∂_s, we find $\partial_s(\ell_q) + \partial_s(b/a) \in C_{\mathbf{q}}.C_q$. This proves the claim.

Now, let us consider an equation of the form

$$\sum_{k=0}^{M} d_k \partial_s^k(b) = \sigma_{\mathbf{q}}(\beta) - \beta,$$

with $\beta \in C_{\mathbf{q}}.C_q$, $d_k \in C_{\mathbf{q}}$ and $d_M \neq 0$, minimal with respect to the maximal order of derivation M of b. We can write this minimal equation as follows

$$d_M \partial_s^M(b) + \sum_{k=0}^{M-1} d_k \partial_s^k(b) = \sigma_{\mathbf{q}}(\beta) - \beta,$$

with $d_M \in C_{\mathbf{q}}^*$. Then dividing by d_M, we find

$$\partial_s^M(b) + \sum_{k=0}^{M-1} \frac{d_k}{d_M} \partial_s^k(b) = \sigma_{\mathbf{q}}\left(\frac{\beta}{d_M}\right) - \frac{\beta}{d_M}.$$

Therefore, we can without loss of assumption assume that $d_M = 1$. Now, if we compute the element $\sigma_q(\sigma_{\mathbf{q}}(\beta) - \beta)) - (\sigma_{\mathbf{q}}(\beta) - \beta))$ and use the fact that $b \in C_q$, we find

$$\sum_{k=0}^{M-1} (\sigma_q(d_k) - d_k) \partial_s^k(b) = \sigma_{\mathbf{q}}(\sigma_q(\beta) - \beta) - (\sigma_q(\beta) - \beta).$$

By minimality, we find that, for all k, the element $d_k \in C_{\mathbf{q}}$ is fixed by σ_q. This means that $d_k \in C$ by Lemma D.7.

Since $\partial_s^M(b) + \sum_{k=0}^{M-1} d_k \partial_s^k(b) \in C_q$ and $\partial_s^M(b) + \sum_{k=0}^{M-1} d_k \partial_s^k(b) = \sigma_{\mathbf{q}}(\beta) - \beta$ with $\beta \in C_{\mathbf{q}}.C_q$, Lemma D.10 shows that we have the existence of $h \in C_q$ such that

$$\partial_s^M(b) + \sum_{k=0}^{M-1} d_k \partial_s^k(b) = \sigma_{\mathbf{q}}(h) - h.$$

□

The results of Appendix D.3 are summarized in the following crucial corollary.

Corollary D.14. *The assumptions of Proposition D.6 are satisfied for*

- *Genus zero case: $K = C_{\mathbf{q}}(s)$ and $b \in C(s)$ with $\mathbf{q} \in C^*$ such that $|\mathbf{q}| \neq 1$;*
- *Genus one case: $K = C_{\mathbf{q}}.C_q(\ell_q)$ and $b \in C_q(\ell_q)$ with $\mathbf{q}, q \in C^*$ such that $|\mathbf{q}|, |q| \neq 1$ and $\mathbf{q}$ and q are multiplicatively independent.*

Proof. The fact that the field K and b satisfy the assumptions **(Hi)** is Lemmas D.5, D.9, D.11, and D.12. □

References

[Abr95] S. A. Abramov, *Indefinite sums of rational functions*, Proceedings of the 1995 International Symposium on Symbolic and Algebraic Computation (1995), 303–308.

[AvdDvdH17] Matthias Aschenbrenner, Lou van den Dries, and Joris van der Hoeven, *Asymptotic differential algebra and model theory of transseries*, Annals of Mathematics Studies, vol. 195, Princeton University Press, Princeton, NJ, 2017.

[BBMR16] Olivier Bernardi, Mireille Bousquet-Mélou, and Kilian Raschel, *Counting quadrant walks via Tutte's invariant method (extended abstract)*, to appear in *Proceedings of FPSAC 2015*, Discrete Math. Theor. Comput. Sci. Proc., 2016.

[BMM10] Mireille Bousquet-Mélou and Marni Mishna, *Walks with small steps in the quarter plane*, Algorithmic probability and combinatorics, Contemp. Math., vol. 520, Amer. Math. Soc., Providence, RI, 2010, pp. 1–39.

[BvHK10] Alin Bostan, Mark van Hoeij, and Manuel Kauers, *The complete generating function for gessel walks is algebraic*, Proc. Amer. Math. Soc. **138** (2010), no. 9, 3063–3078.

[Coh65] Richard M. Cohn, *Difference algebra*, Interscience Publishers John Wiley & Sons, New York-London-Sydeny, 1965.

[CS12] Shaoshi Chen and Michael F. Singer, *Residues and telescopers for bivariate rational functions*, Adv. in Appl. Math. **49** (2012), no. 2, 111–133. MR 2946428

[DHRS17] Thomas Dreyfus, Charlotte Hardouin, Julien Roques, and Michael F Singer, *Walks in the quarter plane, genus zero case*, arXiv preprint arXiv:1710.02848 (2017).

[DHRS18] ———, *On the nature of the generating series of walks in the quarter plane*, Inventiones mathematicae **213** (2018), no. 1, 139–203.

[DR19] Thomas Dreyfus and Kilian Raschel, *Differential transcendence & algebraicity criteria for the series counting weighted quadrant walks*, Publications Mathématiques de Besançon (2019), no. 1, 41–80.

[Dui10] J. Duistermaat, *Discrete integrable systems: Qrt maps and elliptic surfaces*, Springer Monographs in Mathematics, vol. 304, Springer-Verlag, New York, 2010.

[DVH12] Lucia Di Vizio and Charlotte Hardouin, *Descent for differential Galois theory of difference equations: confluence and q-dependence*, Pacific J. Math. **256** (2012), no. 1, 79–104. MR 2928542

[FIM99] Guy Fayolle, Roudolf Iasnogorodski, and Vadim Malyshev, *Random walks in the quarter-plane*, Applications of Mathematics (New York), vol. 40, Springer-Verlag, Berlin, 1999, Algebraic methods, boundary value problems and applications. MR 1691900

[FR10] Guy Fayolle and Kilian Raschel, *On the holonomy or algebraicity of generating functions counting lattice walks in the quarter-plane*, Markov Process. Related Fields **16** (2010), no. 3, 485–496. MR 2759770

[FvdP04] Jean Fresnel and Marius van der Put, *Rigid analytic geometry and its applications*, Progress in Mathematics, vol. 218, Birkhäuser Boston, Inc., Boston, MA, 2004.

[HS08] Charlotte Hardouin and Michael F. Singer, *Differential Galois theory of linear difference equations*, Math. Ann. **342** (2008), no. 2, 333–377.

[Kol68] E. R. Kolchin, *Algebraic groups and algebraic dependence*, Amer. J. Math. **90** (1968), 1151–1164. MR 0240106

[Kol73] Ellis Robert Kolchin, *Differential algebra & algebraic groups*, vol. 54, Academic press, 1973.

[KP11] Manuel Kauers and Peter Paule, *The concrete tetrahedron*, Texts and Monographs in Symbolic Computation, SpringerWienNewYork, Vienna, 2011, Symbolic sums, recurrence equations, generating functions, asymptotic estimates.

[KR12] Irina Kurkova and Kilian Raschel, *On the functions counting walks with small steps in the quarter plane*, Publ. Math. Inst. Hautes Études Sci. **116** (2012), 69–114. MR 3090255

[KY15] Manuel Kauers and Rika Yatchak, *Walks in the quarter plane with multiple steps*, Proceedings of FPSAC 2015, Discrete Math. Theor. Comput. Sci. Proc., Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2015, pp. 25–36.

[MM14] Stephen Melczer and Marni Mishna, *Singularity analysis via the iterated kernel method*, Combin. Probab. Comput. **23** (2014), no. 5, 861–888. MR 3249228

[MR09] Marni Mishna and Andrew Rechnitzer, *Two non-holonomic lattice walks in the quarter plane*, Theoret. Comput. Sci. **410** (2009), no. 38-40, 3616–3630. MR 2553316

[Ost46] Alexandre Ostrowski, *Sur les relations algébriques entre les intégrales indéfinies*, Acta Math. **78** (1946), 315–318. MR 0016764

[OW15] Alexey Ovchinnikov and Michael Wibmer, *σ-Galois theory of linear difference equations*, Int. Math. Res. Not. IMRN (2015), no. 12, 3962–4018.

[Roq70] Peter Roquette, *Analytic theory of elliptic functions over local fields*, no. 1, Vandenhoeck u. Ruprecht, 1970.

[Sil94] Joseph H. Silverman, *Advanced topics in the arithmetic of elliptic curves*, Graduate Texts in Mathematics, vol. 151, Springer-Verlag, New York, 1994.

[Sil09] Joseph H Silverman, *The arithmetic of elliptic curves*, vol. 106, Springer Science & Business Media, 2009.

[vdPS97] Marius van der Put and Michael F. Singer, *Galois theory of difference equations*, Lecture Notes in Mathematics, vol. 1666, Springer-Verlag, Berlin, 1997.

[Wib10] M. Wibmer, *Geometric difference galois theory*, PhD thesis, Heidelberg University (2010).

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