Galoisian approach to differential transcendence

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Contents

Part I
Introduction

Hölder's Theorem asserts that the Gamma function $\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}dt$, which satisfies the functional equation $\Gamma(x+1) = x\Gamma(x)$ does not satisfy a polynomial differential equation over the field $\mathbb{C}(x)$ of rational functions over $\mathbb{C}$. [DV12] gives many references on the distinct proofs of this statement. Hölder's Theorem is what appears in the literature under the various names of "hypertranscendence", or " differential transcendence" as well as "transcendentally transcendental" results. This characterization defines a new class in the hierarchy of special functions. Indeed, one can classify functions over the field $\mathbb{C}$ of complex numbers as

- polynomials, that is elements of $\mathbb{C}[x]$,
- rational functions, that is elements of $\mathbb{C}(x)$,
- algebraic functions, that is functions which satisfy a polynomial equation with coefficients in $\mathbb{C}(x)$,
- and transcendental functions, that is functions which do not satisfy a polynomial equation with coefficients in $\mathbb{C}(x)$.

Among the last class, the notion of hypertranscendence distinguishes the functions which do not satisfy an algebraic differential equation above $\mathbb{C}(x)$ from the so called "differential algebraic functions". For instance, the exponential function is transcendental over $\mathbb{C}(x)$ but obviously satisfies a linear differential equation over $\mathbb{C}$. This classification might seem artificial but each class carries special properties, concerning for instance the analytic regularity of the functions, their growth rate or the existence of algebraic explicit formulas defining the functions as well as their algorithmic implementation.

Therefore, the questions of functional hypertranscendence appear in various mathematical domains. For instance, in [Ber95] and [BB95], the authors study the hypertranscendence of the local conjugacy in complex dynamics in order to get some informations on the regularity of the dynamic. In combinatorics, a famous conjecture is concerned with the holonomicity of the generated functions of walks in the plane. Since a holonomic function satisfies by definition a linear differential equation, this characterization might give linear recurrence relations between the power series coefficients of the generating function as well as some informations on their growth rate (see for instance [BMM10] or [MM14]). One could also quote in combinatorics the study of the hypertranscendence of generating functions of automatic p-sequence (cf for instance [Ran92] and [DHR15]). More surprising, the combination of results in functional transcendence such as Ax-Schanuel Theorem on independence of logarithm of functions and the o-minimality theorem of Pila-Wilkie paved the way to remarkable proofs of famous conjectures in diophantine geometry (see for instance [Pil09]).

Recently, a Galois theory of functional equations with differential parameter has been elaborated (see [CS06] and [HS08]). Starting from a linear functional equation, differential or discrete, the authors construct a Galois group, whose size measures the differential dependencies among the solutions of the first equation. This framework encompasses hypertranscendence problems, where the differential dependency is with respect to the variable of the functions but also problems of differential dependency with respect to an extra parameter such as isomonodromy questions (see for instance [AC91] and [IN86] for introduction to these themes). In this parametrized Galois theory, the Galois groups are differential algebraic groups, that is, groups of matrices whose entries are solutions of differential algebraic equations. The geometry behind is the Ritt-Kolchin geometry (see [Rit50] and [Kol73]). The varieties considered here are defined as zero-sets of differential algebraic equations. Therefore, in order to find enough points in the zero-sets, one is obliged to work with fields of definition that contain enough solutions of differential equations. Such monstrous fields are called differentially closed fields.

Nowadays, the parametrized Galois theory of [HS08] has been generalized in many ways. First, the Ritt-Kolchin geometry, initially developed behind the geometric approach of Weil, admits a modern treatment through a schematic approach. This point of view has allowed one to work with smaller base fields but also to include the situation of a discrete parameter (see [DVHW]). Finally the parametrized Galois theories above encounter a reformulation in the language of Tannakian category (see for instance [Kam12] and [Ovc09]). A sibylline conclusion, one could make out of the work [Kam12], is more or less the following " Starting with a linear functional equation, any operator, that acts on the field of coefficients in a sufficiently nice way, acts on the solutions and endows the group of automorphism of the solutions with an additional structure."

The objective of these notes is to present a simplified version of the theory developed in [HS08]. Precisely, we focus on linear difference equations together with the action of an auxiliary derivation. This is the parametrized version of the Galois theory presented in the notes of M.F. Singer. Our exposition relies heavily on Ritt-Kolchin geometry. Though it is not the most modern treatment of the geometry of differential algebraic equations, we chose this approach for the following reasons. The first one is that the geometry of set of points is more intuitive than the scheme geometry. The second reason comes mainly out of the legacy of Ritt-Kolchin geometry and of its connections with Model Theory and diophantine geometry, which are very active nowadays. We could cite for instance the jet space approach of Buium of Lang's conjecture (see [Bui92]) or the work of Hrushovski on the Manin-Mumford conjecture (see [Hru01]).

Finally, we would like to emphasize that these notes are not a state of the art of the parametrized Galois theories and do not intend to give all the proofs and details. Our objectives are mainly to give some intuitions in this domain, to present each concept from a very basic though perhaps naive point of view but also to give many references in order to help the reader to go further. At the end, the reader should be able to develop an intuition in differential and difference algebra and also to prove by himself some results of hypertranscendence.

The contents of these notes is as follows. In part II, we introduce some basic notions of differential algebra. Part III is concerned with Kolchin geometry. In part IV, we present the parametrized Picard-Vessiot theory as developed in [HS08] and some of its applications to functional hypertranscendence and isomonodromy.

Part II
Differential algebraic equations from an algebraic point of view

In this chapter, we introduce the very basic definitions and constructions of differential algebra. This domain of mathematics focuses on differential equations from an algebraic and geometric point of view. Our preoccupation is to study the intrinsic properties of the algebraic differential equations rather than to solve explicitly these differential equations, which might be very hard. We thus address questions like: Could we find an equivalent differential system in a simpler form? Is an infinite system of differential equations equivalent to a finite one? What are the relations between the solutions of an algebraic differential equation? Are the solutions algebraic over the field of coefficients? Is there a geometry of the set of solutions of an algebraic differential equation? In this part, we introduce the basic algebraic framework whereas Part III is more concerned with the geometric point of view.

The idea of these notes is to give a first flavour of the field and of its techniques rather than a complete exposition. So the reader will find below definitions, statements and proofs presented in some simplified framework: for instance, for one derivation and for differential polynomial ring in one variable. We think that this framework is on one hand very handy and on the other sufficient to understand many of the notions, which hold in greater generality. For the reader interested in a more general presentation, we refer to the following standard books. Kaplansky's introduction to differential algebra is a very pleasant and straightforward introduction. We owe our presentation of the basis theorem to [Kap57, §VII]. The books of Kolchin [Kol73] and Ritt [Rit50] are biblical but they require some technical skills. We recommend also the book of Magid [Mag94] and the many papers contained in [Kol99]. Also, we would like to mention the online notes of P.Cassidy (http://www.sci.ccny.cuny.edu/ ksda/posted.html), which provide plenty of very interesting examples. Finally, we send the reader interested with the connections with model theory to the introductory lecture notes [MMP06].

1 Differential ring

1.1 Differential ring and differential morphism

Definition 1.1. *A differential ring or δ-ring for short, is a pair (R, δ), where R is a commutative ring with unit and $\delta : R \to R$ is a derivation of R, i.e., an additive map satisfying the Leibniz rule*

$$\delta(ab) = \delta(a)b + a\delta(b),$$

for all $(a, b) \in R^2$.

If the ring R is a field, we say that (R, δ) is a differential field or δ-field for short. Moreover, for ease of notation, we shall usually write R instead of (R, δ) and the prefix " δ" instead of " differential"

Definition 1.2. *Let R be a δ-ring. We denote by R^δ the set of δ-constants of R, that is, the set*

$$R^\delta := \{a \in R | \delta(a) = 0\}.$$

Example 1.3.

- *Any commutative ring R endowed with the trivial derivation $\delta = 0$ is a δ-ring. In this case, $R^\delta = R$.*
- *The field of rational functions $\mathbb{C}(x)$ with complex coefficients endowed with the derivation $\delta = \frac{d}{dx}$ is a δ-field. Here, $R^\delta = \mathbb{C}$.*
- *The field $\mathcal{M}er(U)$ of meromorphic functions over an open connected set U of $\mathbb{C}$ together with the usual derivative is a differential field. Its δ-constants are the constant functions i.e., $\mathcal{M}er(U)^\delta = \mathbb{C}$.*
- *Let $\mathbb{C}(x, t)$ be the field of rational functions in two variables x and t. If δ denotes the partial derivative $\frac{\partial}{\partial x}$ with respect to x, the field $\mathbb{C}(x, t)$ is a δ-field, whose field of δ-constants is $\mathbb{C}(t)$.*

Example 1.4. *Let Λ be a lattice in $\mathbb{C}$ and let $\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$ be the Weierstrass $\wp$ function attached to Λ. It is an elliptic function, that is, a meromorphic function, periodic with respect to Λ. The Weierstrass $\wp$ function satisfies the following differential equation*

$$(\frac{d\wp}{dz}(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3,$$

where g_2, g_3 are complex numbers attached to the elliptic curve $E := \mathbb{C}/\Lambda$ satisfying $g_2^3 - 7g_3^2 \neq 0$. One can show that the field C_E of elliptic functions coincides with $\mathbb{C}(\wp(z), \frac{d\wp}{dz}(z))$. Endowed with the derivation $\delta = \frac{d}{dz}$, the field C_E is a δ-field.

Definition 1.5. *A δ-morphism between two δ-rings R and S is a morphism of rings $\phi : R \to S$ that commutes with the derivation i.e., $\phi\delta = \delta\phi$.*

To use the same letter δ for distinct derivations should not lead to confusion if the derivations do not live on the same ring.

If R and S are δ-rings such that R is a sub-ring of S, we say that R is a δ-subring of S if the inclusion map $R \to S$ is a δ-morphism. If, moreover R and S are δ-fields, we say that S is a δ-field extension of R. The notion of R-δ-algebra is defined analogously. A morphism of R-δ-algebras or R-δ-morphism for short, is a morphism of R-algebras which is a δ-morphism.

Exercise 1.6. *Let R be a δ-ring.*

1. *Let s be an invertible element in R. Show that, for any $a \in R$, we have $\delta\left(\frac{a}{s}\right) = \frac{\delta(a)s - a\delta(s)}{s^2}$.*
2. *Show that the previous identity is compatible with the identity $\frac{a}{s} = \frac{ca}{cs}$ for all invertible element $c \in R$.*

Exercise 1.7. *Let $L|K$ be a δ-field extension. Let $a \in L^{\delta}$. We assume that a is algebraic over K. Show that a is algebraic over K^{δ} (hint: show that the minimal monic polynomial of a over K must have constant coefficients).*

Exercise 1.8 (Wronskian lemma)**.** *Let L be a δ-field and $\eta_1, \dots, \eta_r$ be r elements of L. Show that the following facts are equivalent*

1. *$\eta_1, \dots, \eta_r$ are linearly dependent over L^{δ}.*
2. *The wronskian determinant $w(\eta_1, \dots, \eta_s) = \left| \begin{pmatrix} \eta_1 & \dots & \eta_r \\ \delta(\eta_1) & \dots & \delta(\eta_r) \\ \vdots & & \vdots \\ \delta^{r-1}(\eta_1) & \dots & \delta^{r-1}(\eta_r) \end{pmatrix} \right|$ is zero. (hint: set $X_j := \begin{pmatrix} \eta_j \\ \vdots \\ \delta^{r-1}(\eta_j) \end{pmatrix}$ and let $\sum_j \lambda_j X_j = 0$ be a minimal non trivial linear combination over L among the vectors X_j. Up to permutation, one can choose $\lambda_1 = 1$. Show that the coefficients λ_j belong to L^{δ}.)*

Exercise 1.9. *Let R be a δ-ring with $\mathbb{Q} \subset R$. Let $R[[X]]$ be the ring of formal power series with coefficients in R. We set $\delta(aX^i) = iaX^{i-1}$ and $\delta(a) = 0$ for $i \in \mathbb{N}^*$ and $a \in R$. We extend δ by additivity to a derivation on $R[[X]]$. We define the universal Taylor morphism by*

$$T : R \to R[[X]], a \mapsto \sum_{n \in \mathbb{N}} \frac{\delta^{(n)}(a)}{n!} X^n.$$

1. *The universal property of the Taylor morphism*
   - (a) *Show that T is a δ-morphism.*
   - (b) *Let S be a ring endowed with the trivial derivation $\delta = 0$. We endow $S[[X]]$ with a structure of δ-ring as above. Let $\phi : R \to S$ be a morphism of ring. We define $\Phi : R[[X]] \to S[[X]], \sum_{n \in \mathbb{N}} a_n X^n \mapsto \sum_{n \in \mathbb{N}} \phi(a_n) X^n$. Show that $\Phi \circ T : R \to S[[X]]$ is a δ-morphism.*
   - (c) *Conversely, show that any δ-morphism $\Psi : R \to S[[X]]$ arises in this way, i.e., there exists a ring morphism $\phi : R \to S$ such that $\Psi = \Phi \circ T$.*
   - (d) *What is the universal property of the Taylor morphism?*
2. *An example: Let $\mathbb{C}(x, y)$ be a field of rational functions in two variables. Let δ be the $\mathbb{C}$-linear derivation defined by $\delta(x) = 1$ and $\delta(y) = y$.*
   - (a) *Compute the image $Y(y, X)$ of y via the universal Taylor morphism of $(\mathbb{C}(x, y), \delta)$.*
   - (b) *Show that $Y(y, X)$ is the Taylor expansion of the solution of $\frac{dY}{dX} = Y$ which takes y as initial condition, i.e., $Y(y, 0) = y$.*

The universal Taylor morphism together with the above example were introduced by H. Umemura in [Ume96] and are one of the first stones of his Galois theory of non-linear differential equations.

1.2 Localization

Recall that a subset S is multiplicatively closed if $1 \in S$ and $ss' \in S$ whenever s and s' belong to S.

Lemma 1.10. *Let R be a δ-ring and let $S \subset R$ be a multiplicatively closed subset. Then, there exists a unique δ-ring structure on the localization $S^{-1}R$ such that the canonical map*

$$R \to S^{-1}R, a \mapsto \frac{a}{1}$$

is a morphism of δ-ring.

Exercise 1.11 (Proof of Lemma)**.** *We have to prove that one can find a derivation δ extending the one of R and compatible with the equivalence classes defining $S^{-1}R$.*

1. *More precisely, let t, s, s' be elements of S and let $a, a' \in R$ such that $t(as' - a's) = 0$. Show that*

$$t^2\left((\delta(a)s - \delta(s)a)s'^2 - (\delta(a')s' - \delta(s')a')s^2\right) = 0.$$

2. *Conclude with Exercise 1.6.*

2 Differential ideals

In this section, we introduce some basic properties of differential ideals. In algebraic geometry, the geometric characterizations of Zariski closed sets reflect the algebraic properties of their defining ideals. The same holds for differential algebraic geometry and all the properties of δ-ideals detailed below will find their geometric interpretation in Part III.

2.1 Definition and first properties

Definition 2.1. *Let R be a δ-ring and let $\mathfrak{I} \subset R$ be an ideal. Then, $\mathfrak{I}$ is called a δ-ideal if $\delta(\mathfrak{I}) \subset \mathfrak{I}$.*

Example 2.2. *Let R be the δ-ring $\mathcal{C}^\infty(\mathbb{R})$ of smooth functions in the variable x from $\mathbb{R}$ to $\mathbb{R}$ endowed with the derivation $\delta = \frac{d}{dx}$. The algebraic ideal generated by $cos(x)$ and $sin(x)$ is a δ-ideal.*

Exercise 2.3. *Let R be the ring $\mathbb{C}[x]$ of polynomials with complex coefficients endowed with the derivation $\delta = \frac{d}{dx}$. Determine all δ-ideals of R.*

Definition 2.4. *Let R be a δ-ring and let $\Sigma \subset R$ be a subset of R. An intersection of δ-ideals containing Σ is a δ-ideal containing Σ. Thus, there exists a smallest δ-ideal of R containing Σ, which we call the δ-ideal generated by Σ and denote by $[\Sigma]$.*

Exercise 2.5. *Let us consider the ring $\mathcal{H}ol(\mathbb{C}^2)$ of holomorphic functions in two variables (z_1, z_2) endowed with the derivation $\delta = \frac{\partial}{\partial z_2}$. Describe in terms of algebraic generators the δ-ideal $[z_1 \cdot z_2]$.*

2.2 Quotients of δ-ring

It is easy to see that the kernel of a δ-morphism is a δ-ideal. The converse is also true.

Lemma 2.6. *Let R be a δ-ring and let $\mathfrak{I}$ be a δ-ideal of R. There exists a unique structure of δ-ring on the quotient $R/\mathfrak{I}$ such that the canonical map $R \rightarrow R/\mathfrak{I}, a \mapsto \overline{a}$ is a δ-morphism.*

Proof. If $a - b \in \mathfrak{I}$ then $\delta(a-b) = \delta(a) - \delta(b) \in \mathfrak{I}$. This means that the map $\delta : R/\mathfrak{I} \rightarrow \mathfrak{I}, \overline{a} \mapsto \delta(\overline{a})$ is well defined. It is obvious that δ is a derivation of $R/\mathfrak{I}$. □

2.3 Radical differential ideals and saturation

Definition 2.7. *Let R be a δ-ring. Let Σ be a subset of R. We denote by $\{\Sigma\}$ the radical of the δ-ideal generated by Σ in R i.e., $\{a \in R | \exists n \in \mathbb{N}\ a^n \in [\Sigma]\}$.*

Proposition 2.8. *Let R be a δ-ring and let $\mathfrak{I}$ be a δ-ideal of R. If $\mathbb{Q} \subset R$ then the radical $\{\mathfrak{I}\}$ of $\mathfrak{I}$ is a δ-ideal of R*

Proof. Let $a \in R$ such that $a^n \in \mathfrak{I}$ for some $n \in \mathbb{N}$. Then, $n\delta(a)a^{n-1} \in \mathfrak{I}$ since $\mathfrak{I}$ is a δ-ideal. Differentiating again, we find $n\delta^2(a)a^{n-1} + n(n-1)(\delta(a))^2 \cdot a^{n-2} \in \mathfrak{I}$. This implies that $n(n-1)\delta(a)^3 a^{n-2} \in \mathfrak{I}$. It is easy to prove by induction that $r\delta(a)^{2k+1}a^{n-k-1}$ for some $r \in \mathbb{N}$ and for all $k \leq n-1$. Since $\mathbb{Q} \subset R$, we find that $\delta(a)^{2n-1} \in \mathfrak{I}$. Thus $\delta(a) \in \{\mathfrak{I}\}$. □

Remark 2.9. *If R does not contain $\mathbb{Q}$, the radical of a δ-ideal might fail to be a δ-ideal. For instance, let $R = \mathbb{Z}[x]$ endowed with $\delta = \frac{d}{dx}$. Let $\mathfrak{I}$ be the ideal generated by 2 and x^2. Since $\delta(x^2) = 2x$, $\mathfrak{I}$ is a δ-ideal. Assume that $\{\mathfrak{I}\}$ is a δ-ideal. Since $x \in \{\mathfrak{I}\}$, there exists a non-zero integer n such that $\delta(x)^n = 0$ in $R/\mathfrak{I}$. This contradicts the fact that $\delta(x) = 1$.*

Convention! In order to avoid unnecessary complications, we shall always assume that $\mathbb{Q}$ is a sub-ring of the rings, we consider in these notes.

Definition 2.10. *Let R be a δ-ring, $S \subset R$ be a multiplicative subset and $\mathfrak{I} \subset R$ be an ideal. We denote by $\mathfrak{I} : S$ the saturation of $\mathfrak{I}$ with respect to S, i.e., the ideal $\{f \in R | \exists s \in S \text{ with } sf \in \mathfrak{I}\}$. When $S = \{s^n, n \in \mathbb{N}\}$ for some $s \in R$, we write $\mathfrak{I} : s^\infty$ instead of $\mathfrak{I} : S$. If $\mathfrak{I} = \mathfrak{I} : S$, we say that $\mathfrak{I}$ is saturated with respect to S.*

Lemma 2.11. *Let R be a δ-ring. Let $\mathfrak{I}$ be a δ-ideal of R and let S be a multiplicative set such that $S \cap \mathfrak{I} = \emptyset$. There exists a prime δ-ideal $\mathfrak{M}$ of R containing $\mathfrak{I}$ and avoiding S.*

Proof. By Zorn's Lemma, we find one maximal element $\mathfrak{M}$ in the set of δ-ideals of R avoiding S and containing $\mathfrak{I}$. Assume that $\mathfrak{M}$ is not prime. Then, let $a, b \in R$ such that $ab \in \mathfrak{M}$ with $a \notin \mathfrak{M}$ and $b \notin \mathfrak{M}$. Then, the δ-ideals $[a, \mathfrak{M}]$ and $[b, \mathfrak{M}]$ properly contain $\mathfrak{M}$ and must therefore have non-empty intersections with S. Thus, there exist $s \in S \cap [a, \mathfrak{M}]$ and $s' \in S \cap [b, \mathfrak{M}]$. To conclude, we note that $ss' \in [\mathfrak{M}, ab] \subset \mathfrak{M}$, which leads to a contradiction since S is a multiplicative set. $\square$

Exercise 2.12. *Let R be a δ-ring and let $u, v \in R$.*

1. *Show that, for all $j \in \mathbb{N}$, the element $u^{j+1}\delta^j(v)$ belongs to the ideal $[uv]$.*
2. *Let $\mathfrak{I}$ be a radical ideal of R. Assume that $uv \in \mathfrak{I}$.*
   (a) *Show that $u\delta(v)$ and $\delta(u)v \in \mathfrak{I}$.*
   (b) *Conclude that $\delta^i(u)\delta^j(v) \in \mathfrak{I}$ for all $i, j \in \mathbb{N}$.*
3. *Let R be a δ-ring. Let S and T be two subsets of R. We denote by $\{S\}\{T\}$ the set $\{st | s \in \{S\}, t \in \{T\}\}$ and by ST the set $\{st | s \in S, t \in T\}$. Using the questions above, show that $\{S\}\{T\} \subset \{ST\}$.*

Exercise 2.13. *Let R be a δ-ring, $S \subset R$ be a multiplicative subset and $\mathfrak{I} \subset R$ be a δ-ideal. Show that $\mathfrak{I} : S$ is a δ-ideal.*

Exercise 2.14.
1. *Show that a maximal δ-ideal, i.e., maximal for the inclusion among the proper δ-ideals, is prime (Use Lemma 2.11 with $S = \{1\}$ and $\mathfrak{I} = \{0\}$).*
2. *Show that the radical of a δ-ideal $\mathfrak{I}$ is the intersection of all the prime δ-ideals that contains $\mathfrak{I}$ (Reduce to the case $\mathfrak{I} = \{0\}$ and, for a non-nilpotent element x in the intersection of all the prime δ-ideals, use Lemma 2.11 with $S = \{x^n, n \in \mathbb{N}\}$ to get a contradiction).*

3 Differential polynomial rings and differential algebras

In this section, we introduce the notion of differential polynomials. We explain how one can extend to this framework the usual reduction process for polynomial rings. We present all the notions for differential polynomial rings in n variables but we only discuss in details the case $n = 1$.

3.1 Definition

Proposition 3.1. *Let R be a δ-ring. The δ-polynomial ring $R\{y_1, \ldots, y_n\}$ in the δ-variables $(y_1, \ldots, y_n)$ is the polynomial ring over R in the variables $y_i^{(j)}, i = 1, \ldots, n, j \in \mathbb{N}$ turned into a δ-ring by setting*

- *$\delta(a) := \delta(a)$ for $a \in R$,*
- *$\delta(y_i^{(j)}) := y_i^{(j+1)}$.*

For ease of notation, we write $\delta^j(y_i)$ instead of $y_i^{(j)}$ for $j > 1$. By convention, $\delta^0(y_i) = y_i$. We say that $y_1, \ldots, y_n$ are δ-indeterminates.

Remark 3.2. *The derivation δ in $R\{y_1, \ldots, y_n\}$ is more precisely defined as the sum of two derivations δ_1 and δ_2 such that*

- *$\delta_1(a) = \delta(a)$ for all $a \in R$ and $\delta_1\left(\delta^j(y_i)\right) = 0$ for $i = 1, \ldots, n$ and $j \in \mathbb{N}$,*
- *$\delta_2(a) = 0$ for all $a \in R$ and $\delta_2(P) = \sum_{j \in \mathbb{N}, 1 \le i \le n} \frac{\partial P}{\partial \delta^j(y_i)} \delta^{j+1}(y_i)$.*

Example 3.3. *Let $R = \mathbb{C}[x]$ endowed with the derivation $\delta = \frac{d}{dx}$. Let $P \in \mathbb{C}[x]\{y\}$ be the differential polynomial $P(y) = \delta(y) - y^2 - x^2$. Then, $\delta(P)(y) = \delta_1(P) + \delta_2(P) = -2x + \delta^2(y) - 2y\delta(y)$. Thus, if $f(x)$ is a solution of the Riccati equation $P(f) = 0$ then f is also a solution of $\delta(P)(f) = 0$.*

Exercise 3.4. *Let $\mathbb{C}$ be the field of complex numbers equipped with the trivial derivation. Let $g_2, g_3 \in \mathbb{C}$ and let $P(y) := \delta(y)^2 - 4y^3 - g_2y - g_3 \in \mathbb{C}\{y\}$ be the Weierstrass δ-polynomial. Compute $\delta(P)$.*

Exercise 3.5. *Let R be a δ-ring and let $B(y_1, \ldots, y_n) \in R\{y_1, \ldots, y_n\}$ be a δ-polynomial. We say that B is differentially homogeneous of degree $r \in \mathbb{Z}$ if for any differential indeterminate t, we have $B(ty_1, \ldots, ty_n) = t^rB(y_1, \ldots, y_n)$. For instance, $B(y) = y + \delta(y)$ is not differentially homogeneous since $B(ty) = ty + \delta(ty) = tB(y) + \delta(t)y$.*

1. *Show that $y_1\delta(y_2) - \delta(y_1)y_2$ is differentially homogeneous. More generally, the Wronskian determinant is differentially homogeneous.*

2. *Show that $B(y_1, \ldots, y_n) \in R\{y_1, \ldots, y_n\}$ is differentially homogeneous of degree r if and only if there exists a differential polynomial $A(y_2, \ldots, y_n) \in R\{y_2, \ldots, y_n\}$ such that $B(y_1, \ldots, y_n) = y_1^rA(\frac{y_2}{y_0}, \ldots, \frac{y_n}{y_1})$.*

3.2 Reduction procedure and rank on δ-polynomials

In this section, we present a reduction procedure for differential polynomials in one variable. Our presentation follows the lines of [Kap57, Chapter VII]. We refer the interested reader to [Rit50, Chapter 1] for the general setting.

Definition 3.6. *Let R be a δ-ring and let $P \in R\{y\} \setminus R$. The order of P is the largest integer n such that $\delta^n(y)$ occurs in P. By convention, the order of an element of R is -1. If r is the order of P, one refers to $\delta^r(y)$ as the leader of P. The degree of P as a polynomial in $\delta^r(y)$ is called the degree of P itself. We define the initial I_P of P as the leading coefficient of P written as a polynomial in $\delta^r(y)$. The separant S_P of P is the differential polynomial $\frac{\partial P}{\partial \delta^r(y)}$.*

Example 3.7. *The Weierstrass differential polynomial $P(y) := \delta(y)^2 - 4y^3 - g_2y - g_3 \in \mathbb{C}\{y\}$ is of order 1 and of degree 2. We have $I_P = 1$ and $S_P = 2\delta(y)$.*

Now, we introduce a notion of rank on differential polynomials.

Definition 3.8. *Let R be a δ-ring and let $P, Q \in R\{y\} \setminus R$. We say that P is of smaller rank than Q and write $P \ll Q$ if either the order of P is smaller than the order of Q or, in the event that P and Q have the same order, if the degree of P is smaller than the degree of Q. If none of this occurs, we say that P and Q have the same rank. We extend this ranking to the whole $R\{y\}$ by the following convention: any element in R has lower rank than every element in $R\{y\} \setminus R$.*

Remark 3.9. *It is clear that in every non-empty subset of $R\{y\}$, there exists an element, whose rank is lower than or equal to the rank of every element of the subset.*

Now, we can state an analogue of the euclidean division for differential polynomials.

Lemma 3.10. *Let R be a δ-ring and let $B \in R\{y\} \setminus R$. Let A be a δ-polynomial in $R\{y\}$. Then, there exist integers i, s and a δ-polynomial $C \ll B$ such that*

$$S_B^s I_B^i A - C \in [B],$$

where $[B]$ is the δ-ideal generated by B.

Proof. Let us denote by r the order of B and by d its degree. First, let us assume that A is of order $r + k$ with $k \geq 1$. By definition of the separant, $\delta(B) = S_B\delta^{r+1}(y) + C_1$, where C_1 has order strictly less than $r + 1$. Iterating this process, we get that $\delta^k(B) = S_B\delta^{r+k}(y) + C_k$ where C_k is of order strictly less than $r + k$. Then, we use the usual euclidian division of A by $\delta^k(B)$ viewed as polynomial in the variable $\delta^{r+k}(y)$ with coefficients in the ring $R\left[y, \delta(y), \ldots, \delta^{r+k-1}(y)\right]$ [1]. Since $\delta^k(B)$ is of degree 1 as a polynomial in $\delta^{r+k}(y)$, there exists

[1] We just remind that the euclidian division over $S[X]$ where S is an integral domain is as follows. Let $A, B \in S[X]$ and let $c \in S$ be the leading coefficient of B. Then there exist an integer s and a polynomial C of degree strictly smaller than the degree of B such that $c^sA = QB + C$.

$s_k \in \mathbb{N}$ such that $S_B^{s_k} A - \delta^k(B)$ has order less than $r+k-1$. Repeating this process, we reach the case where we can assume that A is of order r. Now, we just have to treat the case where the degree of A is greater than or equal to d. To conclude, we use the usual euclidian division of A by B viewed as polynomials in the variable $\delta^r(y)$ with coefficients in the ring $R\left[y, \delta(y), \ldots, \delta^{r-1}(y)\right]$, noting that the leading coefficient of B, view as a polynomial in $\delta^r(y)$, is by definition its initial I_B. $\square$

Lemma 3.11. *Let K be a δ-field and let $\mathfrak{I} \subsetneq K\{y\}$ be a proper non-zero δ-ideal. There exists $P \in \mathfrak{I}$ of minimal rank such that*

$$[P] \subset \mathfrak{I} \subset [P] : (S_P I_P)^\infty.$$

In particular, if $\mathfrak{I}$ is a prime δ-ideal, then $\mathfrak{I} = [P] : (S_P I_P)^\infty$.

Proof. Let $P \in \mathfrak{I}$ be a non-zero δ-polynomial of minimal rank in $\mathfrak{I}$. Since $\mathfrak{I}$ is a δ-ideal, we have $[P] \subset \mathfrak{I}$. Let $Q \in \mathfrak{I}$. By Lemma 3.13, there exist integers i, s and a δ-polynomial $R \ll P$ such that $S_P^s I_P^i Q - R \in [P]$. Then, R is an element of $\mathfrak{I}$ and of smaller rank than P. Thus R must be in K and thus equal to 0 since $\mathfrak{I} \subsetneq K\{y\}$. This shows that $\mathfrak{I} \subset [P] : (S_P I_P)^\infty$. If $\mathfrak{I}$ is prime then $[P] : (S_P I_P)^\infty \subset \mathfrak{I}$ since neither S_P nor I_P belong to $\mathfrak{I}$. $\square$

A δ-polynomial $P \in K\{y\}$ with coefficients in a δ-field K, is called irreducible if it can not be written in $K\{y\}$ as product of two non-invertible elements. Unlike the case of polynomials, the δ-ideal generated by an irreducible δ-polynomial P does not need to be a prime δ-ideal. However Ritt proved that the δ-ideal $[P] : S_P^\infty$ is prime. These considerations lead to Ritt's low power Theorem and theory of general solutions of algebraic differential equations (see for instance [Kol99, p.584]).

In order to obtain a nice elimination theory in $R\{y_1, \ldots, y_n\}$, one has to order the differential indeterminates $y_1, \ldots, y_n$ and to eliminate not only with respect to one δ-polynomial but with respect to a family of δ-polynomials. To this purpose, we introduce the notions of class, reduced set and chains.

The class $cl(P)$ of a δ-polynomial $P \in R\{y_1, \ldots, y_n\} \setminus R$ is the largest i such that $\delta^j(y_i)$ appears in P for some $j \in \mathbb{N}$. The class of an element in R is zero. Now given two δ-polynomials P and Q, one says that P has larger rank than Q either if $cl(P) > cl(Q)$ or, in the event of $cl(P) = cl(Q) = k$, if $rank_{y_k}(P) > rank_{y_k}(Q)$ where $rank_{y_k}$ is the rank of the δ-polynomials in the single δ-indeterminate y_k.

We say that a δ-polynomial P is *reduced* with respect to a set Σ of δ-polynomials if for all $A \in \Sigma$, we have $rank_{y_p}(P) < rank_{y_p}(A)$ where $p = cl(A)$.

A finite set $B_1, \ldots, B_s$ of δ-polynomials in $R\{y_1, \ldots, y_n\}$ is called a chain if

- either $s = 1$ and $B_1 \neq 0$
- or $s > 1$ and $cl(B_1) > 0$ and for all $j > i$, $cl(B_j) > cl(B_i)$ and B_j is reduced with respect to B_i.

Example 3.12. *In $\mathbb{C}\{y_1, y_2\}$ where $\mathbb{C}$ is a trivial δ-field, the δ-polynomial $B_1 := \delta(y_1) + y_1$ and $B_2 = \delta^2(y_2) y_1^2 + y_2$ form a chain.*

The chain $A_1, \ldots, A_r$ is said of higher rank than the chain $B_1, \ldots, B_s$ if either

- there is a j smaller than r and s such that A_i and B_i are of the same rank for $i < j$ and that A_j has larger rank than B_j,
- $s > r$ and A_i and B_i are of the same rank for $i \leq r$.

In n-variables, the initial and the separant of a δ-polynomial P are defined with respect to the δ-indeterminate y_p with p the class of P. Now, the n-variables version of Lemma 3.13 is the following.

Lemma 3.13. *Let R be a δ-ring and let $\Sigma := \{B_1, \ldots, B_p\}$ be a chain in $R\{y_1, \ldots, y_n\}$. Let I_Σ (resp. S_Σ) be the product of the initials (resp. of the separants) of $B_1, \ldots, B_p$. Let $A \in R\{y_1, \ldots, y_n\}$. Then, there exist integers i, s and a δ-polynomial C reduced with respect to Σ such that*

$$S_\Sigma^s I_\Sigma^i A - C \in [B_1, \ldots, B_p].$$

Proof. For the proof, we refer to [Rit50, Chap. 1,§6]. $\square$

Lemma 3.11 generalizes to the following description of δ-prime ideals of δ-polynomials in n-variables.

Lemma 3.14. *Let K be a δ-field and let $\mathfrak{p} \subsetneq K\{y_1, \dots, y_n\}$ be a proper non-zero prime δ-ideal. There exists a chain $A_1, \dots, A_r \in \mathfrak{p}$ of smallest rank in $\mathfrak{p}$ such that*

$$\mathfrak{p} = [A_1, \dots, A_r] : (S_{A_1} \dots S_{A_r} I_{A_1} \dots I_{A_r})^{\infty}.$$

A chain of lowest rank in $\mathfrak{p}$ is called "characteristic set" of $\mathfrak{p}$.

Proof. See [Kol73, Lemma 3, p136] □

Exercise 3.15 (Clairaut's Equation)**.** *This example is detailed in [Kol99, p. 575]. Consider the δ-polynomial $P(y) = y - x\delta(y) - \frac{1}{4}(\delta(y))^2 \in \mathbb{C}(x)\{y\}$ where $\mathbb{C}(x)$ is endowed with the derivation $\delta = \frac{d}{dx}$.*

1. *Show that P is irreducible.*
2. *Compute $\delta(P)$.*
3. *Show that neither $\delta^2(y)$ nor S_P belong to $[P]$.*
4. *Conclude that $[P]$ is not prime.*

Exercise 3.16. *Let $\mathbb{C}$ be endowed with the trivial derivation. Let $B = \delta(y)^2 + y$ and let $A = \delta^2(y) + y^3\delta(y)$. Compute the reduction of A by B as in Lemma 3.13.*

Exercise 3.17. *Let $\mathbb{C}(x)$ be equipped with the derivation $\delta := \frac{d}{dx}$. Let P_{III} be the δ-polynomial corresponding to the third Painlevé differential equation, that is,*

$$P_{III}(y) := xy\delta^2(y) - x(\delta(y))^2 + y\delta(y) - \delta x - \beta y - \alpha y^3 - \gamma x y^4,$$

where $\alpha, \beta, \delta, \gamma$ are complex parameters.

1. *Find the order and the degree of P_{III}*
2. *Compute the initial and the separant of P_{III}.*

Exercise 3.18. *Let R be a δ-ring and let $R\{y\}$ be the ring of δ-polynomial in one variable.*

1. *Show that the rank is transitive, that is, if $A \ll B$ and $B \ll C$ then $A \ll C$.*
2. *Let $P \in R\{y\} \setminus R$. Show that*
   - *$S_P \ll P$ and $I_P \ll P$.*
   - *$P \ll \delta(P)$.*
3. *Show that the rank is compatible with the derivation of δ-polynomials i.e., $P \ll Q$ implies $\delta(P) \ll \delta(Q)$.*

Exercise 3.19. *Let K be a δ-field and let $P(y) \in K[y]$ be a monic irreducible polynomial of degree $n > 0$. Let S_P be the separant of P. We want to prove that $[P] : S_P^{\infty} \subset K\{y\}$ is a maximal δ-ideal and is thus δ-prime.*

1. *Show that $[P] : S_P^{\infty}$ is a proper δ-ideal (hint: if not then $S_P \in \{P\} \cap K[y]$ and S_P is prime to P).*
2. *Let $\mathfrak{I}$ be a δ-ideal properly containing $[P] : S_P^{\infty}$. Let $Q \in \mathfrak{I} \setminus [P] : S_P^{\infty}$. Show that there exists a non-zero $C \in K[y]$ of degree strictly smaller than P such that $S_P^s Q - C \in [P]$ for some positive integer s.*
3. *Show that $C \in \mathfrak{I}$ and conclude that $1 \in \mathfrak{I}$ (hint: use the fact that P is irreducible and $C \in K[y]$ is of degree strictly smaller than P.)*

3.3 The basis theorem

The differential ideal generated by a set of δ-polynomials $\{P_i(y_1,\dots,y_n)\}_{i\in I}$ corresponds to what Drach and Picard called *the differential consequences* of the system of polynomial differential equations corresponding to $P_i(y_1,\dots,y_n)=0$ for all $i\in I$. Indeed, if $(f_1,\dots,f_n)$ is a collection of functions such that $P_i(f_1,\dots,f_n)=0$ for all $i\in I$ then $(f_1,\dots,f_n)$ is also solution of $\delta(P_i)(y_1,\dots,y_n)=0$ for all $i\in I$ (see Proposition 3.1). Drach and Picard asked the following question: *"Given an infinite set S of differential polynomials, is it possible to find a finite set of δ-polynomial equations such that the differential consequences of this finite system are equivalent to the differential consequences of S ?"* The translation in terms of δ-ideals of this question is as follows " Is a δ-ideal of a δ-polynomial ring always generated as δ-ideal by a finite set of elements?" The algebraic counterpart of this question finds a positive answer in Hilbert's basis theorem ([Lan77, VI, §2]): for a Noetherian ring A, the ideals of the polynomial ring $A[X]$ are finitely generated. For differential polynomial rings, the answer is no longer true and the first counterexample was found by Ritt (see [Rit50, p. 11]), who proved that the differential ideal generated by $y^2,\delta(y)^2,\dots,\delta^k(y)^2,\dots$ in $\mathbb{C}\{y\}$ is not finitely generated as differential ideal. However, Ritt proved also the differential version of Hilbert's basis theorem, which is the object of the present section.

Definition 3.20. *Let R be a δ-ring and let $\mathfrak{I}$ be a radical δ-ideal in R. Let Φ be a finite subset of R. We say that Φ is a basis of $\mathfrak{I}$ if $\{\Phi\}=\mathfrak{I}$. In that case, we also say that $\mathfrak{I}$ is δ-finitely generated. We say that R is Rittian if every radical δ-ideal of R has a basis.*

As expected, the Rittian property translates into a condition about ascending chains of radical δ-ideals. More precisely, we have the following statement.

Lemma 3.21. *Let R be a δ-ring. Then, the following conditions are equivalent:*

1. *R is Rittian.*
2. *R satisfies the ascending chain condition: any sequence $\mathfrak{I}_1\subseteq\mathfrak{I}_2\subseteq\dots\mathfrak{I}_n\dots$ of radical δ-ideals is stationary up to a certain rank.*
3. *Every non-empty set of radical δ-ideals has a maximal element with respect to the inclusion.*

Proof. Exercise. □

Lemma 3.22. *Any quotient, any localization of a Rittian ring is Rittian.*

Proof. Let R be a Rittian δ-ring. Let $\mathfrak{I}$ be a δ-ideal of R and let S be a multiplicatively closed set. The proof relies on the following facts:

- The canonical map $R\to R/\mathfrak{I}$ induces a bijection between the set of radical δ-ideals of R that contains $\mathfrak{I}$ and the set of radical δ-ideals of $R/\mathfrak{I}$.
- The canonical map $R\to S^{-1}R$ induces a bijection between the set of radical δ-ideals of R with empty intersection with S and the set of radical δ-ideals of $S^{-1}R$.

□

The following result is due to Ritt and is crucial for the study of differential algebraic varieties. Indeed, it implies that any differential algebraic variety can be expressed as a finite union of irreducible Kolchin closed sets.

Theorem 3.23 (Ritt basis theorem). *Let R be a Rittian δ-ring with $\mathbb{Q}\subset R$. Let $\mathfrak{I}$ be a radical δ-ideal in $R\{y_1,\dots,y_n\}$. Then $\mathfrak{I}$ has a basis.*

Proof. An easy induction shows that it is sufficient to prove the statement for $n=1$.

Assume that the theorem is false, that is, there exist radical δ-ideals in $R\{y\}$ with no basis. Then, any chain $\{\mathfrak{p}_i\}_{i\in I}$ of radical δ-ideals in $R\{y\}$ with no basis has an upper bound, which is a radical δ-ideal with no basis. Indeed the union $\mathfrak{M}:=\cup_{i\in I}\mathfrak{p}_i$ is an upper bound for the chain and a radical δ-ideal. We claim that $\mathfrak{M}$ has no basis. Suppose to the contrary that Φ is a finite basis of $\mathfrak{M}$. Then Φ belong to some $\mathfrak{p}_j$. This implies that Φ a basis for this $\mathfrak{p}_j$, which is absurd since $\mathfrak{p}_j$ has no basis. Thus, Zorn's lemma secures a radical δ-ideal $\mathfrak{I}$ with no basis, maximal among the radical δ-ideals with no basis.

First, let us show that $\mathfrak{I}$ is prime. Suppose to the contrary that it is not the case and let $a,b\in R\{y\}\setminus\mathfrak{I}$ such that $ab\in\mathfrak{I}$. Then, $\{a,\mathfrak{I}\}$ and $\{b,\mathfrak{I}\}$ must have a basis since they properly contain $\mathfrak{I}$. Let $a_1,\dots,a_p,b_1,\dots,b_n\in\mathfrak{I}$

such that $\{a,\mathfrak{I}\} = \{a, a_1, \dots, a_p\}$ and $\{b,\mathfrak{I}\} = \{b, b_1, \dots, b_n\}$ (see Exercise 3.27 for the justification of this presentation). Now,

$$\{a,\mathfrak{I}\}\{b,\mathfrak{I}\} \subset \{ab, ab_1, \dots a_1 b, \dots, a_p b_n\}.$$

Indeed by Exercise 2.12, we have $\{S\}\{T\} \subset \{ST\}$ for S,T two subsets of $R\{y\}$. Now, if $c \in \mathfrak{I}$ then $c^2 \in \mathfrak{I}\mathfrak{I} \subset \{a,\mathfrak{I}\}\{b,\mathfrak{I}\}$, which is contained in the radical δ-ideal $\{ab, ab_1, a_1 b, \dots, a_p b_n\}$. Thus, $c \in \{ab, ab_1, \dots, a_1 b, \dots, a_p b_n\}$ and $\mathfrak{I} = \{ab, ab_1, \dots, a_1 b, \dots, a_p b_n\}$. We find a contradiction since $\mathfrak{I}$ has no basis.

Now, $\mathfrak{I} \cap R$ is a radical differential ideal of R. Since R is Rittian, $\mathfrak{I} \cap R$ has a basis. The same holds for the radical δ-ideal $\mathfrak{a}$ generated by $\mathfrak{I} \cap R$ in $R\{y\}$. Then, $\mathfrak{a}$ is properly contained in $\mathfrak{I}$ and one can consider an element $P \in \mathfrak{I} \setminus \mathfrak{a}$ of smallest rank. Let $\delta^r(y)$ be the leader of P, d be its degree, S_P be its separant and I_P be its initial (see §3.2). We claim that S_P and I_P do not belong to $\mathfrak{I}$. Since $I_P \ll P$, if I_P were in $\mathfrak{I}$, it should belong to $\mathfrak{a}$. Then, $Q := P - I_P.(\delta^r(y))^d$ would be in $\mathfrak{I} \setminus \mathfrak{a}$ and of smaller rank than P: a contradiction. In the same manner, if $S_P \in \mathfrak{I}$, it should be in $\mathfrak{a}$ since $S_P \ll P$. But then,

$$P - \frac{1}{d}\delta^r(y) S_P = Q - \frac{\partial Q}{\partial \delta^r(y)},$$

would be in $\mathfrak{I} \setminus \mathfrak{a}$ and of smaller rank than P: a contradiction.

Since $\mathfrak{I}$ is a prime ideal, the product $S_P I_P$ does not belong to $\mathfrak{I}$. Thus, the radical δ-ideal $\{S_P I_P, \mathfrak{I}\}$ contains properly $\mathfrak{I}$ and therefore has a basis. Let $F \in \mathfrak{I}$. By Lemma 3.13, we can find integers i, s and $C \ll P$ such that $S_P^s I_P^i F - C \in [P]$. Then, C belongs to $\mathfrak{I}$ and is of smaller rank than P: C must lie in $\mathfrak{a}$. To summarize, we have shown that $S_P I_P \mathfrak{I} \subset \{\mathfrak{a}, P\}$. Let $P_1, \dots, P_s \in \mathfrak{I}$ such that $\{S_P I_P, \mathfrak{I}\} = \{S_P I_P, P_1, \dots, P_s\}$. We find,

$$\mathfrak{I}^2 \subset \mathfrak{I}\{S_P I_P, \mathfrak{I}\} \subset \{S_P I_P \mathfrak{I}, \mathfrak{I}P_1, \dots, \mathfrak{I}P_s\} \subset \{\mathfrak{a}, P, P_1, \dots, P_s\}.$$

We conclude as above that $\mathfrak{I} = \{\mathfrak{a}, P, P_1, \dots, P_s\}$. This proves that $\mathfrak{I}$ is δ-finitely generated, contradicting our assumption. □

Remark 3.24. *To emphasize the importance of the radical hypothesis in the Theorem above, we come back to the counterexample of Ritt. Let $\mathfrak{I}$ be the δ-ideal generated in $\mathbb{C}\{y\}$ by $y^2, \delta(y)^2, \dots, \delta^k(y)^2, \dots$. Ritt showed that there is no finite set $b_1, \dots, b_p$ such that any element of $\mathfrak{I}$ can be expressed as linear combination of the b_i's and their derivatives. However, by Ritt basis theorem, one can find a finite set $b_1, \dots, b_p$ such that, any element in $\mathfrak{I}$ has a certain power, which can be written as linear combination of the b_i's and their derivatives.*

As a direct corollary, we get the following statement.

Corollary 3.25. *Let K be a δ-field. Then, every radical δ-ideal $\mathfrak{I} \subset K\{y_1, \dots, y_n\}$ has a basis.*

Proof. A δ-field is Rittian. □

Corollary 3.26 (δ-analogue of Lasker-Noether Theorem)**.** *Let R be a Rittian ring. Then, every radical δ-ideal $\mathfrak{I}$ is a finite intersection of prime δ-ideals $\mathfrak{p}_i$. If we moreover assume that $\mathfrak{p}_i \subsetneq \mathfrak{p}_j$ for $i \neq j$, this decomposition is unique up to permutation. We call the $\mathfrak{p}_i$ the prime δ-components of $\mathfrak{I}$.*

Proof. Assume this is not the case. Then, by ascending chain condition[2], we can find a radical δ-ideal $\mathfrak{M}$ which is maximal with respect to the property " not a finite intersection of prime δ-ideals". Obviously $\mathfrak{M}$ is not prime and one can find a and b in $R \setminus \mathfrak{M}$ such that $ab \in \mathfrak{M}$. Then, $\{a,\mathfrak{M}\}$ and $\{b,\mathfrak{M}\}$ can be written as finite intersection of prime δ-ideals (since they properly contain $\mathfrak{M}$). By Exercise 2.12, we have

$$\{a,\mathfrak{M}\}\{b,\mathfrak{M}\} \subset \{ab,\mathfrak{M}\} \subset \mathfrak{M}.$$

Then, any element $c \in \{a,\mathfrak{M}\} \cap \{b,\mathfrak{M}\}$ satisfies $c^2 \in \{a,\mathfrak{M}\}\{b,\mathfrak{M}\} \subset \mathfrak{M}$, which gives $\{a,\mathfrak{M}\} \cap \{b,\mathfrak{M}\} \subset \mathfrak{M}$. The reverse inclusion is trivial. We therefore get a contradiction. The irredundancy of the decomposition is a straightforward analogue of [Lan77, VI §5]. □

Exercise 3.27. *Let R be a δ-ring with $\mathbb{Q} \subset R$. Let $\mathfrak{I}$ be a δ-ideal and let $a \in R$. We assume that the radical δ-ideal $\{a,\mathfrak{I}\}$ generated by a and $\mathfrak{I}$ has a basis $\{c_0, c_2, \dots, c_n\}$. Show that one can choose a basis of $\{a,\mathfrak{I}\}$ of the form $\{a, b_1, \dots, b_k\}$ with $b_i \in \mathfrak{I}$ for $i = 1, \dots, k$ (hint: express some power of c_i as an element of $[a,\mathfrak{I}]$.)*

Exercise 3.28 (Around Ritt counterexample)**.** *Let F be a δ-field. Let us consider the δ-ideal $\mathfrak{I} \subset F\{y\}$ generated by $\delta^i(y)\delta^j(y)$ for $(i,j) \in \mathbb{N}^2$.*

[2] Not by Zorn's Lemma!

1. *Show that* $[y]$ *is a radical* δ*-ideal.*

2. *Show that* $[y] \subset \{\mathfrak{I}\}$.

3. *Show that* $\delta^i(y)\delta^j(y) \subset [y]$ *for* $(i,j) \in \mathbb{N}^2$.

4. *Conclude that* y *is a basis of* $\{\mathfrak{I}\}$.

In [Rit50, §15], Ritt showed however that there is no finite set Φ *of* δ*-polynomials such that* $[\Phi] = \mathfrak{I}$.

3.4 Adjunction of elements

Let R be a δ-ring and let S be an R-δ-algebra. Let $B \subset S$ be a subset of S. Obviously, the intersection of R-δ-subalgebras of S is an R-δ-subalgebra. Thus, there exists a smallest R-δ-subalgebra of S which contains B. We denote it by $R\{B\}$ and call it the R-δ-subalgebra generated by B. If $B = \{b_1, \dots, b_n\} \subset S$ is a finite set of elements of S, the R-δ-algebra $R\{B\}$ is easy to describe. It is the R-subalgebra of S generated by $b_1, \dots, b_n$ and all their derivatives. It is also the image of the δ-polynomial ring $R\{y_1, \dots, y_n\}$ under the R-δ-morphism, $y_i \mapsto b_i$. This morphism is called *substitution morphism* anf factorizes through its kernel $\mathfrak{I}$ into an R-δ-isomorphism between $R\{y_1, \dots, y_n\}/\mathfrak{I}$ and $R\{b_1, \dots, b_n\}$. We say that an R-δ-algebra S is δ-finitely generated if there exist $b_1, \dots, b_n \in S$ such that $S = R\{b_1, \dots, b_n\}$. The next lemma summarizes this discussion.

Lemma 3.29. *Let* R *be a* δ*-ring. An* R*-*δ*-algebra* S *is* δ*-finitely generated if and only if there exist a positive integer* n *and a* δ*-ideal* $\mathfrak{I}$ *of* $R\{y_1, \dots, y_n\}$ *such that* $R\{y_1, \dots, y_n\}/\mathfrak{I}$ *is* R*-*δ*-isomorphic to* S.

Corollary 3.30. *Let* K *be a* δ*-field. Let* R *be a* K*-*δ*-algebra* δ*-finitely generated. Then, each radical* δ*-ideal of* R *has a basis.*

Proof. By lemma 3.29, R is a quotient of a δ-polynomial ring $K\{y_1, \dots, y_n\}$. By Corollary 3.25, this latter ring is Rittian. Lemma 3.22 allows to conclude. ☐

Example 3.31. *Let* $K\{Y\} := K\{y_{i,j}, 1 \le i \le n, 1 \le j \le n\}$ *be the* K*-*δ*-algebra of* δ*-polynomials in* n^2 *differential indeterminates. In what follows, we use the matricial notation* $Y = (y_{i,j})_{1 \le i \le n, 1 \le j \le n}$. *Let* S *be the multiplicative set* $\{det(Y)^r | r \in \mathbb{N}\}$. *The localization* $K\{Y, \frac{1}{det(Y)}\}$ *of* $K\{Y\}$ *w.r.t.* S *is a* δ*-finitely generated* K*-*δ*-algebra denoted by* $K\{\mathrm{Gl}_n\}$. *In the following sections, we shall see that* $K\{\mathrm{Gl}_n\}$ *corresponds to the* K*-*δ*-coordinate ring of the group of invertible matrices* Gl_n*, viewed as a differential algebraic variety over* K.

Let $L|K$ be an extension of δ-fields and let $B \subset L$. An intersection of δ-subfields of L containing B and K is a δ-subfield containing B and K. Therefore, there exists a smallest δ-subfield of L containing B and K. We denote it by $K\langle B\rangle \subset L$ and call it the δ-subfield generated by B. We also say that a δ-field extension $L|K$ is δ-finitely generated if $L = K\langle B\rangle$ for some finite set B.

The Lefschetz-Seidenberg Theorem below shows that a δ-finitely generated field extension is a quite common object. In particular, this principle allows us to replace abstract δ-fields extension with fields of meromorphic functions.

Theorem 3.32 (see [Sei69])**.** *Let* F *be a* δ*-finitely generated* δ*-field extension of* $\mathbb{Q}$. *Then, there exist an open set* Ω *in* $\mathbb{C}$ *and a* δ*-isomorphism between* F *and the* δ*-field* $(\mathcal{M}er(\Omega), \frac{d}{dz})$ *of meromorphic functions over* Ω.

Exercise 3.33. *Let* L *be the field of meromorphic functions in the variable* $(x,t) \in (\mathbb{C} \setminus \mathbb{R}^-) \times \mathbb{C}$. *Let* x^t *denote the meromorphic function* $exp(tln(x)) \in L$. *Let* L *be endowed with the derivation* $\delta = \frac{\partial}{\partial t}$. *Describe the* δ*-field generated by* x^t *over* $\mathbb{C}(x,t)$.

Exercise 3.34. *Let* K *be a* δ*-field and let* $L|K$ *be a* δ*-field extension. Let* $a \in L$ *be algebraic over* K. *Show that* $K(a)$*, the field generated by* a *over* K*, is a* δ*-field and thus coincide with* $K\langle a\rangle$. *(Hint: consider the minimal monic polynomial of* a *over* K *to express* $\delta(a)$ *as an element of* $K(a)$.*)*

4 Differential fields

In this section, we study δ-field extensions. We first introduce the notions of δ-algebraic field extensions and δ-transcendence basis, which are very similar to their classical counterparts. The notion of δ-closure of a δ-field is however much more tricky than the notion of algebraic closure.

4.1 δ-algebraic extensions of δ-field

Definition 4.1. *Let $L|K$ be a δ-field extension and let $a_1, \ldots, a_n$ be some elements of L. If there exists a non-zero δ-polynomial $P \in K\{y_1, \ldots, y_n\}$ such that $P(a_1, \ldots, a_n) = 0$, we say that $a_1, \ldots, a_n$ are δ-algebraically dependent over K. Otherwise, we say that they are δ-transcendental over K.*

It is equivalent to saying that $a_1, \ldots, a_n$ are δ-algebraically dependent over K and that the family $\{\delta^j(a_i), 1 \leq i \leq n, j \in \mathbb{N}\}$ is algebraically dependent over K. If $n = 1$, we say, for short, that a_1 is *δ-algebraic* over K.

Definition 4.2. *Let $L|K$ be a δ-field extension. If every element of L is δ-algebraic over K, we say that $L|K$ is a δ-algebraic δ-field extension or for short that L is δ-algebraic over K.*

Example 4.3. *Let $\mathbb{C}(x)$ be equipped with the derivation $\delta = \frac{d}{dx}$. Then,*

- *x is δ-algebraic over $(\mathbb{C}, \delta)$,*
- *$exp(x)$ is δ-algebraic over $\mathbb{C}(x)$ and transcendental over $\mathbb{C}(x)$,*
- *$cos(x)$ is also δ-algebraic over $\mathbb{C}(x)$.*
- *In Exercise 13.6, we will prove that the Gamma function $\Gamma(x) := \frac{e^{-\gamma x}}{x} \prod_{n=1}^{+\infty}(1 + \frac{x}{n})^{-1} e^{x/n}$, where γ is the Euler constant, is δ-transcendental over $\mathbb{C}(x)$.*

Definition 4.4. *Let $L|K$ be a δ-field extension. We define the order of $L|K$ as the transcendence degree of the field extension $L|K$.*

Lemma 4.5. *Let $L|K$ be a δ-field extension and let $a \in L$. The following statements are equivalent.*

1. *The element a is δ-algebraic over K.*
2. *The δ-field extension $K\langle a \rangle |K$ has finite order.*
3. *The element a is contained in a δ-field extension of K of finite order.*

Proof. Let us prove that 1) implies 2). Let $P \in K\{y\}^{\times}$ such that $P(a) = 0$ and let n be the order of P. Then, $\delta^n(a)$ is algebraic over $K\left(a, \delta(a), \ldots, \delta^{n-1}(a)\right)$. Since $\delta^i(P)(a) = 0$, we deduce that, for all $i \in \mathbb{N}$, the element $\delta^{n+i}(a)$ is algebraic over $K\left(a, \delta(a), \ldots, \delta^{n+i-1}(a)\right)$. This shows that $K\langle a \rangle$ is algebraic over $K\left(a, \delta(a), \ldots, \delta^{n-1}(a)\right)$. Thus, the order of $K\langle a \rangle$ over K is less than or equal to n. 2) implies trivially 3). To conclude, if $a \in M$ a δ-field of finite order, say p, over K. Then, the family $a, \delta(a), \ldots, \delta^p(a)$ of elements of M is algebraically dependent over K by definition of the transcendence degree. $\square$

Lemma 4.6. *Let $K \subset L \subset M$ be a tower of δ-field extensions. Then, M is δ-algebraic over K if and only if M is δ-algebraic extension of L and L is δ-algebraic extension of K.*

Proof. One implication is obvious: if M is δ-algebraic over K, then M is δ-algebraic over L and L is δ-algebraic over K as subfield of M. Conversely, let $a \in M$ be δ-algebraic over L. Let $P \in L\{y\}^{\times}$ be an annihilating polynomial of a over L. Let S be the finite set of coefficients of P. Since S is δ-algebraically dependent over K, the δ-field $K\langle S \rangle$ is of finite order over K (it is the composite of fields of finite transcendence degree over K by Lemma 4.5). Then, $K\langle S, a \rangle$ is of finite order over $K\langle S \rangle$ and thus over K by the usual property of the transcendence degree. We conclude once again by Lemma 4.5. $\square$

Exercise 4.7. *Let $q \in \mathbb{C}$ be a non-zero complex number such that $|q| > 1$. Let $L := \mathcal{M}er(\mathbb{C}^{\times})$ be the field of meromorphic functions over $\mathbb{C}^{\times}$ endowed with the derivation $\delta = z\frac{d}{dz}$. Let $C_E := \{f \in L | f(qz) = f(z)\}$ be the field of elliptic function. Let $\theta_q(z)$ be the Jacobi Theta function defined by $\theta_q(z) = \sum_{n \in \mathbb{Z}} q^{-\frac{n(n-1)}{2}} z^n$.*

1. *Show that $\delta(f(qz)) = (\delta(f))(qz)$ for any $f \in L$.*
2. *Show that C_E is a δ-subfield of L.*
3. *Show that θ_q is a holomorphic function over $\mathbb{C}^{\times}$ such that $\theta_q(qz) = qz\theta(z)$.*
4. *Using the questions above, show that $\frac{\delta\theta_q}{\theta_q}(qz) = 1 + \frac{\delta\theta_q}{\theta_q}(z)$.*
5. *Conclude that $\delta(\frac{\delta\theta_q}{\theta_q})(z)$ is an elliptic function and that θ_q is δ-algebraic over C_E.*

Example 4.8. *Let $q \in \mathbb{C}$ be a non-zero complex number such that $|q| > 1$. Let $L := \mathcal{M}er(\mathbb{C}^\times)$ be the field of meromorphic functions over $\mathbb{C}^\times$ in the variable z, endowed with the derivation $\delta = z\frac{d}{dz}$. Let C_E be the field of elliptic functions with respect to the elliptic curve $E := \mathbb{C}^\times/q^{\mathbb{Z}}$. The elliptic curve E can be identified to $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ via the change of variable $z = exp(2\pi ix)$. In Example 1.4, it is shown that the field $C_E(z)$ is δ-algebraic over $\mathbb{C}(z)$. In Exercise 4.7, we proved that the Jacobi Theta function $\theta_q(z) = \sum_{n\in\mathbb{Z}} q^{-\frac{n(n-1)}{2}} z^n$ is δ-algebraic over C_E. By Lemma 4.6, we can conclude that θ_q is δ-algebraic over $\mathbb{C}(z)$.*

Denoting by $\tilde{\theta}_q(q, z)$ the function $\theta_q(q^{1/2}z)$, one recovers the Theta function of the Heat equation by means of the change of variable $q = exp(-2\pi i\tau)$ and $z = exp(2\pi ix)$. Thus, we have proved that θ_q satisfies both a partial differential equation in $\frac{\partial}{\partial q}$ and $\frac{\partial}{\partial z}$, avatar of the heat equation, but also an ordinary non-linear algebraic differential equation in $\frac{\partial}{\partial z}$ with coefficients in $\mathbb{C}(z)$.

Exercise 4.9. *Let $L|K$ be a δ-field extension and let $a \in L^\times$ and $b \in L$ be two δ-algebraic elements over K.*

1. *Show that $a + b$, ab, a^{-1}, $\delta(a)$ are δ-algebraic over K.*
2. *Conclude that the elements of L, which are δ-algebraic over K, form a δ-subfield-extension of $L|K$.*

(hint: use Lemma 4.5 and the following fact: a composite of two field extensions of K of finite transcendence degree is of finite transcendence degree over K.)

4.2 δ-transcendence basis

In this section, we introduce the notion of δ-transcendence basis. This notion is a straightforward analogue of the classical notion. Thus, we only expose here the main definitions and properties without proofs.

Definition 4.10. *Let $L|K$ be a δ-field extension. A subset A is called a δ-transcendence basis of $L|K$ if the elements of A are δ-algebraically independent over K and L is δ-algebraic over $K\langle A\rangle$.*

Proposition 4.11 (Theorem 4 in [Kol73])**.** *Let $L|K$ be a δ-field extension. There exists a δ-transcendence basis of L over K and all distinct δ-transcendence basis have the same cardinality. We denote by $\delta\text{-}\operatorname{trdeg}(L|K)$ this cardinality and call it the δ-transcendence degree of L over K. Moreover if $K \subset L \subset M$ is a tower of δ-fields extension, we have*

$$\delta\text{-}\operatorname{trdeg}(M|K) = \delta\text{-}\operatorname{trdeg}(M|L) + \delta\text{-}\operatorname{trdeg}(L|K).$$

Remark 4.12. *A Rittian ring satisfies the ascending chain condition for radical ideals but not necessarily the descending chain condition: some differential rings might have an infinite descending chain of radical differential ideals (see [Ros59, Proposition 3]). The Krull dimension of a finitely generated algebra R over a field k is defined as the supremum of the length of chains of prime ideals. If R is an integral domain, its Krull dimension equals the transcendence degree of its fraction field over k. Since the descending chain condition on prime differential ideals might fail, one has to carefully adapt the notion of Krull dimension to the differential framework. The interested reader could look at [Joh69] for the notion of large gap chains of ideals and the comparison between δ-Krull dimension and δ-transcendence degree.*

4.3 δ-closed field and δ-closure

Let k be a field. The Zariski closed sets of k^n are precisely the set of zeroes in k^n of a collection of polynomials in $k[y_1, \ldots, y_n]$. Hilbert's Nullstellensatz asserts that if k is algebraically closed, there is a one to one correspondence between Zariski closed sets of k^n and radical ideals in $k[y_1, \ldots, y_n]$. In other terms, if k is algebraically closed, we can get enough information from the points in k^n to recover the equations of the variety. If we replace algebraic equations by differential algebraic equations, an algebraically closed field might be too small. For instance, the equation $\frac{dy}{dx} = y$ has only 0 as point in $\mathbb{C}$ but $\mathbb{C}e^x$ in the field of meromorphic functions. This leads to questions like: what is the definition of differentially closed field or differential closure? What can be the properties of such fields?

This new step in the theory of differential algebra is very subtle and naive generalizations or intuitions fail to be true. For instance, an algebraically closed field is a field with no proper finite algebraic extensions. Straightforward differential analogues are not correct. Indeed, consider a δ-field K and x an indeterminate over K. Then $K(x)$ can be endowed with a structure of δ-field extension of K by setting $\delta(x) = 1$. Then, $L|K$ is a non-trivial δ-algebraic extension of K.

Many of the results on differential algebraic closures and differential algebraic closed fields were proved by model theorists. We won't detail here their proofs since this would require an introduction to Model theory (see for instance [MMP06].)

Definition 4.13 (Robinson's definition)**.** *Let K be a δ-field. We say that K is differentially closed or δ-closed for short if for all $n, r \in \mathbb{N}$ and $P_1, \dots, P_r, Q \in K\{y_1, \dots, y_n\}$ the following holds:*

if the system $P_1(y_1, \dots, y_n) = \cdots = P_r(y_1, \dots, y_n) = 0$ and $Q(y_1, \dots, y_n) \neq 0$ has a solution $(a_1, \dots, a_n) \in L^n$ for some δ-field extension L of K, it has already a solution in K^n.

Remark 4.14. *Let $n, r \in \mathbb{N}$ and $P_1, \dots, P_r, Q \in K\{y_1, \dots, y_n\}$ such that the system $P_1(y_1, \dots, y_n) = \cdots = P_r(y_1, \dots, y_n) = 0$ and $Q(y_1, \dots, y_n) \neq 0$ has a solution $a = (a_1, \dots, a_n) \in L^n$ in some δ-field extension L of K. Then, $Q \notin Ker(\sigma_a)$ and $[P_1, \dots, P_r] \subset Ker(\sigma_a)$, where $Ker(\sigma_a)$ is the kernel of the substitution morphism $\sigma_a : K\{y_1, \dots, y_n\} \to L, y_i \mapsto a_i$. Note that $Ker(\sigma_a)$ is a proper prime δ-ideal of $K\{y_1, \dots, y_n\}$.*

Conversely let $\mathfrak{p}$ be a prime δ-ideal in $K\{y_1, \dots, y_n\}$ and $Q \notin \mathfrak{p}$. Then by Corollary 3.25, there exist $P_1, \dots, P_r \in K\{y_1, \dots, y_n\}$ such that $\mathfrak{p} = \{P_1, \dots, P_r\}$. The δ-ring $K\{y_1, \dots, y_n\}/\mathfrak{p}$ is an integral domain R. Let L be the fraction field of R. Then, the image a of $y_1, \dots, y_n$ in L^n satisfies $P_1(a) = \cdots = P_r(a) = 0$ and $Q(a) \neq 0$.

To summarize, a δ-field K is δ-closed if for all $n \in \mathbb{N}$ and all prime δ-ideals $\mathfrak{p} \subset K\{y_1, \dots, y_n\}$ and $Q \notin \mathfrak{p}$, there exists $a = (a_1, \dots, a_n) \in K^n$ such that $\mathfrak{p} \subset Ker(\sigma_a)$ and $Q(a_1, \dots, a_n) \neq 0$.

Example 4.15. *Let k be a δ-closed field and let $A \in k^{n\times n}$. Then, there exists a fundamental solution matrix of $\delta(Y) = AY$ with coefficients in k, that is , there exists $U \in \mathrm{Gl}_n(k)$ such that $\delta(U) = AU$. We just need to find a δ-field extension K of k and $V \in \mathrm{Gl}_n(K)$ such that $\delta(V) = AV$. We proceed as follows. Let $k[Y]$ be a polynomial ring in n^2 indeterminates where $Y := (y_{i,j})$ is a matrix of indeterminates. We endow $k[Y]$ with a structure of k-δ-algebra by setting $\delta(Y) := AY$. Then, Lemma 1.11 shows that the localization R of $k[Y]$ with respect to $det(Y)$ inherits a unique structure of δ-ring. Since R is an integral domain, one can consider its fraction field K. The image V of Y in K is obviously a fundamental solution matrix of $\delta(Y) = AY$. In this context, the equation $det(Y) \neq 0$ is, what Kolchin calls, the constraint attached to the differential equation $\delta(Y) = AY$.*

As explained above, a δ-closed field may have proper δ-algebraic extensions. In [Kol74], Kolchin introduced the notion of constrained differential field extension and proved that a δ-closed field has no proper constrained differential field extension (see Exercise 4.21).

The notion of differential closure or δ-closure of a δ-field is also quite tricky. Using classical arguments, one can show the following.

Proposition 4.16. *Every δ-field k can be embedded in a δ-closed field.*

One can now define the notion of δ-closure of a δ-field as follows.

Definition 4.17. *Let K be a δ-field. We say that $\tilde{K}|K$ is a δ-closure of K if $\tilde{K}$ is δ-closed and for any δ-closed field extension $L|K$, there exists a δ-embedding $\iota : \tilde{K} \to L$.*

The existence of a δ-closure of a given δ-field has eluded differential algebraist for a while and the first positive answer turned out to come from model theory. In her Ph. D. Thesis, L. Blum proved the existence of a δ-closure. We won't give this proof here and refer to [MMP06] and in particular to the second chapter of *Model theory of differential fields* by D. Marker, for more details on δ-closed fields.

Proposition 4.18. *Let K be a δ-field. Then, K has a δ-closure and any two δ-closures of K are K-δ-isomorphic. Moreover, the elements of a δ-closure of K are δ-algebraic over K.*

An intriguing fact about δ-closures is that they are not necessarily minimal. Kolchin [Kol74], Rosenlicht [Ros74] and Shelah [She73] proved independently that there exist non-minimal δ-closures of $\mathbb{Q}$. This means that there exists a δ-closure L of $\mathbb{Q}$ having a non-trivial $\mathbb{Q}$-δ-isomorphism with a proper δ-subfield. In these notes, we will stay away from these subtleties. But since we want to work with points of differential algebraic varieties, we can not avoid completely δ-closures. We need one more result:

Proposition 4.19. *Let K be a δ-field and let $\tilde{K}$ be a δ-closure of K. Then, the δ-constants $\tilde{K}^\delta$ are algebraic over the δ-constants K^δ of K. In particular, if K^δ is algebraically closed, then $\tilde{K}^\delta = K^\delta$.*

A δ-closed field is an enormous field. In fact, even a δ-closure $\tilde{\mathbb{Q}}$ of $\mathbb{Q}$ is a monstrous object. The geometric point of view developed in the following chapters will however consider varieties whose points lie in δ-closed fields. This is the consequence of the "naive " approach, which defines the differential algebraic varieties as set of zeroes of δ-polynomial equations. Nowadays, a schematic version of differential algebraic geometry has been initiated, mainly by J. Kovacic (see [Kov02]). In this schematic setting, one focusses on the differential algebraic equations rather than on their zeroes. Then, one can avoid δ-closed fields. Our presentation might thus look a little bit old-fashioned but can be justified both by the evident similarity with the classical algebraic geometry of Zariski and by historical reasons: Kolchin developed its geometry following the work of Weil and used intensively the notions of field of definition and specialization. This point of view seems to be also the one of model theory.

We conclude this section by stating a last result on the extension of differential specializations, which is the differential analogue of [Spr09, Proposition 1.9.4]. One of the many consequences of the extension of δ-specializations is the differential analogue of Chevalley's theorem about the image of constructible sets by algebraic morphisms.

Proposition 4.20 (Theorem 3, p. 140 in [Kol73])**.** *Let R be an integral δ-ring and let R_0 be a δ-subring of R over which R is δ-finitely generated. Given $u \in R^\times$, there exists $u_0 \in R_0^\times$ such that any δ-morphism ϕ from R_0 into a δ-closed field $\tilde{k}$ with $\phi(u_0) \neq 0$ can be extended to a δ-morphism $\phi : R \to \tilde{k}$ with $\phi(u) \neq 0$.*

Exercise 4.21. *Let $L|K$ be a δ-field extension. Let $\eta = (\eta_1, \dots, \eta_n) \in L^n$ and let $\mathfrak{p}$ be the prime δ-ideal, kernel of the substitution morphism $\sigma_\eta : K\{y_1, \dots, y_n\} \to L, y_i \mapsto \eta_i$. We say that the family η is constrained over L with constraint $C \in K\{y_1, \dots, y_n\}$ if*

1. *$\mathfrak{p} \neq \{0\}$ and $C \notin \mathfrak{p}$*
2. *for all prime δ-ideal $\mathfrak{q}$ strictly containing $\mathfrak{p}$, we have $C \in \mathfrak{q}$.*

We say that a δ-field extension $L|K$ is constrained over K if every element of L is constrained over K.

Let $\eta \in L^n$ for some δ-field extension L of K. Let $F|K$ be a δ-field extension of K. We say that $\beta = (\beta_1, \dots, \beta_n) \in F^n$ is a differential specialization of η if $Ker(\sigma_\eta) \subset Ker(\sigma_\beta)$. One says that β is a generic specialization if $Ker(\sigma_\eta) = Ker(\sigma_\beta)$.

1. *Let $\eta \in L^n$ be δ-algebraic over K and let $C \in K\{y_1, \dots, y_n\}$ such that $C(\eta_1, \dots, \eta_n) \neq 0$. Show that there exists a differential specialization of η constrained over K with constraint C (Hint: use Lemma 2.11).*
2. *Let K be a δ-closed field and let $L|K$ be a δ-field extension. Let $\eta = (\eta_1, \dots, \eta_n) \in L^n$ be constrained over K with constraint C.*
   - *(a) Let $\mathfrak{p} = ker(\sigma_\eta)$ be the kernel of the substitution morphism σ_η. Show that there exist $P_1, \dots, P_r \in K\{y_1, \dots, y_n\}$ such that $\mathfrak{p} = \{P_1, \dots, P_r\}$.*
   - *(b) Show that there exists $c = (c_1, \dots, c_n) \in K^n$ such that $\mathfrak{p} \subset Ker(\sigma_c) = \{(y_1 - c_1), \dots, (y_n - c_n)\}$ and $C(c) \neq 0$.*
   - *(c) Conclude that $\eta = c$ and that a δ-closed field has no proper constrained δ-field extension.*

Exercise 4.22. *Let K be a δ-closed field and let x be transcendental over K. Let $L = K(x)$ endowed with the derivation satisfying $\delta(x) = 1$. Show that L is not constrained over K.*

Exercise 4.23. *Let $\mathbb{C}(x)$ be the field of rational functions over $\mathbb{C}$ endowed with the derivation $\delta = \frac{d}{dx}$. Let $\eta := exp(x)$ be the exponential function. We want to show that η is constrained over $\mathbb{C}(x)$ with constraint $C(y) := y$.*

1. *Show that $\delta(y) - y \subset Ker(\sigma_\eta)$ and that $C(\eta) \neq 0$. We admit that the δ-ideal $\mathfrak{p}$ generated by $\delta(y) - y$ in $\mathbb{C}(x)\{y\}$ is prime (it is in fact always the case for δ-ideals generated by linear δ-polynomials).*
2. *Let $\mathfrak{q}$ be a prime δ-ideal containing properly $\mathfrak{p}$. We want to show that $C \in \mathfrak{q}$*
   - *(a) Let $P \in \mathfrak{q} \setminus \mathfrak{p}$. Show that there exists a non-zero polynomial $R \in \mathbb{C}(x)[y]$ such that $P - R \in [\delta(y) - y]$. Show that $R \in \mathfrak{q}$.(hint: use the reduction procedure)*
   - *(b) Now, let us choose $R \in \mathfrak{q} \cap \mathbb{C}(x)[y]$ be a non-zero polynomial of minimal degree.*
     - *i. If R is a non-zero constant, conclude that $C(y) \in \mathfrak{q}$.*

   ii. *If $R(0) = 0$ and R is non-constant, conclude that $C(y) \in \mathfrak{q}$. (hint: R is of minimal degree in $\mathfrak{q}$ and $\mathfrak{q}$ is prime).*
   iii. *If $R(0) \neq 0$, show that $\delta(\frac{R}{R(0)})$ is congruent modulo $[\delta(y) - y]$ to a non-zero polynomial R' of the same degree than R satisfying $R'(0) = 0$ (hint: there is no $c \in \mathbb{C}(x)^\times$ such that $\delta(c) + nc = 0$ for $n \in \mathbb{N}^\times$). Conclude that $C(y) \in \mathfrak{q}$.*

3. *Conclude that $Ker(\sigma_\eta) = [\delta(y) - y]$ and that $exp(x)$ is constrained over $\mathbb{C}(x)$ with constraint $y \neq 0$.*

Exercise 4.24. *Let $L|K$ be a δ-field extension. Let $a \in L$ algebraic over K. We want to prove that a is constrained over K with constraint $C = 1$. Let $P \in K[y]$ be the minimal monic annihilating polynomial of a over K.*

1. *Show that the defining ideal of a over K, i.e., the kernel of $K\{y\} \to, L, y \mapsto a$ is $[P] : S_P^\infty$ (hint: use the reduction procedure and the minimality of P as annihilating polynomial).*

2. *Use Exercise 3.19 to conclude that a is constrained over K of constraint $C = 1$.*

Part III
Differential algebraic geometry

In this chapter, we introduce some basic notions in ordinary differential algebraic geometry. This is the geometry of varieties defined as zero sets of algebraic differential equations in one derivation. The differential algebraic geometry has been initially developed by Ritt [Rit50] and Kolchin [Kol73]. Among their motivations, one could find the interpretation of the irreducibility of algebraic differential equations in terms of components of the corresponding differential algebraic variety as well as a Galois theory for non-linear equations. Differential algebraic geometry has now grown in many directions including some striking applications to diophantine equations, for instance the proof of the Mordell-Lang conjecture by Hrushovski [Hru96] but also new results in transcendence theory (see for instance [BP10]).

In this section, we first describe differential algebraic varieties from the very basic point of view of their points in a δ-closed field in the wake of classical algebraic geometry. Then, we focus on differential algebraic groups and give some classification results on differential algebraic subgroups of vector groups and tori. As in the previous sections, we choose not to present all the proofs and all the achievements, that a standard book in differential algebraic geometry should contain and we refer either to [Kol73] or to [Cas72] for a more complete exposition. Our intention is to give a first insight into the zoology of differential algebraic geometry but also to understand some of the analogies as well as some of the differences between Zariski and Kolchin geometry.

Throughout this chapter, k denotes a δ-closed field of characteristic zero.

5 Differential algebraic sets

5.1 The Kolchin topology

In this section, we introduce the notion of Kolchin closed sets and show that this topology is very similar to the Zariski topology.

Definition 5.1. *Let $S \subset k\{y_1, \dots, y_n\}$ be a set of δ-polynomials. We denote $\mathbb{V}(S)$ the zero-set of S in k^n. That is $\mathbb{V}(S) = \{(\eta_1, \dots, \eta_n) \in k^n | f(\eta_1, \dots, \eta_n) = 0$ for all $f \in S\}$.*

Proposition 5.2. *We have the following properties*

1. *$\mathbb{V}(0) = k^n$ and $\mathbb{V}(1) = \emptyset$,*

2. *$S \subset T$ implies $\mathbb{V}(T) \subset \mathbb{V}(S)$*

3. *$\mathbb{V}(S) = \mathbb{V}([S]) = \mathbb{V}(\{S\})$*

4. *$\mathbb{V}(\cup S_i) = V(\sum[S_i]) = \cap \mathbb{V}(S_i)$*

5. *$\mathbb{V}([S] \cap [T]) = \mathbb{V}([S][T]) = \mathbb{V}(S) \cup \mathbb{V}(T)$*

Definition 5.3. *A set $X \subset k^n$ is Kolchin closed or k-δ-closed for short if there exists $S \subset k\{y_1, \ldots, y_n\}$ such that $X = \mathbb{V}(S)$.*

Remark 5.4. *In these notes, we only consider differential algebraic varieties contained in an affine space k^n for some $n \in \mathbb{N}^\times$. Therefore, we will use equally the terms differential algebraic variety over k, k-δ-variety or k-δ-closed set to refer to a Kolchin closed set.*

By Proposition 5.2, the Kolchin closed sets form the closed sets of a topology on k^n, called the Kolchin topology. A subset $U \subset k^n$ is called k-δ-open if it is an open set for the Kolchin topology.

Example 5.5. *Let $P \in k\{y\}$ be a δ-polynomial. If P is a polynomial then its zero set $\mathbb{V}(P)$ is a finite set of points. This is not the case for a generic δ-polynomial. For instance, for $P = \delta(y) - y$, we have $\mathbb{V}(P) = \{cy_0 | \text{ for all } c \in k^\delta\}$ where y_0 denotes a non-zero solution of P in k (see Example 4.15). This basic example is just an illustration of the fact that the Kolchin topology is finer than the Zariski topology.*

If $X, Y \subset k^n$ are k-δ-varieties, we say that Y is k-δ-subvariety of X if $Y \subset X$.

Lemma 5.6. *Let $X \subset k^n$. Let $\mathfrak{I}(X)$ be the set $\{P \in k\{y_1, \ldots, y_n\} | P(\eta_1, \ldots, \eta_n) = 0 \text{ for all } (\eta_1, \ldots, \eta_n) \in X\}$. Then, $\mathfrak{I}(X)$ is a radical δ-ideal in $k\{y_1, \ldots, y_n\}$, called the defining k-δ-ideal of X.*

Example 5.7. *The defining k-δ-ideal of a point $(\eta_1, \ldots, \eta_n) \in k^n$ is the kernel of the substitution morphism $k\{y_1, \ldots, y_n\} \to k, y_i \mapsto \eta_i$.*

Proposition 5.8. *Let X_1, X_2 be two subsets of k^n. Then,*

- *If $X_1 \subset X_2$ then $\mathfrak{I}(X_2) \subset \mathfrak{I}(X_1)$.*
- *$\mathfrak{I}(X_1 \cup X_2) = \mathfrak{I}(X_1) \cap \mathfrak{I}(X_2)$.*

The field of definition of the defining k-δ-ideal of a k-δ-variety might be smaller than k. Therefore, we say that a k-δ-variety $X \subset k^n$ is defined over a δ-field $F \subset k$ if there exists a set $S \subset F\{y_1, \ldots, y_n\}$ such that $\mathfrak{I}(X) = \{S\}$.

The differential analogue of Hilbert's Nullstellensatz is as follows.

Theorem 5.9. *Let k be a δ-closed field. The application $\mathfrak{a} \mapsto \mathbb{V}(\mathfrak{a})$ is a one to one correspondence between radical δ-ideals of $k\{y_1, \ldots, y_n\}$ and k-δ-closed sets of k^n. Its inverse is given by $X \mapsto \mathfrak{I}(X)$.*

Proof. Let $\mathfrak{a} \subset k\{y_1, \ldots, y_n\}$ be a radical δ-ideal. It is clear that $\mathfrak{a} \subset \mathfrak{I}(\mathbb{V}(\mathfrak{a}))$. Now, let $f \notin \mathfrak{a}$. By Corollary 3.25, the ring $k\{y_1, \ldots, y_n\}$ is Rittian and by Corollary 3.26, the radical δ-ideal $\mathfrak{a}$ is the intersection of a finite number of prime δ-ideals. Since $f \notin \mathfrak{a}$, there exists a prime δ-ideal $\mathfrak{p}$ containing $\mathfrak{a}$ such that $f \notin \mathfrak{p}$. By Remark 4.14, there exists $a \in k^n$ such that $a \in \mathbb{V}(\mathfrak{p}) \subset \mathbb{V}(\mathfrak{a})$ and $f(a) \neq 0$. This shows that $\mathfrak{a} = \mathfrak{I}(\mathbb{V}(\mathfrak{a}))$. ☐

Exercise 5.10. *Let k be a δ-closed field. Determine the k-δ-subvariety of k^2 defined by $y_1\delta(y_2) + y_2\delta(y_1) = 0$ and $y_1 + \delta(y_1) = 0$*

Exercise 5.11. *Prove proposition 5.2 and 5.8.*

5.2 Kolchin closure

Definition 5.12. *Let $X \subset k^n$. We denote by $\overline{X}$ the closure of X with respect to the Kolchin topology.*

Exercise 5.13. *Show that the Kolchin closure of a subset $X \subset k^n$ is $\mathbb{V}(\mathfrak{I}(X))$.*

Exercise 5.14. *Let k be a δ-closed field and let $C = k^\delta$ be its field of δ-constants. We want to compute the Kolchin closure of the field of rational numbers $\mathbb{Q} \subset k$.*

1. *Show that $\delta(y) \in \mathfrak{I}(\mathbb{Q}) \subset k\{y\}$.*
2. *Let $f \in \mathfrak{I}(\mathbb{Q})$. Show that there exists $R \in k[y] \cap \mathfrak{I}(\mathbb{Q})$ such that $f - R \in [\delta(y)]$ (hint: use the reduction procedure).*
3. *Conclude that $\mathfrak{I}(\mathbb{Q}) = [\delta(y)]$ and determine the Kolchin closure of $\mathbb{Q}$.*

5.3 Irreducible components

The geometric consequence of Ritt's basis theorem is that the Kolching topology is Noetherian.

Proposition 5.15. *Let k^n be endowed with the Kolchin topology. For any descending chain of k-δ-closed sets $X_1 \supseteq X_2 \supseteq X_i \supseteq \dots$ in k^n, there exists $s \in \mathbb{N}$ such that $X_i = X_s$ for all $i \geq s$.*

Proof. For all $i \in \mathbb{N}$, let $\mathfrak{I}(X_i)$ be the defining k-δ-ideal of X_i. Then, $(\mathfrak{I}(X_i))_{i\in\mathbb{N}}$ is an ascending chain of radical δ-ideals in $k\{y_1, \dots, y_n\}$. Lemma 3.21 implies that there exists $s \in \mathbb{N}$ such that $\mathfrak{I}(X_i) = \mathfrak{I}(X_s)$ for all $i \geq s$. This ends the proof. □

In a Noetherian topological space, any non-empty closed set can be expressed as finite union of irreducible closed subsets. Applied to the Kolchin topology, we find the following statement.

Definition 5.16. *Let $X \subset k^n$ a non-empty set endowed with the induced Kolchin topology. We say that X is irreducible if X can not be written as the union of two proper k-δ-closed subsets of X.*

The following proposition states some classical properties on irreducible sets.

Proposition 5.17.
1. *A non-empty open subset of an irreducible space is irreducible and dense.*
2. *X is irreducible if and only if its closure $\overline{X}$ is irreducible.*
3. *$X \subset k^n$ is irreducible if and only if $\mathfrak{I}(X)$ is a prime δ-ideal.*

Proposition 5.18. *Let X be a non-empty k-δ-closed set in k^n. Then X can be expressed as a finite union $X_1 \cup \dots \cup X_n$ of irreducible k-δ-closed sets X_i. If we require that $X_i \subsetneq X_j$ for $i \neq j$ then the X_i are uniquely determined and we call them the irreducible k-δ-components of X.*

Proof. The irreducible k-δ-components of X correspond to the prime δ-components of $\mathfrak{I}(X)$ in Corollary 3.26. □

Exercise 5.19. *Show that the irreducible k-δ-components are the maximal irreducible k-δ-closed subsets of X.*

Exercise 5.20. *Let k be a δ-closed field.*

1. *Show that k^n is irreducible.*
2. *Show that the k-δ-variety $k^\delta = \mathbb{V}(\delta(y)) \subset k$ is irreducible.*

The irreducibility in differential algebraic geometry is a very strong property, which does not rely only on the irreducibility of the defining equations. In Exercise 3.15, we gave the example of an irreducible differential polynomial, which defines a non-irreducible k-δ-variety. However, given an irreducible δ-polynomial P, one can show that the δ-ideal $[P] : S_P^\infty$ is prime and thus is an irreducible k-δ-component of $\mathbb{V}(P)$, called the " general component" (see for instance [MMP06, p44 Cor 1.7]). The interpretation of the irreducible k-δ-components of $\mathbb{V}(P)$ in terms of distinguished sets of solutions of P corresponds to the " Low power Theorem" by Ritt and provides an answer to problems addressed by Laplace and Lagrange (see [Kol99, p 584] for a detailed exposition).

Exercise 5.21 (Clairaut's equation)**.** *Let $f \in k[x]$ be a polynomial. We consider a more general form of Clairaut's equation given by the δ-polynomial $P = x\delta(y) + f(\delta(y)) - y$.*

1. *Show that $\delta(P) = S_P\delta^2(y)$.*
2. *Show that $\{P\} = \{P, \delta^2(y)\} \cap \{P, S_P\}$ (hint : use Exercice 2.12 and the fact that if $x \in \{P, \delta^2(y)\} \cap \{(P, S_P)\}$ then $x^2 \in \{P, \delta^2(y)\}.\{P, S_P\}$).*
3. *Show that $\mathbb{V}(P, \delta^2(y)) = \{cx + f(c)$ with $c \in k^\delta\}$.*

6 k-δ-coordinate rings and k-δ-morphism

The k-δ-coordinate ring is the differential analogue of the coordinate ring in affine algebraic geometry and is defined in a similar way. As its algebraic counterpart, the k-δ-coordinate ring carries many important informations on the k-δ-closed set X.

Definition 6.1. *Let X be a k-δ-closed subset of k^n. The ring $k\{y_1,\ldots,y_n\}/\mathfrak{I}(X)$ is called the differential coordinate ring of X or k-δ-coordinate ring and is denoted by $k\{X\}$. Its total ring of fractions (i.e., its localization w.r.t. the non-zero divisors) is denoted by $k\langle X\rangle$ and called the k-δ-ring of δ-rational functions on X.*

Remark 6.2.

1. *Explicitly, $k\langle X\rangle$ is the total ring of fractions with denominators in the multiplicative set of δ-polynomials, that do not vanish identically on any component of X.*
2. *If $X_1,\ldots,X_n$ are the irreducible k-δ-components of X, then $k\langle X\rangle$ is isomorphic to $\prod k\langle X_i\rangle$, that is to a product of δ-fields.*
3. *$k\{X\}$ is an integral domain if and only if X is irreducible.*

Let $X\subset k^n$ be a k-δ-variety and let $\eta\in X$. The substitution morphism $\sigma_\eta : k\{y_1,\ldots,y_n\}\to k$ factors through $k\{X\}$ and determines a k-δ-morphism from $k\{X\}$ to k. This gives a natural bijection between the points of X and the set of k-δ-morphism from $k\{X\}$ to k, that is with the set of maximal δ-ideals of $k\{X\}$. More generally, we can extend Theorem 5.9 to obtain the following bijection.

Theorem 6.3. *Let k be a δ-closed field and let $X\subset k^n$ be a k-δ-variety. The map*

$$Y\to\mathfrak{I}_X(Y):=\{f\in k\{X\}|f(a)=0 \text{ for all } a\in Y\},$$

is an inclusion reversing bijection between the set of k-δ-subvarieties of X and the set of radical δ-ideals of $k\{X\}$. Its inverse attaches to any radical δ-ideal $\mathfrak{a}\subset k\{X\}$ the subset $\mathbb{V}_X(\mathfrak{a})=\{x\in X|f(x)=0 \text{ for all } f\in\mathfrak{a}\}$.

Proof. The image $\mathfrak{I}_X(Y)$ of Y under the above map is $\pi(\mathfrak{I}(Y))$ where $\pi : k\{y_1,\ldots,y_n\}\to k\{X\}=k\{y_1,\ldots,y_n\}/\mathfrak{I}(X)$ is the canonical projection. Theorem 5.9 and the canonical bijection between the radical δ-ideals of $k\{y_1,\ldots,y_n\}$ that contain $\mathfrak{I}(X)$ and the radical δ-ideals of $k\{X\}$ allows us to prove the claim. ∎

One defines the k-δ-morphisms from X to k as the elements of $k\{X\}$. More generally, we have the following definition.

Definition 6.4. *Let $X\subset k^n$ and $Y\subset k^m$ be k-δ-varieties. A morphism $\phi : X\to Y$ is called morphism of differential algebraic varieties over k or k-δ-morphism for short, if there exist $\phi_1,\ldots,\phi_m\in k\{X\}$ such that $\phi(a)=(\phi_1(a),\ldots,\phi_m(a))\in Y$ for all $a\in X$.*

One remarks that the composition of k-δ-morphisms is a k-δ-morphism. Thus, one can speak of the category of k-δ-varieties whose objects are the differential algebraic varieties over k and whose morphisms are the k-δ-morphisms.

A very important class of k-δ-morphisms is provided by the logarithmic derivatives (see [Cas72, p 923]). Exercise 6.11 below gives an example of such a map.

Instead of considering δ-polynomials maps, one could consider morphisms between k-δ-varieties defined by quotients of δ-polynomials. In other terms, we could study δ-rational functions for short. However, unlike to classical algebraic geometry, an everywhere defined δ-rational function does not need to be a δ-polynomial. This is a big difficulty, when one tries to have a schematic approach of differential algebraic geometry. Exercise 6.12 below gives an example of an everywhere defined non-polynomial δ-rational function.

We introduce now the notion of dual morphism of a k-δ-morphism.

Definition 6.5. *Let $X\subset k^n$ and $Y\subset k^m$ be k-δ-varieties and let $\phi : X\to Y$ be a k-δ-morphism such that $\phi(a)=(\phi_1(a),\ldots,\phi_m(a))$ for all $a\in X$ and $\phi_1,\ldots,\phi_m\in k\{X\}$. The k-δ-morphism $\sigma_\phi : k\{y_1,\ldots,y_m\}\to k\{X\}, y_i\mapsto\phi_i$ induces a k-δ-morphism $\phi^* : k\{Y\}\to k\{X\}$ called the dual morphism of ϕ.*

Proof. We just have to show that $\mathfrak{I}(Y)\subset Ker(\sigma_\phi)$. Let $h\in\mathfrak{I}(Y)$ then $\phi^*(h)=h(\phi_1,\ldots,\phi_m)\in k\{X\}$. For all $a\in X$, we have $(\phi_1(a),\ldots,\phi_m(a))\in Y$ and, since $h\in\mathfrak{I}(Y)$, we get $\phi^*(h)(a)=0$. This means that $\phi^*(h)=0\in k\{X\}$. ∎

Similarly to classical algebraic geometry, one has a dictionary between k-δ-varieties and k-δ-algebras.

Theorem 6.6. *Let k be a δ-closed field. We define a functor F from the category of k-δ-varieties to the category of reduced k-δ-algebras, δ-finitely generated over k as follows. F sends a k-δ-variety X onto its k-δ-coordinate ring $k\{X\}$ and a morphism $\phi : X \to Y$ onto its dual ϕ^*. The functor F is an antiequivalence of category.*

Proof. The proof is essentially a differential analogue of [Har77, Prop. 3.5]. □

The notion of dual morphism allows us to define precisely the notion of embedding of a differential algebraic variety.

Definition 6.7. *Let X and Y be k-δ-varieties. We say that $\phi : X \to Y$ is a k-δ-closed embedding if $\phi(X)$ is a k-δ-subvariety of Y and $X \to \phi(X)$ is an isomorphism of k-δ-varieties.*

As for algebraic geometry, one can prove that a k-δ-morphism is a k-δ-closed embedding if and only if its dual morphism is surjective.

Lemma 6.8. *Let X, Y be k-δ-varieties and let $\phi : X \to Y$ be a k-δ-morphism. Let $Z = \mathbb{V}_X(\mathfrak{a})$ be the k-δ-subvariety of X defined by the radical δ-ideal $\mathfrak{a} \subset k\{X\}$. Then, the Kolchin closure $\overline{\phi(Z)}$ of $\phi(Z)$ equals $\mathbb{V}_Y((\phi^*)^{-1}(\mathfrak{a}))$. In particular, we find $\overline{\phi(X)} = \mathbb{V}_Y(Ker(\phi^*))$.*

Proof. Let $f \in k\{Y\}$. Since k-δ-morphisms are continuous for the Kolchin topology, we get that

$$f(\phi(Z)) = 0 \text{ if and only if } f(\overline{\phi(Z)}) = 0.$$

By Theorem 6.3, this implies that $f \in \mathfrak{I}_Y(\overline{\phi(Z)})$ if and only if $\phi^*(f) = f \circ \phi \in \mathfrak{a}$. This proves that $\mathfrak{I}_Y(\overline{\phi(Z)}) = (\phi^*)^{-1}(\mathfrak{a})$. For $Z = X$, we have $\mathfrak{a} = \{0\}$, which ends the proof. □

Proposition 6.9. *Let $V \subset k^n$ and $W \subset k^p$ be k-δ-varieties and let $\alpha : V \to W$ be a k-δ-morphism. The following hold.*

1. *If V is irreducible then $\overline{\alpha(V)}$ is irreducible,*
2. *α^* is injective if and only if $\alpha(V)$ is dense in W. A morphism, whose image is dense in its target set is called dominant.*

Proof. 1. If V is irreducible then $k\{V\}$ is an integral domain and $Ker(\phi^*)$ is a prime δ-ideal. We conclude with Lemma 6.8.

2. By Lemma 6.8, we have $\overline{\alpha(X)} = \mathbb{V}_Y(Ker(\phi^*))$ which gives the conclusion.

□

In the sequel, we prove the differential analogue of Chevalley theorem which asserts that the image of an algebraic variety V under a polynomial map contains a dense open set of its closure. This theorem is a direct consequence of the extension of δ-specializations.

Theorem 6.10. *Let $V \subset k^n$ and let $W \subset k^p$ be k-δ-varieties. Let $\phi : V \to W$ be a k-δ-morphism. Then, $\phi(V)$ contains a k-δ-open set dense in its closure.*

Proof. Let us write $k\{V\} = k\{v_1, \ldots, v_n\}$ and $k\{W\} = k\{w_1, \ldots, w_p\}$.

First let us assume that V is irreducible. By 6.9, $\overline{\phi(V)}$ is irreducible and replacing W by $\overline{\phi(V)}$, one can assume that ϕ^* is injective. Thus, we can identify $k\{W\}$ with a k-δ-subalgebra of $k\{V\}$ and ϕ^* with the inclusion map $k\{W\} \to k\{V\}$. Since both $k\{V\}$ and $k\{W\}$ are k-δ-finitely generated and $k\{V\}$ is an integral domain, we can apply Proposition 4.20 with $u = 1$. Then, there exists $u_0 \in k\{W\}^\times$ such that any $\psi : k\{W\} \to k$ with $\psi(u_0) \neq 0$ can be extended in a k-δ-morphism $\overline{\psi} : k\{V\} \to k$. Now, $\mathbf{D}(u_0) := \{(\eta_1, \ldots, \eta_p) \in W | u_0(\eta_1, \ldots, \eta_p) \neq 0\} \subset W$ is a non-empty k-δ-open set. Now, let $\eta \in \mathbf{D}(u_0)$. The substitution morphism $\sigma_\eta : k\{W\} \to k$ can be extended to a morphism $\overline{\sigma_\eta} : k\{V\} \to k$. Let us denote, for all $i = 1, \ldots, n$, by α_i the image of v_i by $\overline{\sigma_\eta}$. Then, $\eta = \phi(\alpha_1, \ldots, \alpha_n)$ which proves that $\mathbf{D}(u_0) \subset \phi(V)$. We conclude by noticing that an open set is dense in an irreducible set.

If V is reducible then we denote by $V_1, \ldots, V_q$ its irreducible k-δ-components (see Proposition 5.18). Then, $\overline{\phi(V)} = \cup \overline{\phi(V_i)}$ and the theorem can be proved by Rittian induction. □

Exercise 6.11. *Let k be a δ-closed field. $\mathrm{Gl}_n(k)$ is embedded in $k^{n \times n} \times k$ via $A \mapsto (A, \frac{1}{det(A)})$. This allows us to see $\mathrm{Gl}_n(k)$ as a k-δ-subvariety of $k^{n \times n} \times k$ defined by $det(Y)z - 1 = 0$ in $k\{k^{n \times n} \times k\} = k\{Y, z\}$. For $A \in \mathrm{Gl}_n(k)$, we define $ld(A) := A^{-1} \cdot \delta(A)$.*

1. *Show that $ld : \mathrm{Gl}_n(k) \to k^{n\times n}, A \mapsto ld(A)$ is a k-δ-morphism.*

2. *Show that $ld(AB) = B^{-1} \cdot ld(A) \cdot B + ld(B)$.*

Exercise 6.12 (p.901 in [Cas72])**.** *Let k be a δ-closed field. Let $X = \mathbb{V}(\delta(y) - y) \subset k$.*

1. *Show that $k\{X\}$ is k-δ-isomorphic to the polynomial ring $k[\overline{y}]$ endow with $\delta(\overline{y}) = \overline{y}$.*

2. *Show that, for all $c \in k^{\delta}$, the polynomial $\overline{y} - c$ is not invertible in $k[\overline{y}]$.*

3. *Let $c \in k^{\delta} \setminus \{0\}$. Show that the k-δ-rational map $\phi_c : X \to k, a \mapsto \frac{1}{a-c}$ is everywhere defined and does not belong to $k\{X\}$.*

Exercise 6.13. *Let X be a k-δ-variety.*

1. *Let $Y = \mathbb{V}_X(\mathfrak{a})$ be a k-δ-subvariety of X. Let $i : Y \to X$. Compute $i^* : k\{X\} \to k\{Y\}$.*

2. *Conversely, let R be a k-δ-algebra and let $\pi : k\{X\} \to R$ be a surjective k-δ-morphism. Show that there exists a k-δ-subvariety $Y \subset X$ such that R is k-δ-isomorphic to $k\{Y\}$.*

Exercise 6.14. *Let X, Y be k-δ-varieties and let $\phi : X \to Y$ be a k-δ-morphism. Let $Z = \mathbb{V}_Y(\mathfrak{a}) \subset Y$ for some radical δ-ideal $\mathfrak{a} \subset k\{Y\}$.*

Show that $\phi^{-1}(Z) = \mathbb{V}_X(\phi^*(\mathfrak{a}))$. In particular, this proves that k-δ-morphisms are continuous for the Kolchin topology.

Exercise 6.15. *Let k be a δ-closed field. We consider the k-δ-variety $k^{\times} \subset k^2$ and $\mathrm{Gl}_2(k) \subset k^4 \times k$ (see Exercise 6.11). Let $\phi : k^{\times} \to \mathrm{Gl}_2(k), a \mapsto \begin{pmatrix} a & \delta(a) \\ 0 & a \end{pmatrix}$*

1. *Show directly that ϕ is a k-δ-closed embedding such that $\phi(ab) = \phi(a)\phi(b)$.*

2. *Compute $\phi^* : k\{Y, \frac{1}{det(Y)}\} \to k\{y, \frac{1}{y}\}$ and show that it is surjective.*

Exercise 6.16. *Let X, Y be k-δ-varieties and let $\phi : X \to Y$ be a k-δ-closed embedding.*

1. *Show that $\phi(X) = \mathbb{V}_Y(Ker(\phi^*))$.*

2. *Show that ϕ^* is surjective. (hint: Compute the dual map of $\phi : X \to \phi(X)$)*

7 Dimension of a k-δ-closed set

Let k be a δ-closed field.

Definition 7.1. *Let $V \subset k^n$ be a k-δ-variety.*

1. *If V is irreducible then $k\langle V\rangle$ is a δ-field and we define the k-δ-dimension of V as $\delta\text{-}\mathrm{trdeg}(k\langle V\rangle|k)$.*

2. *If V is reducible then we define the k-δ-dimension of V as the supremum of the k-δ-dimensions of its irreducible k-δ-components.*

We denote the k-δ-dimension of X by $\delta\text{-}\dim_k(X)$.

An irreducible k-δ-variety has dimension 0 if its field of δ-rational functions $k\langle V\rangle$ is δ-algebraic over k. For instance, if $V = \mathbb{V}(P)$ for a δ-polynomial P, the vanishing of the k-δ-dimension of V means that the solutions of P depends on a finite number of arbitrary constants.

The k-δ-dimension defines an invariant, up to k-δ-isomorphism, of a k-δ-variety. However, the k-δ-dimension is not sufficient to measure the size of a k-δ-variety. Indeed, even if $V \subset W$ and V is a proper k-δ-subvariety of W, it might happen that $\delta\text{-}\dim_k(V) = \delta\text{-}\dim_k(W)$. One has to introduce finer invariants, for instance the type and the Kolchin polynomial (see [Kol73, p.115]). In fact, the Kolchin polynomial distinguishes between two k-δ-varieties properly included in one-another and for instance counts the number of arbitrary constant functions attached to these varieties.

Finally, we quote a proposition on products of k-δ-varieties.

Proposition 7.2 (§7 in [Cas72])**.** *Let $V \subset k^n$ and let $W \subset k^p$ be k-δ-varieties. Then,*

1. *The cartesian product $V \times W$ is k-δ-closed in k^{n+p};*

2. *The k-δ-coordinate ring of $V \times W$ is the tensor product $k\{V\} \otimes_k k\{W\}$;*

3. *If V and W are irreducible so is $V \times W$ and the k-δ-dimension of the product equals the sum of the k-δ-dimensions of V and W;*

4. *If V and W are defined over a δ-subfield F of k, the same holds for $V \times W$.*

Exercise 7.3. 1. *What is the k-δ-dimension of k^n?*

2. *Let $V = \mathbb{V}(\delta(y) - y))$. Show that V is irreducible and compute the k-δ-dimension of V.*

Exercise 7.4. *Let $V = \mathbb{V}(\delta(y) - y)$ and $W = \mathbb{V}(\delta^2(y) - \delta(y))$. Let y_0 be a non-zero point of V. Show that $V = \{cy_0 | c \in k^\delta\} \subsetneq W = \{cy_0 + d | c, d \in k^\delta\}$. Compute the k-δ-dimension of V and W.*

8 From algebraic geometry to differential algebraic geometry and vice versa

In this section, we investigate the relations between classical algebraic geometry and differential algebraic geometry.

Let $X \subset k^n$ be an algebraic variety over k. That is X is the zero-set of a collection $S \subset k[y_1, \ldots, y_n]$ of polynomials over k. We denote by $k[X]$ its coordinate ring, that is, $k[X] := k[y_1, \ldots, y_n]/[S]$. We can attach to X the differential algebraic variety $\mathbf{X} := \mathbb{V}(\{S\})$, where $\{S\}$ denotes the radical δ-ideal generated by S in $k\{y_1, \ldots, y_n\}$. The k-δ-variety $\mathbf{X}$ and the algebraic variety X share the same points in k^n but $\mathbf{X}$ has more open sets than X. When there is no confusion, we shall still denote by X the k-δ-variety $\mathbf{X}$. The transition from algebraic geometry to differential algebraic geometry can be made much more intrinsic via a schematic approach (see [Gil02]) but we won't detail this approach here.

Kolchin irreducibility theorem states that if the algebraic variety X is irreducible as Zariski closed set, the same holds for $\mathbf{X}$. Moreover, $\delta\text{-}\dim_k(\mathbf{X})$ coincides with the dimension of X as algebraic variety (see [Kol73, Chapter IV, proposition 10] or [Gil02, Section 2] for a schematic approach).

Conversely, given a k-δ-variety $X \subset k^n$, one can also consider the Zariski closure $\overline{X}^Z$ of X. This is the closure of X as subset of k^n endowed with the Zariski topology. We have the following lemma.

Lemma 8.1. *Let $X \subset k^n$ be a k-δ-variety and let $\mathfrak{I}(X) \subset k\{y_1, \ldots, y_n\}$ be its defining k-δ-ideal. Let $\overline{X}^Z$ denote its Zariski closure. Then, $\overline{X}^Z$ is the zero set in k^n of the polynomial ideal $\mathfrak{I}(X) \cap k[y_1, \ldots, y_n] \subset k[y_1, \ldots, y_n]$.*

Proof. First of all, $\mathbb{V}(\mathfrak{I}(X) \cap k[y_1, \ldots, y_n])$ is Zariski closed as zero set of polynomials and contains obviously X and thereby its Zariski closure. Conversely, if $S \subset k[y_1, \ldots, y_n]$ and $X \subset \mathbb{V}(S)$ then by theorem 5.9, we have $\{S\} \subset \mathfrak{I}(X)$, which implies

$$S \subset \{S\} \cap k[y_1, \ldots, y_n] \subset \mathfrak{I}(X) \cap k[y_1, \ldots, y_n].$$

Then $\mathbb{V}(\mathfrak{I}(X) \cap k[y_1, \ldots, y_n]) \subset \mathbb{V}(S)$, which ends the proof. □

Unlike to the situation of Kolchin irreducibility theorem, the δ-dimension of a k-δ-variety and the dimension of its Zariski closure can differ as well as their irreducible character.

Example 8.2. *Let $X = \mathbb{V}(\delta^2(y) + 2y) \subset k$. Then, $\overline{X}^Z = k$ and $\delta\text{-}\dim_k(X) = 0$ whereas $\dim(\overline{X}^Z) = 1$.*

Exercise 8.3. *Let $X = \{\begin{pmatrix} a & \delta(a) \\ 0 & a \end{pmatrix} | a \in k^\times\} \subset \mathrm{Gl}_2(k)$. By Exercice 6.15, we know that $k^\times$ is k-δ-isomorphic to X.*

1. *Compute the Zariski closure of X and its dimension.*

2. *Compare with the Zariski closure of $k^\times$.*

This exercise shows that Zariski closures do not behave well with respect to k-δ-isomorphisms.

9 Linear k-δ-groups

The study of linear differential algebraic groups was principally developed by Cassidy ([Cas72], [Cas89], [Cas75]). These groups, which are groups of matrices whose entries are solutions of differential algebraic equations, are connected to many interesting problems such as non-linear Galois theory (see for instance [Mal01] or [Ume96]) as well as diophantine geometry (see [Kol99, §2]). First introduced in the language of Weil, they encounter now a modern development using Hopf algebraic and Tannakian approach ([Kov02] and [Ovc09]).

In this section, we recall some basic definitions and give some classification results for differential algebraic subgroups of vector groups, tori and quasi simple algebraic groups.

In this section, k again denotes a δ-closed field of characteristic zero.

9.1 Definition and first properties

Definition 9.1. *A k-δ-group $G \subset k^n$ is a k-δ-variety, such that G is a group and the group laws, the multiplication $\mu : G \times G \rightarrow G$ and the inverse $\iota : G \rightarrow G$ are k-δ-morphisms. A k-δ-subgroup of a k-δ-group is a subgroup which is k-δ-closed.*

Remark 9.2. *In [Cas72, p905]), the group laws are allowed to be everywhere defined δ-rational functions. We restrict ourselves to an easier framework, where all k-δ-morphisms are regular and all k-δ-varieties are affine, i.e., k-δ-subvarieties of some k^n. Moreover, Cassidy shows that k-δ-groups whose laws are given by δ-polynomials can be embedded as k-δ-subgroups of some $\mathrm{Gl}_n(k)$. Therefore, all the k-δ-groups, considered in these notes, are linear k-δ-groups. This means that they can be embedded in some $\mathrm{Gl}_n(k)$.*

Example 9.3.

1. *Let $\mathbf{G}_a^n$ denotes $(k^n, +)$ endowed with the addition component wise.*
2. *Let $L_1, \dots, L_s \in k\{y_1, \dots, y_n\}$ be linear homogeneous δ-polynomials. Then, $\mathbb{V}(L_1, \dots, L_s)$ is a k-δ-subgroup of $\mathbf{G}_a^n$.*
3. *Let $\mathrm{Sl}_n(k)$ be the k-δ-subgroup of $k^{n\times n}$ of matrices of determinant 1.*
4. *$\mathrm{Gl}_n(k)$ can be viewed as a k-δ-subgroup of $\mathrm{Sl}_{n+1}(k)$ by associating to the matrix $S = (s_{i,j})_{1\leq i,j\leq n}$ the matrix $(s'_{i,j})_{1\leq i,j\leq n+1} = \begin{pmatrix} S & 0 \\ 0 & \frac{1}{det(S)} \end{pmatrix}$. With this identification, the k-δ-coordinate ring of $\mathrm{Gl}_n(k)$ is isomorphic to $k\{\mathrm{Gl}_n\} = k\{Y, \frac{1}{det(Y)}\}$, where $Y = (y_{i,j})_{1\leq i,j\leq n}$ is a matrix of n^2 differential indeterminates.*
5. *Let $\mathbf{G}_m^n$ be $((k^\times)^n, *)$ endowed with the multiplication component wise. $\mathbf{G}_m^n$ can be seen as a k-δ-subgroup of $\mathrm{Gl}_n(k)$ with respect to the diagonal embedding $\mathbf{G}_m^n \rightarrow \mathrm{Gl}_n(k), (\lambda_1, \dots, \lambda_n) \mapsto \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{pmatrix}$.*
6. *Let G be a Zariski closed subgroup in $\mathrm{Gl}_n(k)$, i.e., a Zariski closed set endowed with group laws given by polynomials over k. Then, since polynomial maps are δ-polynomials map, the k-δ-variety $\mathbf{G}$ (see §8) is a k-δ-subgroup of $\mathrm{Gl}_n(k)$.*

Definition 9.4. *Let H, G be linear k-δ-groups. A map $\phi : H \rightarrow G$ is called k-δ-group morphism if it is a k-δ-morphism and a group morphism.*

Example 9.5. *The map $dlog : \mathbf{G}_m^n \rightarrow \mathbf{G}_a^n, (s_1, \dots, s_n) \mapsto (\frac{\delta(s_1)}{s_1}, \dots, \frac{\delta(s_n)}{s_n})$ is a k-δ-group morphism called the logarithmic derivative. This example is crucial since there is no polynomial map from a torus to a vector group. This logarithmic derivative is a baby example. This notion can be generalized for algebraic group defined over differential fields (see [Pil04]).*

Definition 9.6. *Let G be a linear k-δ-group and let $g \in G$. We denote by λ_g (resp. ρ_g) the left (resp. right) translation $\lambda_g : G \rightarrow G, x \mapsto gx$ (resp. $\rho_g : G \rightarrow G, x \mapsto xg^{-1}$) and by ι_g the inner automorphism $\iota_g = \lambda_g \circ \rho_g$. The maps λ_g and ρ_g are k-δ-morphisms of G.*

Remark 9.7. *Let $g \in G$ and let $\rho_g^* : k\{G\} \rightarrow k\{G\}$ be the adjoint map of ρ_g. Then, g is the neutral element in G if and only if ρ_g^* is the identity map on $k\{G\}$.*

Lemma 9.8. *Let G be a k-δ-group. Let U and V two dense k-δ-open subsets in G. Then $UV = G$*

Proof. See Exercise 9.14. ☐

Proposition 9.9. *Let G be a k-δ-group.*

1. *There is a unique irreducible k-δ-component G^0 of G which contains the identity element e. It is a k-δ-closed normal subgroup of G of finite index. G^0 is called the identity component of G.*
2. *The irreducible k-δ-components are the cosets of G^0 and they are disjoint.*
3. *G^0 is connected and the irreducible k-δ-components of G coincide with the connected components of G for the Kolchin topology.*

Proof. See Exercise 9.14. ☐

Proposition 9.10. *Let G, G' be k-δ-groups. Let $\alpha : G \mapsto G'$ be a k-δ-group morphism. Then,*

1. *$Ker(\alpha)$ is a normal k-δ-subgroup of G;*
2. *$\alpha(G)$ is a k-δ-subgroup of G*
3. *$\alpha(G^0) = \alpha(G)^0$*

Proof. See Exercise 9.15 ☐

Exercise 9.11 (p.911 in [Cas72])**.** *Let us consider the k-δ-variety $G := \mathbb{V}(y_1^2 - \delta(y_1) - y_1, y_1y_2 - 1) \subset k^2$.*

1. *Show that G is a group w.r.t. the group law given by $\mu((s_1, s_2), (t_1, t_2)) = (s_1t_1/(s_1 + t_1 - s_1t_1), (s_1 + t_1 - s_1t_1)/s_1t_1)$.*
2. *Show that $\mu : G \times G \to G$ is not a δ-polynomial map.*

Exercise 9.12. *Let $G \in \mathrm{Gl}_n(k)$ be a Zariski closed subgroup and let us denote by $\mathbf{G}$ its corresponding k-δ-variety. We want to show that the set $G^\delta = \{\eta \in k^n | \eta \in \mathbf{G}$ and $\delta(\eta) = 0\}$ of constant points of $\mathbf{G}$ is a k-δ-subgroup of $\mathrm{Gl}_n(k)$.*

1. *Let $S \subset k[y_1, \ldots, y_n]$ such that $G = \mathbb{V}(S)$. Show that $G^\delta = \mathbb{V}(\{S, \delta(y_1), \ldots, \delta(y_n)\})$.*
2. *Show that the multiplication μ and the inverse ι of $\mathrm{Gl}_n(k)$ stabilizes G^δ (hint: they are defined over the constants k^δ.)*
3. *Let $u \in k^\times$ such that $\delta(u) = u$. Consider $G = \{(y_1, y_2) \in k^2 | uy_1 + y_2 = 0\}$. Compute $\mathbf{G}$, G^δ and the Zariski closure of G^δ.*
4. *Show that, if G is defined above k^δ then the Zariski closure of G^δ is G.*

Exercise 9.13. *Show that the set $\{ \begin{pmatrix} a & 0 & c \\ 0 & 1 & b \\ 0 & 0 & a \end{pmatrix} | a, b, c \in k$ such that $\delta(a) = c\}$ is a k-δ-subgroup of $\mathrm{Gl}_3(k)$.*

Exercise 9.14. *Prove Lemma 9.8 and Proposition 9.9. (Hint: Use the map ρ_g, ι_g and follow the lines of the algebraic analogue of the statement above, which can be found either in Michael Singer's lecture or [Spr09, Prop. 2.2.1]).*

Exercise 9.15. *Prove Proposition 9.10. (Hint: Follow the lines of their algebraic analogue, which can be found either in Michael Singer's lecture or [Spr09, Prop. 2.2.5]) and the differential analogue of Chevalley's Theorem 6.10.)*

9.2 k-δ-Hopf algebras

A k-Hopf algebra over k is a k-algebra A together with k-algebra maps, a comultiplication $\Delta : A \to A \otimes_k A$, a coinverse $S : A \to A$ and a counit $\epsilon : A \to k$ satisfying certain types of diagram (see [Wat79, p 8]). If the Hopf algebra has a derivation compatible with its structure, we obtain the following definition.

Definition 9.16. *A differential Hopf algebra over k, or k-δ-Hopf algebra for short, is a k-δ-algebra A together with k-δ-morphisms: a comultiplication $\Delta : A \to A \otimes_k A$ [3], a coinverse $S : A \to A$ and a counit $\epsilon : A \to k$ such that the diagrams*

$$\begin{tikzcd} A \otimes_k A \otimes_k A & A \otimes_k A \arrow[l, "id\otimes\Delta"'] \\ A \otimes_k A \arrow[u, "\Delta\otimes id"] & A \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} k \otimes_k A & A \otimes_k A \arrow[l, "\epsilon\otimes id"'] \\ & A \arrow[ul, "\simeq"] \arrow[u, "\Delta"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A & A \otimes_k A \arrow[l, "S\otimes id"'] \\ k \arrow[u] & A \arrow[l, "\epsilon"] \arrow[u, "\Delta"'] \end{tikzcd}$$

commute. Morphisms of k-δ-Hopf algebras are k-δ-morphisms that commute with the Hopf structural maps.

Example 9.17. *Let us consider $k\{\mathbf{G}_a\} = k\{y\}$ together with the k-δ-morphisms Δ, ϵ and S given by their action on y via $\Delta(y) := y \otimes 1 + 1 \otimes y$, $S(y) = -y$ and $\epsilon(y) = 0$.*

Let G be an algebraic group over k, i.e., a subgroup of some $\mathrm{Gl}_n(k)$, which is a Zariski closed. We denote by μ its multiplication, and by ι its inverse. The neutral element e is views as a map from the point $\{e\}$ to G. The coordinate ring $k[G]$ of G is a k-Hopf algebra over k where Δ, S and ϵ are the dual maps of μ, ι and e. This gives a one to one correspondence between algebraic groups over k and Hopf algebras over k (see [Wat79, Theorem p9]). In the differential framework, the antiequivalence of Theorem 6.6 restricts to the following Theorem.

Theorem 9.18. *Let k be a δ-closed field. The functor $G \mapsto k\{G\}$ from the category of k-δ-groups to the category of reduced k-δ-Hopf algebras, δ-finitely generated over k is an antiequivalence of categories.*

The point of view of k-δ-Hopf algebras is more intrinsic than the point of view of differential algebraic groups. Moreover, the theory of Hopf algebras has been developed in a great generality, so that many existing results in this domain hold for non finitely generated k-algebras (see [Abe80] for an introduction). Such kind of results can be applied directly to k-δ-algebras. Then, one just has to check the compatibility with the derivation. This kind of strategy is used to prove Proposition 9.19 and Theorem 9.20.

Proposition 9.19. *Let G be a k-δ-subgroup of $\mathrm{Gl}_n(k)$. The Zariski closure $\overline{G}^Z$ of G inside $\mathrm{Gl}_n(k)$ is an algebraic group over k.*

Proof. Let Δ, S and ϵ be the comultiplication, the coinverse and counit of $k\{\mathrm{Gl}_n\} = k\{Y, \frac{1}{det(Y)}\}$ as k-δ-Hopf algebra. They restrict to the comultiplication, the coinverse and counit of the Hopf algebra $k[\mathrm{Gl}_n] = k[Y, \frac{1}{det(Y)}]$.

Since G is a k-δ-subgroup of $\mathrm{Gl}_n(k)$, let us denote by $\pi : k\{Gl_n\} \to k\{G\}$ the quotient map of $k\{\mathrm{Gl}_n\}$ by $\mathfrak{I}_{\mathrm{Gl}_n(k)}(G)$. Since G is a k-δ-subgroup of $\mathrm{Gl}_n(k)$, the morphism π is a morphism of k-δ-Hopf algebra. Thus, the morphisms Δ, S and ϵ pass to the quotient and induce the comultiplication, the coinverse and counit of $k\{G\}$. Now, one can easily see that the algebraic coordinate ring $k[\overline{G}^Z]$ of $\overline{G}^Z$ is $\pi(k[Y, \frac{1}{det(Y)}])$. Since Δ, S and ϵ stabilize $k[Y, \frac{1}{det(Y)}]$, when passing to quotient on $k\{G\}$, they stabilize also $k[\overline{G}^Z]$. This proves that $k[\overline{G}^Z]$ is a Hopf algebra and that $\pi : k[Y, \frac{1}{det(Y)}] \to k[\overline{G}^Z]$ is a surjective morphism of k-Hopf algebras. By algebraic correspondence [Wat79, Theorem p9], we get that $\overline{G}^Z$ is an algebraic subgroup of $\mathrm{Gl}_n(k)$. ☐

A more subtle and difficult theorem on differential algebraic groups, which can be proved by Hopf algebra methods, is the existence of quotients by normal subgroups. More precisely, we have the following result.

Theorem 9.20. *Let G be a k-δ-group and let $N \subset G$ be a normal k-δ-subgroup. Then, there exists a quotient of G modulo N. More precisely, there exists a k-δ-group morphism $\pi : G \to G/N$ satisfying the universal property of quotients in the category of k-δ-groups.*

[3] The derivation on $A \otimes_k A$ is given by $\delta(a \otimes b) = \delta(a) \otimes b + a \otimes \delta(b)$

Proof. We give a very very sketchy proof. The full proof is the exact differential analogue of [DVHW, Theorem A.43]. The main idea is to use an already existing correspondence between Hopf sub-algebras of $k\{G\}$ and the quotients of affine group schemes over k view as proalgebraic groups (see [DVHW, Proposition A.42] as well as the Hopf theoretic results contained in [Tak72]). In this correspondence the algebra

$$k\{G\}(\mathfrak{I}(N)) := \{r \in k\{G\} | \Delta(r) - r \otimes 1 \in k\{G\} \otimes \mathfrak{I}(N)\}$$

is a k-Hopf sub-algebra of $k\{G\}$, which corresponds to the Hopf algebra of the universal quotient, but in the category of affine group schemes. To conclude, one has just to prove that $k\{G\}(\mathfrak{I}(N))$ is stable under the derivation and thus represents a k-δ-group. This is obvious since Δ is a k-δ-morphism and $\mathfrak{I}(N)$ is a δ-ideal. ☐

Exercise 9.21. *For $n \in \mathbb{N}$, let us consider $k\{\mathrm{Gl}_n\} = k\{Y, \frac{1}{det(Y)}\}$ for $Y = (y_{i,j})$ a $n \times n$ matrix of δ-indeterminates. We define the k-δ-morphism Δ via its action on Y through $\Delta(y_{i,j}) := \sum_{k=1}^{n} y_{i,k} \otimes y_{k,j}$. What is the definition of S and ϵ in order to recover the classical inverse and unit in $\mathrm{Gl}_n(k)$.*

9.3 Differential algebraic torsors

The notion of torsor is central in any Galois theory. We refer the interested reader to [DD77] for the usual Galois theory but also to the notes of Michael Singer for the Galois theory of linear difference equations. The algebraic notion of G-torsor (see [Spr09, §2.3]) has a straightforward analogue in the differential context. However, any differential algebraic torsor is trivial over a differentially closed field so that we have to consider differential algebraic varieties defined over smaller fields. We introduce the following notation. Let $F \subset k$ be a δ-sub-field of k. We recall that a k-δ-variety X is defined over F if $V = \mathbb{V}(\Sigma)$ for some set $\Sigma \subset F\{y_1, \ldots, y_n\}$. In that situation, we denote by $F\{X\}$ the F-δ-algebra $F\{y_1, \ldots, y_n\}/\{\Sigma\}$. For any F-δ-algebra S, we denote by $X(S)$ the zero set of Σ in S^n.

Now, we are able to give the definition of a differential algebraic torsor.

Definition 9.22. *Let k be a δ-closed field and let $F \subset k$ be a δ-subfield. Let G a linear k-δ-group defined over F. A* G-torsor *over F is a k-δ-variety X defined over F together with a differential polynomial map $f : X \times_F G \to X \times_F X$ (denoted by $f : (x, g) \mapsto xg$) defined over F such that*

1. *for any F-δ-algebra S and $x \in V(S), g_1, g_2 \in G(S)$, $x.e_G = x$, $x(g_1 g_2) = (xg_1)g_2$ with $e_G \in G(S)$ the neutral element ;*
2. *the homomorphism $F\{X\} \otimes_F F\{X\} \to F\{X\} \otimes_F F\{G\}$ is an isomorphism (or equivalently, for any F-δ-algebra S, the map $X(S) \times G(S) \to X(S) \times X(S), (x, g) \mapsto (x, xg)$ is a bijection).*

Remark 9.23. *For $L|K$ a finite Galois extension, one has a fundamental isomorphism*

$$L \otimes_K L \simeq L \otimes_K K^{\mathrm{Gal}(L|K)}, \tag{9.1}$$

where $\mathrm{Gal}(L|K)$ *is the usual Galois group of the finite field extension $L|K$ and $K^{\mathrm{Gal}(L|K)}$ is the ring of functions from* $\mathrm{Gal}(L|K)$ *to K. This fundamental isomorphism is an algebraic torsor isomorphism in disguise. Indeed, $K^{\mathrm{Gal}(L|K)}$ is the Hopf algebra of* $\mathrm{Gal}(L|K)$ *view as a constant group scheme (see [Wat79, 2.3]) and L can be interpreted as the field of fractions of the coordinate ring of some* $\mathrm{Gal}(L|K)$*-torsor X.Then,* (9.1) *becomes*

$$K[X] \otimes_K K[X] \simeq K[X] \otimes_K K[\mathrm{Gal}(L|K)]. \tag{9.2}$$

9.4 Classification of linear k-δ-subgroups of $\mathbf{G}_a^n$, $\mathbf{G}_m^n$ and of quasi-simple algebraic groups

The spirit of Galois theory is to draw a dictionary between the structure of the Galois group and the type of relations satisfied by the solutions: for instance, in usual Galois theory, solutions in radicals correspond to a solvable Galois group. In parametrized Galois theory, the Galois groups are differential algebraic groups. Therefore, we need some classification results for differential algebraic groups. All of the results presented below are due to Phyllis Cassidy and proved in a more general framework in [Cas72].

Theorem 9.24 (Proposition 11 in [Cas72])**.** *A subset V of k^n is a proper k-δ-subgroup of $\mathbf{G}_a^n$ if and only if it is the zero set of a finite number of non-zero linear homogeneous δ-polynomials in $k\{y_1, \ldots, y_n\}$.*

The logarithmic derivative (see example 9.5) maps $\mathbf{G}_m^n$ into $\mathbf{G}_a^n$. This fundamental map allows to classify the k-δ-subgroup of $\mathbf{G}_m^n$. We have the following result.

Theorem 9.25 (Chapter IV in [Cas72])**.** *Let G be a proper k-δ-subgroup of $\mathbf{G}_m^n$. Then, the defining k-δ-ideal of G is generated by equations of the following form*

- $y_1^{m_1}\dots y_n^{m_n}-1$ *for some* $(m_1,\dots,m_n)\in\mathbb{Z}^n$, *or*
- $L(\frac{\delta(y_1)}{y_1},\dots,\frac{\delta(y_n)}{y_n})$ *for some non-zero linear homogeneous δ-polynomial $L\in k\{y_1,\dots,y_n\}$.*

In any of the cases above, there exists a non-zero linear homogeneous δ-polynomial $L\in k\{y_1,\dots,y_n\}$ such that $G\subset\mathbb{V}\left(L(\frac{\delta(y_1)}{y_1},\dots,\frac{\delta(y_n)}{y_n})\right)$.

For $n=1$, one can make the above theorem even more precise.

Corollary 9.26. *Let G be a k-δ-subgroup of $\mathbf{G}_m$. Then, either*

- *G is finite and cyclic, or*
- $G=\mathbb{V}(L(\frac{\delta(y)}{y}))$ *for some linear homogeneous δ-polynomial $L\in k\{y\}$.*

Finally, we want to quote a difficult result in the classification of k-δ-groups, which is highly connected to the notion of integrability of differential systems and Lax pairs. This is one of the main results of [Cas89]. It uses a differential analogue of the Lie algebra of an algebraic group and strong classification results for Chevalley groups. The Lie counterpart of this theorem was proved simultaneously by Kiso using the language of connections on foliations (see for instance [Trá10, p 308] for a discussion of the two proofs). We recall that a simple algebraic group is an algebraic group, with no non-trivial normal subgroup. A quasi-simple algebraic group H is an algebraic group which is a central extension of a simple algebraic group by a finite central subgroup. A standard example of quasi-simple group is $\mathrm{Sl}_n(k)$. We recall also that we say that a k-δ-group G is a k-δ-subgroup of an algebraic group H whenever G is a k-δ-subgroup of $\mathbf{H}$, the k-δ-group attached to H.

Theorem 9.27. *Let G be a proper k-δ-subgroup of a quasi-simple algebraic group $H\subset\mathrm{Gl}_n(k)$ Assume that G is a Zariski dense subset of H [4]. Then, there exists $P\in H(k)$ such that*

$$PGP^{-1}=\mathrm{Gl}_n(k^\delta)\cap H=\{g\in H|\delta(g)=0\}.$$

Remark 9.28. *In the notation of the above theorem, let us denote by C the logarithmic derivative of P, that is, $P^{-1}\delta(P)$. Since $PGP^{-1}\subset\mathrm{Gl}_n(k^\delta)$, we have $\delta(B)+CB-BC=0$ for all $B\in G$. This equality means that G is contained in the solution space of the linear differential equation*

$$\delta(Y)+CY-YC=0. \tag{9.3}$$

This linear differential equation is precisely the connection of Kiso.

Finally, the differential equation (9.3) corresponds in the language of integrable systems to what is called a Lax equation. We will show in the last section of these notes, how Lax equations are connected to compatible systems of functional equations.

Exercise 9.29 (Kolchin and Zariski closed subgroups of $\mathbf{G}_a$)**.** *We want to prove Theorem 9.24 for $n=1$. Moreover, we study separately Zariski and Kolchin closed subgroups.*

1. *Let G be a Zariski closed subgroup of $\mathbf{G}_a$, i.e., $G=\mathbb{V}(S)$ where S is a subset of $k[y]$. We want to prove that $G=\{0\}$ or $G=\mathbf{G}_a$*
   (a) *Show that there exists $P\in k[y]$ such that $G=\mathbb{V}(P)$.*
   (b) *Let us assume that $G\neq\{0\}$ and let $x\in G\subsetneq\{0\}$. Show that $P(nx)=0$ for all $n\in\mathbb{Z}$. Conclude that $G=\mathbf{G}_a$.*
2. *Let G be a proper non-zero k-δ-subgroup of $\mathbf{G}_a$.*
   (a) *Show that G is a k^δ-vector space (hint: Let $P\in\mathfrak{I}(G)$ and let $x\in G\subsetneq\{0\}$. For $t\in k^\delta$, develop $P(tx)$ as polynomial in $k[t]$ and use the fact that $P(nx)=0$ for all $n\in\mathbb{Z}$.)*

[4] This means that $\overline{G}^Z=H$

(b) Prove that G is irreducible and conclude that there exists a non-zero $P \in \mathfrak{I}(G)$ of minimal rank such that $\mathfrak{I}(G) = [P] : (S_P I_P)^\infty$, with S_P the separant of P and I_P its initial.

(c) Since $S_P I_P \notin \mathfrak{I}(G)$, there exists $x \in G$ such that $S_P(x) I_P(x) \neq 0$. Let us consider the linearisation morphism

$$\mathcal{L}_x : k\{y\} \to k\{y\}, Q \mapsto \sum_{k \in \mathbb{N}} \frac{\partial Q}{\partial(\delta^k(y))}(x)\delta^k(y).$$

i. Let $Q \in \mathfrak{I}(G)$. Show that $\mathcal{L}_x(Q) \in \mathfrak{I}(G)$ (hint: Let $y \in G \subsetneq \{0\}$. For all $t \in k^\delta$, consider $Q(x+ty)$ as a polynomial in $k[t]$ and consider its term of order 1. Use $P(x+ny) = 0$ for all $n \in \mathbb{Z}$).

ii. Show that P and $\mathcal{L}_x(P)$ have same leader and that the separant of $\mathcal{L}_x(P)$ is $S_P(x)$.

iii. Conclude that $\mathfrak{I}(G) = [\mathcal{L}_x(P)]$.

The proof in n-variables is similar but one has to use the notion of characteristic set and Lemma 3.14.

Exercise 9.30. *Let dlog : $\mathbf{G_a} \to \mathbf{G}_m$ be the logarithmic derivative*

1. *Show that the kernel of dlog is ${\mathbf{G_m}}^\delta := \{c \in k^\delta | c \neq 0\}$.*

2. *We want to show that the proper k-δ-subgroups of ${\mathbf{G_m}}^\delta$ are finite and cyclic. Let H be a proper k-δ-subgroup of ${\mathbf{G_m}}^\delta$.*

   (a) Show that $\mathfrak{I}(H)$ is generated by non zero polynomials in $k[y, \frac{1}{y}]$ and thus $H \subset (k^\delta)^\times$ is a finite set.

   (b) Show that there exists $N \in \mathbb{N}$ such that $h^N = 1$ for all $h \in H$.

   (c) Use the differential Nullstellensatz to conclude that $H \subset \mathbb{V}(y^N - 1)$ and thus is a cyclic group.

3. *Let G be a k-δ-subgroup of $\mathbf{G}_m$ and assume that the image of G by dlog is a proper k-δ-subgroup and show that $G \subset \mathbb{V}(L(\frac{\delta(y)}{y}))$ for some linear homogeneous δ-polynomial $L \in k\{y\}$ (use Theorem 9.24).*

Part IV
Parametrized Picard-Vessiot theory

In this part, we present a simplified version of the parametrized Galois theory developed in [HS08]. We focus here on linear difference systems with a single continuous parameter whereas [HS08] is concerned with compatible systems of differential and difference equations with finitely many continuous parameters. The theory of [HS08] encompasses the Galois theory developed in [vdPS97], [vdPS03] when the set of differential parameters is empty and [CS06], which considers compatible differential systems with continuous parameters. Once again, the objective of these notes is to give the fundamental ideas and intuitions for this parametrized theory rather than to give a full presentation. Our main goal is to let the reader play with examples, with parametrized Galois groups and finally to allow him to prove via galoisian methods Hölder's Theorem on the hypertranscendence of the Gamma function (see Exercise 13.6).

10 The framework

So what are we interested in? For instance for the Gamma function $\Gamma(x)$, we would like to understand the $\frac{d}{dx}$-algebraic behaviour of a function $y(x)$, which satisfies the linear difference equation $y(x+1) = xy(x)$. Therefore, the context of our study combines difference and differential algebra. These theories might look quite similar but they have deep differences. For instance, whereas the language of field is perfectly convenient for differential algebra, it is too restrictive for difference algebra and one has to allow rings with zero divisors. We can not introduce difference algebra in a great generality but we will try to keep our notes as self contained as possible. We refer the interested reader to the founding book of Cohn ([Coh65]) for an introduction to difference algebra and to Michael Singer's notes and Singer-van der Put reference book ([vdPS97]) for a detailed exposition of Galois theory of linear difference systems.

The algebraic framework of our study is as follows.

Definition 10.1. *A $\sigma\delta$-ring is a commutative ring R with unit together with an automorphism σ and a derivation δ satisfying $\sigma(\delta(r)) = \delta(\sigma(r))$ for all $r \in R$. A $\sigma\delta$-field is a $\sigma\delta$-ring which is a field.* [5].

Remark 10.2. *We largely use standard notation of difference and differential algebra. Since we don't want to bother the reader with many similar definitions, we recall the basic conventions: Algebraic attributes always refer to the underlying ring whereas the operator suffix means that the algebraic attributes commutes with the operator. For instance, a σ-ideal is an ideal stable by σ, a $\sigma\delta$-morphism is a ring morphism which commutes with σ and δ.*

Example 10.3.

1. *$\mathbb{C}(x)$ endowed with $\sigma(f(x)) = f(x+1), \delta = \frac{d}{dx}$ is a $\sigma\delta$-field.*
2. *Let $q \in \mathbb{C} \setminus \{0,1\}$. We endow $\mathbb{C}(x)$ with a structure of $\sigma\delta$-field via $\sigma(f(x)) = f(qx)$, $\delta = x\frac{d}{dx}$. This $\sigma\delta$-field corresponds to a parametric counterpart of the Galois theory of q-difference equations, as developed by Jacques Sauloy (see for instance, Sauloy's notes or [Sau04]).*
3. *Let $\mathbb{C}(x,t)$ be a field of rational bivariate functions endowed with $\sigma(f(x,t)) = f(x+1,t)$ and $\delta = \frac{\partial}{\partial t}$. This $\sigma\delta$-field is suitable to study linear difference equations with a continuous parameter t.*
4. *Let $ln(x)$ be a principal determination of the logarithm and let p be an integer greater than 1. We endow the field $\mathbb{C}(x, ln(x))$ with a structure of $\sigma\delta$-field via $\sigma(f(x, ln(x))) := f(x^p, pln(x))$ and $\delta := xln(x)\frac{d}{dx}$. This $\sigma\delta$-field was used in [DHR15] to study the hypertranscendence of solutions of Mahler difference equations.*

One should warn the reader that the word " parameter" might be tricky. In the examples above, the parameter designates the variable itself (Examples 10.3.1., 10.3.2. and 10.3.4.) and sometimes it is really an extra indeterminate (Example 10.3.3.). Therefore, one should think to the parameter as an auxiliary operator, precisely the derivation δ.

Remark 10.4. *In a $\sigma\delta$-ring R, one might be interested by the differential constants R^δ as well as by the σ-constants $R^\sigma := \{c \in R | \sigma(c) = c\}$. A priori, both rings differ. It is not difficult to see that since σ and δ commute, the ring R^σ is a δ-ring. This is precisely R^σ, which is predestined to be our "parameter space".*

Example 10.5. *Let $q \in \mathbb{C} \setminus \{0,1\}$ not a root of unity. Let $\sigma_q : \mathcal{M}er(\mathbb{C}^\times) \to \mathcal{M}er(\mathbb{C}^\times), x \mapsto qx$ be the q-difference automorphism of the field of meromorphic functions $\mathcal{M}er(\mathbb{C}^\times)$ over $\mathbb{C}^\times$ in one variable z endowed with the derivation $\delta = x\frac{d}{dx}$. Then, $\mathcal{M}er(\mathbb{C}^\times)^{\sigma_q}$ coincides with the field C_E of elliptic functions w.r.t. the elliptic curve $(E) : \mathbb{C}^\times/q^{\mathbb{Z}}$.*

Exercise 10.6. *For each $\sigma\delta$-field in Example 10.3, compute the fields of δ and σ-constants.*

11 Parametrized Picard-Vessiot rings

For K a $\sigma\delta$-field, we shall consider linear difference systems, that is systems of the form,

$$\sigma(Y) = AY, \; A \in \mathrm{Gl}_n(K). \tag{11.1}$$

Two difference systems $\sigma(Y) = AY$ and $\sigma(Y) = BY$ with $A, B \in \mathrm{Gl}_n(K)$ are equivalent if there exists $P \in \mathrm{Gl}_n(K)$ such that $B = \sigma(P)^{-1}AP$. The matrix P is called a gauge transformation.

We recall that a linear difference equation of order n with coefficients in K is an equation of the form

$$L(y) := \sigma^n(y) + a_{n-1}\sigma^{n-1}(y) + \cdots + a_0 y = 0,$$

with $a_0 \neq 0$. It corresponds to the following difference system of order n

$$\sigma(Y) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 0 & 1 \\ -a_0 & -a_1 & \dots & \dots & -a_{n-1} \end{pmatrix} Y = A_L Y.$$

[5] All fields considered in this paper are of characteristic 0.

The matrix A_L is called the companion matrix of the difference equation L. Conversely, under the assumption that K contains a non-periodic element, any difference system $\sigma(Y) = AY$ is equivalent to a system of the form $\sigma(Y) = A_L Y$ with L a linear difference operator over K (see [HS99, Appendix B]).

Example 11.1. *1. For $K = \mathbb{C}(x)$ as in Example 10.3.1), a solution of the difference equation $\sigma(y) = xy$ is the Gamma function $\Gamma(x) := \frac{e^{-\gamma x}}{x} \prod_{n=1}^{+\infty}(1 + \frac{x}{n})^{-1} e^{x/n}$.*

2. For $K = \mathbb{C}(x)$ as in Example 10.3.2), the q-hypergeometric function ${}_2\phi_1(a, a, q; x) := \sum_{n=0}^{\infty} \left(\frac{(1-a)(1-aq)\ldots(1-aq^{n-1})}{(1-q)\ldots(1-q^n)} \right)^2 x^n$ satisfies the q-difference equation

$$\sigma^2(y) - \frac{2ax - 2}{a^2x - 1}\sigma(y) + \frac{x - 1}{a^2x - 1} y = 0$$

where $a \notin q^{\mathbb{Z}}$ and $a^2 \in q^{\mathbb{Z}}$. In a matricial form, this gives

$$\sigma(Y) = \begin{pmatrix} \frac{2ax-2}{a^2x-1} & -\frac{x-1}{a^2x-1} \\ 1 & 0 \end{pmatrix} Y$$

(for more details see [Roq08]).

3. Let $q \in \mathbb{C} \setminus \{0, 1\}$. Let $K = \mathbb{C}(x, t)$ be endowed with $\sigma(f(x, t)) = f(qx, t)$ and $\delta = \frac{\partial}{\partial t}$. In [AR13], the authors give algorithms to find for which values of t, certain q-difference equations with coefficients in K have a rational solution. They consider for instance the following equation

$$(xq + q^2 + t)\sigma(y) + (-x - q^2 - t)y = 0.$$

11.1 Construction of parametrized Picard-Vessiot ring and first properties

Starting with a $\sigma\delta$-field K and a linear difference system $\sigma(Y) = AY$ with $A \in \mathrm{Gl}_n(K)$, we want to solve the system and derive the solutions with respect to δ in a consistent way, that is, such that the identity $\sigma \circ \delta = \delta \circ \sigma$ still holds on the solution space.

Classical Picard-Vessiot theory (see [vdPS97] or Singer's notes) attaches to any linear difference system, a Picard-Vessiot ring, which contains a full set of solutions. We recall here the definition of a Picard-Vessiot ring.

Definition 11.2. *Let K be a σ-field and let $A \in \mathrm{Gl}_n(K)$. A Picard-Vessiot ring, or PV-ring for short, over K for $\sigma(Y) = AY$ is a K-σ-algebra R_0 such that*

1. *R_0 is a simple σ-ring,that is, R_0 has no ideals other than $\{0\}$ and R_0, that are invariant under σ;*
2. *there exists a matrix $Z \in \mathrm{Gl}_n(R_0)$ such that $\sigma(Z) = AZ$;*
3. *$R_0 = K[Z, \frac{1}{det(Z)}]$.*

Picard-Vessiot rings always exist. If one assume moreover that $k := K^{\sigma}$ is an algebraically closed field, a Picard-Vessiot ring R_0 has no new constant, i.e., $R_0^{\sigma} = k$ and two PV-rings attached to the same system are K-σ-isomorphic.

Our first goal is to understand how one can derive the fundamental solution matrix $Z \in \mathrm{Gl}_n(R_0)$ as above. In general, this is not completely obvious.

Exercise 11.3. *Let $K = \mathbb{C}(x)$ endowed with $\sigma(f(x)) = f(x + 1)$ and $\delta = \frac{d}{dx}$. We consider the difference equation*

$$\sigma(y) = \lambda y \tag{11.2}$$

for $\lambda \in \mathbb{C}^*$. We want to understand how one could extend δ to a Picard-Vessiot ring R_0 for (11.2) so that $\sigma \circ \delta = \delta \circ \sigma$ still holds on $R_0 = \mathbb{C}(x)[z, \frac{1}{z}]$, where z is a non-zero solution of (11.2).

1. *First of all, we want to show very naively the following fact: z is algebraic over $\mathbb{C}[x]$ if and only if λ is a root of unity.*
   (a) If λ is a root of unity of order m, show that z^m is a σ-constant and prove one direction.

(b) *Assume that z is algebraic over $\mathbb{C}[x]$. Let $P(X) = X^n + a_{n-1}(x)X^{n-1} + \cdots + a_0(x)$ be its minimal annihilating polynomial above $\mathbb{C}(x)$. Since $z \neq 0$ and P is of minimal order we must have $a_0(x) \neq 0$. Use the fact that $\lambda^n P(z) - \sigma(P(z)) = 0$ and the minimality of P to deduce that $(\lambda^n a_0(x) - a_0(x+1) = 0$. Conclude from this last equation that $\lambda^n = 1$.*

2. *Assume that $\lambda^n = 1$ and show that the only suitable derivation δ on R_0 satisfies $\delta(z) = 0$ (hint use the fact that $z^n \in \mathbb{C}$).*

3. *Assume that λ is not a root of unity. What should be the relation between $\delta(z)$ and z?*

One can show that the Picard-Vessiot ring of the equation $\sigma(y) = -y$ is of the form $\mathbb{C}(x)[z-1] \oplus \mathbb{C}(x)[z+1]$ where $z^2 = 1$ and $\sigma(z) = -z$. How could we define $\delta(z)$?

The first natural idea to understand how to derive a fundamental solution matrix, is, as often in mathematics, to start from the conclusion. Assume that Z is a fundamental solution matrix of $\sigma(Y) = AY$, that one can derive infinitely many times. Then, by using the commutativity of σ and δ and $\sigma(Z) = AZ$, one finds that $\sigma(\delta(Z)) = \delta(A)Z + A\delta(Z)$. This means that the matrix $\begin{pmatrix} Z & 0 \\ \delta(Z) & Z \end{pmatrix}$ is a fundamental solution matrix of the new linear difference system $\sigma(Y_1) = A_1 Y_1$ where $A_1 = \begin{pmatrix} A & 0 \\ \delta(A) & A \end{pmatrix} \in \mathrm{Gl}_{2n}(K)$. Repeating this process , one finds that, for all $s \in \mathbb{N}$, the matrix

$$Z_s = \begin{pmatrix} Z & 0 & 0 & \cdots & 0 \\ \delta Z & Z & 0 & \ldots & 0 \\ \delta^2 Z & \delta Z & Z & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \delta^{s-1}Z & \delta^{s-2}Z & \delta^{s-3}Z & \ldots & 0 \\ \delta^s Z & \delta Z^{s-1} & \delta^{s-2}Z & \ldots & Z \end{pmatrix} \tag{11.3}$$

satisfies the linear difference system $\sigma(Y_s) = A_s Y_s$ where

$$A_s = \begin{pmatrix} A & 0 & 0 & \cdots & 0 \\ \binom{s}{1}\delta A & A & 0 & \ldots & 0 \\ \binom{s}{2}\delta^2 A & \binom{s}{1}\delta A & A & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{s}{s-1}\delta^{is-1}A & \binom{s}{s-2}\delta^{s-2}A & \binom{s}{s-3}\delta^{s-3}A & \ldots & 0 \\ \binom{s}{s}\delta^s A & \binom{s}{s-1}\delta A^{s-1} & \binom{s}{s-2}\delta^{s-2}A & \ldots & A \end{pmatrix} \in \mathrm{Gl}_{(s+1)n}(K). \tag{11.4}$$

Thereby, we get a sequence of linear difference systems $\sigma(Y) = A_s Y$ where $A_0 := A$. Then, one could try to compare the usual Picard-Vessiot rings R_s of all these linear difference systems above K. Unfortunately, if we proceed brutally, there is a priori no criteria to guarantee that $R_s \subset R_{s+1}$, which is more or less equivalent to say that a fundamental solution matrix of $\sigma(Y_s) = A_s Y_s$ is of the form Z_s above. We try to explain in §11.2 how, after careful choices, this method can however be very efficient if we run it carefully.

But it is still the combination of this naive idea and of the classical construction of Picard-Vessiot ring, that gives the correct definition and construction of a parametrized Picard-Vessiot ring.

Definition 11.4. *Let K be a $\sigma\delta$-field and let $A \in \mathrm{Gl}_n(K)$. A $\sigma\delta$-Picard-Vessiot ring, or $\sigma\delta$-PV-ring for short, over K for $\sigma(Y) = AY$ is a K-$\sigma\delta$-algebra R such that*

1. *R is a simple $\sigma\delta$-ring, that is, R has no ideals, other than $\{0\}$ and R, that are invariant under σ and δ;*

2. *there exists a matrix $Z \in \mathrm{Gl}_n(R)$ such that $\sigma(Z) = AZ$;*

3. *$R = K\{Z, \frac{1}{det(Z)}\}$.*

One defines also the total $\sigma\delta$-Picard-Vessiot ring as the total ring of fractions $Quot(R)$ [6] of R.

[6] It is the localization of R with respect to the set of non-zero divisors.

Remark 11.5. *We just want to emphasize that a σ-simple $\sigma\delta$-ring is obviously $\sigma\delta$-simple. This remark shall make more sense in §11.2.*

Also, if δ is the trivial derivation, that is $\delta = 0$, the notions of $\sigma\delta$-PV ring and PV ring coincide.

Now, we can prove an unconditional existence theorem.

Proposition 11.6. *Let K be a $\sigma\delta$-field and let $A \in \mathrm{Gl}_n(K)$. There exists a $\sigma\delta$-Picard-Vessiot ring over K for $\sigma(Y) = AY$.*

Proof. Let $K\{Y, \frac{1}{det(Y)}\}$ be the ring of δ-polynomials in the matrix $Y = (y_{i,j})_{i,j=1,\dots,n}$ of δ-indeterminates, localized in the determinant. We endow this δ-ring with a structure of K-$\sigma\delta$-algebra as follows

1. σ acts on K via its natural action,
2. $\sigma(\delta^k(Y)) := \sum_{j=0}^{k} \binom{k}{j} \delta^{k-j}(A)\delta^j(Y)$ for all $k \in \mathbb{N}$.

The last equalities come from the generalized Leibnitz rule and the commutativity of σ and δ. It also implies that $\sigma(Y_s) = A_s Y_s$ in the notation of (11.4). In particular, we have $\sigma(Y) = AY$. Now, Zorn Lemma allows us to find a maximal $\sigma\delta$-ideal $\mathfrak{m}$ in $K\{Y, \frac{1}{det(Y)}\}$ (maximal among the $\sigma\delta$-ideals). Finally, the quotient ring $K\{Y, \frac{1}{det(Y)}\}/\mathfrak{m}$ is a $\sigma\delta$-Picard-Vessiot ring for $\sigma(Y) = AY$. □

Remark 11.7. *A maximal $\sigma\delta$-ideal does not need to be a maximal ideal. For instance on $\mathbb{C}[x]$, with $\sigma =$ identity and $\delta = \frac{d}{dx}$, the only $\sigma\delta$-ideal is $\{0\}$.*

Exercise 11.8. *Let R be a $\sigma\delta$-ring.*

1. *Show that the radical of a $\sigma\delta$-ideal is a $\sigma\delta$-ideal.*
2. *Show that a maximal $\sigma\delta$-ideal is radical*

Exercise 11.9. *Let $\lambda \in \mathbb{C}^*$ not a root of unity. Let $K = \mathbb{C}(x)$ endowed with $\delta = \frac{d}{dx}$ and $\sigma(f(x)) = f(x+1)$. We consider the difference equation $\sigma(y) = \lambda y$. We endow the δ-polynomial ring $K\{y, \frac{1}{y}\}$ by letting σ act via $\sigma(y) = \lambda y$, $\sigma(\delta(y)) = \lambda\delta(y)$ etc. Let $c \in \mathbb{C}^*$.*

1. *Show that the δ-ideal $\mathfrak{M}_c := \{\delta(y) - cy\}$ is stable by σ.*
2. *We want to show that the $\sigma\delta$-ideal $\mathfrak{M}_c$ is maximal.*
   (a) *Let $\mathfrak{a}$ be a $\sigma\delta$-ideal containing properly $\mathfrak{M}_c$. Let $A \in \mathfrak{a} \setminus \mathfrak{M}_c$. Since y is invertible in $K\{y, \frac{1}{y}\}$, one can choose A in $K\{y\}$. Show that there exists $C \in K[y]$ such that $A - C \in \mathfrak{M}_c$. This implies that $\mathfrak{a} \cap K[y] \neq \{0\}$.*
   (b) *Choose $D \in \mathfrak{a} \cap K[y]$ non zero, monic and of smaller degree say n. Since y is invertible, we can assume that $D(0) \neq 0$. Show that $n = 0$ and conclude. (hint: use the fact that $\lambda^n D - \sigma(D) \in \mathfrak{a} \cap K[y]$ and that λ is not a root of unity)*
3. *Conclude that for all $c \in \mathbb{C}^*$, the K-$\sigma\delta$-algebra $R_c := K\{z_c, \frac{1}{z_c}\} := K\{y, \frac{1}{y}\}/\mathfrak{M}_c$ is a $\sigma\delta$-PV ring.*
4. *Show that R_c is a PV-ring and deduce that $R_c^\sigma = \mathbb{C}$.*
5. *Let $c_1, c_2 \in \mathbb{C}^*$ with $c_1 \neq c_2$.*
   (a) *Suppose that there exists a K-$\sigma\delta$-isomorphism $\phi : R_{c_1} \to R_{c_2}$. Show that there exists $\mu \in \mathbb{C}^*$ such that $\phi(z_{c_1}) = \mu z_{c_2}$ and use $\delta \circ \phi = \phi \circ \delta$ to find a contradiction.*
   (b) *Now we extend the σ-constants from $\mathbb{C}$ to some δ-closed field extension k of $\mathbb{C}$. All the constructions above still work. Now, we can choose $\mu \in k$. Find the differential equation that μ should satisfy in order to guaranty the existence of the isomorphism ϕ.*

The structure of a $\sigma\delta$-PV ring is very similar to the structure of a usual PV-ring (see [vdPS97, Corollary 1.16]).

Lemma 11.10 (Lemma 6.8 in [HS08])**.** *Let K be a $\sigma\delta$-field. Let R be a K-$\sigma\delta$-algebra, δ-finitely generated and $\sigma\delta$-simple. Then, there exist $e_0, \dots, e_{t-1} \in R$ such that*

1. $e_0 + \cdots + e_{t-1} = 1, e_i^2 = e_i, e_i e_j = 0$ *for* $i \neq j$,

2. $R = R_0 \oplus \ldots \oplus R_{t-1}$ *with* $R_i = e_i R$ *and* $\sigma(R_i) = R_{i+1}$ *modulo* t,

3. R_i *is an integral* δ*-ring and a* $\sigma^t\delta$*-simple ring.*

Proof. Since R is $\sigma\delta$-simple and the radical of a $\sigma\delta$-ideal is a $\sigma\delta$-ideal, the zero ideal is radical and R is reduced. Since R is K-δ-finitely generated, Corollary 3.26 implies that $\{0\} = \cap_i \mathfrak{p}_i$ where the $\mathfrak{p}_i$'s are prime δ-ideals of R, none containing another. This representation is unique up to permutation and one can show that, after a possible renumbering, $\sigma(\mathfrak{p}_i) = \mathfrak{p}_{i+1}$ modulo t. This proves that $\mathfrak{p}_i$ is a $\sigma^t\delta$-ideal and that the ring $R/\mathfrak{p}_i$ is an integral $\sigma^t\delta$-ring. One can then show that, for all $i = 0, \ldots, t-1$, the ring R has no proper $\sigma^t\delta$-ideals properly containing $\mathfrak{p}_i$ (see [HS08, lemma 6.8]). This implies that $R/\mathfrak{p}_i$ is $\sigma^t\delta$-simple and that the $\mathfrak{p}_i$'s are pairwise co-maximals, i.e., $\mathfrak{p}_i + \mathfrak{p}_j = R$ if $i \neq j$. The Chinese remainder Theorem implies that $\pi : R \simeq \oplus_{i=0}^{t-1} R/\mathfrak{p}_i, r \mapsto (r_0, \ldots, r_{t-1})$ where r_i denotes the class of r in $R/\mathfrak{p}_i$. One concludes the proof by setting $e_i = \pi^{-1}(1_{R_{\mathfrak{p}_i}})$ and $R_i := \pi^{-1}(R/\mathfrak{p}_i)$. ☐

Applied to $\sigma\delta$-PV ring, the Lemma above yields to the following structural description.

Corollary 11.11. *Let K be a $\sigma\delta$-field and let $A \in \mathrm{Gl}_n(K)$. Let R be a $\sigma\delta$-PV ring for $\sigma(Y) = AY$ and let L be its total ring of fractions. Then,*

- *$L = L_0 \oplus \cdots \oplus L_{t-1}$ where L_i are $\sigma^t\delta$-fields, all isomorphic as δ-fields. In particular, L has no nilpotent element and every non-zero divisor in L is invertible.*

- *If S is a σ-subring of L, then any $s \in S$ that is a zero divisor in L, is a zero divisor in S. In particular, we can embed the total ring of fractions of S in L.*

Proof.

- By definition, R fulfils the hypothesis of Lemma 11.10. In the notation of Lemma 11.10, let us denote by L_i the fraction field of R_i. It is then almost immediate that these last fields are δ-isomorphic and that, in a product of fields, one has no nilpotent element and every non-zero divisor is invertible.

- See [HS08, Corollary 6.9]

☐

Definition 11.12. *In the notation of Corollary 11.11, we define the δ-transcendence degree, $\delta\text{-}\mathrm{trdeg}(L|K)$ (resp. $\delta\text{-}\mathrm{trdeg}(R|K)$) of L (resp. R) over K as the common value of the differential transcendence degrees of the L_i's over K.*

Example 11.13. *Let us come back to some of the situations studied in Exercises 11.3 and 11.9. Let $K = \mathbb{C}(x)$ endowed with $\sigma(f(x)) = f(x+1)$ and $\delta = \frac{d}{dx}$.*

- *We would like to construct a $\sigma\delta$-PV ring for $\sigma(y) = -y$. We proceed as in Proposition 11.6. We endow $\mathbb{C}(x)\{y, \frac{1}{y}\}$ with the σ-structure given by $\sigma(\delta^s(y)) = -\delta^s(y)$ for all $s \in \mathbb{N}$. Let $\mathfrak{m}$ be the radical δ-ideal generated by $y^2 - 1$ in $\mathbb{C}(x)\{y, \frac{1}{y}\}$. It is a $\sigma\delta$-ideal and the only δ-ideals that contain $\mathfrak{m}$ properly are $\mathfrak{p}_1 := [y-1]$ and $\mathfrak{p}_2 := [y+1]$, which are both prime δ-ideals. Since none of them is stable by σ, the $\sigma\delta$-ideal $\mathfrak{m}$ is maximal. Thus, $R := \mathbb{C}(x)\{z, \frac{1}{z}\} := \mathbb{C}(x)\{y, \frac{1}{y}\}/\mathfrak{m}$ is a $\sigma\delta$-PV ring. One finds also that $R \simeq \mathbb{C}(x)\{y, \frac{1}{y}\}/\mathfrak{p}_1 \oplus \mathbb{C}(x)\{y, \frac{1}{y}\}/\mathfrak{p}_2$. Now, let z denote the image of y in R. Since z is invertible in $\mathbb{C}(x)\{z, \frac{1}{z}\}$, one finds that $\delta(z) = \frac{\delta(z^2-1)}{2z}$ belongs to $\mathfrak{m}$. This implies that $\delta(z) = 0$ and that $R := \mathbb{C}(x)\{z, \frac{1}{z}\}$ coincides with the usual Picard-Vessiot ring $\mathbb{C}(x)[z, \frac{1}{z}]$. By [vdPS97, Lemma 1.8], the σ-constants of R are exactly $\mathbb{C}$. However $R^\delta = \mathbb{C}[z, \frac{1}{z}]$*

- *Let $q \in \mathbb{C} \setminus \{0, 1\}$ not a root of unity. Let $\mathbb{C}(x)$ be endowed with a structure of $\sigma\delta$-field via $\sigma(f(x)) = f(qx)$ and $\delta = x\frac{d}{dx}$. Let $\mathcal{M}er(\mathbb{C}^*)$ be the field of meromorphic functions over $\mathbb{C}^*$ and let C_E be the field of elliptic functions, that is $C_E = \mathcal{M}er(\mathbb{C}^*)^\sigma$. In Exercise 4.7, we have shown that the Jacobi Theta function θ_q, that satisfies $\sigma(\theta_q) = qx\theta_q$, verifies also $\delta(\frac{\delta(\theta_q)}{\theta_q}) \in C_E$. One can show by looking at the poles of θ_q that $\delta(\frac{\delta(\theta_q)}{\theta_q})$ is not a constant complex function. This proves that the σ-constants of the $\mathbb{C}(x)$-$\sigma\delta$-algebra $\mathbb{C}(x)\{\theta_q, \frac{1}{\theta_q}\} \subset \mathcal{M}er(\mathbb{C}^*)$ strictly contains $\mathbb{C}$.*

To guarantee the non-increase of the σ-constants when passing to the Picard-Vessiot ring, we have to assume that the σ-constants K^σ of the base field K form an algebraically closed field (see [vdPS97, Lemma 1.8]). The proof relies on the extension of algebraic specializations. The differential counterpart of this result requires that K^σ is a δ-closed field and relies on the extension of differential specialization (see Theorem 4.20). Then, we have the following statement.

Proposition 11.14. *[Proposition 6.14 in [HS08]] Let K be a $\sigma\delta$-field such that $k = K^\sigma$ is δ-closed. Let R be a K-$\sigma\delta$-algebra, δ-finitely generated over K and $\sigma\delta$-simple. Then, $R^\sigma = K^\sigma = k$.*

We deduce the Corollary below.

Corollary 11.15. *Let K be a $\sigma\delta$-field such that $k = K^\sigma$ is δ-closed. Let $A \in \mathrm{Gl}_n(K)$ and let R be a $\sigma\delta$-PV ring for $\sigma(Y) = AY$ and let L be its total ring of fractions. Then,*

- *Two fundamental solution matrices $U, V \in \mathrm{Gl}_n(R)$ differ by a constant matrix, that is, there exists $C \in \mathrm{Gl}_n(k)$ such that $U = VC$.*
- *$L^\sigma = k$.*

Proof.

- By Proposition 11.14, we have $R^\sigma = k$. Since $\sigma(UV^{-1}) = UV^{-1}$, there exists $C \in \mathrm{Gl}_n(k)$ such that $UV^{-1} = C$.
- Let $c = \frac{a}{b} \in L^\sigma$ where $a, b \in R$ and b is not a zero divisor. We will show that $R\{c\}$ is a simple $\sigma\delta$-ring, since, by Proposition 11.14, this will imply that $c \in k$. Let $\mathfrak{J}$ be a nonzero $\sigma\delta$-ideal of $R\{c\}$. We claim that $\mathfrak{J} \cap R$ contains a non-zero element. Assuming that this is the case. Since R is $\sigma\delta$-simple, we get $\mathfrak{J} \cap R = R$. Thus, $\mathfrak{J} = R\{c\}$. To prove the claim let $0 \neq u \in \mathfrak{J}$. We can write u as a polynomial in the $\delta^i(c)$'s with coefficients in R. Using $\delta(\frac{a}{b}) = \frac{\delta(a)b - a\delta(b)}{b^2}$, we see that for each i, there is a positive integer n_i such that $b^{n_i}\delta^i(c) \in R$. Therefore there exists a positive integer n such that $b^n u \in R$. Since b is not a zero divisor, $b^n u$ is a nonzero element of $\mathfrak{J} \cap R$.

$\square$

From the non-increase of the σ-constants, one gets the uniqueness of $\sigma\delta$-PV ring.

Corollary 11.16. *Let K be a $\sigma\delta$-field such that $k = K^\sigma$ is δ-closed. Let $A \in \mathrm{Gl}_n(K)$ and let R_1, R_2 be two $\sigma\delta$-PV rings for $\sigma(Y) = AY$. Then, R_1 and R_2 are isomorphic as K-$\sigma\delta$-algebras.*

Proof. Choose a maximal $\sigma\delta$-ideal $\mathfrak{m}$ in $R_1 \otimes_K R_2$ and set $R_3 := R_1 \otimes_K R_2/\mathfrak{m}$. By $\sigma\delta$-simplicity of R_1 (resp R_2) the morphism $\phi_1 : R_1 \to R_3$ (resp. $\phi_2 : R_2 \to R_3$), which maps r_1(resp r_2) on the class of $r_1 \otimes 1$ (resp. $1 \otimes r_2$) is injective. Moreover, for $i = 1, 2$, the image of ϕ_i is K-δ-generated by $B_i = \phi_i(Z_i)$ and $\phi_i(det(Z_i)^{-1})$ where Z_i stands for a fundamental solution matrix in R_i. Now, B_1 and B_2 are fundamental solution matrices in R_3 for the same equation. Since R_3 satisfies the hypothesis of Proposition 11.14, there exists $C \in \mathrm{Gl}_n(k)$ such that $B_1 = B_2 C$. This implies $\phi_1(R_1) = \phi_2(R_2)$. Then, $\phi_2^{-1} \circ \phi_1$ is the required isomorphism. $\square$

Remark 11.17. *In Exercise 11.9, we illustrate how the hypothesis of δ-closure of k is crucial in order to find isomorphic $\sigma\delta$-PV rings.*

11.2 On the field of σ-constants

To assume that the field k of σ-constants of the base field K is δ-closed is a rather restrictive hypothesis. Since δ-closed fields are gigantic fields, they barely appear in natural examples. The strategy used in [HS08], was first to get results on δ-closed field and then to use some methods of descents to reach algebraically closed fields of constants. Recently, many works, from a Tannakian point of view ([GGO13]), from a model theoretic approach ([Kam]) as well as from a direct Picard-Vessiot construction ([DVH12] or [Wib12]), have proved that, if one assumes k to be algebraically closed, one can construct a $\sigma\delta$-PV ring R such that $R^\sigma = k$.

We would like to discuss a little bit the Picard-Vessiot approach. Given $A \in \mathrm{Gl}_n(K)$, one considers the sequence of linear systems $\sigma(Y) = A_s Y$, obtained by deriving repeatedly the initial system (see Equation 11.4 for the precise form of A_s in terms of derivatives of $A = A_0$). Using differential kernels, one can secure by induction on $s \in \mathbb{N}$ a usual PV-ring R_s for the system $\sigma(Y) = A_s Y$ such that $R_s \subset R_{s+1}$ and $\delta(R_s) \subset R_{s+1}$. In terms of solutions, one can, by choosing carefully the fundamental matrices Z_s of $\sigma(Y) = A_s Y$, construct a derivation δ on $\cup_{s\in\mathbb{N}} R_s$ inductively so that for instance Z_1 is of the form $\begin{pmatrix} Z_0 & 0 \\ \delta(Z_0) & Z_0 \end{pmatrix}$. Then, the direct limit $R := \cup_{s\in\mathbb{N}} R_s = K\{Z_0, \frac{1}{det(Z_0)}\}$ is a σ-simple $\sigma\delta$-PV ring for $\sigma(Y) = AY$. Indeed, since any R_s is σ-simple by definition, the same holds for R. Remark 11.5 becomes much more consistent now. Also, the gain with this limit construction is that one can use classical results. For instance, if k is algebraically closed, Picard-Vessiot rings over K have no new constants, i.e., $R_s^\sigma = k$ for all $s \in \mathbb{N}$ ([vdPS97, Lemma 1.8]). This trivially implies that $R^\sigma = k$.

The moral of this construction is threefold. First, one can see in a certain sense, the parametrized Picard-Vessiot theory as a limit process of the classical Picard-Vessiot theory. But also, hidden in this process is the idea that the only things that matters is not to increase the σ-constants. This is the definition adopted by Amano and Masuoka in their Hopf algebraic approach of Picard-Vessiot theory ([AM05]). Finally, one can with a functorial and schematic approach rewrite all the parametrized Picard-Vessiot theory of [HS08] with the only assumption that the field of σ-constants is algebraically closed, which is not a restrictive assumption at all (see [DVH12] or [Wib12]). However, one has to take care that the uniqueness of Picard-Vessiot ring is valid only after a δ-closed field extension of k (see Exercice 11.9 for a counterexample).

The construction detailed above is somehow very abstract. In many concrete situations, one can appeal to analysis to secure this non-increase of the σ-constants. We detail below two of these situations:

Example 11.18.

- *$\mathbb{C}(x)$ endowed with $\sigma(f(x)) = f(x+1), \delta(x) = \frac{d}{dx}$. In [Pra86], it is proved that, given a difference system $\sigma(Y) = Y(x+1) = A(x)Y(x)$ with $A \in \mathrm{Gl}_n(\mathbb{C}(x))$, one can find a fundamental solution matrix $Z \in \mathrm{Gl}_n(\mathcal{M}er(\mathbb{C}))$, where $\mathcal{M}er(\mathbb{C})$ is the field of meromorphic functions over $\mathbb{C}$ in the variable x. One can extend σ and δ to $\mathcal{M}er(\mathbb{C})$. Let $C_1 := \mathcal{M}er(\mathbb{C})^\sigma$ be the field of 1-periodic functions. Then, $R := C_1(x)\{Z, \frac{1}{det(Z)}\} \subset \mathcal{M}er(\mathbb{C})$ is a $C_1(x)$-$\sigma\delta$-algebra and satisfies $R^\sigma = C_1$.*
- *Let $q \in \mathbb{C}\setminus\{0,1\}$ not a root of unity. We endow $\mathbb{C}(x)$ with a structure of $\sigma\delta$-field via $\sigma(f(x)) = f(qx)$, $\delta = x\frac{d}{dx}$. In [Pra86], it is proved that, given a q-difference system $Y(qx) = A(x)Y(x)$ with $A \in \mathrm{Gl}_n(\mathbb{C}(x))$, one can find a fundamental solution matrix $Z \in \mathrm{Gl}_n(\mathcal{M}er(\mathbb{C}^\times))$, where $\mathcal{M}er(\mathbb{C}^\times)$ is the field of meromorphic functions over $\mathbb{C}$ in the variable x. Let $C_E = \mathcal{M}er(\mathbb{C}^*)^\sigma$ be the field of elliptic functions. One can extend σ and δ to $\mathcal{M}er(\mathbb{C}^\times)$. Then, $R' := C_E(x)\{Z, \frac{1}{det(Z)}\} \subset \mathcal{M}er(\mathbb{C}^\times)$ is a $C_E(x)$-$\sigma\delta$-algebra and satisfies $R'^\sigma = C_E$.*

Exercise 11.19. *Let K be a $\sigma\delta$-field such that $k = K^\sigma$ is algebraically closed. Let $A \in \mathrm{Gl}_n(K)$. We want to show that if a $\sigma\delta$-ring L satisfies the following conditions*

1. *L has no nilpotent elements and every non-zero divisor in L is invertible;*
2. *$L^\sigma = k$,*
3. *there exists $Z \in \mathrm{Gl}_n(L)$ solution of $\sigma(Y) = AY$ such that $L = Quot(K\{Z, \frac{1}{det(Z)}\})$;*

then $R := K\{Z, \frac{1}{det(Z)}\}$ is σ-simple and therefore a $\sigma\delta$-PV ring for $\sigma(Y) = AY$.
We will assume the algebraic counterpart of this statement, i.e.,

Proposition 11.20 (Proposition 1.23 in [vdPS97])**.** *Let K be a σ-field such that $k = K^\sigma$ is algebraically closed. Let $B \in \mathrm{Gl}_n(K)$. If a σ-ring F satisfies the following conditions*

1. *F has no nilpotent elements and every non-zero divisor in F is invertible;*
2. *$F^\sigma = k$,*
3. *there exists $U \in \mathrm{Gl}_n(F)$ solution of $\sigma(Y) = BY$ such that $F = Quot(K[U, \frac{1}{det(U)}])$,*

then $K[U, \frac{1}{det(U)}]$ is σ-simple and therefore a PV ring.

1. *For all $s \in \mathbb{N}$, let $R_s := K[Z, \ldots, \delta^s(Z), \frac{1}{det(Z)}]$ and let $L_s = Quot(R_s)$. Use Corollary 11.11 to show that $L_s \subset L$ and Proposition 11.20 to show that R_s is a PV-ring of $\sigma(Y_s) = A_s Y_s$ where A_s is as in (11.4).*
2. *Use the fact that $R = \cup_{s\in\mathbb{N}} R_s$ to conclude that R is σ-simple and thus a $\sigma\delta$-PV-ring for $\sigma(Y) = AY$.*

12 The parametrized Galois group

In this section, we introduce the parametrized Galois group. We show that this group has a structure of differential algebraic group and state the parametrized Galois correspondence.

12.1 A differential algebraic group

Definition 12.1. *Let K be a $\sigma\delta$-field such that $k = K^{\sigma}$ is δ-closed. Let $A \in \mathrm{Gl}_n(K)$ and let R be a $\sigma\delta$-PV ring for $\sigma(Y) = AY$. The δ-Galois group $\delta\text{-}\mathrm{Gal}(R|K)$ of R is the set of $\sigma\delta$-automorphisms ϕ of R that are the identity on K.*

Remark 12.2. *Let L be the total ring of fractions of R as above. We let $\delta\text{-}\mathrm{Gal}(R|K)$ acts on L as follows. Let $c = \frac{a}{b} \in L$ with $a, b \in R$ and b a non-zero divisor. For $\phi \in \delta\text{-}\mathrm{Gal}(R|K)$, we define $\phi(c) := \frac{\phi(a)}{\phi(b)}$. One can show that this action is well defined and that ϕ is a K-$\sigma\delta$-morphism of L.*

Let $R = K\{Z, \frac{1}{\det(Z)}\}$ be as above with $Z \in \mathrm{Gl}_n(R)$ a fundamental solution matrix. If $\phi \in \delta\text{-}\mathrm{Gal}(R|K)$ then $\phi(Z) \in \mathrm{Gl}_n(R)$ also satisfies $\sigma(Y) = AY$. By Corollary 11.15, there exists a matrix $[\phi]_Z \in \mathrm{Gl}_n(k)$ such that $\phi(Z) = Z[\phi]_Z$. The map $\delta\text{-}\mathrm{Gal}(R|K) \to \mathrm{Gl}_n(k), \phi \mapsto [\phi]_Z$ is a group homomorphism. The following proposition shows that this morphism identifies $\delta\text{-}\mathrm{Gal}(R|K)$ with a k-δ-subgroup of $\mathrm{Gl}_n(k)$

Proposition 12.3. *[Proposition 6.18 in [HS08]] Let K be a $\sigma\delta$-field and assume that $k = K^{\sigma}$ is a δ-closed field. Let $A \in \mathrm{Gl}_n(K)$ and let $R = K\{Z, \frac{1}{\det Z}\}$ be a $\sigma\delta$-PV ring of $\sigma(Y) = AY$ over K. The group morphism*

$$\iota_Z : \delta\text{-}\mathrm{Gal}(R|K) \to \mathrm{Gl}_n(k), \phi \mapsto [\phi]_Z$$

identifies $\delta\text{-}\mathrm{Gal}(R|K)$ with a k-δ-subgroup of $\mathrm{Gl}_n(k)$

Proof. As in the lecture of M.F. Singer, we shall restrict ourselves to $n = 2$ and produce a differential analogue of Kovacic's proof ([Kov86]).

First of all, the morphism ι_Z is injective since the only K-$\sigma\delta$-morphism ϕ of $R = K\{Z, \frac{1}{\det Z}\}$ such that $\phi(Z) = Z$ is the identity.

Now, let $Z = \begin{pmatrix} z_{1,1} & z_{1,2} \\ z_{2,1} & z_{2,2} \end{pmatrix}$ be a fundamental solution matrix and let $R = K\{z_{1,1}, z_{1,2}, z_{2,1}, z_{2,2}, \frac{1}{det(Z)}\}$. We can write $R = K\{y_{1,1}, y_{1,2}, y_{2,1}, y_{2,2}, \frac{1}{det(Y)}\}/\mathfrak{I}$ where the $y_{i,j}$ are δ-indeterminates over K and $\mathfrak{I}$ is a radical δ-ideal. We denote by Y the matrix $\begin{pmatrix} y_{1,1} & y_{1,2} \\ y_{2,1} & y_{2,2} \end{pmatrix}$. We let

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Gl}_n(k) \tag{12.1}$$

acts on $K\{y_{1,1}, y_{1,2}, y_{2,1}, y_{2,2}, \frac{1}{det(Y)}\}$ as follows. First, M acts on the $y_{i,j}$ via

$$Y \mapsto Y.M$$

and on the $\delta^k(y_{i,j})$ by deriving successively the last equation, that is,

$$\delta^k(Y) \mapsto \sum_{j=0}^{k} \binom{k}{j} \delta^{k-j}(Y)\delta^j(M).$$

For instance, $y_{1,1}$ is sent on $ay_{1,1} + cy_{1,2}$ and $\delta(y_{1,1})$ maps to $\delta(a)y_{1,1} + \delta(c)y_{1,2} + a\delta(y_{1,1}) + c\delta(y_{1,2})$. Then, we extend the action of M to the whole $K\{y_{1,1}, y_{1,2}, y_{2,1}, y_{2,2}, \frac{1}{det(Y)}\}$ in order to get a K-δ-algebra morphism. This action commutes trivially with σ since M is a matrix with σ-constants coefficients.

Now, the matrix M is in the image of $\delta\text{-}\mathrm{Gal}(R|K)$ by ι if and only if its action on $K\{y_{1,1}, y_{1,2}, y_{2,1}, y_{2,2}, \frac{1}{det(Y)}\}$ takes $\mathfrak{I}$ to itself. Since $\mathfrak{I}$ is a radical δ-ideal, Corollary 3.25 shows that $\mathfrak{I} = \{q_1, \dots, q_r\}$ where q_i is an element of $K\{y_{1,1}, y_{1,2}, y_{2,1}, y_{2,2}, \frac{1}{det(Y)}\}$. Let m denote the maximum of the orders of derivation of the $y_{i,j}$ in the q_i's, let s be the maximum of the degree in the variables $\delta^k(y_{i,j})$ for $i, j = 1, 2$ and $k = 0, \dots, m$ and in $\frac{1}{det(Y)}$. Let $W \subset K\{y_{1,1}, y_{1,2}, y_{2,1}, y_{2,2}, \frac{1}{det(Y)}\}$ be the finite dimensional K-vector space of polynomials in the variables $\delta^k(y_{i,j})$ for $i, j = 1, 2$ and $k = 0, \dots, m$ and of the degree in $\frac{1}{det(Y)}$ of degree at most s. Let $\{p_i\}_{i \in I}$ be a K-basis of $W \cap \mathfrak{I}$. Extend $\{p_i\}_{i \in I}$ to a K-basis $\{p_j\}_{j \in J}$ of W. For any $M \in \mathrm{Gl}_n(k)$ as in (12.1) and $i \in I$, we can develop

$$p_i(YM) = p_i\left(ay_{1,1} + cy_{1,2}, \dots, by_{2,1} + dy_{2,2}, \dots, \sum_{j=0}^{s} \binom{s}{j} \delta^{s-j}(b)\delta^j(y_{2,1}) + \sum_{j=0}^{s} \binom{s}{j} \delta^{s-j}(d)\delta^j(y_{2,2}), \frac{1}{det(M)}\right)$$

in the basis $\{p_j\}_{j\in J}$ as follows

$$p_i(YM) = \sum_{j\in J} P_{i,j}(a,b,c,d,\frac{1}{det(M)})p_j, \tag{12.2}$$

where the $P_{i,j}$ are δ-polynomials with coefficients in K of order less than or equal to s. Therefore, $M \in \mathrm{Gl}_n(k)$ stabilizes $\mathfrak{I}$ if and only if $P_{i,j}(a,b,c,d,\frac{1}{det(M)}) = 0$ for $i \in I$ and $j \in J \setminus I$. If $\{a_\alpha\}_{\alpha\in A}$ is a k-basis of K, we can write each $P_{i,j} = \sum_{\alpha\in A} P_{i,j,\alpha}a_\alpha$ for finitely many non-zero δ-polynomials $P_{i,j,\alpha} \in k\{Y, \frac{1}{det(Y)}\}$. Therefore, the set of zeroes of the δ-polynomials $\{P_{i,j,\alpha} | i \in I, j \in J\setminus I, \alpha \in A\}$ coincides with the image of $\delta\text{-}\mathrm{Gal}(R|K)$ in $\mathrm{Gl}_n(K)$. □

Remark 12.4. *The proposition above shows that $\delta\text{-}\mathrm{Gal}(R|K)$ can be identified through the choice of a fundamental solution matrix to a differential algebraic subgroup of $\mathrm{Gl}_n(K)$. Of course, any different choice will lead to a conjugated δ-subgroup of $\mathrm{Gl}_n(K)$. If $U = ZD$ for some $D \in \mathrm{Gl}_n(k)$ then $[\phi]_U = D^{-1}[\phi]_Z D$.*

One can have a more intrinsic approach based on k-δ-scheme. This approach does not require that k is δ-closed but only algebraically closed. Then, as detailed in §11.2, one can show that there exists a $\sigma\delta$-PV ring R with $R^\sigma = k$. Using a functorial approach, one is able to show that the parametrized Galois group scheme is represented by the δ-ring $(R \otimes_K R)^\sigma$ (see [Wib12] for instance).

Example 12.5. *Let $K = \tilde{\mathbb{C}}(x)$, where $\tilde{\mathbb{C}}$ is a δ-closed field, endowed with $\sigma(f(x)) = f(x+1)$ and $\delta = \frac{d}{dx}$. Let us consider the linear difference system $\sigma(y) = xy$ and let $R = K\{z, \frac{1}{z}\}$ be a $\sigma\delta$-PV ring for this equation. Then, $\delta\text{-}\mathrm{Gal}(R|K) \rightarrow \mathbf{G}_m, \phi \mapsto [\phi]_Z$ such that $\phi(z) = z[\phi]_Z$ identifies $\delta\text{-}\mathrm{Gal}(R|K)$ with a δ-subgroup of the multiplicative group $\mathbf{G}_m$.*

Exercise 12.6. *Let $K = \tilde{\mathbb{C}}(x)$, where $\tilde{\mathbb{C}}$ is a δ-closed field, endowed with $\sigma(f(x)) = f(qx)$ and $\delta = x\frac{d}{dx}$ for some $q \in \tilde{\mathbb{C}} \setminus \{0,1\}$. Let us consider the linear difference system $\sigma(y) = qxy$ and let $R = K\{z, \frac{1}{z}\}$ be a $\sigma\delta$-PV ring for this equation, where z is a non-zero solution.*

1. *Show that $\delta(\frac{\delta(z)}{z})$ is a σ-constant and thus in k.*
2. *Show that $\delta\text{-}\mathrm{Gal}(R|K) \rightarrow \mathbf{G}_m, \phi \mapsto [\phi]_Z$ identifies $\delta\text{-}\mathrm{Gal}(R|K)$ with a k-δ-subgroup of $\mathbb{V}(\delta(\frac{\delta(y)}{y}))$.(hint: derive $\phi(z) = z[\phi]_z$ and use the fact that ϕ is the identity on K and commutes with δ.)*

12.2 The parametrized Galois correspondence

The main ingredient of a Galois correspondence is that the elements fixed by the Galois group are in the base field. This is also the statement that we use mainly in the applications.

We fix the notation for this section. Let K be a $\sigma\delta$-field and let $k = K^\sigma$ be a δ-closed field. Let $A \in \mathrm{Gl}_n(K)$ and let $R = K\{Z, \frac{1}{det(Z)}\}$ be a $\sigma\delta$-PV for $\sigma(Y) = AY$. We denote by L the total ring of fractions of R. For any subset F of L, we denote by $F^{\delta\text{-}\mathrm{Gal}(R|K)}$ the set $\{c \in F | \phi(c) = c \text{ for all } \phi \in \delta\text{-}\mathrm{Gal}(R|K)\}$. In this setting, we have the following result.

Proposition 12.7 (Proposition 2.3.18 in [HS08])**.** *In the notation above, we have $L^{\delta\text{-}\mathrm{Gal}(R|K)} = K$.*

Remark 12.8. *Carrying on the discussions of Remark 12.4, one should mention that Proposition 12.7 still holds with the only assumption that k is algebraically closed. Of course, in the schematic approach, one has to introduce a functorial notion of invariants (see [Dyc] in the differential context, [Wib12] and [DVH12]).*

Relying on Proposition 12.7, we can state the whole parametrized Galois correspondence.

Theorem 12.9. *[Theorem 6.20 in [HS08]] Let K, k, R and L be as above. Let $\delta\text{-}\mathrm{Gal}(R|K)$ be the parametrized Galois group of R. Let*

$$\mathcal{F} := \{F | F \text{ is a } \sigma\delta\text{-ring }, K \subset F \subset L \text{ and every non-zero divisor of } F \text{ is invertible in } F\},$$

and let

$$\mathcal{G} := \{H | H \text{ is a } \delta\text{-subgroup of } \delta\text{-}\mathrm{Gal}(R|K)\}.$$

Then, $\alpha : \mathcal{F} \rightarrow \mathcal{G}$ given by $\alpha(F) := \delta\text{-}\mathrm{Gal}(L|F) := \{\phi \in \delta\text{-}\mathrm{Gal}(R|K) \ | \phi(u) = u \text{ for all } u \in F\}$ is a reversing inclusion bijective correspondence. The map $\beta : \mathcal{G} \rightarrow \mathcal{F}$ given by $\beta(H) := L^H := \{u \in L \mid \phi(u) = u \text{ for all } \phi \in H\}$ is the inverse of α.

12.3 A differential algebraic torsor

Another formulation of the parametrized Galois correspondence is given by the notion of differential algebraic torsor (see §9.3). When we appeal to this notion, we identify, via the choice of a fundamental solution matrix, the parametrized Galois group with a k-δ-closed subgroup of $\mathrm{Gl}_n(k)$ (see Proposition 12.3).

Proposition 12.10. *[Proposition 6.24 in [HS08]]*
Let K be a $\sigma\delta$-field such that $k = K^\sigma$ is δ-closed. Let R be a $\sigma\delta$-PV ring. Then, R is the K-δ-coordinate ring of a $\delta\text{-}\mathrm{Gal}(R|K)$-torsor. In particular, we have $\delta\text{-}\mathrm{trdeg}(R|K) = \delta\text{-}\dim_k(\delta\text{-}\mathrm{Gal}(R|K)) = \delta\text{-}\mathrm{trdeg}(L|K)$.

Remark 12.11. *From the point of view of the k-δ-coordinate rings, this Theorem reflects the fundamental R-δ-isomorphism,*

$$R \otimes_k (R \otimes R)^\sigma \simeq R \otimes_K R, r \otimes u \mapsto (r \otimes 1)u$$

where $(R \otimes_K R)^\sigma$ is, in fact, the k-δ-coordinate ring of $\delta\text{-}\mathrm{Gal}(R|K)$ (see for instance [Dyc, Lemma 2.4] for a proof in the differential context). From the differential algebraic variety point of view, this isomorphism is nothing more than

$$\delta\text{-}\mathrm{Gal}(R|K) \times X \simeq X \times X, (g, x) \mapsto (x, gx),$$

where X is a δ-variety defined over K such that $K\{X\} = R$.

12.4 Comparison with the usual Galois group

In this section, we compare the classical and the parametrized Galois theory and we show that the parametrized Galois group contains in fact all the informations of the usual Galois theory and even more.

Let K be a $\sigma\delta$-field such that $k = K^\sigma$ is δ-closed. Let $A \in \mathrm{Gl}_n(K)$. Let $R = K\{Z, \frac{1}{det(Z)}\}$ be a $\sigma\delta$-PV ring for $\sigma(Y) = AY$, where Z denotes a fundamental solution matrix. We denote by L the quotient ring of R, by $R_0 := k[Z, \frac{1}{det(Z)}]$ and by L_0 the total ring of fractions of R_0. Then, by Corollary 11.11 and 11.15, we get that $L_0 \subset L$ has no nilpotent elements and any non-zero divisor is invertible and $L_0^\sigma = k$. Then, by [vdPS97, Proposition 1.23][7]for usual Picard-Vessiot ring, $R_0 = K[Z, \frac{1}{det(Z)}]$ is a Picard-Vessiot ring for $\sigma(Y) = AY$ in the classical sense, that is in the sense of Definition 11.2. The usual Galois group $\mathrm{Gal}(R_0|K)$ of R_0 consists in the K-σ-automorphism of R_0 and it can be identified, via its action on the fundamental solution matrix Z, to an algebraic subgroup of $\mathrm{Gl}_n(k)$ (see Singer's notes or [vdPS97, §1.2]).

A very natural question is then the comparison between the usual Galois group $\mathrm{Gal}(R_0|K)$ and the parametrized Galois group $\delta\text{-}\mathrm{Gal}(R|K)$. Obviously, any $\phi \in \delta\text{-}\mathrm{Gal}(R|K)$ stabilizes R_0 and thus defines an element of $\mathrm{Gal}(R_0|K)$. Thus, we have a map $\iota : \delta\text{-}\mathrm{Gal}(R|K) \to \mathrm{Gal}(R_0|K), \phi \mapsto [\phi]|_{R_0}$. This map is injective since $\phi(Z) = Z$ implies that ϕ is the identity on R. We have an even more precise statement which also summarizes the discussion above.

Proposition 12.12. *[Proposition 6.21 in [HS08]] Let K be a $\sigma\delta$-field such that $k = K^\sigma$ is δ-closed. Let $A \in \mathrm{Gl}_n(K)$. Let $R = K\{Z, \frac{1}{det(Z)}\}$ be a $\sigma\delta$-PV ring for $\sigma(Y) = AY$, where Z denotes a fundamental solution matrix. Then,*

- *The σ-ring $R_0 := K[Z, \frac{1}{det(Z)}]$ is a PV-ring for $\sigma(Y) = AY$,*
- *The group $\delta\text{-}\mathrm{Gal}(R|K)$ is a Zariski dense subgroup of* $\mathrm{Gal}(R_0|K)$[8].

Proof. We give an intuitive proof in the case $n = 2$. Following the notation and the lines of the proof of Proposition 12.3, we fix a fundamental solution matrix $Z = \begin{pmatrix} z_{1,1} & z_{1,2} \\ z_{2,1} & z_{2,2} \end{pmatrix}$ for $\sigma(Y) = AY$ and we identify $\delta\text{-}\mathrm{Gal}(R|K)$ and $\mathrm{Gal}(R_0|K)$, via their action on Z, with their image in $\mathrm{Gl}_n(k)$. Let $\mathfrak{I}(\delta\text{-}\mathrm{Gal}(R|K)) \subset k\{Y, \frac{1}{det(Y)}\}$ (resp $\mathfrak{I}(\mathrm{Gal}(R_0|K) \subset k[Y, \frac{1}{det(Y)}])$ be the defining ideal of $\delta\text{-}\mathrm{Gal}(R|K)$ (resp. $\mathrm{Gal}(R_0|K)$) as k-δ-subgroup (resp. algebraic subgroup) of $\mathrm{Gl}_n(k)$. Since $\delta\text{-}\mathrm{Gal}(R|K) \subset \mathrm{Gal}(R_0|K)$, we find that $\mathfrak{I}(\mathrm{Gal}(R_0|K)) \subset \mathfrak{I}(\delta\text{-}\mathrm{Gal}(R|K)) \cap k[Y, \frac{1}{det(Y)}]$. To show that $\delta\text{-}\mathrm{Gal}(R|K)$ is Zariski dense in $\mathrm{Gal}(R_0|K)$ is nothing else than to prove that the last inclusion is an equality (see §8). Now, if one compares the proof of the algebraicity of the usual Galois group in Singer's notes and the proof of Proposition 12.3, one can see that the polynomial elements in $\{P_{i,j,\alpha} | i \in I, j \in J \setminus I, \alpha \in A\}$ coincide exactly with the defining equations of $\mathrm{Gal}(R_0|K)$.

[7] See also Exercise 11.19

[8] We identify the groups with their image in $\mathrm{Gl}_n(k)$ via their action on Z and $\mathrm{Gal}(R_0|K)$ with its corresponding differential algebraic group (see §8).

A precise proof of this statement relies on the classical Galois correspondence (see [vdPS97, §1.3]). If one denotes by L_0 the total ring of fractions of R_0. One can show that L_0 is embedded in L. Since $L_0^{\delta\text{-}\mathrm{Gal}(R|K)} = L^{\delta\text{-}\mathrm{Gal}(R|K)} = K$, the usual Galois correspondence proves that $\delta\text{-}\mathrm{Gal}(R|K)$ is a Zariski dense subgroup of $\mathrm{Gal}(R_0|K)$. $\square$

13 Applications to differential transcendence and Isomonodromic problems

In this section, we show how the parametrized Galois theory developed above combined with classification results on k-δ-group allows to predict the differential behaviour of solutions of linear difference systems. Among all possible differential algebraic relations, we distinguish one, that we call δ-integrability.

13.1 Towards a galoisian treatment of hypertranscendence

In this paragraph, we show how the defining equations of the parametrized Galois group determine the differential algebraic relations satisfied by the solutions of the linear difference system and vice versa. We focus here on linear difference systems, whose parametrized Galois group is a k-δ-subgroup of $\mathbf{G}_m^n$ or $\mathbf{G}_a^n$. In these simple situations, the hypertranscendence results obtained in this section can be deduced from classical results such as the Kolchin or Ostrowski Theorems (see Remark 13.3). Indeed, for the difference systems below, one can study the hypertranscendence of the solutions of the initial system $\sigma(Y) = AY$ in terms of the transcendence of the solutions of the iterated systems $\sigma(Y) = A_sY$ (see (11.4)) for all $s \in \mathbb{N}$. For more complicated difference systems, it might be very difficult to work directly with the systems $\sigma(Y) = A_sY$ because of their size and complexity. In these situations, the parametrized Galois theory might be much more efficient and concise.

However, if we still state the following results and proofs, it is mainly because these simple cases emphasize that the differential algebraic relations between the solutions are the reflection of the defining equations of the parametrized Galois group.

Proposition 13.1. *Let K be a $\sigma\delta$-field such that $k = K^\sigma$ is δ-closed. Let $a_1, \dots, a_n \in K$. Let L be a K-$\sigma\delta$-algebra such that*

- *$L^\sigma = k$,*
- *L has no nilpotents and any non-zero divisor of L is invertible in L,*
- *there exist $z_1, \dots, z_n \in L$ such that*

$$\sigma(z_i) - z_i = a_i \quad \textit{for} \quad i = 1, \dots, n \; .$$

Then $z_1, \dots, z_n$ are δ-algebraically dependent over K if and only if there exist a non-zero homogeneous linear differential polynomial $L(Y_1, \dots, Y_n)$ with coefficients in k and an element $f \in K$ such that

$$L(a_1, \dots, a_n) = \sigma(f) - f.$$

Proof. and that is detailed in the last section of these notes

Assuming there exist such an L and f, we see that $L(z_1, \dots z_n) - f$ is left fixed by σ and so lies in $k = L^\sigma$. This yields a relation of differential dependence over k among the z_i. Now assume that the z_i are δ-algebraically dependent over k. Let $R := K\{z_1, \dots, z_n\} \subset L$. Since $L^\sigma = k$ and L has no nilpotent element and any non-zero divisor of L is invertible in L, Corollary 11.11, shows that the total ring of fractions $Quot(R)$ can be embedded in L. Exercise 11.19 implies that R is a $\sigma\delta$-PV ring for $\sigma(Y) = AY$ where

$$A = \begin{pmatrix} 1 & a_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & a_2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 1 & a_n \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix} \text{ and } Z = \begin{pmatrix} 1 & z_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & z_2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 1 & z_n \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

is a fundamental solution matrix. Let $\phi \in \delta\text{-}\mathrm{Gal}(R|K)$. Then, for all $i = 1, \dots, n$, we have $\sigma(z_i - \phi(z_i)) = z_i - \phi(z_i)$. This means that there exist $c_i \in k$ such that $\phi(z_i) = z_i + c_i$ for all $i = 1, \dots, n$. Now, $\delta\text{-}\mathrm{Gal}(R|K)$ can be identified via its action on Z to a k-δ-subgroup of $\mathbf{G}_a^n$, where $\mathbf{G}_a^n$ is embedded in $\mathrm{Gl}_n(k)$ via the k-δ-group morphism $(c_1, \dots, c_n) \mapsto \begin{pmatrix} 1 & c_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & c_2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 1 & c_n \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}$. By hypothesis, the z_i's are δ-algebraically dependent over K and thus $\delta\text{-}\mathrm{trdeg}(R|K) < n$. By Proposition 12.10, we have $\delta\text{-}\dim_k(\delta\text{-}\mathrm{Gal}(R|K)) = \delta\text{-}\mathrm{trdeg}(R|K) < n$, which means that $\delta\text{-}\mathrm{Gal}(R|K)$ is a proper k-δ-subgroup of $\mathbf{G_a}^n$. By Proposition 9.24, there exist a non-zero homogeneous linear differential polynomial $L(Y_1, \dots, Y_n)$ with coefficients in k such that $\delta\text{-}\mathrm{Gal}(R|K) \subset \{(c_1, \dots, c_n) \in k^n \mid L(c_1, \dots, c_n) = 0\}$.

We claim that $L(z_1, \dots, z_n) = f \in K$. To prove this, it is enough to show that this element is left fixed by $\delta\text{-}\mathrm{Gal}(R|K)$. Let $\phi \in \delta\text{-}\mathrm{Gal}(R|K)$. We have that $\phi(L(z_1, \dots, z_n)) = L(z_1 + c_1, \dots, z_n + c_n) = L(z_1 \dots, z_n)$ so the claim is proved. Finally we have that $L(a_1, \dots, a_n) = L(\sigma(z_1) - z_1, \dots, \sigma(z_n) - z_n)) = \sigma(f) - f$. □

Remark 13.2. *By taking a closer look at the proof, we see that we have a more precise information than the one stated. If the z_i's are δ-algebraically dependent over K, then we know exactly the form of the δ-algebraic relation. It must be a linear differential equation.*

Remark 13.3. *Proposition 13.1 can be deduced from a discrete version of Kolchin's Theorem:" let $L|K$ be a difference field extension of K such that $L^\sigma = k$. Let $d_1, \dots, d_m$ be elements in K and let $z_1, \dots, z_m$ be non-zero elements of L such that $\sigma(z_i) = z_i + d_i$. Then, $z_1, \dots, z_m$ are algebraically dependent over K if and only if there exist $c_1, \dots, c_m \in k$ and $f \in K$ such that $\sum_{i=1}^m c_i z_i = \sigma(f) - f$." Then, we just have to remark that $\sigma(\delta^j(z_i)) - \delta^j(z_i) = \delta^j(a_i)$. In that case, Ostrowski's Theorem says that if $z_1, \dots, z_n, \dots, \delta^m(z_1), \dots, \delta^m(z_n)$ are algebraically dependent over K then there exists $c_{i,j} \in k$ and $f \in K$ such that $\sum_{i=1,j=1}^{n,m} c_{i,j}\delta^j(z_i) = \sigma(f) - f$. This shows that Proposition 13.1 can be obtained without parametrized Galois theory. However, one has to underline that the existence of the K-$\sigma\delta$-algebra L containing the solutions and such that $L^\sigma = k$ is not at all guaranteed by classical Galois theory. We need to appeal to the parametrized theory to prove the existence of such an algebra.*

Of course, a similar result holds for diagonal linear systems. Appealing to the classification of k-δ-subgroups of $\mathbf{G}_m^n$ (see Theorem 9.25), one can obtain the following statement.

Proposition 13.4. *Let K be a $\sigma\delta$-field such that $k = K^\sigma$ is δ-closed. Let $a_1, \dots, a_n \in K^\times$. Let L be a K-$\sigma\delta$-algebra such that*

- *$L^\sigma = k$,*
- *L has no nilpotent and any non-zero divisor of L is invertible in L,*
- *there exist $z_1, \dots, z_n \in L^\times$ such that*

$$\sigma(z_i) = a_i z_i \quad \textit{for} \quad i = 1, \dots, n\ .$$

Then $z_1, \dots, z_n$ are δ-algebraically dependent over K if and only if there exist a non-zero homogeneous linear differential polynomial $L(Y_1, \dots, Y_n)$ with coefficients in k and an element $f \in K^\times$ such that

$$L\left(\frac{\delta(a_1)}{a_1}, \dots, \frac{\delta(a_n)}{a_n}\right) = \frac{\sigma(f)}{f}.$$

Proof. Left as an Exercise. Follow the lines of Proposition 13.1 and use Theorem 9.25. □

Remark 13.5. *Following Remark 12.8, we want to underline that Propositions 13.1 and 13.4 remain true if one replaces the assumption k is a δ-closed field by k is an algebraically closed field.*

Exercise 13.6 (Hypertranscendance of Γ)**.** *Let $K := \tilde{C_1}(x)$ endowed with $\sigma(f(x)) = f(x+1), \delta(x) = \frac{d}{dx}$ where $\tilde{C_1}$ is a differential closure of the field of 1-periodic functions C_1. Let us consider the linear difference system $\sigma(y) = xy$. Let $L := K\langle\Gamma(x)\rangle$ be a K-δ-sub-field generated by the Γ function. One can show that L is a total $\sigma\delta$-PV ring for $\sigma(y) = xy$ over K. The logarithmic derivative of the Gamma function satisfies $\sigma(\frac{\delta(\Gamma)}{\Gamma}) = \frac{\delta(\Gamma)}{\Gamma} + \frac{1}{x}$. Then, Γ is δ-algebraic over $\tilde{C_1}(x)$ if and only if there exist a linear homogeneous operator $L(y) = \sum_{i=0}^{s} a_i\delta^i(y) \in \tilde{C_1}\{y\}$ with $a_s \neq 0$ and $f(x) \in \tilde{C_1}(x)$ such that*

$$L(\frac{1}{x}) = \sum_{i=0}^{s} \frac{(-1)^i a_i i!}{x^{i+1}} = f(x+1) - f(x) \tag{13.1}$$

Now, some descent arguments (see [HS08, Corollary 3.2]) shows that in Equation (13.1), one can replace the field $\tilde{C_1}$ by $\mathbb{C}$. We will show that(13.1) with $L \in \mathbb{C}\{y\}$ and $f \in \mathbb{C}(x)$ can not hold and thereby prove that the Gamma function is δ-transcendental over $\tilde{C_1}(x)$. By partial fraction decomposition, $f(x) = P(x) + \sum_{i,j} \frac{a_{i,j}}{(x-c_i)^j}$ where $P(x) \in \mathbb{C}[x]$ and finitely many $a_{i,j}$ are non-zero complex numbers.

1. *What are the poles of $L(\frac{1}{x})$?*
2. *Show that $P(x) = 0$ (hint: use (13.1) and uniqueness of partial fraction decomposition.)*
3. *Let c_i be a pole of f of order r and let $n = max\{m | c_i - m \text{ is a pole of } f(x)\}$.*
   (a) *Show that $c_i - n - 1$ is a pole of $f(x+1)$*
   (b) *Use the first question and (13.1) to find a contradiction.*

Exercise 13.7. *Let $K := \mathbb{C}(x)$ endowed with $\sigma(f(x)) = f(x+1), \delta(x) = \frac{d}{dx}$. Let $a_1, a_2 \in \mathbb{C}$. Let us consider the system of equations $\sigma(y_1) = (x+a_1)y_1$ and $\sigma(y_2) = (x+a_2)y_2$. Use the ideas of Exercise 13.6 to find a sufficient and necessary condition for $\Gamma(x+a_1)$ and $\Gamma(x+a_2)$ to be δ-algebraically dependent over $\mathbb{C}(x)$.*

Exercise 13.8. *Let $\tilde{C}$ be a δ-closed field and let us endow $\tilde{C}(x)$ with a structure of $\sigma\delta$-field via $\sigma(f(x)) = f(qx)$, $\delta = x\frac{d}{dx}$. Let us consider the linear difference system $\sigma(y) = qxy$. Let $R = K\{z, \frac{1}{z}\}$ be a $\sigma\delta$-PV ring for this last system. In Exercise 12.6, we have proved that $\delta\text{-}\mathrm{Gal}(R|K) \subset \mathbb{V}(\delta(\frac{\delta(y)}{y})) \subset \mathbf{G}_m$. We want to show that we have equality in the first inclusion.*

1. *Prove that the only k-δ-subgroup H of $\mathbf{G}_m$ properly contained in $\mathbb{V}(\delta(\frac{\delta(y)}{y}))$ are $\mathbf{G}_m^\delta = \{c \in (k^\delta)^\times\}$ and its finite and cyclic subgroups (hint: use the logarithmic derivative and show that the image of H is a subgroup of $\mathbf{G}_a^\delta$.)*
2. *Assume that $\delta\text{-}\mathrm{Gal}(R|K) \subset \mathbf{G}_m^\delta$.*
   (a) *Show by the parametrized Galois correspondence that $\frac{\delta(z)}{z} \in K$.*
   (b) *Show that $\sigma(y) = 1 + y$ has no solution in K (hint: use once again partial fraction decomposition)*
   (c) *Conclude.*
3. *Show that the Zariski closure of $\delta\text{-}\mathrm{Gal}(R|K) = \mathbf{G}_m$ and conclude that z is transcendental over K but satisfies a linear differential equation with coefficients in K.*

All these results are valid if we replace $\tilde{C}$ by $\mathbb{C}$ and z by the function θ_q (see Exercise 4.7).

13.2 Integrability and parametrized Galois group

Starting with a linear difference system

$$\sigma(Y) = A(x,t)Y \tag{13.2}$$

with parameter t, one could study how the behaviour of the solutions depend on the parameter t. This is equivalent to asking how the usual Galois group of (13.2) above $\overline{\mathbb{C}(t)}$ varies when we vary t. These kind of questions are highly connected to isomonodromic deformations, as well as integrability problems and Painlevé equations (see for instance [AC91] and [IN86] for introduction to these themes.) The parametrized Galois group aims at controlling this dependence with respect to the parameter t.

In our framework, we introduce the following definition.

Definition 13.9. *Let K be a $\sigma\delta$-field and let $A \in \mathrm{Gl}_n(K)$. We say that the system $\sigma(Y) = AY$ is δ-integrable over K if there exists $B \in K^{n\times n}$ such that*

$$\sigma(B) = ABA^{-1} + \delta(A)A^{-1} \tag{13.3}$$

The condition (13.3) is the compatibility condition of the systems of equations

$$\left\{ \begin{array}{lcl} \sigma(Y) & = & AY \\ \delta(Y) & = & BY \end{array} \right. \tag{13.4}$$

The following proposition interprets the compatibility relation (13.3) in terms of solutions of the system (13.4).

Proposition 13.10. *Let K be a $\sigma\delta$-field, $\sigma(Y) = AY$ be a linear difference equation with $A \in \mathrm{Gl}_n(K)$ and L be a $\sigma\delta$-field extension of K.*

1. *If there exist $B \in \mathrm{Gl}_n(K)$ and $Z \in \mathrm{Gl}_n(L)$ such that $\sigma(Z) = AZ$ and $\delta(Z) = BZ$ (i.e., Z is a fundamental solution of* (13.4)*), then B satisfies* (13.3).
2. *Conversely, assume that L is a $\sigma\delta$-Picard-Vessiot extension for $\sigma(Y) = AY$ and that $k = K^\sigma = L^\sigma$ is linearly δ-closed* [9]*. If there exists a matrix $B \in \mathrm{Gl}_n(K)$ verifying* (13.3)*, then there exists a fundamental solution $Z \in \mathrm{Gl}_n(L)$ of* (13.4).

Proof. For (i) observe that

$$\delta(\sigma(Z)) = \delta(A)Z + A\delta(Z) = \delta(A)Z + ABZ$$

and

$$\sigma(\delta(Z)) = \sigma(BZ) = \sigma(B)AZ.$$

Because $\delta(\sigma(Z)) = \sigma(\delta(Z))$, this implies (13.3).

To prove (ii), fix $Z \in \mathrm{Gl}_n(L)$ with $\sigma(Z) = AZ$. Equation (13.3) implies that

$$\sigma(\delta(Z) - BZ) = \delta(\sigma(Z)) - \sigma(B)AZ = A(\delta(Z) - BZ).$$

We conclude that there exists $C \in \mathrm{Gl}_n(k)$ such that $\delta(Z) - BZ = ZC$. Since k is linearly δ-closed, there exists $D \in \mathrm{Gl}_n(k)$ such that $\delta(D) + CD = 0$. The matrix $ZD \in \mathrm{Gl}_n(L)$ is a fundamental solution of (13.4). □

The system (13.4) is called a semi-discrete Lax pair, as introduced in the papers [AL76] and [AL75]. The Bessel functions J_α, Y_α [10] satisfy such a semi-discrete Lax pair over $\mathbb{C}(\alpha, x)$ endowed with $\sigma(f(\alpha, x)) = f(\alpha+1, x)$ and $\delta := \frac{\partial}{\partial x}$. The matrix

$$Y = \begin{pmatrix} J_\alpha(x) & Y_\alpha(x) \\ \delta(J_\alpha(x)) & \delta(Y_\alpha(x)) \end{pmatrix}$$

is a fundamental solution matrix $\sigma(Y) = BY$, where

$$B = \begin{pmatrix} \frac{\alpha}{x} & -1 \\ \frac{-\alpha(\alpha+1)}{x^2} + 1 & \frac{\alpha+1}{x} \end{pmatrix} \in \mathrm{Gl}_n(K).$$

and $\delta(Y) = AY$ where

$$A = \begin{pmatrix} 0 & 1 \\ \frac{\alpha^2}{x^2} - 1 & \frac{-1}{x} \end{pmatrix}.$$

Now, we can state our galoisian criteria for δ-integrability.

Proposition 13.11. *Let K be a $\sigma\delta$-field such that $k = K^\sigma$ is δ-closed. Let $A \in \mathrm{Gl}_n(K)$ and let $R = K\{Z, \frac{1}{det(Z)}\}$ be a $\sigma\delta$-PV ring. We identify $\delta\text{-}\mathrm{Gal}(R|K)$ with a k-δ-subgroup of $\mathrm{Gl}_n(k)$ via its action on Z. The following statements are equivalent:*

- *There exist $D \in \mathrm{Gl}_n(k)$ such that $D^{-1}\delta\text{-}\mathrm{Gal}(R|K)D \subset \mathrm{Gl}_n(k^\delta) := \{M \in \mathrm{Gl}_n(k) | \delta(M) = 0\}$.*
- *The linear difference system $\sigma(Y) = AY$ is δ-integrable over K.*

[9] This means that every linear differential system with coefficients in k has a fundamental solution matrix in k.

[10] For generalities on Bessel functions we refer the reader to [Wat95].

Proof. Assume that such a B exists. By proposition 13.10, one can secure a fundamental solution matrix $V = Z.D$ where $D \in \mathrm{Gl}_n(k)$ such that $\delta(V) = BV$. We claim that $[\phi]_V \in \mathrm{Gl}_n(k^\delta)$. To see this note that $\delta(\phi(V)) = BV[\phi]_V + V\delta([\phi]_V)$ and $\phi(\delta(V)) = \phi(BV) = BV[\phi]_V$. This implies that $V\delta([\phi]_V) = 0$ so $\delta([\phi]_V) = 0$. Now assume that there exists a $D \in \mathrm{Gl}_n(k)$ such that $D^{-1}\delta\text{-}\mathrm{Gal}(R|K)D \subset \mathrm{Gl}_n^\delta$. For $\phi \in \delta\text{-}\mathrm{Gal}(R|K)$, let $[\phi]_Z \in \mathrm{Gl}_n(k)$ be the matrix such that $\phi(Z) = Z[\phi]_Z$. Let $U = ZD$. For any $\phi \in \delta\text{-}\mathrm{Gal}(R|K)$, we have that $\phi(U) = Z[\phi]_Z D = ZD(D^{-1}[\phi]_Z D)$. Therefore $\phi(U) = U[\phi]_U$ for some $[\phi]_U \in \mathrm{Gl}_n(k^\delta)$. This implies that $B = \delta(U)U^{-1}$ is left fixed by $\delta\text{-}\mathrm{Gal}(R|K)$ and so $B \in K^{n\times n}$. A calculation shows that $\sigma(B) = \sigma(\delta(U)U^{-1}) = ABA^{-1} + \delta(A)A^{-1}$. □

Remark 13.12. *The proposition above is still true if k is only an algebraically closed field (see Remark 12.8). But, in that case, one has to allow the coefficients of the matrix D to be in some δ-closed field extension of k. Then, if $\tilde{k}$ denotes a δ-field extension of k, view as constant σ-field, one can show that if $\sigma(Y) = AY$ is δ-integrable over the fraction field of $K \otimes_k \tilde{k}$, it was already δ-integrable on K.*

The criteria of Proposition 13.11 might not be very easy to verify a priori, that is, in terms of the coefficients of the initial linear system $\sigma(Y) = AY$. However, in many cases of interest, one can use the ambient structure given by the usual Galois group to have a more rigid situation. It is the case when the usual Galois group is quasi simple (e.g. $\mathrm{Sl}_n(k)$).

Proposition 13.13. *Let K be a $\sigma\delta$-field such that $k = K^\sigma$ is δ-closed. Let $A \in \mathrm{Gl}_n(K)$ and let $R = K\{Z, \frac{1}{det(Z)}\}$ be a $\sigma\delta$-PV ring. We assume that the usual Galois group of $\sigma(Y) = AY$ is a quasi simple algebraic group H of dimension t. Then,*

- *either $\delta\text{-}\mathrm{Gal}(R|K) = H$,*
- *or $\delta\text{-}\mathrm{trdeg}(R|K) < t$ and $\sigma(Y) = AY$ is δ-integrable over K.*

Proof. By Proposition 12.12, we know that

- The σ-ring $R_0 := K[Z, \frac{1}{det(Z)}]$ is a PV-ring for $\sigma(Y) = AY$,
- The group $\delta\text{-}\mathrm{Gal}(R|K)$ is a Zariski dense subgroup of $\mathrm{Gal}(R_0|K)$.

This means that $\delta\text{-}\mathrm{Gal}(R|K)$ is a Zariski dense subgroup of the quasisimple algebraic group H (or of one of its conjugate in $\mathrm{Gl}_n(k)$). By Theorem 9.27, we know that either $\delta\text{-}\mathrm{Gal}(R|K) = H$ or there exists $P \in \mathrm{Gl}_n(k)$ such that $PGP^{-1} = \{M \in \mathrm{Gl}_n(k^\delta) \cap H\}$. Proposition 13.11 ends the proof. □

Remark 13.14. *The assumption that k is δ-closed is rather superfluous and Proposition 13.13 is still true with the hypothesis k algebraically closed.*

Now, we have a strategy. Starting with a linear difference system $\sigma(Y) = AY$ with quasi-simple usual Galois group H, we can

- either find a δ-algebraic relation between the solutions. This proves that the parametrized Galois group is a proper subgroup H and we conclude that the system $\sigma(Y) = AY$ is δ-integrable.
- or prove that the compatibility equation $\sigma(B) = ABA^{-1} + \delta(A)A^{-1}$ has no rational solution $B \in K^{n\times n}$. This means that the system $\sigma(Y) = AY$ is not δ-integrable and that the solutions are δ-transcendental over K.

Exercise 13.15 (Example 3.13 in [HS08])**.** *Let $K = \mathbb{C}(x)$ endowed with $\sigma(f(x)) = f(x+1)$ and $\delta = \frac{d}{dx}$. In [vdPS97, p. 42], it is shown that the difference equation $Y(x+1) = A(x)Y(x)$, where*

$$A(x) = \begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix}$$

has Galois group over $\mathbb{C}(x)$ equal to $\mathrm{Sl}_2(\mathbb{C})$. We shall show that the corresponding differential transcendence degree of the $\sigma\delta$-PV ring is 3. By Remark 13.14, one has just to prove that there is no $B \in \mathbb{C}(x)^{n\times n}$ such that $\sigma(B) = ABA^{-1} + \delta(A)A^{-1}$.

1. *Assume that such B exists and write $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{C}(x)^{2\times 2}$. Find a linear difference equation of the form*

$$\alpha(x)\sigma^3(b) + \beta(x)\sigma^2(b) + \gamma(x)\sigma(b) + \delta(x)b = \epsilon(x). \tag{13.5}$$

   (Use computer algebra system or do it by hand)

2. *We want to show that* (13.5) *has no rational solutions.*
   (a) *Show that b is a polynomial (hint: assume that b has a pole and, reasoning by cancellation of poles modulo $\mathbb{Z}$ as in Exercise 13.6, find a contradiction.)*
   (b) *Prove that* (13.5) *has no polynomial solution (hint: consider the term of highest degree).*

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