GALOISIAN STRUCTURE OF LARGE STEPS WALKS CONFINED IN THE FIRST QUADRANT

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1. Introduction

We consider 2-dimensional lattice weighted walks confined to the quadrant $\mathbb{N}^2$ as illustrated in Figure 1.1. In recent years, the enumeration of such walks has attracted a lot of attention involving many new methods and tools. This question is ubiquitous since lattice walks encode several classes of mathematical objects in discrete mathematics (permutations, trees, planar maps, . . .), in statistical physics (magnetism, polymers, . . .), in probability theory (branching processes, games of chance . . .), in operations research (birth-death processes, queueing theory).

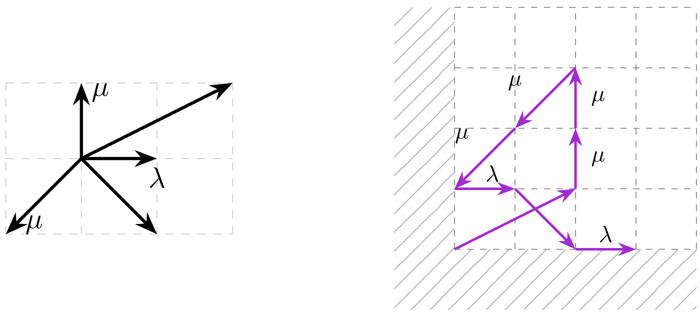


Figure 1.1. The weighted model $\mathcal{G}_3^{\lambda,\mu}$ along with an example of a walk of size 8, total weight $\mu^4\lambda^2$ and ending at $(3, 0)$

Given a finite set $\mathcal{S}$ of allowed steps in $\mathbb{Z}^2$ and a family of $\mathcal{W} = (w_s)_{s\in\mathcal{S}}$ of non-zero weights, the question consists in enumerating the weighted lattice walks in $\mathbb{N}^2$ with steps in $\mathcal{S}$. To this end, we study the generating function

$$Q(X, Y, t) = \sum_{i,j,n} q_n^{(i,j)} X^i Y^j t^n$$

where $q_n^{(i,j)}$ is the sum of the weights of all walks in $\mathbb{N}^2$ of n steps taken in $\mathcal{S}$ that start at $(0, 0)$ and end at (i, j). One natural question for this class of walks is to decide where $Q(X, Y, t)$ fits in the classical hierarchy of power series:

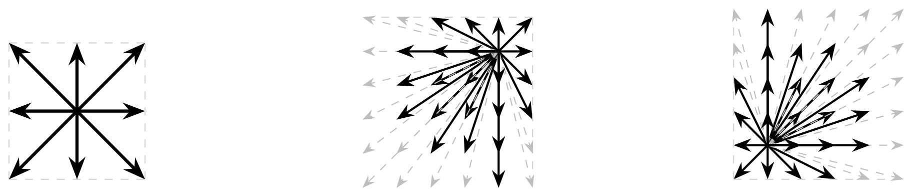
$$\text{algebraic } \subset D\text{-finite} \subset D\text{-algebraic}.$$

Here, one says that the series $Q(X, Y, t)$ is D-finite if it satisfies a linear differential equation in each variable X,Y,t, and D-algebraic if it satisfies a polynomial differential equation in each of the variables X, Y, t.

Walks with small steps. For *unweighted small steps* walks (that is $\mathcal{S} \subset \{-1, 0, 1\}^2$ and weights all equal to 1), the algebraic classification is now complete. It required almost ten years of research and the contribution of many mathematicians, combining a large variety of tools: elementary power series algebra [BMM10], computer algebra [BvHK10], probability theory ([DW15]), complex uniformization ([KR12]), Tutte invariants [BBMR21] as well as differential Galois theory ([DHRS18]).

A certain group G of birational transformations introduced in [BMM10] associated with the model $\mathcal{W}$ plays a crucial role in the nature of $Q(X,Y,t)$. Indeed, the series $Q(X,Y,t)$ is D-finite if and only if G, called here the *classical group of the walk*, is a finite group (see [BMM10, BvHK10, KR12, MR09, DHRS20]).

When the group G is finite, the algebraic nature of the generating function is intrinsically related to the existence of certain rational functions in X, Y, t called in this paper *Galois invariants* and *Galois decoupling pairs*. These notions were introduced in [BBMR21] where the authors proved that the finiteness of the group G is equivalent to the existence of non-trivial Galois invariants (see [BBMR21, Theorem 4.6]) and that the algebraicity of the model is equivalent to the existence of Galois invariants and decoupling pairs for the fraction XY (see [BBMR21, §4.5]).



(A) Small steps models (B) Large steps models with small forward steps (C) Large steps model with small backward steps

FIGURE 1.2. Models of walks

Walks with arbitrary steps. Contrasting with the precision of the classification for small steps walks, the study of walks with large steps is still at its infancy. In [BBMM21], Bostan, Bousquet-Mélou and Melczer lay the foundation of the study of large steps walks. To this purpose, they attach to any model with large steps, a graph called *the orbit of the walk* whose edges are pairs of algebraic elements over $\mathbb{Q}(x,y)$. When all the steps of the walk are small, the *orbit of the walk* coincides with the orbit of (x,y) under the action of the group G of birational transformations introduced in [BMM10].

Bostan, Bousquet-Mélou and Melczer started a thorough classification of the associated 13110 nonequivalent models whose steps have coordinates in $\{1,0,-1,-2\}$ (which are instances of walk with *small forward steps*, see Figure 1.2b). They ended up with a partial classification of the algebraic nature of the associated generating functions (see [BBMM21, Figure 7]). Among the 240 models with finite orbit, they were able to prove that all but 9 were D-finite via orbit sums constructions or Hadamard products. For the 12870 models with an infinite orbit, they proved that all but 16 were non-D-finite by exhibiting some wild asymptotics for the associated generating functions.

Content of the paper. When the steps set contains at least one large step, the authors of [BBMM21] deplored that, within their study, the group of the walk " is lost, but the associated orbit survives". In this paper, we show that one can generalize the notion of *group of the walk* to models with large steps as well as many objects and results related to the small steps framework. The novelty of our approach lies in the use of tools from graph theory, in particular graph homology and their combination with a Galois theoretic approach. We list below our contributions.

- We attach to any model $\mathcal{W}$ a group G, which we call the *group of the walk*. This group generalizes the *classical group of the walk* in many ways. First, G is the group of automorphisms of a certain field extension. It is generated by Galois automorphisms and extends thereby the definition of the *classical group of the walk* as in [FIM99, §2.4] (see Theorem 3.8 below). Moreover, we also prove that the *orbit of the walk* is the orbit of (x, y) under the faithful action of G viewed as group of graph automorphisms (see Theorem 3.13). Finally, Appendix A studies the geometric interpretation of the group G as group of birational transformations of a certain algebraic curve.
- The Galoisian structure of the *group of the walk* enables us to characterize algebraically the existence of *Galois invariants*. To any model $\mathcal{W}$, one can attach a *kernel polynomial* $\widetilde{K}(X,Y)$ in $\mathbb{C}[X,Y,t]$. *Galois invariants* consist in a pair of rational fractions $(F(X), G(Y)$ in $\mathbb{C}(X,t) \times \mathbb{C}(Y,t)$ such that
$$\widetilde{K}(X,Y)R(X,Y) = F(X) - G(Y),$$
where R is a rational fraction in $\mathbb{C}(X,Y,t)$ whose denominator is not divisible by $\widetilde{K}$. We prove that the existence of non-trivial *Galois invariants* is equivalent to the finiteness of the group G, itself equivalent to the finiteness of the orbit. This extends to any model of walk the result of [BBMR21] for small steps walks (see Theorem 4.3). Finally, we give an explicit way of obtaining the Galois invariants out of the data of a finite orbit (see §4.3).
- This Galoisian setting also sheds a new light on the existence of *Galois decoupling pairs*. Given a rational fraction $H(X,Y)$ in $\mathbb{C}(X,Y,t)$, we say that H admits a *Galois decoupling pair* if and only if there exists a pair $(H(X), G(Y))$ in $\mathbb{C}(X,t) \times \mathbb{C}(Y,t)$ such that
$$\widetilde{K}(X,Y)R(X,Y) = H(X,Y) - F(X) - G(Y),$$
where R is a rational fraction in $\mathbb{C}(X,Y,t)$ whose denominator is not divisible by $\widetilde{K}$.

  When the *group of the walk* of the model $\mathcal{W}$ is finite, we give a general criterion for the Galois decoupling of a rational fraction H, which amounts to evaluate H on some well chosen linear combination of pairs of the orbit (from Theorem 5.10). Our procedure is entirely effective, and admits an efficient implementation under a small assumption depending only on the graph structure of the orbit (see §5.5). This result generalizes [BBMR21, Theorem 4.11] to the large steps framework and proposes some algorithmic answer to the more general question of separating variables in principal bivariate polynomial ideals, as studied in [BKP20].

  As an application, we study the existence of Galois decoupling pairs of the function XY for weighted models with steps in $\{-1,0,1,2\}^2$. The finite orbit-types, that is, the graph structure of the *orbit of the walk* of these models, have been classified in [BBMM21]. We compute the decoupling of (x, y) for weighted models having an orbit-type $\mathcal{O}_{18}$, $\widetilde{\mathcal{O}_{12}}$, Hadamard or Fan. We prove that, for an unweighted model with steps in $\{-1,0,1,2\}$ with orbit-type ${}_{c}O_{18}$ and any model with orbit-type Hadamard or fan, the function XY had no decoupling pair. Finally, we show that XY decouples for any choice of weights on the model $\mathcal{G}_3^{\lambda,\mu}$, of orbit type $\mathcal{O}_{12}$ (see Figure 1.1).

- Generating functions associated to models with small backward steps (see Figure 1.2c) satisfy a functional equation in two catalytic variables of the form
$$\widetilde{K}(X,Y)Q(X,Y,t) = XY + F(X) + G(Y),$$
where $F(X)$ (resp. $G(Y)$) involves only the section $Q(X,0,t)$ (resp. $Q(0,Y,t)$) of the generating function. In [BBMR21] for small steps walks and [BM21] for walks confined in the three-quadrant, the authors develop a strategy to prove the algebraicity of the generating function. When XY admits a decoupling pair and when there exist nontrivial Galoisian invariants, they were able to obtain from the functional equation above two functional equations in one catalytic variable each, whose solutions are known to be algebraic by [BMJ06]. Thanks to our systematic approach to Galoisian invariants and decoupling, we apply their strategy to prove the algebraicity of the excursion series of two Gessel models $\mathcal{G}_3$ and $\mathcal{G}_2$ with small forward steps ([BBMM21, Table 4, second and third row]). These models were conjectured algebraic by Bostan, Bousquet-Mélou and Melczer on [BBMM21, Page 57]. In particular, we prove that the excursion generating function $Q(0,0,t)$ of $\mathcal{G}_3^{\lambda,\mu}$ is algebraic of degree 32 over $\mathbb{Q}(\lambda,\mu)(t)$.

The paper is organized as follows. In Section 2, we prove the algebraicity of the model $\mathcal{G}_3^{\lambda,\mu}$. In Section 3, we recall the construction of the *orbit of the walk* and define the group of the walk as a group of field automorphisms. Section 4 is concerned with the notion of *Galois invariants* and their properties. In Section 5, we define the notion of *Galois decoupling of the pair* (x,y) in the orbit and prove the unconditional existence of such a decoupling when the orbit is finite. This yields a criterion to test the decoupling of any rational fraction including XY. We also study its expression via the notion of *level lines* of the graph of the orbit, allowing a more effective computation. Appendix A is for the geometry inclined reader since it presents the Riemannian geometry behind the large steps models. Appendix B studies the algebraicity of the excursion series of the model $\mathcal{G}_3^{\lambda,\mu}$. Appendix C completes the discussion on the algorithmic aspects of our decoupling procedure. Appendix D studies the decoupling of (x,y) for some important orbit-types. Appendix E gives an explicit description a the group of the walk for a subclass of Hadamard models.

Note that, in this paper, we consider a weighted model $\mathcal{W}$ which is entirely determined by a set of directions $\mathcal{S}$ together with a set of weights $(w_s)_{s\in\mathcal{S}}$. The weights are always non-zero and they belong to a certain field extension of $\mathbb{Q}$ which is not necessarily algebraic, allowing the choice of indeterminate weights. Without loss of generality, one can assume that $\mathbb{Q}(w_s, s\in\mathcal{S}) \subset \mathbb{C}$. For ease of presentation, we consider polynomials, rational fractions with coefficients in $\mathbb{C}$. However, the reader must keep in mind that our results are valid if one replace $\mathbb{C}$ by the algebraic closure of $\mathbb{Q}(w_s, s\in\mathcal{S})$

2. A step by step proof of algebraicity

In this section, we fix a weighted model $\mathcal{W}$ with small backward steps. We explain how one can combine the approach of Bousquet-Mélou and Jehanne on equations with one catalytic variables [BMJ06] and the notion of decoupling and invariants of a model to address the question of algebraicity for models with small backwards steps. This strategy is not yet entirely algorithmic and follows the one developed in the small steps case in [BBMR21, Section 5] and in [BM21] for walks in the three-quadrant. We summarize its main steps in Figure 2.1. In subsection 2.2, we apply this strategy to prove that the generating function of the weighted model $\mathcal{G}_3^{\lambda,\mu}$ defined in Example 2.1 is algebraic. Therefore, the same holds for its excursion series. Since excursion series are preserved under central symmetry, the excursion series of the reversed model of $\mathcal{G}_3^{\lambda,\mu}$ is also algebraic. Thereby, we prove two of the four conjectures of Bostan, Bousquet-Mélou and Melczer on [BBMM21, Section 8.4.2]. More precisely, the authors of [BBMR21] consider the models $\mathcal{G}_3$ and $\mathcal{G}_2$ which are obtained by reversing the step sets of $\mathcal{G}_3^{1,1}$ and $\mathcal{G}_3^{0,1}$. As a corollary of the algebraicity of the model $\mathcal{G}_3^{\lambda,\mu}$, we find that the generating functions for excursions of $\mathcal{G}_3$ and $\mathcal{G}_2$ are algebraic.

2.1. **Walks and functional equation in two catalytic variables.** Recall that we do not only study the number of walks of size n that corresponds to the series $Q(1,1)$. We record in the enumeration the coordinates where they end, encoded in the generating function as the exponents associated with the variables X and Y. The variables X and Y in $Q(X,Y,t)$ are called *catalytic*, as they provide an easy way to write a functional equation for $Q(X,Y,t)$ from the recursive description of walks: either a walk is the trivial walk (with no steps), either one adds a step to an existing walk, provided the new walk does not leave the quarter plane. This is that boundary constraint which forces to consider the final coordinates (i,j) of the walk to form a functional equation. This inductive description yields a functional equation for the generating function $Q(X,Y,t)$.

Thereby, we encode the model in two Laurent polynomials which are the *step polynomial* of the model $S(X,Y) = \sum_{(i,j)\in\mathcal{S}} w_{i,j}X^iY^j$ and the *kernel polynomial* $K(X,Y,t) = 1 - tS(X,Y)$. This Laurent polynomial can be normalized into a polynomial $\widetilde{K}(X,Y,t) = X^{m_x}Y^{m_y}K(X,Y,t)$ where $-m_x, -m_y$ are the smallest moves of the walk in the X and Y-direction. By an abuse of terminology, we also call $\widetilde{K}$ the kernel polynomial. We shall sometimes write $Q(X,Y), K(X,Y)$ and $\widetilde{K}(X,Y)$ instead of $Q(X,Y,t), K(X,Y,t), \widetilde{K}(X,Y,t)$ in order to lighten the notation. We now illustrate the construction of the functional equation for the model $\mathcal{G}_3^{\lambda,\mu}$.

Example 2.1. Consider the weighted model

$$\mathcal{G}_3^{\lambda,\mu} = \{((-1,-1),\mu),((0,1),\mu),(1,-1),(2,1),((1,0),\lambda)\}$$

together with its step polynomial $S(X,Y) = \frac{\mu}{XY} + \mu Y + \frac{X}{Y} + X^2Y + \lambda X$, and kernel polynomial $\widetilde{K}(X,Y,t) = XY - t(\mu + \mu XY^2 + X^2 + X^3Y^2 + \lambda X^2Y)$. The weights μ and λ are non-zero complex numbers.

Now, to form a functional equation, observe that the steps $(1,0)$, $(0,1)$ and $(2,1)$ can be concatenated to any existing walk, whereas the step $(1,-1)$ can only be concatenated to a walk that does not terminate on the X-axis, and the step $(-1,-1)$ can only be concatenated to a walk that does not terminate on the X-axis or the Y-axis. These conditions translate directly into the following functional equation:

$$\begin{aligned} Q(X,Y) = {} & 1 + t\mu y Q(X,Y) + tX^2YQ(X,Y) + \lambda tXQ(X,Y) \\ & + t\tfrac{X}{Y}\left(Q(X,Y) - Q(X,0)\right) \\ & + t\tfrac{\mu}{XY}\left(Q(X,Y) - Q(X,0) - Q(0,Y) + Q(0,0)\right). \end{aligned} \tag{2.1}$$

Note that we can express the generating function for walks ending on the X-axis, the Y-axis or at $(0,0)$ as specializations of the generating function $Q(X,Y)$. For instance, the series $Q(X,Y) - Q(X,0)$ counts the walks that do not end on the X-axis.

Grouping terms in $Q(X,Y)$ to the left-hand side and multiplying by XY to have polynomial coefficients, we finally obtain the following equation for $Q(X,Y)$:

$$\widetilde{K}(X,Y)Q(X,Y) = XY - t(X^2+\mu)Q(X,0) - t\mu Q(0,Y) + t\mu Q(0,0).$$

The general form of the functional equation satisfied by the generating function of a weighted model might be quite complicated [BBMM21, Equation (11)]. For models with small backward steps, the functional equation satisfied by $Q(X,Y)$ simplifies as follows:

$$\widetilde{K}(X,Y)Q(X,Y) = XY + A(X) + B(Y), \tag{2.2}$$

where $A(X) = \widetilde{K}(X,0)Q(X,0) + t\epsilon Q(0,0)$ and $B(Y) = \widetilde{K}(0,Y)Q(0,Y)$ where ϵ is one if $(-1,-1) \in \mathcal{S}$ and 0 otherwise. Thus, (2.2) only involves the sections $Q(X,0)$ and $Q(0,Y)$ which makes it easier to study.

Remark 2.2. Consider a lattice walk on $\mathcal{G}_3^{\lambda,\mu}$ taking a times the step $(0,1)$, b times the step $(2,1)$, c times the step $(1,0)$, d times the step $(1,-1)$ and e times the step $(-1,-1)$. This lattice path contributes to the generatung function $Q(X,Y,t)$ via the monomial

$$\lambda^c \, \mu^{a+e} \, X^{2b+c+d-e} \, Y^{a+b-d-e} \, t^{a+b+c+d+e}.$$

One then remarks that the knowledge of the exponents of λ, X, Y and t completely determines the exponent of μ, for

$$a+e = -\tfrac{1}{4}c - \tfrac{1}{2}\left(2b+c+d-e\right) + \tfrac{1}{4}\left(a+b-d-e\right) + \tfrac{3}{4}\left(a+b+c+d+e\right).$$

Thus, the series $Q(X,Y,t)$ for $\mathcal{G}_3^{\lambda,\mu}$ can be expressed as $Q'(X\mu^{-\frac{1}{2}}, Y\mu^{\frac{1}{4}}, t\mu^{\frac{3}{4}})$, where Q' is the generating function for walks using the steps $\{(-1,-1),(0,1),(1,-1),(2,1),((1,0),\lambda\mu^{-\frac{1}{4}})\}$. Thus, the weight μ is combinatorially redundant when considering the full generating series $Q(X,Y)$ or the excursion generating series $Q(0,0)$. Note however that the exponent of the weight λ cannot be deduced from the exponents of X, Y and t alone. We still keep the weight μ as a demonstration of the robustness of our methods, and because the redundancy doesn't affect the generating function $Q(1,1)$ counting walks regardless of their ending point.

2.2. **Algebraicity strategy.** We recall some results on equations with one catalytic variable. In [BMJ06], Bousquet-Mélou and Jehanne proved the algebraicity of power series solution of *well founded* polynomial equations in one catalytic variable. Their method has been further extended recently to the case of systems of discrete differential equations by Notarantonio and Yurkevich in [NY23]. These algebraicity results are in fact particular cases of an older result in commutative algebra of Popescu [Pop86] but the strength of the strategy developed in [BMJ06, NY23] lies in the efficiency of their approach.

We now recall the results of [BMJ06, Section 4]. Let $\mathbb{L}$ be a field of characteristic zero. For an unknown bivariate function $F(u,t)$ denoted for short $F(u)$, we consider the functional equation

$$F(u) = F_0(u) + t\,Q\left(F(u), \Delta F(u), \Delta^{(2)}F(u), \ldots, \Delta^{(k)}F(u), t, u\right), \tag{2.3}$$

where $F_0(u) \in \mathbb{L}[u]$ is given explicitly and Δ is the *discrete derivative*: $\Delta F(u) = \frac{F(u)-F(0)}{u}$. One can easily show that the equation (2.3) has a unique solution $F(u,t)$ in $\mathbb{L}[u][[t]]$, the ring of formal power series in t with coefficients in the ring $\mathbb{L}[u]$. Such an equation is called *well-founded*. Here is one of the main result of [BMJ06].

Theorem 2.3 (Theorem 3 in [BMJ06])**.** *The formal power series $F(u,t)$ defined by* (2.3) *is algebraic over $\mathbb{L}(u,t)$.*

We shall use Theorem 2.3 as a black box in order to establish the algebraicity of power series solutions of a polynomial equation in one catalytic variable.

In order to eliminate directly trivial algebraic models, we make the following assumption on the step sets. Write $-m_x$, M_x (resp. $-m_y$, M_y) for the smallest and largest move in the x direction (resp. y direction) of the model $\mathcal{W}$ (the m_x, M_x, m_y and M_y are non-negative). Now, consider the class of models where one of these quantities is zero. All the models in this class are algebraic. Indeed, the corresponding models are essentially one dimensional. More precisely, if $M_x = 0$, an easy induction shows that a walk based upon such a model is included in the half-line $x = 0$. Similarly, if $m_x = 0$, then the walks on this model have only the y constraint. Reasoning analogously to [BMM10, Section 2.1] or [BBMM21, Section 6], one proves that the series is algebraic. Thus, we may assume from now on that none of these parameters are zero.

This assumption being made, the series $Q(X, Y)$ satisfies naturally an equation with two catalytic variables, and therefore does not fall directly into the conditions of Theorem 2.3. However, the functional equation (2.2) implies that the generating function $Q(X, Y)$ is algebraic over $\mathbb{C}(X, Y, t)$ if and only if the series $A(X)$ and $B(Y)$ are algebraic over $\mathbb{C}(X, t)$ and $\mathbb{C}(Y, t)$ respectively. Therefore, we set ourselves to find two well founded polynomial equations with one catalytic variable: one for $A(X)$ and the other for $B(Y)$.

In order to produce these two equations from the functional equation (2.2), we now present a method which goes back to Tutte [Tut95] and was further adapted by Bernardi, Bousquet-Mélou and Raschel in the context of small steps walks ([BBMR21]) and by Bousquet-Mélou in the context of three quadrant walks [BM21]. We reproduce here the method of [BM21] which relies on suitable notion of t-invariants and an *invariant lemma* for multivariate power series. The strategy developed in [BM21] adapts the approach already introduced in Section 4.3 in [BBMR21].

Definition 2.4. We denote by $\mathbb{C}(X, Y)((t))$ the field of Laurent series in t with coefficients in the field $\mathbb{C}(X, Y)$. The subring of $\mathbb{C}(X, Y)((t))$ formed by the series of the form

$$H(X, Y, t) = \sum_t \frac{p_n(X, Y)}{a_n(X)b_n(Y)} t^n,$$

where $p_n(X, Y) \in \mathbb{C}[X, Y]$, $a_n(X) \in \mathbb{C}[X]$ and $b_n(Y) \in \mathbb{C}[Y]$ is denoted by $\mathbb{C}_{\text{mul}}(X, Y)((t))$.

Definition 2.5 (Definition 2.4 in [BM21])**.** Let $H(X, Y, t)$ be a Laurent series in $\mathbb{C}_{\text{mul}}(X, Y)((t))$. The series H is said to have *poles of bounded order at 0* if the collection of its coefficients (in the t-expansion) have poles of bounded order at $X = 0$ and $Y = 0$. In other words, this means that, for some natural numbers m and n, the coefficients in t of the series $X^m Y^n H(X, Y)$ have no pole at $X = 0$ nor at $Y = 0$.

Given a model $\mathcal{W}$, one can use the notion of poles of bounded order at zero to construct an equivalence relation in the ring $\mathbb{C}_{\text{mul}}(X, Y)((t))$. To this purpose, we slightly adapt Definition 2.5 in [BM21] to encompass the large step case. Moreover, in the following definition, we consider division by $\widetilde{K}$ and not by K as in [BM21] but one easily checks that Definition 2.8 below and Definition 2.3 in [BM21] coincide.

Definition 2.6 (t-equivalence)**.** Let $F(X, Y)$ and $G(X, Y)$ be two Laurent series in $\mathbb{C}_{\text{mul}}(X, Y)((t))$. We say that these series are *t-equivalent*, and we write $F(X, Y) \equiv G(X, Y)$ if the series $\frac{F(X,Y)-G(X,Y)}{\widetilde{K}(X,Y)}$ has poles of bounded order at 0.

The t-equivalence is compatible with the ring operations on Laurent series applied pairwise as stated below.

Proposition 2.7 (Lemma 2.5 in [BM21])**.** *If* $A(X,Y) \equiv B(X,Y)$ *and* $A'(X,Y) \equiv B'(X,Y)$*, then* $A(X,Y)+B(X,Y) \equiv A'(X,Y)+B'(X,Y)$ *and* $A(X,Y)B(X,Y) \equiv A'(X,Y)B'(X,Y)$*.*

The notion of t-equivalence allows us to define the notion of t-invariants as follows.

Definition 2.8 (Invariants (Definition 2.3 in [BM21]))**.** Let $I(X)$ and $J(Y)$ be two Laurent series in t with coefficients lying respectively in $\mathbb{C}(X)$ and $\mathbb{C}(Y)$. If $I(X) \equiv J(Y)$, then the pair $(I(X), J(Y))$ is said to be a *pair of* t*-invariants* (with respect to the model $\mathcal{W}$).

By Proposition 2.7, pairs of t-invariants are also preserved under sum and product applied pairwise. We now state the main result on t-invariants (Lemma 2.6 in [BM21]*):

Lemma 2.9 (Invariant Lemma)**.** *Let* $(I(X), J(Y))$ *be a pair of* t*-invariants. If the coefficients in the* t*-expansion of* $\frac{I(X)-J(Y)}{\widetilde{K}(X,Y)}$ *have no pole at* $X=0$ *nor* $Y=0$*, then there exists a Laurent series* $A(t)$ *with coefficients in* $\mathbb{C}$ *such that* $I(X)=J(Y)=A(t)$*.*

Note that the equations $I(X)=A(t)$ and $J(Y)=A(t)$ involve only one catalytic variable. In other words, the invariant lemma allows us to produce nontrivial equations with one catalytic variable from one pair of t-invariants satisfying a certain analytic regularity.

Still assuming that the negative steps are small, we now combine the notions of t-invariants and the invariant lemma with the functional equation satisfied by $Q(X,Y)$ in order to obtain two equations in one catalytic variable.

First, we find a pair of t-invariants which involves the specializations of $Q(X,Y)$. Thus, Lemma 2.9 provide the desired equations for $Q(X,0)$ and $Q(0,Y)$. One way to obtain such a pair of invariants is by looking at the functional equation (2.2), namely:

$$\widetilde{K}(X,Y)Q(X,Y) = XY + A(X) + B(Y).$$

Assume that there exist some fractions $F(X)$ in $\mathbb{C}(X,t)$, $G(Y)$ in $\mathbb{C}(Y,t)$, and $H(X,Y)$ in $\mathbb{C}(X,Y,t)$ having poles of bounded order at 0 such that that XY can be written as

$$XY = F(X) + G(Y) + \widetilde{K}(X,Y)H(X,Y).$$

We call such a relation a t-decoupling of XY. Combining the above expression for XY with the functional equation for $Q(X,Y)$, one obtains the following rewriting

$$\widetilde{K}(X,Y)\left(Q(X,Y) - H(X,Y)\right) = \left(F(X) + A(X)\right) + \left(G(Y) + B(Y)\right).$$

Note now that the right-hand side has separated variables from the t-decoupling of XY.

Since $Q(X,Y)$ is a generating function for walks in the quarter plane, the coefficients of its t-expansion are polynomials in $\mathbb{C}[X,Y]$ (the coefficient of t^n is $\sum_{i,j\geq 0} q_n^{(i,j)} X^i Y^j$), so the power series $Q(X,Y)$ has poles of bounded order at 0. By assumption on $H(X,Y)$, this is also the case for the series $Q(X,Y)-H(X,Y)$. Therefore, $(I_1(X), J_1(Y)) = (F(X)+A(X), -G(Y)-B(Y))$ is a pair of t-invariants. It is noteworthy that this pair of t-invariants involve the sections $Q(X,0)$ and $Q(0,Y)$.

We must note that the writing of XY as the sum of two univariate fractions modulo $\widetilde{K}$ was the only condition to the existence of the above pair.

In Section 5, we introduce the notion of Galois decoupling of XY which is a weaker though easier to test than the notion of t-decoupling. The existence and computation of a Galois decoupling for XY or, more generally, for any rational fraction in $\overline{\mathbb{Q}}(X,Y)$ is one of the main

*In [BM21], Lemma 2.6 requires that the coefficients in the t-expansion of $\frac{I(X)-J(Y)}{K(X,Y)}$ vanish at $X=0$ and $Y=0$. This is equivalent to the condition stated in Lemma 2.9.

results of this paper, and is covered in full generality in Section 5. Provided the orbit of the walk defined in Section 3 is finite, our Galois decoupling procedure is entirely algorithmic. Thus, one can search for a t-decoupling of XY by first looking for a Galois decoupling and then by checking if this Galois decoupling is a t-decoupling.

We now illustrate this step on the model $\mathcal{G}_3^{\lambda,\mu}$:

Example 2.1 (continued)**.** Recall that the functional equation obtained for $\mathcal{G}_3^{\lambda,\mu}$ is:

$$\widetilde{K}(X,Y)Q(X,Y) = XY - t(X^2+\mu)Q(X,0) - t\mu Q(0,Y) + t\mu Q(0,0),$$

with $A(X) = -t(X^2+\mu)Q(X,0) + t\mu Q(0,0)$ and $B(Y) = -t\mu Q(0,Y)$. One can check that XY admits a t-decoupling of the following form:

$$XY = -\frac{3\lambda X^2 t - \mu\lambda t - 4X}{4t(X^2+\mu)} + \frac{-\lambda Y - 4}{4Y} - \frac{\widetilde{K}(X,Y)}{(X^2+\mu)Yt}.$$

Combining this identity with the functional equation, one obtains the following pair of t-invariants:

$$(I_1(X), J_1(Y)) = \left(\frac{3\lambda t\, X^2 - \lambda\mu t - 4X}{-4t\, X^2 - 4\mu t} - t\left(X^2+\mu\right)Q(X,0) + t\mu Q(0,0)\, , t\mu Q(0,Y) + \frac{\lambda Y + 4}{4Y}\right).$$

In general, the pair of t-invariants $(I_1(X), J_1(Y))$ that can be obtained through the combination of the functional equation and a decoupling equation does not satisfy the conditions of Lemma 2.9, as the coefficients of the series in t of $\frac{I_1(X)-J_1(Y)}{\widetilde{K}(X,Y)}$ might have poles at 0. In order to remove these poles, we want to combine the pair $(I_1(X), J_1(Y))$ with another pair of t-invariants $(I_2(X), J_2(Y))$ by means of Proposition 2.7, where $I_2(X)$ and $J_2(Y)$ will be assumed to be respectively in $\mathbb{C}(X,t)$ and $\mathbb{C}(Y,t)$.

Currently, this pole elimination process requires a case by case treatment. We detail it for our running example $\mathcal{G}_3^{\lambda,\mu}$:

Example 2.1 (continued)**.** The pair $(I_2(X), J_2(Y))$ below is a pair of t-invariant for $\mathcal{G}_3^{\lambda,\mu}$:

$$(I_2, J_2) = \left(\frac{\left(-\lambda^2\mu\, X^3 - \mu\, X^4 - X^6 + \mu^2 X^2 + \mu^3\right)t^2 - X^2\lambda\left(X^2-\mu\right)t + X^3}{t^2 X\left(X^2+\mu\right)^2}, \frac{-\mu t\, Y^4 + \lambda t Y + Y^3 + t}{Y^2 t}\right).$$

Analogously to the t-decoupling, we first search for a pair of Galois invariants, which amounts to use the semi-algorithm presented in Section 4, and then check that this pair is a pair of t-invariants.

As we now have two pairs of t-invariants $P_1 = (I_1(X), J_1(Y))$ and $P_2 = (I_2(X), J_2(Y))$, we perform some algebraic combinations between them in order to eliminate their poles. To lighten notation, we write the component-wise operations on the pairs P_i of t-invariants. Computations can be checked in the joint Maple worksheet [Wora].

Consider the Taylor expansions of the first coordinates:

$$I_1(X) = \frac{\lambda}{4} + O(X)\,,$$
$$I_2(X) = \mu X^{-1} + O(X)\,.$$

Out of these two pairs of t-invariants, we first produce a pair of t-invariants without a pole at $X = 0$ as follows:

$$P_3 = (I_3, J_3) := P_2\left(P_1 - \frac{\lambda}{4}\right).$$

The first coordinate of the pairs P_1 and P_3 do not have a pole at $X = 0$. The Taylor expansion of their second coordinates $J_1(Y)$ and $J_3(Y)$ at $Y = 0$ is as follows:

$$J_3(Y) = Y^{-3} + (t\mu Q(0,0) + \lambda)\, Y^{-2} + \mu t \left(Q(0,0)\, \lambda + \frac{\partial^2 Q}{\partial Y^2}(0,0) \right) Y^{-1} + O\big(Y^0\big)\,,$$
$$J_1(Y) = Y^{-1} + O\big(Y^0\big)\,.$$

In order to produce a pair of t-invariants satisfying the assumption of Lemma 2.9, we need to combine P_1 and P_3 in order to eliminate the pole at $Y = 0$. Note that, since the first coordinate of P_1 and P_3 have no pole at zero, the first coordinate of any sum or product between these two pairs have no pole at $X = 0$.

Using the simple pole at $Y = 0$ of J_3, we produce a new pair P_4 whose coordinates have no pole at X and Y equal zero by setting

$$P_4 = (I_4, J_4) := P_3 - P_1^3 + \left(2t\mu Q(0,0) - \frac{\lambda}{4} \right) P_1^2 + \left(2\mu t \frac{\partial^2 Q}{\partial Y^2}(0,0) - t^2\mu^2 Q(0,0)^2 + \frac{5\lambda^2}{16} \right) P_1.$$

In order to use the Invariants Lemma, it remains to check that $\frac{I_4(X) - J_4(Y)}{\widetilde{K}(X,Y)}$ has no poles at $X = 0$ and $Y = 0$. This is done in the Maple Worksheet [Wora]. Therefore, the invariant lemma 2.9 yields the existence of a series $C(t)$ in $\mathbb{C}((t))$ such that $I_4(X) = C(t)$ and $J_4(Y) = C(t)$.

Once we have found a pair of t-invariants satisfying the conditions of Lemma 2.9, we end up with two nontrivial equations of one catalytic variable involving the sections $Q(X, 0)$ and $Q(0, Y)$. If these equations are well-founded, then Theorem 2.3 allows us to conclude that the series $Q(X, 0)$ and $Q(0, Y)$ are algebraic over $\mathbb{C}(X, t)$ and $\mathbb{C}(Y, t)$ respectively, and therefore that $Q(X, Y)$ is algebraic over $\mathbb{C}(X, Y, t)$.

Example 2.1 (continued)**.** The value of $C(t)$ can be deduced from the values of $Q(0, Y)$ and its derivatives at $(0, 0)$ by looking at the Taylor expansion of $J_4(Y)$ in Y. The verification that the polynomial equations $I_4(X) = C(t)$ and $J_4(Y) = C(t)$ are well-founded is done in the Maple worksheet [Wora]. We only give here the form of the well-founded equation for $F(Y, t) := Q(0, Y)$:

$$F(y) = 1 + t \Big(\mu^2 t^2 y F(y) \left(\Delta^{(1)} F(y) \right)^2 + \lambda \mu t F(y)\, \Delta^{(1)} F(y) + \mu t \left(\Delta^{(1)} F(y) \right)^2$$
$$+ 2\mu t F(y)\, \Delta^{(2)} F(y) + \mu y F(y) + \lambda \Delta^{(2)} F(y) + 2\Delta^{(3)} F(y) \Big)\,.$$

Theorem 2.3 with $\mathbb{L} = \mathbb{Q}(\lambda, \mu)$ implies that the generating function of the weighted model $\mathcal{G}_3^{\lambda,\mu}$ is algebraic over $\mathbb{Q}(\lambda, \mu)(X, Y, t)$. Moreover, one can show that, at any step of our reasoning, one may have taken the weight λ to be zero. In particular, the generating functions of the reversed models of $\mathcal{G}_2$ ($\mu = 1, \lambda = 0$) and of $\mathcal{G}_3$ ($\mu = 1, \lambda = 1$) are algebraic. Thus, their excursion series are algebraic. In Appendix B, we apply the method of Bousquet-Mélou and Jehanne to the catalytic equation for $F(y)$, giving an explicit minimal polynomial for the series $Q(0, 0)$ (with λ and μ). This proves a conjecture of Bostan, Bousquet-Mélou and Melczer ([BBMM21, Table 4]).

In the small steps case, one can show that the generating function is algebraic in the variables X and Y if and only if the model admits some non-trivial Galois invariants and XY has a Galois decoupling (see discussion in Section 4.2). We conjecture that this equivalence is still valid in

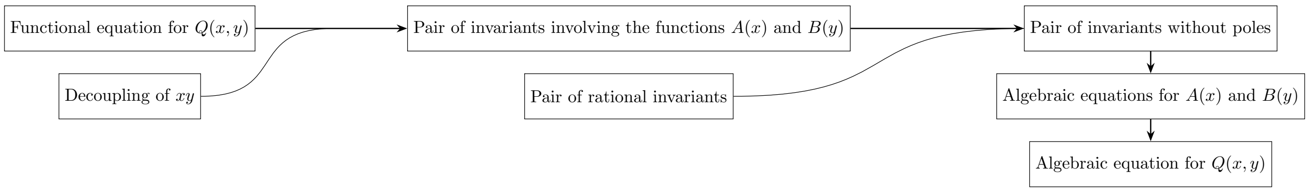


FIGURE 2.1. Summary of the strategy for proving algebraicity

the large steps case. The general strategy we used in this section is summarized in Figure 2.1 and motivates the above conjecture. It is the first attempt at finding uniform proofs for the algebraicity of generating functions of large steps models. Our strategy is entirely algorithmic, except for the poles elimination process detailed above on one example. Nonetheless, we think that this last step could be made constructive via for instance the generalization of the notion of weak invariants ([BBMR21, Section 5.2]) to the large steps framework. The rest of the paper is devoted to the systematic and algorithmic study of the notions of Galois invariants and decoupling.

3. The orbit of the walk and its Galoisian structure

In the context of small steps models, the *group of the walk* (which we qualify *classic* in this paper for disambiguation) has been initially introduced in [BMM10, Section 3]. It is the group generated by two birational involutions Φ and Ψ of $\mathbb{C} \times \mathbb{C}$ defined as follows. Assuming that the model has at least a negative and a positive X and Y-steps, one writes its step polynomial $S(X,Y) = \sum_{(i,j)\in\mathcal{S}} w_{(i,j)} X^i Y^j$ as

$$S(X,Y) = A_{-1}(X)\frac{1}{Y} + A_0(X) + A_1(X)Y = B_{-1}(Y)\frac{1}{X} + B_0(X) + B_1(Y)X,$$

where the A_i and B_i's are Laurent polynomials. The birational transformations Φ and Ψ are then defined as

$$\Phi : (x,y) \mapsto \left(\frac{B_{-1}(y)}{xB_1(y)}, y\right) \text{ and } \Psi : (x,y) \mapsto \left(x, \frac{A_{-1}(x)}{yA_1(x)}\right).$$

When the classic group of the walk is infinite, its action on the variables X and Y produces an infinite amount of singularities for the generating function $Q(X,Y)$ proving that the series is not D-finite (see [MM14] or [KR12] for instance). When the group of the walk is finite, one can describe in certain cases the generating function as a diagonal of a rational function, called the (alternating) orbit sum. To such a group, one can attach a graph, called the *orbit*, whose vertices are the orbit in $\mathbb{C}(x,y)^2$ of the pair (x,y) under the action of Φ and Ψ and whose edges correspond to the action of Φ and Ψ (see [BMM10, §3]).

In [BBMM21], the authors generalized the notion of the orbit of the walk to arbitrary large steps models but did not attempt to find a group of transformations which generates this orbit. In this section, we show how one can associate to a weighted model $\mathcal{W}$ a group, called in this paper *the group of the walk*, which is generated by Galois automorphisms of two field extensions. In this section, we prove that the group of the walk acts faithfully and transitively on the orbit analogously to the classic group. When the orbit is finite, this group is itself presented as a Galois group. We interpret in the next two sections the notions of invariants and decoupling in this Galoisian framework. Moreover, for finite orbits, one can interpret the group of the walk as a group of automorphisms of an algebraic curve (see Appendix A). This point of view generalizes the notion of the classic group of the walk in the small step case used in [KR12, DHRS18, DHRS20].

From now on, we fix $\mathcal{W}$ a weighted model, and we assume that the step polynomial $S(X,Y)$ is not univariate, which is the case when considering models with both positive and negative steps in each direction as in Section 2.1. In order to distinguish the coordinates of the orbit from the coordinates of X, Y, t of the functional equation in Section 2.1, we introduce two new variables x and y that are taken algebraically independent over $\mathbb{C}$. We also denote by k the field $\mathbb{C}(S(x,y))$. As x, y and $S(x,y)$ satisfy by definition the polynomial relation $\widetilde{K}(x,y,1/S(x,y)) = 0$, the condition that S is not univariate implies the following lemma.

Lemma 3.1. *The variables x, y and the polynomial $S(x,y)$ satisfy the following relations:*

(1) *x and $S(x,y)$ are algebraically independent over $\mathbb{C}$, and so are y and $S(x,y)$,*
(2) *x is algebraic over $k(y)$ and y is algebraic over $k(x)$*

The orbit as well as the associated group, invariants and Galois decoupling pairs are constructed for $S(X,Y)$ arising from a model of walk. These constructions should pass directly to the case where $S(X,Y)$ is an arbitrary bivariate rational fraction by letting $\widetilde{K}(X,Y,t)$ be $(1 - tS(X,Y))Q(X,Y)$ with $S = \frac{P}{Q}$ for P, Q two relatively prime polynomial in $\mathbb{C}[X,Y]$.

In Section 3.1, we recall the definition of the orbit of a model $\mathcal{W}$ with large steps. We give it a Galois structure in Section 3.2. In Section 3.3, we define the group of the walk and prove that it acts faithfully and transitively by graph automorphisms on the orbit. Finally, we investigate the evaluation of fractions in $\mathbb{C}(X,Y,t)$ on the orbit.

3.1. **The orbit.**

We recall below the definition of the orbit introduced in [BBMM21, Section 3], and we also fix once and for all an algebraic closure $\mathbb{K}$ of $\mathbb{C}(x,y)$.

Definition 3.2 (Definition 3.1 in [BBMM21])**.** Let (u,v) and (u',v') be in $\mathbb{K} \times \mathbb{K}$.

If $u = u'$ and $S(u,v) = S(u',v')$, then the pairs (u,v) and (u',v') are called *x-adjacent*, and write $(u,v) \sim^x (u',v')$. Similarly, if $v = v'$ and $S(u,v) = S(u',v')$, then the pairs (u,v) and (u',v') are called *y-adjacent*, and write $(u,v) \sim^y (u',v')$. Both relations are equivalence relations on $\mathbb{K} \times \mathbb{K}$.

If the pairs (u,v) and (u',v') are either x-adjacent or y-adjacent, they are called *adjacent*, and we write $(u,v) \sim (u',v')$. Finally, denoting by $\sim^*$ the reflexive transitive closure of $\sim$, the *orbit of the walk*, denoted by $\mathcal{O}$, is the equivalence class of the pair (x,y) under the relation $\sim^*$.

The orbit $\mathcal{O}$ has a graph structure: the vertices are the elements of the orbit and the edges are adjacencies, colored here by their adjacency type. In the figures, the x-adjacencies are represented in red and the y-adjacencies in blue. As the x and y adjacencies come from equivalence relations, the monochromatic connected components of $\mathcal{O}$ are *cliques* (any two vertices of such a component are connected by an edge). Moreover, by definition of the transitive closure, the graph $\mathcal{O}$ is *connected*, that is, every two vertices of the graph are connected by a path. In the sequel, we denote by $\mathcal{O}$ either the set of pairs in the orbit or the induced graph. The structure considered should be clear from the context. For a model $\mathcal{W}$, its *orbit type* corresponds to the class of its orbit modulo graph isomorphisms.

Example 3.3. For small steps models, the orbit when finite is always isomorphic to a cycle whose vertices all belong to $\mathbb{C}(x,y)^2$. Example $\mathcal{C}_6$ in Figure 3.1 is for instance the unlabelled orbit of the unweighted small steps model $\mathcal{S} = \{(-1,0),(0,1),1,-1)\}$ ([BMM10, Example 2]).

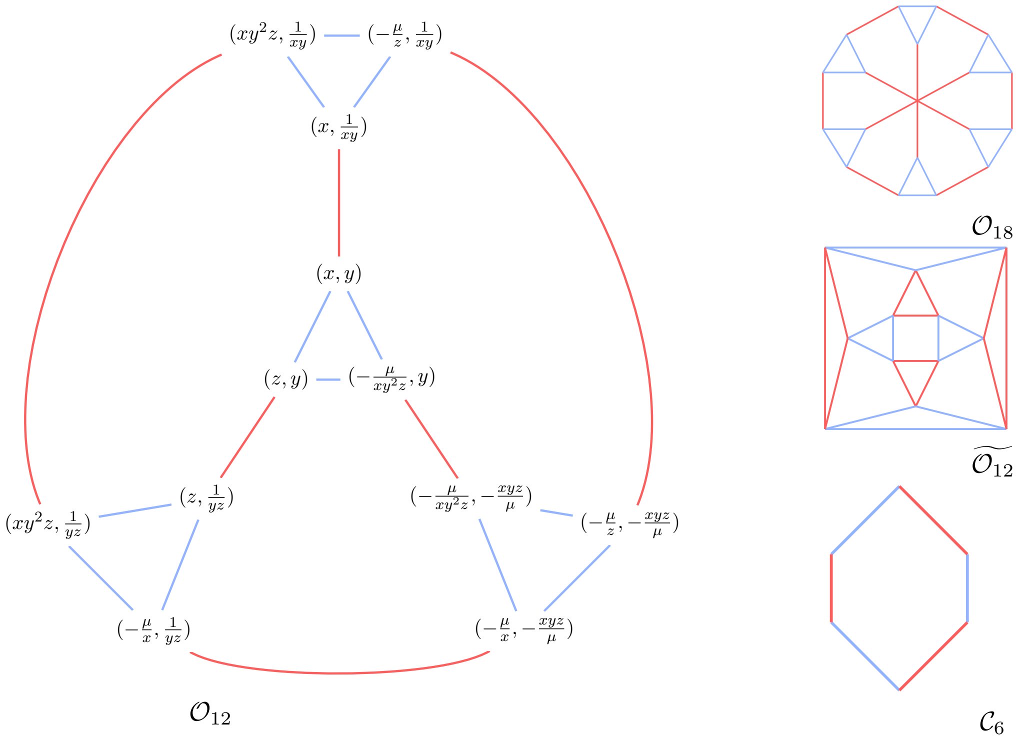


FIGURE 3.1. A sample of finite orbits

The orbit type being preserved when one reverses the model, Section 10 in [BBMM21] lists the distinct orbit types for models with steps in $\{-1,0,1,2\}^2$ with at least one large step. For these models, the finite orbit types are exactly $\mathcal{O}_{12}, \widetilde{\mathcal{O}_{12}}$ and $\mathcal{O}_{18}$ in Figure 3.1 and the so-called *Hadamard models* whose orbit type is finite and a cartesian product (see [BBMM21, Section 6] or Section 5.6.3).

Example 2.1 (continued)**.** For the model $\mathcal{G}_3^{\lambda,\mu}$, an element $(z,y) \in \mathbb{K}^2$ distinct from (x,y) is y-adjacent to (x,y) if $S(z,y) = S(x,y)$. Then, z is a root of $\widetilde{K}(Z,y,1/S(x,y)) \in k(y)[Z]$. The polynomial $\widetilde{K}(Z,y,1/S(x,y))$ is reducible over $k(x,y)[Z]$ and factors as

$$\frac{(Z-x)y(x\,y^2Z^2 + (x^2y^2 + \lambda xy + x)Z - \mu)}{x^3y^2 + (\lambda y + 1)x^2 + \mu y^2 x + \mu}.$$

The polynomial $x\,y^2Z^2 + (x^2y^2 + \lambda xy + x)Z - \mu \in \mathbb{C}(x,y)[Z]$ is irreducible (see Example 2.1 for more details). Its roots are taken in a quadratic field extension of $\mathbb{C}(x,y)$ and do not belong to $\mathbb{C}(x,y)$. They are of the form $z, \frac{-\mu}{xy^2z}$ by the relation between the roots and the coefficients of a degree two polynomial. One can then show that the orbit $\mathcal{O}_{12}$ in Figure 3.1 is the orbit of the model $\mathcal{G}_3^{\lambda,\mu}$. Since none of the vertices depend from λ, the graph $\mathcal{O}_{12}$ is also the orbit of the reversed model of $\mathcal{G}_2$.

Example 2.1 shows that, unlike the small steps case, one has to go to a finite non-trivial field extension of $\mathbb{C}(x,y)$ in order to build the orbit of a large steps model.

Finally, we would like to discuss the finiteness of the orbit. For small steps walks, the finiteness of the orbit depends only on the order of $\Phi \circ \Psi$. Some number theoretic considerations on the torsion subgroup of the Mordell-Weil group of a rational elliptic surface prove that this order, when finite, is bounded by 6, which provides a very easy algorithm to test the finiteness of the group of the walk. This bound is valid for any choice of weights contained in an algebraically closed field of characteristic zero (see [HS08, Remark 5.1] and [SS19, Corollary 8.21]). For models with arbitrarily large steps, there does not exist currently a general criterion to determine whether the orbit is finite or not, but only a semi-algorithm [BBMM21, Section 3.2]. We hope that analogously to the small steps case a geometric interpretation of the notion of orbit will provide some bounds on the potential order of the orbit.

3.2. **The Galois extension of the orbit.** In the remaining of the article, we denote by $k(\mathcal{O})$ the subfield of $\mathbb{K}$ generated over $k = \mathbb{C}(S(x, y))$ by all coordinates of the orbit $\mathcal{O}$. Note that $k(\mathcal{O})$ coincides with $\mathbb{C}(\mathcal{O})$ since x, y belong to the orbit.

We start this subsection with some terminology on field extensions. Our main reference is [Sza09] which is a concise exposition of the Galois theory of field extensions of finite and infinite degree. A field extension $M \subset L$ is denoted by $L|M$. The degree of the field extension $L|M$ is the dimension of L as M-vector space. When this degree is finite, we denote it $[L : M]$. For $L|M$ and $L'|M$ two field extensions, an M-algebra homomorphism of L into L' is a ring homomorphism from L to L' that is the identity on M. An algebraic closure of a field M is an algebraic extension of M that is algebraically closed. Let us recall some of its properties.

Proposition 3.4 (Proposition 1.1.3 in [Sza09])**.** *Let M be a field.*

1. *There exists an algebraic closure $\overline{M}$ of M. It is unique up to isomorphism.*
2. *For an algebraic extension L of M, there exists an embedding from L to $\overline{M}$ leaving M elementwise fixed. Moreover, any M-algebra homomorphism from L into $\overline{M}$ can be extended to an M-algebra isomorphism of $\overline{L}$ to $\overline{M}$*

The field $\mathbb{K}$ introduced in Section 3.1 is an algebraic closure of $\mathbb{C}(x, y)$. By definition of the orbit, $k(\mathcal{O}) = \mathbb{C}(\mathcal{O})$ is an algebraic field extension of $\mathbb{C}(x, y)$. Moreover, since y is algebraic over $k(x)$ and x is algebraic over $k(y)$ by Lemma 3.1, then $\mathbb{C}(x, y)$ is an algebraic field extension of $k(x)$ and $k(y)$. Therefore, $k(\mathcal{O})$ is algebraic over $k(x)$ and $k(y)$. Proposition 3.4 implies that $\mathbb{K}$ is an algebraic closure of $k(x), k(y)$ and $k(\mathcal{O})$.

Let $L|M$ be a field extension. Any M-algebra endomorphism of L is an automorphism and we denote by $\mathrm{Aut}(L|M)$ the set of M-algebra endomorphisms of L. An algebraic field extension $L|M$ is said to be Galois if the set $L^{\mathrm{Aut}(L|M)}$ of elements of L that remain fixed under the action of $\mathrm{Aut}(L|M)$ coincides with M (see [Sza09, Definition 1.2.1]). In this case, $\mathrm{Aut}(L|M)$ is denoted by $\mathrm{Gal}(L|M)$. By [Sza09, Proposition 1.2.4], an algebraic field extension $L|M$ is Galois if and only if, fixing an algebraic closure $\overline{M}$ of M, we have $\sigma(L) \subset L$ for any automorphism σ in $\mathrm{Aut}(\overline{M}|M)$ †. The Galois group $\mathrm{Gal}(L|M)$ of a finite Galois extension $L|M$ has order $[L : M]$ ([Sza09, Corollary 1.2.7]). It is clear that any sub-extension $L|M'$ of a Galois extension $L|M$‡ Finally, we recall the following result.

Lemma 3.5 (Lemma 1.22 in [Sza09])**.** *Let $L|M$ be a Galois extension and $\mu \in M[X]$ an irreducible polynomial with some root α in L. Then μ splits in L, and the group* $\mathrm{Gal}(L|M)$ *acts transitively on its roots.*

†Since we are in characteristic zero, the separable closure of M coincides with the algebraic closure of M (see [Sza09, page 12]).

‡By subextension, we mean that $M \subset M' \subset L$ is Galois.

We let any $\mathbb{C}$-algebra endomorphism σ of $\mathbb{K}$ act on $\mathbb{K} \times \mathbb{K}$ coordinate-wise by

$$\sigma \cdot (u,v) \overset{\text{def}}{=} (\sigma(u), \sigma(v)).$$

The following lemma establishes the compatibility of the equivalence relation $\sim^*$ with the action of $\mathbb{C}$-algebra endomorphisms of $\mathbb{K}$.

Lemma 3.6. *Let (u,v) and (u',v') be two pairs in $\mathbb{K} \times \mathbb{K}$ and $\sigma\colon \mathbb{K} \to \mathbb{K}$ be a $\mathbb{C}$-algebra endomorphism. Then $(u,v) \sim^x (u',v')$ (resp. $(u,v) \sim^y (u',v')$) implies that $\sigma \cdot (u,v) \sim^x \sigma \cdot (u',v')$ (resp. $\sigma(u,v) \sim^y \sigma(u',v')$). The same holds therefore for $\sim^*$.*

Proof. Since σ is a $\mathbb{C}$-algebra endomorphism, we have $\sigma S(u,v) = S(\sigma u, \sigma v)$ for any u,v in $\mathbb{K}$. Therefore, if $(u,v) \sim^x (u,v')$ then $S(\sigma(u), \sigma(v)) = \sigma(S(u,v)) = \sigma(S(u,v')) = S(\sigma(u), \sigma(v'))$, so $\sigma \cdot (u,v) \sim^x \sigma \cdot (u,v')$. The same argument applies if $(u,v) \sim^y (u',v)$. The general case of $(u,v) \sim^* (u',v')$ follows by induction. □

As a direct corollary, we find the following lemma which ensures the setwise stability of the orbit under certain endomorphisms of $\mathbb{K}$.

Lemma 3.7. *Let $\sigma_x\colon \mathbb{K} \to \mathbb{K}$ be a $k(x)$-algebra endomorphism. Then, for all (u,v) in the orbit, $\sigma_x \cdot (u,v)$ is in the orbit. Similarly, the orbit is also stable under $k(y)$-algebra endomorphisms of $\mathbb{K}$.*

Proof. Let (u,v) be in the orbit, i.e. $(u,v) \sim^* (x,y)$. By Lemma 3.6, we find that

$$\sigma_x \cdot (u,v) \sim^* \sigma_x \cdot (x,y) = (x, \sigma_x(y)).$$

By transitivity, we only need to prove that $(x, \sigma_x(y))$ is in the orbit. This is true because $S(x, \sigma_x(y)) = \sigma_x S(x,y) = S(x,y)$ since σ_x fixes $\mathbb{C}(x, S(x,y))$ so $(x, \sigma_x(y)) \sim^x (x,y)$. □

The above two lemmas imply that any $k(x)$ or $k(y)$-algebra automorphism of $\mathbb{K}$ induces a permutation of the vertices of $\mathcal{O}$ which preserves the colored adjacencies, and is therefore a *graph automorphism* of $\mathcal{O}$.

The stability result of Lemma 3.7 translates as a field theoretic statement.

Theorem 3.8. *The extensions $k(\mathcal{O})|k(x), k(\mathcal{O})|k(y)$ and $k(\mathcal{O})|k(x,y)$ are Galois.*

Proof. We first prove that $k(\mathcal{O})|k(x)$ is a Galois extension. Recall that the field extension $k(\mathcal{O})|k(x)$ is algebraic and $\mathbb{K}$ is an algebraic closure of $k(\mathcal{O})$ and $k(x)$. Thus, we only need to prove that $\sigma(k(\mathcal{O})) \subset k(\mathcal{O})$ for every automorphism σ in $\mathrm{Aut}(\mathbb{K}|k(x))$. This follows directly from Lemma 3.7. The proof for $k(\mathcal{O})|k(y)$ is entirely symmetric and the field extension $k(\mathcal{O})|k(x,y)$ is Galois as subextension of $k(\mathcal{O})|k(x)$. □

Theorem 3.8 gives a Galoisian framework to the orbit, which will be central in our study of invariants and decoupling. Remark that the algebraic extension $k(\mathcal{O})|k(x,y)$ may be of infinite degree. In Figure 3.2, we represent the different Galois extensions involved in Theorem 3.8 and we denote their Galois groups $G_x = \mathrm{Gal}(k(\mathcal{O})|k(x))$, $G_y = \mathrm{Gal}(k(\mathcal{O})|k(y))$ and $G_{xy} = \mathrm{Gal}(k(\mathcal{O})|k(x,y))$. Note that $G_{xy} = G_y \cap G_x$.

Example 3.9. For small steps models, we have $k(\mathcal{O}) = k(x,y) = \mathbb{C}(x,y)$. Moreover, the field extensions $k(\mathcal{O})|k(x)$ and $k(\mathcal{O})|k(y)$ are both of degree 2 so that G_x and G_y are groups of order 2 and thereby isomorphic to $\mathbb{Z}/2\mathbb{Z}$. In the notation of the beginning of Section 3, consider the endomorphisms ϕ, ψ of $\mathbb{C}(x,y)$ defined as follows: for $f(x,y) \in \mathbb{C}(x,y)$, we set $\phi(f) = f(\Phi(x,y))$ and $\psi(f) = f(\Psi(x,y))$. It is easily seen that $\psi \in G_x$ and that $\phi \in G_y$ and that they both are non-trivial involutions. Thus, we have $G_{xy} = 1$, $G_x = \langle \psi \rangle \simeq \mathbb{Z}/2\mathbb{Z}$ and $G_y = \langle \phi \rangle \simeq Z/2\mathbb{Z}$.

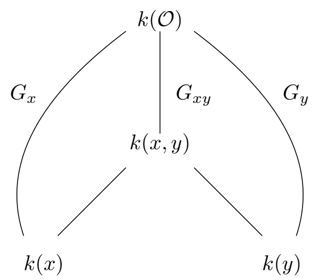


FIGURE 3.2. The field extensions attached to the orbit

Example 2.1 (continued)**.** In the case of $\mathcal{G}_3^{\lambda,\mu}$, we have $k(\mathcal{O}) = \mathbb{C}(x,y,z)$ where z is a root of the polynomial $P(Z) = xy^2Z^2 + (x^2y^2 + \lambda xy + x)Z - \mu$. The polynomial $P(Z)$ is irreducible in $\mathbb{C}(x,y)[Z]$ since its discriminant is

$$x\left(x^3y^4 + 2\lambda x^2y^3 + \lambda^2 x\, y^2 + 2x^2y^2 + 2\lambda xy + 4\mu y^2 + x\right).$$

Because of the irreducible factor x, this discriminant cannot be a square in $k(x,y) = \mathbb{C}(x,y)$. Therefore, $z \notin k(x,y)$ and the extension $k(\mathcal{O})|k(x,y)$ is of degree 2. As above, we find that $G_{xy} \simeq \mathbb{Z}/2\mathbb{Z}$.

The field extension $k(\mathcal{O})|k(y)$ is of degree 6 so that its Galois group is either S_3 or $\mathbb{Z}/6\mathbb{Z}$. In this last case, the group G_{xy} would be a normal subgroup of G_y. As $k(x,y) = k(\mathcal{O})^{G_{xy}}$, the extension $k(x,y)|k(y)$ would be Galois by [Sza09, Theorem 1.2.5]. This is impossible since the root z of $\widetilde{K}(Z, y, 1/S(x,y))$ is not in $k(x,y)$. Hence, we find that $G_y \simeq S_3$.

The extension $k(\mathcal{O})|k(x)$ is of degree 4. Its Galois group is of order four and therefore either isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or to $\mathbb{Z}/4\mathbb{Z}$. If G_x were $\mathbb{Z}/4\mathbb{Z}$ then there would exist a $k(x)$-algebra endomorphism σ of $k(\mathcal{O})$ of order 4. Since G_{xy} is of order 2, the automorphism σ can not fix y and we must have $\sigma(y) = \frac{1}{xy}$, which is the other root of $\widetilde{K}(x, Y, 1/S(x,y)) \in k(x)[Y]$. Since the orbit is setwise invariant by σ and (z,y) is in the orbit, the same holds for $\sigma(z,y) = (\sigma(z), \frac{1}{xy})$. From the description of the orbit of $\mathcal{G}_3^{\lambda,\mu}$ in Figure 3.1, we find that $\sigma(z) \in \{\frac{\mu}{z}, xy^2z\}$. In both cases, we find that $\sigma^2(z) = z$ which implies that σ^2 is the identity on $k(\mathcal{O})$. A contradiction. Hence, we conclude that the group G_x is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

3.3. **The group of the walk.** In this section, we prove that the orbit $\mathcal{O}$ is the orbit of the pair (x,y) under the action of a certain group which generalizes the one introduced in the small steps case by Bousquet-Mélou and Mishna ([BMM10, §3]).

For this group, we take $G = \langle G_x, G_y\rangle$, that is, the subgroup of $\mathrm{Aut}(k(\mathcal{O})|k)$ generated by G_x and G_y, and we call it the *group of the walk*.

Recall the discussion on small steps models. By definition and Example 3.9, the group G is generated by the automorphisms ψ, ϕ, as they generate G_x and G_y. Thereby, G is isomorphic to the classic group of the walk $\langle \Phi, \Psi\rangle$.

As explained in Section 3.2, every element of G induces a *graph automorphism* of $\mathcal{O}$, that is, a permutation of the vertices of $\mathcal{O}$ which preserves the colored adjacencies on the orbit $\mathcal{O}$. In Theorem 3.13 below, we prove that there exists a finitely generated subgroup of G whose action on $\mathcal{O}$ is faithful and transitive, which is a notable property of the classic group of the walk.

It is clear that the group G acts faithfully on the orbit $\mathcal{O}$. Indeed, if an element σ of G is the identity on any element of the orbit then σ is the identity on $k(\mathcal{O})$. Therefore, σ is the identity. The construction of a finitely generated subgroup of G with a transitive action on the orbit requires a bit more work.

We first prove two lemmas on the polynomial $\widetilde{K}(X,Y,t)$.

Lemma 3.10. *The kernel polynomial $\widetilde{K}(X,Y,t)$ is irreducible in $\mathbb{C}[X,Y,t]$. Therefore, it is irreducible as a polynomial in $\mathbb{C}(X,t)[Y], \mathbb{C}(Y,t)[X]$ and $\mathbb{C}(t)[X,Y]$.*

Proof. The kernel polynomial is a degree 1 polynomial in t, therefore it is irreducible in $\mathbb{C}(X,Y)[t]$. Moreover, its content is one by construction. Therefore, by Gauss Lemma ([Lan02, chap. V par. 6 Theorem 10]), the kernel polynomial is irreducible in $\mathbb{C}[X,Y][t] = \mathbb{C}[X,Y,t]$. Since $S(X,Y)$ is not univariate, the polynomial $\widetilde{K}$ does not belong to $\mathbb{C}[X,t]$, Gauss Lemma asserts that $\widetilde{K}$ being irreducible in $\mathbb{C}[X,t][Y]$ is also irreducible in $\mathbb{C}(X,t)[Y]$. The same reasoning holds for the irreducibility of $\widetilde{K}$ in $\mathbb{C}(Y,t)[X]$. It is clear that since $\widetilde{K}$ is irreducible in $\mathbb{C}[X,Y,t]$ and not in $\mathbb{C}(t)$, it is irreducible in $\mathbb{C}(t)[X,Y]$. □

Lemma 3.11. *The specializations of the kernel polynomial $\widetilde{K}\,(x,Y,1/S(x,y))$ and $\widetilde{K}\,(X,y,1/S(x,y))$ are respectively irreducible as polynomials in $k(x)[Y]$ and in $k(y)[X]$.*

Proof. We only prove the first assertion by symmetry of the roles of x and y. Consider the $\mathbb{C}$-algebra homomorphism $\phi : \mathbb{C}[X,t] \rightarrow k(x)$ defined by $\phi(X) = x$ and $\phi(t) = 1/S(x,y)$. Since $S(X,Y)$ is not univariate, the fractions x and $1/S(x,y)$ are algebraically independent over $\mathbb{C}$. Therefore the morphism ϕ is one-to-one, so it extends to a field isomorphism $\phi : \mathbb{C}(X,t) \rightarrow k(x)$ (onto by definition of $k(x)$), which extends to a $\mathbb{C}$-algebra isomorphism ϕ from $\mathbb{C}(X,t)[Y]$ to $k(x)[Y]$. Moreover, by Lemma 3.10, $\widetilde{K}(X,Y,t)$ is irreducible as a polynomial in $\mathbb{C}(X,t)[Y]$. Therefore, since $\widetilde{K}\,(x,Y,1/S(x,y)) = \phi(\widetilde{K}(X,Y,t))$ and $\phi(\mathbb{C}(X,t)) = k(x)$, we conclude that the polynomial $\widetilde{K}\,(x,Y,1/S(x,y))$ is irreducible over $k(x)$. □

For large steps models, the extensions $k(\mathcal{O})|k(x)$ and $k(\mathcal{O})|k(y)$ might be of infinite degree, hence the groups G_x and G_y might not be finite, not even finitely generated (unlike the small steps case where they are always cyclic of order 2). However, note that G_{xy} is the stabilizer of the pair (x,y) in the orbit. Therefore, the action of G on (x,y) factors through the left quotients G_x/G_{xy} and G_y/G_{xy} which are proved to be finite in the following lemma.

Lemma 3.12. *The group G_{xy} is of finite index in G_x and in G_y with $[G_x : G_{xy}] = m_y + M_y$ and $[G_y : G_{xy}] = m_x + M_x$.*

Proof. The orbit Ω of y under the action of G_x is a subset of the roots of the polynomial $\widetilde{K}(x,Z,1/S(x,y)) \in k(x)[Z]$. This polynomial is irreducible by Lemma 3.10, so G_x acts transitively on its roots by Lemma 3.5, hence Ω coincides with the set of roots of $\widetilde{K}(x,Z,1/S(x,y))$ which is a finite set of cardinal $\deg_Y \widetilde{K} = M_y + m_y$. Moreover, the stabilizer of y for this action is precisely the group G_{xy}. Therefore, the quotient G_x/G_{xy} can be identified with Ω, wich proves that G_{xy} is of finite index in G_x with $[G_x : G_{xy}] = M_y + m_y$. The proof for the subgroup G_y is analogous. □

Therefore, we fix once and for all a set $I_x = \{\mathrm{id}, \iota_1^x, \dots, \iota_{m_y+M_y}^x\}$ of representatives of the left cosets of G_x/G_{xy}, and a set $I_y = \{\mathrm{id}, \iota_1^y, \dots, \iota_{m_x+M_x}^y\}$ of representatives of the left cosets of G_y/G_{xy}. By construction,

$$G_x = \langle I_x, G_{xy}\rangle\,, G_y = \langle I_y, G_{xy}\rangle\,,\ \text{and}\ G = \langle I_x, I_y, G_{xy}\rangle\,.$$

We now have all the ingredients to prove the transitivity of the action of a finitely generated subgroup of G on $\mathcal{O}$. We only recall that the distance between two vertices of a graph is the number of edges in a shortest path connecting them.

Theorem 3.13 (Transitivity of the action)**.** *The subgroup of G generated by I_x and I_y acts transitively on the orbit $\mathcal{O}$.*

Proof. We show that for all pairs (u, v) of $\mathcal{O}$ there exists an element σ in $\langle I_x, I_y \rangle$ such that $\sigma \cdot (x, y) = (u, v)$. As the graph of the orbit is connected, the proof is done by induction on the distance between (x, y) and (u, v). If (u, v) is at distance zero to (x, y) then $(u, v) = (x, y)$ and we set $\sigma = \mathrm{id}$.

Let (u, v) be in $\mathcal{O}$ of positive distance d to (x, y). Then there exists a pair (u', v') at distance $d - 1$ to (x, y) that is adjacent to (u, v). Without loss of generality, one can assume that (u', v') is x-adjacent to (u, v), that is, $u = u'$. By induction hypothesis, there exists σ in $\langle I_x, I_y \rangle$ such that $\sigma \cdot (x, y) = (u, v')$. Therefore, since $(u, v') \sim^x (u, v)$, the application of σ^{-1} implies by Lemma 3.6 that $(x, y) \sim^x (x, \sigma^{-1}(v))$. Thus, y and $\sigma^{-1}(v)$ satisfy the equation $S(x, y) - S(x, Y)$, so they are roots of the polynomial $\widetilde{K}(x, Y, 1/S(x, y))$ which is an irreducible polynomial over $k(x)$ by Lemma 3.11. Therefore, by Lemma 3.5, there is an element σ_x in G_x such that $\sigma_x(y) = \sigma^{-1}(v)$. Let ι_i^x in I_x be the representative of the left coset $\sigma_x G_{xy}$. Then, $(\sigma\, \iota_{x,1}) \cdot (x, y) = \sigma \cdot (\sigma_x \cdot (x, y)) = \sigma \cdot (x, \sigma^{-1}(v)) = (u, v)$. This concludes the proof. □

This result shows that the orbit $\mathcal{O}$ is actually the orbit of the pair (x, y) under the action of a finitely generated subgroup of G. As a direct corollary, one finds that the extensions $k(x, y)|k$ and $k(u, v)|k$ are isomorphic for any pair (u, v) in the orbit. Indeed, let σ in G such that $\sigma \cdot (x, y) = (u, v)$ then σ induces a k-algebra isomorphism between $k(x, y)$ and $k(u, v)$.

For large steps models with an infinite orbit, it might be quite difficult to give a precise description of the automorphisms in I_x and I_y. Indeed, they act as a permutation on the infinite orbit $\mathcal{O}$ and their action on x or y is not in general a rational fraction in x and y as in the small steps case. When the steps are small or when the orbit is finite, one is able to give a more precise description of these generators.

Example 3.14. In the small steps case and in the notation of Example 3.9, one can choose $I_x = \{\mathrm{id}, \psi\}$ and $I_y = \{\mathrm{id}, \phi\}$.

Example 2.1 (continued)**.** For $\mathcal{G}_3^{\lambda,\mu}$, the group G_y is isomorphic to S_3, G_{xy} to $\mathbb{Z}/2\mathbb{Z}$ and G_x to $\mathbb{Z}/2\mathbb{Z}$. We give below the expression of automorphisms ι^x, ι^y and τ such that

$$I_x = \{\mathrm{id}, \iota^x\},\ I_y = \{\mathrm{id}, \iota^y\}, G_{xy} = \langle \tau \rangle.$$

They satisfy the relations $(\iota^x)^2 = (\iota^y)^3 = \tau^2 = \mathrm{id}$. We represent below their action on the orbit.

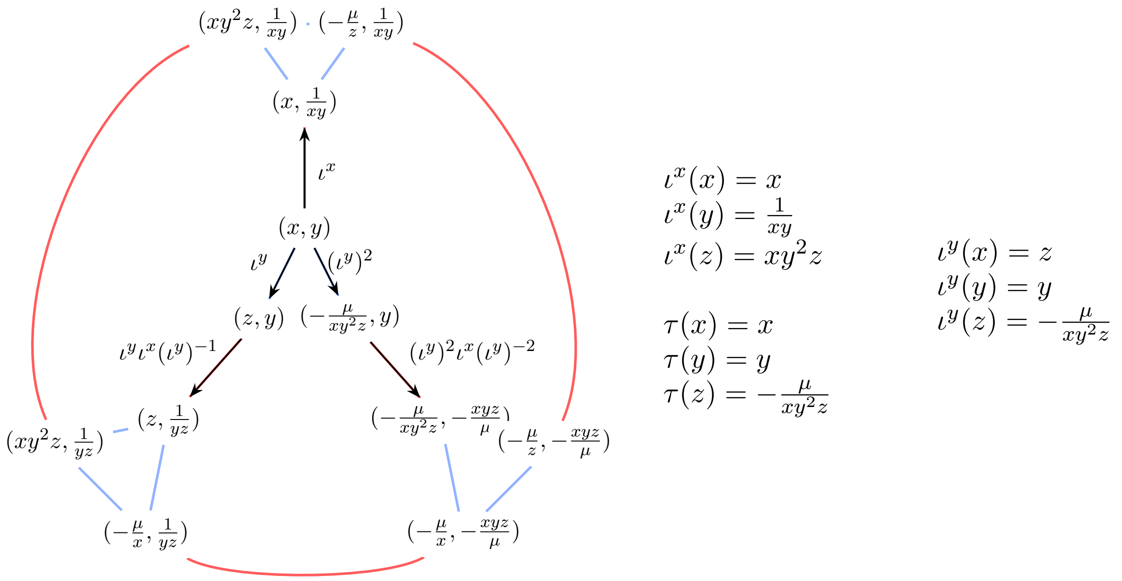


FIGURE 3.3. The elements of I_x and I_y.

Note that an x-adjacency in the orbit corresponds to the action of an element of G that is conjugate to an element of G_x. Indeed, for (u,v) in the orbit, Theorem 3.13 yields the existence of $\sigma \in G$ such that $\sigma(u,v) = (x,y)$. Then, if $(u,v) \sim^x (u,v')$, Lemma 3.6 proves that $\sigma(u,v') \sim^x (x,y)$. As explained above, any x-adjacency to (x,y) corresponds to the action of an automorphism in G_x so that there exists σ_x in G_x such that $\sigma(u,v') = \sigma_x(x,y)$. We conclude that $(u,v') = \sigma^{-1}\sigma_x\sigma(u,v)$. In the above example for the model $\mathcal{G}_3^{\lambda,\mu}$, one sees that $(z,y) \sim^x (z, \frac{1}{xz})$ and that $\iota^y\iota^x(\iota^y)^{-1} \cdot (z,y) = (z, \frac{1}{xz})$. The automorphism ι^x belong to G_x but it is not the case of $\iota^y\iota^x(\iota^y)^{-1}$ since $\iota^y\iota^x(\iota^y)^{-1}(x) = xy^2z$.

Moreover, the transitivity of the action of G on the orbit also implies the following minimality result for the extension $k(\mathcal{O})$.

Proposition 3.15. *The field $k(\mathcal{O})$ is the smallest field in $\mathbb{K}$ that is a Galois extension of $k(x)$ and a Galois extension of $k(y)$.*

Proof. Let $M \subset \mathbb{K}$ be a Galois extension of $k(x)$ and a Galois extension of $k(y)$. Proposition 3.4 shows that $\mathbb{K}$ is an algebraic closure for M. Let (u,v) be an element of $\mathcal{O}$. To prove that $k(\mathcal{O}) \subset M$, we only need to show that u and v belong to M. By Theorem 3.13, there exists σ in G such that $\sigma \cdot (x,y) = (u,v)$. Let us first assume that σ belongs to G_x. Since $\mathbb{K}$ is an algebraic closure for $k(x)$, Proposition 3.4 shows that σ extends as a $k(x)$-algebra endomorphism of $\mathbb{K}$ still denoted σ. The field extension $M|k(x)$ is Galois and $\mathbb{K}$ is an algebraic closure of M so that $\sigma(M) \subset M$. Since x and y belong to M, the same holds for (u,v). The proof is analogous if σ belong to G_y. Since G is generated by G_x and G_y, an easy induction concludes that $u = \sigma(x)$ and $v = \sigma(y)$ both belong to M for any σ in G. □

3.4. **Orbit sums.** One of the purposes of the orbit is to provide a nice family of changes of variables, in the sense that the kernel polynomial $K(X,Y,t)$ is constant on the orbit: for all pairs (u,v) of the orbit, $K(u,v,t) = K(x,y,t)$ (because $S(x,y) = S(u,v)$) This polynomial being a factor of the left-hand side of the functional equation satisfied by the generating function, one can evaluate the variables (X, Y) at any pair (u, v) of the orbit and obtain what is called an *orbit equation*. Indeed, the generating function $Q(X, Y)$ and its sections $Q(X, 0)$ and $Q(0, Y)$ belong to the ring of formal power series in t with coefficients in $\mathbb{C}[X, Y]$ so that their evaluation at (u, v) belong to the ring $\mathbb{C}[\mathcal{O}][[t]]$. Note that such an evaluation leaves the variable t fixed. The strategy developed in [BBMM21, Section 4] for models with small forward steps consists in forming linear combinations of these orbit equations so that the resulting equation is free from sections. From the section-free equation, Bostan, Bousquet-Mélou and Melczer sometimes succeed in isolating the generating function $Q(X, Y)$ and expressing it as a diagonal of algebraic fractions which leads to its D-finitness by [Lip88]. For models with small backward steps, it is quite easy to produce a section-free equation from (2.2) when the orbit contains a cycle. However, it is very unlikely that, for models with small backward steps and at least one large step, such a section free equation suffices to characterize the generating function.

In this paper, we want to evaluate the variables X, Y, t at $(u, v, 1/S(x, y))$ for (u, v) an element of the orbit. Since $\widetilde{K}(u, v, 1/S(x, y)) = 0$ for any element (u, v) of the orbit $\mathcal{O}$, such an evaluation is similar to the kernel method used in [KR12] for models with small steps. More precisely, let us define a 0-*chain* as a formal $\mathbb{C}$-linear combination of elements of the orbit $\mathcal{O}$. This terminology is borrowed from graph homology (see Section 5 for some basic introduction). Let $\gamma = \sum_{(u,v)\in\mathcal{O}} c_{(u,v)}(u, v)$ be a zero chain. Since the coefficients $c_{(u,v)}$ are complex and almost all zero, the evaluation P_γ of a polynomial $P(X, Y, t) \in \mathbb{C}[X, Y, t]$ at γ is defined as

$$P_\gamma = \sum_{(u,v)\in\mathcal{O}} c_{(u,v)} P(u, v),$$

and belongs to $\mathbb{C}[\mathcal{O}]$. The evaluation of $\widetilde{K}(X, Y, t)$ at any 0-chain vanishes so that one can not evaluate a rational fraction in $\mathbb{C}(X, Y, t)$ whose denominator is divisible by $\widetilde{K}$. This motivate the following definition.

Definition 3.16. Let $H(X, Y, t) = \frac{A(X,Y,t)}{B(X,Y,t)}$ be a rational fraction in $\mathbb{C}(X, Y, t)$ where $A(X, Y, t)$ and $B(X, Y, t)$ are relatively prime polynomials in $\mathbb{C}[X, Y, t]$.

We say that $H(X, Y, t)$ is a *regular fraction* if $B(X, Y, t)$ is not divisible by the kernel polynomial $\widetilde{K}(X, Y, t)$ in $\mathbb{C}[X, Y, t]$.

Example 3.17. Since $S(X, Y)$ is not univariate, the kernel polynomial involves all three variables X,Y and t, so does a multiple of $\widetilde{K}(X, Y, t)$ (by a simple degree argument). Therefore, any fraction in $\mathbb{C}(X, t)$ or $\mathbb{C}(Y, t)$ is regular.

We endow the set of regular fractions in $\mathbb{C}(X, Y, t)$ with the following equivalence relation: two regular fractions H, G are *equivalent* if there exists a regular fraction R such that $H - G = \widetilde{K}(X, Y, t)R$. We denote by $\mathcal{C}$ the set of equivalence classes. Since the equivalence relation is compatible with the addition and multiplication of fractions, one easily notes that $\mathcal{C}$ can be endowed with a ring structure. Moreover, since $\widetilde{K}(X, Y, t)$ is irreducible in $\mathbb{C}[X, Y, t]$, any non-zero class is invertible proving that $\mathcal{C}$ is a field. Indeed, if H is a regular fraction that is not equivalent to zero, then one can write $H = \frac{P}{Q}$ with $P, Q \in \mathbb{C}[X, Y, t]$ relatively prime and $\widetilde{K}$ does not divide P. Thus, the fraction $\frac{Q}{P}$ is regular and its class in $\mathcal{C}$ is an inverse of the class of $\frac{P}{Q}$. Moreover, since $\widetilde{K}$ is not univariate, any non-zero element in $\mathbb{C}(X, t)$ or $\mathbb{C}(Y, t)$ is a regular fraction which is not equivalent to zero. Therefore, the fields $\mathbb{C}(X, t)$ and $\mathbb{C}(Y, t)$ embed into $\mathcal{C}$. By an abuse of notation, we denote by $\mathbb{C}(X, t)$ and $\mathbb{C}(Y, t)$ their image in $\mathcal{C}$.

Proposition 3.18. *For a fraction H in $\mathbb{C}(X,Y,t)$ and (u,v) in $\mathcal{O}$, the evaluation $H_{(u,v)} = H(u,v,1/S(x,y))$ is a well defined element of $\mathbb{K}$ if and only if H is a regular fraction.*

The $\mathbb{C}$-algebra homomorphism $\phi : \mathcal{C} \to k(x,y), P(X,Y,t) \mapsto P(x,y,1/S(x,y))$ is well defined and is a field isomorphism which maps isomorphically $\mathbb{C}(t)$ onto $k = \mathbb{C}(S(x,y))$, $\mathbb{C}(X,t)$ onto $k(x)$ and $\mathbb{C}(Y,t)$ onto $k(y)$.

Proof. Recall that by Theorem 3.13, given a pair $(u,v) \in \mathcal{O}$, there exists $\sigma \in G$ such that $\sigma \cdot (x,y) = (u,v)$. The automorphism σ induces a k-algebra isomorphism between $k(x,y)$ and $k(u,v)$ so that the evaluation at $(x,y,1/S(x,y))$ composed by σ is the evaluation at $(u,v,1/S(x,y))$. Thereby, we only need to prove the first part of the proposition for the evaluation at $(x,y,1/S(x,y))$.

Since $\widetilde{K}(x,y,1/S(x,y)) = 0$, it is clear that one can not evaluate a fraction that is not regular. Thus, we only need to show that the evaluation of a regular fraction at $(x,y,1/S(x,y))$ is well defined. Let us write $H(X,Y,t) = \frac{A(X,Y,t)}{B(X,Y,t)}$ where $A(X,Y,t)$ and $B(X,Y,t)$ are relatively prime in $\mathbb{C}[X,Y,t]$, and the kernel polynomial $\widetilde{K}(X,Y,t)$ does not divide $B(X,Y,t)$. There exist two polynomials $U,V \in \mathbb{C}[X,Y,t]$ such that

$$B(X,Y,t)U(X,Y,t) + \widetilde{K}(X,Y,t)V(X,Y,t) = R(X,Y)$$

with $R(X,Y) \in \mathbb{C}[X,Y]$ the resultant of $\widetilde{K}(X,Y,t)$ and $B(X,Y,t)$ for the variable t. Since $\widetilde{K}(X,Y,t)$ is an irreducible polynomial that does not divide $B(X,Y,t)$, the resultant $R(X,Y)$ is a nonzero polynomial. Since x,y are algebraically independent over $\mathbb{C}$, one finds that $R(x,y) \neq 0$ and $K(x,y,1/S(x,y)) = 0$ which implies that $B(x,y,1/S(x,y)) \neq 0$, so $H(x,y,1/S(x,y))$ is well defined.

By Lemma 3.10, the kernel polynomial $\widetilde{K}(X,Y,t)$ is irreducible as a polynomial in $\mathbb{C}(t)[X,Y]$. The ring $R = \mathbb{C}(t)[X,Y]/(\widetilde{K}(X,Y,t))$ is therefore an integral domain. By [Mat80, page 9, (1K)], its fraction field is precisely $\mathcal{C}$. Now, the evaluation map from $\mathbb{C}(t)[X,Y]/(\widetilde{K}(X,Y,t))$ to $k[x,y]$ is a ring isomorphism which maps isomorphically $\mathbb{C}(t)$ onto k. The latter ring isomorphism extends to an isomorphism between the fraction fields $\mathcal{C}$ of $\mathbb{C}(t)[X,Y]/(\widetilde{K}(X,Y,t))$ and the fraction field $k(x,y)$ of $k[x,y]$ which concludes the proof. □

If H is a regular fraction, we denote $H_{(u,v)}$ its evaluation at an element (u,v) of the orbit and we can extend this evaluation by $\mathbb{C}$-linearity to any 0-chain γ. We denote by H_γ the corresponding element in $k(\mathcal{O})$. Such an evaluation is called an *orbit sum*. We let the group G act on 0-chains by $\mathbb{C}$-linearity, that is, $\sigma\left(\sum_{(u,v)\in\mathcal{O}} c_{(u,v)}(u,v)\right) = \sum_{(u,v)\in\mathcal{O}} c_{(u,v)}\sigma \cdot (u,v)$. The following lemma shows that the evaluation morphism is compatible with the action of G on $k(\mathcal{O})$ and on 0-chains.

Lemma 3.19. *Let σ be an element of G, γ be a 0-chain, and $H(X,Y,t)$ be a regular fraction in $\mathbb{C}(X,Y,t)$. Then $\sigma(H_\gamma) = H_{\sigma\cdot\gamma}$.*

Proof. Let (u,v) be an element in the orbit. Since σ fixes $k = \mathbb{C}(S(x,y))$, we have

$$\sigma(H_{(u,v)}) = \sigma\left(H(u,v,1/S(x,y))\right) = H\left(\sigma(u),\sigma(v),1/S(x,y)\right) = H_{\sigma\cdot(u,v)}.$$

The general case follows by $\mathbb{C}$-linearity. □

Two equivalent regular fractions have the same evaluation in $k(\mathcal{O})$. Thereby, certain class of regular fractions can be characterized by the Galoisian properties of their evaluation in $k(\mathcal{O})$. This idea underlies the Galoisian study of invariants and decoupling in Sections 4 and 5. To conclude, we want to compare the equivalence relation among regular fractions that are elements of $\mathbb{C}_{\text{mul}}(X,Y)((t))$ and the t-equivalence (see Section 2.2 for notation).

Proposition 3.20. *Let $F \in \mathbb{C}_{mul}(X,Y)((t))$ that is also a regular fraction in $\mathbb{C}(X,Y,t)$. If F is t-equivalent to 0, that is, the t-expansion of $F/\widetilde{K}$ has poles of bounded order, then the fraction $F/\widetilde{K}$ is regular so that the regular fraction F is equivalent to zero by definition.*

Proof. Our proof starts by following the lines of the proof of Lemma 2.6 in [BM21]. Assume that F is t-equivalent to 0, so that there exists $H(X,Y,t) \in \mathbb{C}_{\text{mul}}(X,Y)((t))$ with poles of bounded order at 0 such that

$$F(X,Y) = \widetilde{K}(X,Y,t)H(X,Y,t). \tag{3.1}$$

Analogous arguments to Lemma 2.6 in [BM21] show that there exists a root $\mathcal{X}$ of $K(.,Y,t) = 0$ that is a formal power series in t with coefficients in an algebraic closure of $\mathbb{C}(Y)$ and with constant term 0. Since H and F have poles of bounded order at 0, one can specialize (3.1) at $X = \mathcal{X}$ and find $F(\mathcal{X},Y,t) = 0$. Writing $F = \frac{P}{Q}$ where $P,Q \in \mathbb{C}[X,Y,t]$ are relatively prime, one finds that $P(\mathcal{X},Y,t) = 0$. Since $\widetilde{K}(.,Y,t)$ is an irreducible polynomial over $\mathbb{C}(Y,t)$ by Lemma 3.10, we conclude that $\widetilde{K}$ divides P. Because P and Q are relatively prime, we find that $\widetilde{K}$ doesn't divide Q which concludes the proof. □

Clearly, the regular fraction $\frac{\widetilde{K}(X,Y,t)}{Y-t}$ is equivalent to zero but not t-equivalent to zero, so the converse of Proposition 3.20 is false. With the strategy presented in Section 2 in mind, we will use in the next sections the notion of equivalence on regular fractions and its Galoisian interpretation to produce *Galois invariants* and *Galois decoupling pairs*. For each Galois invariants and decoupling functions constructed in Section 5.6, it happens that any equivalence relation among these regular fractions is actually a t-equivalence. Unfortunately, we do not have any theoretical arguments yet to explain this phenomenon.

The rest of the paper is devoted to the Galoisian interpretation of the notions of invariants and decoupling. Their construction relies on the evaluation of regular fractions on suitable 0-chains.

4. Galois invariants

In this section, we prove that the finiteness of the orbit is equivalent to the existence of non-constant Galois invariants (see Theorem 4.3 below). This result generalizes [BBMR21, Theorem 7] in the small steps case and was proved in the more general context of finite algebraic correspondences in [Fri78, Theorem 1]. Fried's framework is geometric, but his proof is essentially Galois theoretic. We give here an alternative presentation which does not require any algebraic geometrical background. Moreover, we show in this section that if the orbit is finite, the field of Galois invariants is of the form $k(c)$ for some element c transcendental over k. In addition, we give an algorithmic procedure to effectively construct c.

4.1. Galois invariants.

In §2, we aimed at constructing invariants that were rational fractions, that is, pairs $(I(X,t), J(Y,t))$ satisfying an equation of the form $I(X,t) - J(Y,t) = \widetilde{K}(X,Y,t)R(X,Y,t)$ with R having poles of bounded order at zero (I and J are t-equivalent). With the philosophy of §3.4, we introduce the weaker notion of *Galois invariants* based on rational equivalence. Our definition extends Definition 4.3 in [BBMR21] to the large steps context.

Definition 4.1. Let $(I(X,t), J(Y,t))$ be a pair of rational fractions in $\mathbb{C}(X,t) \times \mathbb{C}(Y,t)$ (hence regular, as they are univariate). Then this pair is a *pair of Galois invariants* if there exists a regular fraction $R(X,Y,t)$ such that $I(X,t) - J(Y,t) = \widetilde{K}(X,Y,t)R(X,Y,t)$, that is, the regular fractions $I(X,t)$ and $J(Y,t)$ are equivalent.

From Proposition 3.20, a pair of rational invariants is pair of Galois invariants. Therefore, it is justified to look for a pair of Galois invariants first, and then to check by hand if their difference is t-equivalent to 0. Moreover, the notion of Galois invariant is purely algebraic while the notion of invariants involves some analytic considerations which might be difficult to handle. Using Lemma 3.18, the set of Galois invariants corresponds to a subfield of $k(\mathcal{O})$ which can be easily described.

Proposition 4.2. *Let $P = (I(X,t), J(Y,t))$ be a pair of fractions in $\mathbb{C}(X,t) \times \mathbb{C}(Y,t)$. Then P is a pair of Galois invariants if and only if the evaluations $I_{(x,y)}$ and $J_{(x,y)}$ are equal, and thus belongs to $k(x) \cap k(y) \subset k(\mathcal{O})$. Moreover, the pair P is a pair of constant invariants if and only if $I_{(x,y)} = J_{(x,y)}$ is in k.*

Therefore we denote the field $k(x) \cap k(y)$ as k_{inv} and, by an abuse of terminology, call its elements Galois invariants. The definition of the group G and the Galois correspondence applied to $k(\mathcal{O})|k(x)$ and $k(\mathcal{O})|k(y)$ show that f in $k(\mathcal{O})$ is a Galois invariant if and only if f is fixed by G. Moreover, Proposition 4.2 reduces the question of the existence of a pair of nonconstant Galois invariants to the question of deciding whether $k_{\text{inv}} = k$ or not.

4.2. **Existence of nontrivial invariants and finiteness of the orbit.** The existence of non-constant Galois invariants is equivalent to the finiteness of the orbit as proved in Theorem 1 and Lemma p 470 in [Fri78] which holds also in positive characteristic and in a higher dimensional context. Theorem 4.3 below is a rephrasing of Fried's Theorem in pure Galois theoretic arguments, so that our proof simplifies slightly Fried's proof.

Theorem 4.3. *The following are equivalent:*

(1) *The orbit $\mathcal{O}$ is finite.*
(2) *There exists a finite Galois extension M of $k(x)$ and $k(y)$ such that* $\mathrm{Gal}(M|k(x))$ *and* $\mathrm{Gal}(M|k(y))$ *generate a finite group* $\langle \mathrm{Gal}(M|k(x)), \mathrm{Gal}(M|k(y))\rangle$ *of automorphisms of M.*
(3) *There exists nontrivial Galois invariants, that is, $k \subsetneq k_{inv}$.*

Proof. (1) $\Rightarrow$ (2): Set $M = k(\mathcal{O})$. The group $G = \langle G_x, G_y\rangle$ acts faithfully on the orbit, so it embeds as a subgroup of $S(\mathcal{O})$, of permutations of the pairs of the orbit $\mathcal{O}$. The orbit is finite, therefore G is finite.

(2) $\Rightarrow$ (3): Write $H = \langle \mathrm{Gal}(M|k(x)), \mathrm{Gal}(M|k(y))\rangle$. By the same argument as in the beginning of §3.3, the field M^H is the field k_{inv} of Galois invariants. Since H is finite, the extension $M|k_{\text{inv}}$ is finite of degree $|H|$, hence the subextension $k(x)|k_{\text{inv}}$ is also finite. Since the extension $k(x)|k$ is transcendental by hypothesis on $\mathcal{W}$, we conclude that $k \subsetneq k_{\text{inv}}$. Proposition 4.2 yields the existence of a pair of nontrivial Galois invariants.

(3) $\Rightarrow$ (1): Let $(I(X,t), J(Y,t))$ be a pair of nontrivial Galois invariants. By the assumption on the model, $S(x,y)$ and x are algebraically independent over $\mathbb{C}$. Since $I(x, 1/S(x,y))$ is not in $\mathbb{C}(1/S(x,y))$ by Lemma 3.18, this implies that the extension $k(I(x,1/S(x,y)))|k$ is transcendental. As the transcendence degree of $k(x)$ over k is 1, this implies that the extension $k(x)|k(I(x,1/S(x,y)))$ is algebraic, hence x is algebraic over k_{inv}, with minimal polynomial $P(X)$.

The group G leaves k_{inv} fixed. Thus the orbit of x in $\mathbb{K}$ under the action of G is a subset of the roots of $P(X)$. By Theorem 3.13, the action of G is transitive on the orbit, hence the set $G \cdot x = \{u \in \mathbb{K} \,|\, \exists \sigma \in G, u = \sigma x\} = \{u \in \mathbb{K} \,|\, \exists v \in \mathbb{K}, (u, v) \in \mathcal{O}\}$ is finite. As there are $\deg_Y \widetilde{K}(X, Y, t)$ pairs of the orbit with first coordinate u for each u in $G \cdot x$, we conclude that $\mathcal{O}$ is finite. □

In the rest of the paper, we assume that the orbit is finite. Theorem 4.3 implies that the extension $k(\mathcal{O})|k_{\mathrm{inv}}$ is finite and Galoisian with Galois group $G = \langle G_x, G_y \rangle$.

4.3. **Effective construction.** In order to apply the algebraic strategy presented in §2, we want to find explicit nonconstant rational invariants. As already mentioned, we shall first construct explicitly the field of Galois invariant and then search among these Galois invariants the potential rational invariants.

In the small steps case, an orbit sum argument was used to construct a pair of rational invariants ([BBMR21, Theorem 4.6]). This construction generalizes mutatis mutandis to the large steps case, and is reproduced here to show one way to exploit the group of the walk.

Lemma 4.4. *Let ω be the 0-chain $\frac{1}{|\mathcal{O}|}\sum_{a \in \mathcal{O}} a$. Then, for any regular fraction $H \in \mathbb{Q}(X, Y, t)$ the element H_ω is a rational invariant.*

Proof. Let $H(X, Y, t)$ be a regular fraction. Since, by Theorem 3.13, the group G acts faithfully on $\mathcal{O}$, the 0-chain ω is invariant by the action of G. Thus, by Lemma 3.19, for all σ in G, $\sigma(H_\omega) = H_{\sigma \cdot \omega} = H_\omega$. Therefore, by the Galois correspondence, H_ω is a rational invariant. □

Unfortunately, a non-constant regular fraction H might have a constant evaluation, that is, H_ω might belong to k. Thus, one has to choose carefully H in order to avoid this situation which is precisely the strategy used in [BBMR21, Theorem 4.6]. Below, we describe an alternative construction which is easier to compute effectively and yields a complete description of the field k_{inv}.

Consider first this simple observation. Since x is algebraic over k_{inv}, we can consider its minimal polynomial $\mu_x(Z)$ in $k_{\mathrm{inv}}[Z]$. One of its coefficients must be in $k_{\mathrm{inv}} \setminus k$ because x is transcendental over k. Thus, such a coefficient is a non-trivial rational invariant.

A more sophisticated argument using a constructive version of Lüroth's Theorem says actually much more about such a coefficient.

Theorem 4.5 (Lüroth's Theorem ([Rot15], Th. 6.66))**.** *Let $k(x)$ be a field with x transcendental over k and $k \subset K \subset k(x)$ a subfield. If x is algebraic over K, then any coefficient c of its minimal polynomial $\mu_x(Z)$ over K that is not in k is such that $K = k(c)$.*

Applying this result to the tower $k \subset k_{\mathrm{inv}} \subset k(x)$, not only can we find nontrivial Galois invariants among the coefficients of μ_x, but any one of them generates the field of Galois invariants. In one sense, these coefficients contain all the information on the Galois invariants attached to the model. Therefore, all that remains is to compute effectively the polynomial $\mu_x(Z)$.

By irreducibility of the polynomial $\mu_x(Z)$ in $k_{\mathrm{inv}}[Z]$, the Galois group $G = \mathrm{Gal}(k(\mathcal{O})|k_{\mathrm{inv}})$ acts transitively on its roots. By Theorem 3.13, the orbit of x under the action of G is the set of left coordinates of the orbit. Therefore, $\mu_x(Z)$ is precisely the vanishing polynomial of the left coordinates of the orbit, which is exactly computed in the construction of the orbit in [BBMM21, Section 3.2]. We detail this construction in Appendix C.

In order to find an explicit pair of non-constant Galois invariant $(I(X, t), J(Y, t))$, we only need to apply Proposition 3.18 to lift to $\mathbb{C}(X, t)$ and to $\mathbb{C}(Y, t)$ any non-constant coefficient of the polynomial $\mu_x(Z) \in k_{\text{inv}}[Z]$. The lifts of the polynomial $\mu_x[Z]$ to $\mathbb{C}(X,t)[Z]$ and to $\mathbb{C}(Y,t)[Z]$ can be computed directly when constructing the orbit, see C.1.

Example 4.6. Consider the model $\mathcal{G}_3^{\lambda,\mu}$. Its orbit type is $\mathcal{O}_{12}$. We compute the lift of $\mu_x(Z)$ in $\mathbb{C}(X,t)[Z]$ as

$$Z^6 \boxed{-\frac{\left(\lambda^2\mu X^3 + X^6 + \mu X^4 - \mu^2 X^2 - \mu^3\right)t^2 + X^2\lambda\left(X^2-\mu\right)t - X^3}{t^2 X\left(X^2+\mu\right)^2}} Z^5 + \frac{\mu t + \lambda}{t} Z^4$$
$$-2\frac{X^6 t^2\mu + \left(-\frac{\lambda^2\mu t^2}{2} + \frac{1}{2}\right)X^5 + t\mu\left(\mu t + \lambda\right)X^4 + \left(-t^2\mu^3 - \lambda\mu^2 t\right)X^2 - \frac{\mu^2\left(\lambda^2\mu t^2 - 1\right)X}{2} - t^2\mu^4}{t^2 X\left(X^2+\mu\right)^2} Z^3$$
$$-\frac{\mu\left(\mu t+\lambda\right)Z^2}{t} - \frac{\left(\left(\lambda^2\mu X^3 + X^6 + \mu X^4 - \mu^2 X^2 - \mu^3\right)t^2 + X^2\lambda\left(X^2-\mu\right)t - X^3\right)\mu^2}{t^2 X\left(X^2+\mu\right)^2} Z - \mu^3$$

and in $\mathbb{C}(Y,t)[Z]$ as

$$Z^6 + \boxed{\frac{-\mu t Y^4 + \lambda t Y + Y^3 + t}{t Y^2}} Z^5 + \frac{\mu t + \lambda}{t} Z^4 - 2\frac{\mu\left(Y^4\mu - \frac{1}{2}Y^2\lambda^2 - Y\lambda - 1\right)t^2 - \mu t Y^3 + \frac{Y^2}{2}}{t^2 Y^2} Z^3$$
$$-\frac{\mu\left(\mu t + \lambda\right)}{t} Z^2 + \frac{\mu^2\left(-\mu t Y^4 + \lambda t Y + Y^3 + t\right)}{t Y^2} Z - \mu^3.$$

The coefficient of Z^5 is nonconstant, hence we have the following pair of non-trivial Galois invariants $(I(X,t), J(Y,t))$

$$\left(-\frac{\left(\lambda^2\mu X^3 + X^6 + \mu X^4 - \mu^2 X^2 - \mu^3\right)t^2 + X^2\lambda\left(X^2-\mu\right)t - X^3}{t^2 X\left(X^2+\mu\right)^2}, \frac{-\mu t Y^4 + \lambda t Y + Y^3 + t}{t Y^2}\right).$$

We check that $\frac{I(X,t)-J(X,t)}{\widetilde{K}(X,Y,t)}$ has poles of bounded order at 0, hence it is a pair of invariants. Moreover, Theorem 4.5 says that $k_{\text{inv}} = k\left(I(x, 1/S(x,y))\right)$, so any pair of Galois invariants for $\mathcal{G}_3^{\lambda,\mu}$ is a fraction in the pair $(I(X,t), J(Y,t))$.

Example 4.7. The orbit type of the model with step polynomial $S(X,Y) = X + \frac{X}{Y} + \frac{Y}{X^2} + \frac{1}{X^2}$ is $\mathcal{O}_{18}$ (see Figure 3.1). With our method, we find the following pair of Galois invariants

$$\left(\frac{\left(-X^9 - 3X^3 + 1\right)t^2 + \left(X^8 + X^5 - 2X^2\right)t + X^4}{X^6 t^2}, \frac{\left(Y^3 + 3Y + 1\right)\left(Y+1\right)^3 t^3 + Y^4}{Y^2 t^3\left(Y+1\right)^3}\right).$$

One can also check by looking at the t-expansions that it is a pair of invariants.

5. Decoupling

In this section, we study the Galoisian formulation of the notion of decoupling introduced in Section 2.2. In particular, assuming the finiteness of the orbit, we show how the Galois decoupling of a rational fraction $H(X,Y,t)$ can be, analogously to the small steps case, tested and constructed if it exists via the evaluation on certain 0-chains on the orbit.

5.1. **Galois formulation of decoupling.** As in the previous section, we adapt the notion of decoupling introduced in Section 2.2 to our Galoisian framework. The definition below is the straightforward analogue of Definition 4.7 in [BBMR21] in the large steps framework.

Definition 5.1 (Galois decoupling of a fraction)**.** Let $H(X,Y,t)$ be a regular fraction in $\mathbb{C}(X,Y,t)$. A pair of fractions $(F(X,t), G(Y,t))$ in $\mathbb{C}(X,t)\times\mathbb{C}(Y,t)$ is called a *Galois decoupling pair* for the fraction H if there exists a regular fraction $R(X,Y,t)$ satisfying

$$H(X,Y,t) = F(X,t) + G(Y,t) + \widetilde{K}(X,Y,t)R(X,Y,t).$$

We call such an identity a *Galois decoupling* of the fraction H.

Thanks to Proposition 3.20, if a regular fraction admits a decoupling with respect to the t-equivalence then it admits a Galois decoupling. Analogously to the notion of Galois invariants and as a corollary of Proposition 3.18, one can interpret the Galois decoupling as an identity in the extension $k(\mathcal{O})$.

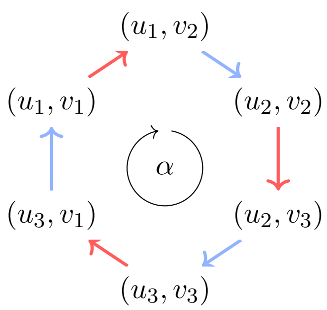
Proposition 5.2. *Let H be a regular fraction in $\mathbb{C}(X,Y,t)$. Then H admits a Galois decoupling if and only if $H_{(x,y)}$ can be written as $f+g$ with f in $k(x)$ and g in $k(y)$.*

By an abuse of terminology, we call any identity $H_{(x,y)} = f+g$ with f in $k(x)$ and g in $k(y)$ a Galois decoupling of H. Furthermore, these last two conditions can be reformulated algebraically via the Galois correspondence applied to the extensions $k(\mathcal{O})|k(x)$ and $k(\mathcal{O})|k(y)$: $H_{(x,y)} = f+g$ with f fixed by G_x and g fixed by G_y.

Given a regular fraction H, one could try to use the normal basis theorem (see [Lan02, chapter 6, § 13]) to test the existence of a Galois decoupling for H. The normal basis theorem shows that there exists a k_{inv}-basis of $k(\mathcal{O})$ of the form $(\sigma(\alpha))_{\sigma\in G}$ for some $\alpha \in k(\mathcal{O})$. The action of G_x and G_y on this basis is given by permutation matrices, and thus the linear constraints for the Galois decoupling of $H_{(x,y)}$ is equivalent to a system of linear equations. Unfortunately the computation of a normal basis requires *a priori* a complete knowledge of the Galois group G, whose computation is a difficult problem. Therefore, we present in the rest of the section a construction of a Galois decoupling test which relies entirely on the orbit and its Galoisian structure.

5.2. **The decoupling of (x,y) in the orbit.**

Definition 5.3. Let α be a 0-chain of the orbit. We say that α *cancels decoupled fractions* if $H_\alpha = 0$ for any regular fraction $H(X,Y,t)$ of $\mathbb{C}(X,t)+\mathbb{C}(Y,t)$.



$$\alpha = (u_1,v_2) - (u_1,v_1) + (u_2,v_3) - (u_2,v_2) + (u_3,v_1) - (u_3,v_3)$$

$$F(X)_\alpha = F(u_1) - F(u_1) + F(u_2) - F(u_2) + F(u_3) - F(u_3) = 0$$

$$G(Y)_\alpha = G(v_2) - G(v_1) + G(v_3) - G(v_2) + G(v_1) - G(v_3) = 0$$

FIGURE 5.1. The 0-chain induced by a bicolored loop cancels decoupled fractions

We recall that a *path* in the graph of the orbit is a sequence of vertices $(a_1, a_2, \dots, a_{n+1})$ such that $a_i \sim a_{i+1}$ for all $0 \leq i \leq n$. The length of $(a_1, a_2, \dots, a_{n+1})$ is the number of adjacencies (that is n). A path is called a *loop* [§] if $a_{n+1} = a_1$. A loop is called *simple* if only its first and last vertices are equal.

Example 5.4. A *bicolored loop* is a loop $(a_1, a_2, \dots, a_{2n+1})$ of even length such that for all i, $a_{2i} \sim^x a_{2i-1}$ and $a_{2i+1} \sim^y a_{2i}$. One associates to $(a_1, a_2, \dots, a_{2n+1})$ the 0-chain

$$\alpha = \sum_{i=1}^{2n} (-1)^i a_i = \sum_{i=1}^{n} (a_{2i} - a_{2i-1}) = \sum_{i=1}^{n} (a_{2i} - a_{2i+1}).$$

Taking $F(X,t)$ a regular fraction in $\mathbb{C}(X,t)$, one observes that for all i, $F_{a_{2i}} - F_{a_{2i-1}} = 0$, as vertices a_{2i} and a_{2i-1} share their first coordinate. Symmetrically, taking $G(Y,t)$ a regular fraction in $\mathbb{C}(Y,t)$, $G_{a_{2i+1}} - G_{a_{2i}} = 0$. Therefore, $F_\alpha = G_\alpha = 0$. Hence, the 0-*chains induced by bicolored loops* cancel decoupled fractions. Figure 5.1 illustrates this observation.

Example 5.4 is fundamental for picturing the 0-chains that cancel decoupled fractions because of the following stronger result:

Proposition 5.5. *A* 0*-chain cancels decoupled fractions if and only if it can be decomposed as a* $\mathbb{C}$*-linear*[¶] *combination of* 0*-chains induced by bicolored loops.*

There exists an elementary graph theoretic proof of this fact. However, we choose to postpone the proof of Proposition 5.5 after the proof of Theorem 5.24, which is an algebraic reformulation of the condition for a 0-chain to cancel decoupled fractions.

Example 5.6. A straightforward application of this observation, is the following obstruction for the existence of a Galois decoupling of XY. Consider an orbit whose graph contains a bicolored square (bicolored loop of length 4), with associated 0-chain $\alpha = (u_1, v_1) - (u_1, v_2) + (u_2, v_2) - (u_2, v_1)$ (thus with $u_1 \neq u_2$ and $v_1 \neq v_2$). The evaluation of XY on this 0-chain factors as $(XY)_\alpha = (u_1 - u_2)(v_1 - v_2)$, which is always nonzero. Therefore, if the orbit of a model $\mathcal{W}$ contains such a cycle, then XY never admits a Galois decoupling and thereby a decoupling in the sense of the t-equivalence. Thus, we can conclude that for models with orbit $\widetilde{\mathcal{O}_{12}}$ (see Figure 3.1) or Hadamard (see § 5.6.3), or the "Fan model" (see § D.0.3), the fraction XY never admits a decoupling.

For now, we only saw that the canceling of a regular fraction on 0-chains that cancel decoupled fraction is just a necessary condition for the Galois decoupling of this fraction. We prove in this section that this condition is sufficient and that one only needs to consider the evaluation on a precise 0-chain.

For small steps walks with finite orbit, there is only one bicolored loop and thereby only one 0-chain α induced by a bicolored loop. Theorem 4.11 in [BBMR21] shows that a regular fraction admits a Galois decoupling if and only its evaluation on α is zero. More precisely, Bernardi, Bousquet-Mélou and Raschel proved an explicit identity in the algebra of the group of the walk. Rephrasing their equality in terms of 0-chains in the orbit, we introduce the notion of decoupling of the pair (x, y) in the orbit as follows:

[§] We know that the terminology *loop* is unorthodox, however we follow [Gib81, Definition 1.8].

[¶] Note that if the 0-chain is with integer coefficients, one can choose the combination with integer coefficients as well.

Definition 5.7 (Decoupling of (x,y))**.** We say that (x,y) *admits a decoupling in the orbit* if there exist 0-chains $\widetilde{\gamma_x}$, $\widetilde{\gamma_y}$, α such that

- $(x,y) = \widetilde{\gamma_x} + \widetilde{\gamma_y} + \alpha$
- $\sigma_x \cdot \widetilde{\gamma_x} = \widetilde{\gamma_x}$ for all $\sigma_x \in G_x$
- $\sigma_y \cdot \widetilde{\gamma_y} = \widetilde{\gamma_y}$ for all $\sigma_y \in G_y$
- the 0-chain α cancels decoupled fractions

In that case, we call the identity $(x,y) = \widetilde{\gamma_x} + \widetilde{\gamma_y} + \alpha$ a *decoupling of* (x,y).

Note that if $(\widetilde{\gamma_x}, \widetilde{\gamma_y}, \alpha)$ is a decoupling of (x,y) then the 0-chain α is equal to $(x,y) - \widetilde{\gamma_x} - \widetilde{\gamma_y}$. Hence, when giving such a decoupling, we will often state explicitly only $\widetilde{\gamma_x}$ and $\widetilde{\gamma_y}$.

Example 5.8. For the orbit of the model $\mathcal{G}_3^{\lambda,\mu}$, a decoupling equation is as follow, as constructed in Subsection 5.6.2: $(x,y) = \widetilde{\gamma_x} + \widetilde{\gamma_y} + \alpha$ with

$$\widetilde{\gamma_x} = \Bigg(\frac{1}{2} \left((x,y) + (x, \overline{xy}) \right) - \frac{1}{8} \Big((z,y) + (-\mu \overline{xy^2 z}, y) + (xy^2 z, \overline{xy}) + (-\mu \overline{z}, \overline{xy}) \Big)$$
$$+ \frac{1}{8} \Big((xy^2 z, \overline{yz}) + (z, \overline{yz}) + (-\mu \overline{xy^2 z}, -\overline{\mu} xyz) + (-\mu \overline{z}, -\overline{\mu} xyz) \Big) \Bigg)$$
$$\text{and } \widetilde{\gamma_y} = \left(\frac{1}{4} \Big((x,y) + (z,y) + (-\mu \overline{xy^2 z}, y) \Big) - \frac{1}{4} \Big((x, \overline{xy}) + (z, \overline{yz}) + (-\mu \overline{xy^2 z}, -\overline{\mu} xyz) \Big) \right).$$

The 0-chain α is represented in Figure 5.2. It is the sum of the two 0-chains α_1 and α_2 induced by the two bicolored loops where the weights of the 0-chains α_1 and α_2 are written in grey next to their corresponding vertex.

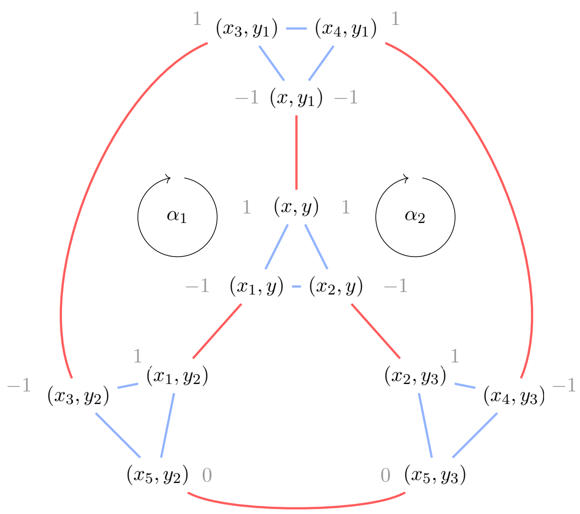


FIGURE 5.2. A 0-chain of $\mathcal{O}_{12}$ characterizing decoupled fractions for the model $\mathcal{G}_3^{\lambda,\mu}$ where the weights are written in grey next to their corresponding vertex.

The relation between the notion of decoupling of (x,y) in the orbit and the notion of Galois decoupling is the detailed in the following proposition.

Proposition 5.9. *Assume that* $(x,y) = \widetilde{\gamma_x} + \widetilde{\gamma_y} + \alpha$ *is a decoupling of* (x,y), *and let* $H(X,Y,t)$ *be a regular fraction. Then the following assertions are equivalent:*

1. H *admits a Galois decoupling*
2. $H_\alpha = 0$
3. $H_{(x,y)} = H_{\widetilde{\gamma_x}} + H_{\widetilde{\gamma_y}}$ *is a Galois decoupling of* H.

Proof. (1) $\Rightarrow$ (2): By definition of a Galois decoupling of (x,y), α cancels decoupled fractions.

(2) $\Rightarrow$ (3): Evaluating H on the decoupling of (x,y) yields the identity $H_{(x,y)} = H_{\widetilde{\gamma_x}} + H_{\widetilde{\gamma_y}}$. Moreover, since $\widetilde{\gamma_x}$ (resp. $\widetilde{\gamma_y}$) is fixed by G_x (resp. G_y), then Lemma 3.19 and the Galois correspondence in the extensions $k(\mathcal{O})|k(x)$ and $k(\mathcal{O})|k(y)$ ensure that $H_{\widetilde{\gamma_x}}$ and $H_{\widetilde{\gamma_y}}$ belong respectively to $k(x)$ and $k(y)$, hence $H_{\widetilde{\gamma_x}} + H_{\widetilde{\gamma_y}}$ is a Galois decoupling of H.

(3) $\Rightarrow$ (1) is obvious. □

Therefore, if we solve the decoupling problem of (x,y) in the orbit, we also solve the Galois decoupling problem for rational fractions: an explicit decoupling of (x,y) will grant us with a simple test to check whether a regular fraction admits a Galois decoupling (some orbit sum is zero), and an effective way to construct the associated Galois decoupling based on orbit sum computations.

We now state the main result of this section, whose proof will follow from Theorem 5.25:

Theorem 5.10 (Decoupling)**.** *If the orbit* $\mathcal{O}$ *is finite, then* (x,y) *always admits a decoupling in the orbit.*

Note that the construction of the decoupling in the next sections gives a decoupling whose 0-chains have rational coefficients.

The rest of this section is dedicated to the proof of Theorem 5.10 and to the effective construction of the decoupling of (x,y) in the orbit.

5.3. **Pseudo-decoupling.** We define here a more flexible notion of decoupling in the orbit called *pseudo-decoupling*, mainly used in the proof of the main theorem.

Definition 5.11 (Pseudo-decoupling)**.** Let γ_x and γ_y be two 0-chains. We call the pair (γ_x, γ_y) a *pseudo-decoupling* of (x,y) if for every regular fraction $H(X,Y,t)$ that admits a Galois decoupling, the equation $H_{(x,y)} = H_{\gamma_x} + H_{\gamma_y}$ is a Galois decoupling of H, that is, $H_{\gamma_x} \in k(x)$ and $H_{\gamma_y} \in k(y)$.

For instance, if $(x,y) = \widetilde{\gamma_x} + \widetilde{\gamma_y} + \alpha$ is a decoupling of (x,y), then the pair $(\widetilde{\gamma_x}, \widetilde{\gamma_y})$ is a pseudo-decoupling of (x,y) by Proposition 5.9.

The following theorem shows how a pseudo-decoupling yields a decoupling.

Let G' be a subgroup of G. We denote by $[G']$ the formal sum $\frac{1}{|G'|}\sum_{\sigma \in G'} \sigma$. From a Galois theoretic point of view, if G' is the Galois group of some subextension $k(\mathcal{O})|M$, then $[G']$ is the trace of the field extension $k(\mathcal{O})|M$.

Theorem 5.12. *If a pair* (γ_x, γ_y) *is a pseudo-decoupling of* (x,y), *then* (x,y) *admits a decoupling of the form*

$$(x,y) = \widetilde{\gamma_x} + \widetilde{\gamma_y} + \alpha$$

where $\widetilde{\gamma_x} = [G_x] \cdot \gamma_x$ *and* $\widetilde{\gamma_y} = [G_y] \cdot \gamma_y$.

Proof. By construction, the 0-chains $\widetilde{\gamma_x}$ and $\widetilde{\gamma_y}$ are fixed under the respective actions of G_x and G_y. Therefore, we only need to prove that α cancels decoupled fractions, for which purpose we rewrite it as the sum of three terms

$$\alpha = ((x,y) - \gamma_x - \gamma_y) + (\gamma_x - [G_x] \cdot \gamma_x) + (\gamma_y - [G_y] \cdot \gamma_y).$$

Let H be a regular fraction that admits a Galois decoupling. Then

- $H_{(x,y)} - H_{\gamma_x} - H_{\gamma_y} = 0$ by definition of the pseudo-decoupling (γ_x, γ_y).
- For σ_x in G_x, we compute $H_{\gamma_x - \sigma_x \cdot \gamma_x} = H_{\gamma_x} - \sigma_x(H_{\gamma_x})$ thanks to Lemma 3.19. As H_{γ_x} is in $k(x)$, it turns out that $H_{\gamma_x - \sigma_x \cdot \gamma_x}$ is zero. Since $\frac{1}{|G_x|} \sum_{\sigma_x \in G_x} (\gamma_x - \sigma_x \cdot \gamma_x) = \gamma_x - [G_x] \cdot \gamma_x$, we obtain that $\gamma_x - [G_x] \cdot \gamma_x$ cancels H.
- The argument for $\gamma_y - [G_y]\gamma_y$ is similar.

Thus $H_\alpha = 0$, which concludes the proof. □

We finish this subsection with two important lemmas.

Lemma 5.13. *If the pair (γ_x, γ_y) is a pseudo-decoupling of (x, y), and α and α' are 0-chains that cancel decoupled fractions, then $(\gamma_x + \alpha, \gamma_y + \alpha')$ is also a pseudo-decoupling of (x, y).*

Proof. Let $H(X, Y, t)$ be a regular fraction that admits a Galois decoupling. By definition of α and α', we have $H_\alpha = H_{\alpha'} = 0$, which by linearity proves that $H_{\gamma_x+\alpha} = H_{\gamma_x}$ and $H_{\gamma_y+\alpha'} = H_{\gamma_y}$. Since (γ_x, γ_y) is a pseudo-decoupling of (x, y), the equation $H_{(x,y)} = H_{\gamma_x} + H_{\gamma_y} = H_{\gamma_x+\alpha} + H_{\gamma_x+\alpha'}$ is a Galois decoupling of H proving that $(\gamma_x+\alpha, \gamma_y+\alpha')$ is also a pseudo-decoupling of (x, y). □

Lemma 5.14. *If two 0-chains γ_x and γ_y satisfy the following conditions*

- $(x, y) = \gamma_x + \gamma_y$
- *for all $\sigma_x \in G_x$, the 0-chain $\sigma_x \cdot \gamma_x - \gamma_x$ cancels decoupled fractions*
- *for all $\sigma_y \in G_y$, the 0-chain $\sigma_y \cdot \gamma_y - \gamma_y$ cancels decoupled fractions*

then (γ_x, γ_y) is a pseudo-decoupling of (x, y).

Proof. Let H be a regular fraction which admits a Galois decoupling. As $H_{(x,y)} = H_{\gamma_x} + H_{\gamma_y}$ from the first point, one only needs to show that H_{γ_x} is in $k(x)$ and that H_{γ_y} is in $k(y)$. Let σ_x be in G_x, then $\sigma_x(H_{\gamma_x}) = H_{\sigma_x \cdot \gamma_x} = H_{(\sigma_x \cdot \gamma_x - \gamma_x) + \gamma_x} = H_{\gamma_x}$ because $(\sigma_x \cdot \gamma_x - \gamma_x)$ cancels decoupled fractions. Therefore, the Galois correspondence proves that H_{γ_x} is in $k(x)$. The same argument proves that H_{γ_y} is in $k(y)$. □

5.4. **Graph homology and construction of the decoupling.** Our construction of a decoupling relies on the graph structure of the orbit $\mathcal{O}$, and in particular on the formalism of graph homology.

5.4.1. *Basic graph homology.* We recall here the basic definitions of graph homology and the properties that will be used in the construction of the decoupling (see [Gib81] for a comprehensive introduction to graph homology).

Definition 5.15. A *graph* (undirected) is a pair $\Gamma = (V, E)$ where V is the set of vertices and $E \subset \{\{a, a'\} \mid a, a' \in V, a \neq a'\}$ is the set of edges. A *subgraph* of Γ is a graph $\Gamma' = (V', E')$ such that $V' \subset V$ and $E' \subset E$.

An *oriented graph* is a pair $\Gamma = (V, E^+)$ where V is the set of vertices and $E^+ \subset \{(a, a') \mid a, a' \in V, a \neq a'\}$ the set of *arcs* (oriented edges) such that if $(a, a') \in E^+$ then $(a', a) \notin E^+$. An *orientation* of a graph $\Gamma = (V, E)$ is an oriented graph $\Gamma' = (V, E^+)$ such that the map $E^+ \to E$ which maps (a, a') to $\{a, a'\}$ is a bijection.

Note that every graph can be given an orientation by freely choosing an origin for each edge. Conversely, given an oriented graph $\Gamma = (V, E^+)$, one can consider the associated undirected graph (V, E) where $E = \{\{a, a'\}$ such that $(a, a') \in E^+$ or $(a', a) \in E^+\}$. In what follows, the notions of graph homomorphism, path, connected components concern the structure of undirected graph.

Example 5.16. The graphs considered here are the graph induced by the orbit $(\mathcal{O}, \sim)$ still denoted $\mathcal{O}$, and the two subgraphs of the orbit restricted to each individual type of adjacency, which are $\mathcal{O}^x = (\mathcal{O}, \sim^x)$ and $\mathcal{O}^y = (\mathcal{O}, \sim^y)$.

We now introduce the chain complex attached to an oriented graph.

Definition 5.17. Let $\Gamma = (V, E^+)$ be an oriented graph and K a field. The space $C_0(\Gamma)$ of 0-chains of Γ is the free K-vector space spanned by the vertices of V. Similarly, the space $C_1(\Gamma)$ of 1-chains of Γ is the free K-vector space spanned by the arcs of E^+. We turn $C_*(\Gamma)$ into a chain complex by defining the *boundary* homomorphism, which is the K-linear map defined by

$$\begin{array}{rccc} \partial\colon & C_1(\Gamma) & \longrightarrow & C_0(\Gamma) \\ & (a, a') \in E^+ & \longmapsto & a' - a \end{array}$$

As the reader notices, the chain complex has only been defined for an oriented graph. Nonetheless, if (V, E_1^+) and (V, E_2^+) are two orientations of a graph Γ, it is easy to see that the associated chain complexes are isomorphic [Gib81, 1.21 (3)]. When the context is clear, we shall abuse notation and define a chain complex $C_*(\Gamma)$ of a graph Γ as the chain complex of the oriented graph (V, E^+) where E^+ is an arbitrary orientation of Γ.

We make the following convention. Let a and a' be two adjacent vertices of Γ. Given an orientation E^+ of Γ, we abuse notation and denote by (a, a') the 1-chain

$$(a, a') = \begin{cases} (a, a') & \text{if } (a, a') \text{ is in } E^+ \\ -(a', a) & \text{otherwise} \end{cases}$$

This notation is extremely convenient, because for two adjacent vertices of Γ, the boundary homomorphism always satisfies $\partial((a, a')) = a' - a$ and $(a, a') = -(a', a)$.

Definition 5.18. Let $\Gamma = (V, E^+)$ be an oriented graph. A 1-chain c which satisfies $\partial(c) = 0$ is called a 1-*cycle*.

Example 5.19 (1-chain induced by a path)**.** Let $\Gamma = (V, E)$ be a graph and let $(a_1, a_2, \ldots, a_{n+1})$ be a path in Γ, that is, a sequence of vertices such that a_i is adjacent to a_{i+1} for $i = 1, \ldots, n$. Given an arbitrary orientation E^+ of Γ, we define the 1-chain $p = \sum_{i=1}^{n} (a_i, a_{i+1})$, and we call it the 1-*chain induced by the path* $(a_1, a_2, \ldots, a_{n+1})$. By telescoping, $\partial(p) = a_{n+1} - a_1$, therefore if the path is a loop of Γ then p is a 1-cycle, hence the name. Every 1-cycle is a linear combination of 1-cycles induced by the simple loops of the graph, that is, loops with no repeated vertex (see [Gib81, Theorem 1.20]).

We recall that a graph is called *connected* if any two vertices are joined by a path. The reader should note that the notion of path does not take into account a potential orientation of the edges. Every finite graph is the disjoint union of finitely many connected components which are maximal connected subgraphs. Any orientation of a graph induces by restriction an orientation on its subgraphs and thereby on its connected components. With this convention, it turns out that the chain complex of a finite oriented graph is isomorphic to the direct sum of the chain complexes of its connected components. Hence, it is harmless to extend Theorem 1.23 in [Gib81] to the case of a non-connected graph.

Proposition 5.20. *Let $\Gamma = (V, E)$ be a graph, and let $(\Gamma_i = (V_i, E_i))_{i=1,\dots,r}$ be its connected components. Define the* augmentation map $\varepsilon\colon C_0(\Gamma) \to K^r$ *by* $\varepsilon(\sum_{a\in V} \lambda_a a) = (\sum_{a\in V_i} \lambda_a)_{i=1,\dots,r}$. *Then,* $\operatorname{Ker} \varepsilon = \operatorname{Im} \partial$.

Let $\Gamma = (V, E)$ be a graph and let σ be a graph endomorphism of Γ. Fixing an orientation E^+ on Γ, we let σ act on the space of 0 and 1-chains by K-linearity via:

$$\sigma \cdot a = \sigma(a) \text{ for any } a \text{ in } V \text{ and } \sigma \cdot (a, a') = (\sigma(a), \sigma(a')) \text{ for any } (a, a') \text{ in } E^+.$$

The reader should note that the action on the space of 1-chains uses the convention on the arc notation introduced at the beginning of the section. It is easily seen that the action of a graph endomorphism of Γ is compatible with the boundary homomorphism of the chain complex $C_*(\Gamma)$.

Proposition 5.21. *Let $\Gamma = (V, E)$ be a graph and σ be a graph endomorphism of Γ. Then σ induces a* chain map *on $C_*(\Gamma)$, which means that the following diagram of K-linear maps is commutative.*

$$\begin{array}{ccc} C_1(\Gamma) & \xrightarrow{\partial} & C_0(\Gamma) \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} \\ C_1(\Gamma) & \xrightarrow{\partial} & C_0(\Gamma) \end{array}$$

5.4.2. *The chain complex of the orbit and algebraic description of bicolored loops.* We now apply the homological formalism to the graphs associated with the orbit $\mathcal{O}$ with base field $\mathbb{C}$ (see Example 5.16). We fix once for all an orientation on $\mathcal{O}$ which induces an orientation on the subgraphs $\mathcal{O}^x$ and $\mathcal{O}^y$. Quoting [Gib81, Remark 1.21],“ the choice of this orientation is just a technical device introduced to enable the computation of the boundary homomorphisms”. We denote by ∂ (resp. ∂^x, ∂^y) the boundary homomorphism on the connected graph $\mathcal{O}$ (resp. the non-connected graphs $\mathcal{O}^x$, $\mathcal{O}^y$). Moreover, we denote by ε (resp. ε^x, ε^y) the augmentation map defined in Proposition 5.20 for $\mathcal{O}$ (resp. $\mathcal{O}^x$, $\mathcal{O}^y$).

Lemma 5.22. *The $\mathbb{C}$-vector space $C_1(\mathcal{O})$ is equal to $C_1(\mathcal{O}^x) \oplus C_1(\mathcal{O}^y)$ and the boundary homomorphism ∂ coincides with $\partial^x + \partial^y$ where one has extended ∂^x (resp. ∂^y) by zero on $C_1(\mathcal{O}^y)$ (resp. $C_1(\mathcal{O}^x)$).*

Proof. Every edge $\{a, a'\}$ of $\mathcal{O}$ is either an x-adjacency or an y-adjacency, and not both. Therefore, the set of arcs of an orientation of $\mathcal{O}$ is the disjoint union of the arcs of the orientations of $\mathcal{O}^x$ and $\mathcal{O}^y$, which thus induces a direct sum decomposition on the free vector space $C_1(\mathcal{O})$. The decomposition of the homomorphism ∂ follows directly. □

The action of the Galois group G on the vertices of $\mathcal{O}$ preserves the adjacency types of the edges (see Lemma 3.6). Therefore G acts by graph automorphisms on $\mathcal{O}^x$ and $\mathcal{O}^y$. Thus, Proposition 5.21 allows us to define the action of G on the chains of $\mathcal{O}^x$ and $\mathcal{O}^y$ in a compatible way with the decomposition of Lemma 5.22.

Proposition 5.23. *Let σ be in G. Then σ induces automorphisms of the chain complexes $C_*(\mathcal{O}), C_*(\mathcal{O}^x)$ and $C_*(\mathcal{O}^y)$ such that $\sigma \circ \partial^x = \partial^x \circ \sigma$ and $\sigma \circ \partial^y = \partial^y \circ \sigma$.*

The boundary homomorphisms ∂^x, ∂^y allow us to rewrite the 0-chains induced by bicolored loops as boundaries. If α is the 0-chain associated to a bicolored loop as in Example 5.4, then it is easily seen that $\alpha = \partial^x(p) = \partial^y(-p)$ with p the 1-chain as in Example 5.19. The homology formalism generalizes the above description to any 0-chain that cancels decoupled fractions.

Theorem 5.24. *Let α be a 0-chain. Then the following statements are equivalent:*

(1) *α cancels decoupled fractions.*
(2) *$\varepsilon^x(\alpha) = 0$ and $\varepsilon^y(\alpha) = 0$.*
(3) *There exists a 1-cycle c of $\mathcal{O}$ such that $\alpha = \partial^x(c)$.*
(3') *There exists a 1-cycle c of $\mathcal{O}$ such that $\alpha = \partial^y(c)$.*

Proof. (1) $\Rightarrow$ (2): Let α be a 0-chain that cancels decoupled fractions. The connected components of the graph $\mathcal{O}^x$ are of the form $\mathcal{O}^x_u = \{(u', v') \in \mathcal{O} \mid u' = u\}$ for the distinct left coordinates u of $\mathcal{O}$. Therefore, we decompose $\alpha = \sum_u \alpha_u$ where $\alpha_u = \sum_{v'} \lambda^u_{v'}(u, v')$ is a 0-chain with vertices in $\mathcal{O}^x_u$. Now, we consider the family of monomials $(X^i)_i$ which are obviously decoupled. Since α cancels decoupled fractions, the following holds for all i:

$$0 = (X^i)_\alpha = \sum_u (X^i)_{\alpha_u} = \sum_u \sum_{v' \,/\, (u,v') \in \mathcal{O}^x_u} \lambda^u_{v'} (X^i)_{(u,v')} = \sum_u \left(\sum_{v' \,/\, (u,v') \in \mathcal{O}^x_u} \lambda^u_{v'} \right) u^i = \sum_u \varepsilon^x(\alpha)_u u^i.$$

Because the elements u are distinct, this is a Vandermonde system, in which the unknowns are the $\varepsilon^x(\alpha)_u$, therefore we deduce that they are all equal to 0. Thus, $\varepsilon^x(\alpha) = 0$. The same argument yields $\varepsilon^y(\alpha) = 0$.

(2) $\Rightarrow$ (3) and (3'): Assume that $\varepsilon^x(\alpha) = 0$ and $\varepsilon^y(\alpha) = 0$. By Proposition 5.20, there exist c_x in $C_1(\mathcal{O}^x)$ and c_y in $C_1(\mathcal{O}^y)$ such that $\partial^x(c_x) = \alpha$ and $\partial^y(c_y) = \alpha$. Moreover,

$$\partial(c_x - c_y) = \partial(c_x) - \partial(c_y) = \partial^x(c_x) - \partial^y(c_y) = \alpha - \alpha = 0.$$

Therefore, $c = c_x - c_y$ is a 1-cycle of $\mathcal{O}$ which satisfies $\partial^x(c) = \alpha$ and $\partial^y(-c) = \alpha$.

(3) $\Leftrightarrow$ (3'): Let c be a 1-cycle of $\mathcal{O}$, then $\partial^x(c) = \partial(c) - \partial^y(c) = \partial^y(-c)$. This proves the equivalence.

(3) $\Rightarrow$ (1): Assume that $\alpha = \partial^x(c) = \partial^y(-c)$ for c a cycle of $\mathcal{O}$. Now, let $e = ((u, v), (u, v'))$ be an arc of $\mathcal{O}^x$ and take $F(X, t) \in \mathbb{C}(X, t)$. Then $F_{\partial^x(e)} = F(u, 1/S(x, y)) - F(u, 1/S(x, y)) = 0$. Therefore, by $\mathbb{C}$-linearity, this implies that $F_\alpha = F_{\partial^x(c)} = 0$. Symmetrically, if $G(Y, t) \in \mathbb{C}(Y, t)$, then we deduce that $G_\alpha = G_{\partial^y(-c)} = 0$, which concludes the proof. □

We now apply this pleasant characterization to prove our earlier claim that 0-chains that cancel decoupled fractions are induced by $\mathbb{C}$-linear combinations of 1-cycles induced by bicolored loops.

Proof of Proposition 5.5. Let α be a 0-chain which cancels decoupled fractions, then by (3) of Theorem 5.24, we can write it $\alpha = \partial^x(c) = -\partial^y(c)$ with c a 1-cycle of $\mathcal{O}$.

Since the 1-cycles induced by the simple loops of $\mathcal{O}$ generate the 1-cycles of $\mathcal{O}$ (see [Gib81, Theorem 1.20]), we can assume without loss of generality that c is induced by a simple loop $p = (a_1, a_2, \dots, a_n)$ of $\mathcal{O}$.

Moreover, if consecutive arcs $e_i, \dots, e_{i+k-1} = (a_i, a_{i+1}), (a_{i+1}, a_{i+2}), \dots, (a_{i+k-1}, a_{i+k})$ of p are of the same adjacency type (say x), then since the monochromatic components of $\mathcal{O}$ are cliques, (a_i, a_{i+k}) is an arc of $\mathcal{O}$. Therefore, $\partial^x(e_i + \cdots + e_{i+k-1}) = \partial^x(e_i + \cdots + e_{i+k-1} + (a_{i+k}, a_i)) + \partial^x((a_i, a_{i+k}))$, the first term being zero because it is the boundary of a monochromatic cycle. The exact same reasoning can be done for consecutive y-adjacencies. Thus, replacing consecutive arcs of the same adjacency type by one single arc of the same adjacency type, we can assume without loss of generality that c is the 1-chain induced by a simple bicolored loop. This proves that α is the 0-chain induced by a bicolored loop, finishing the proof. □

5.4.3. *Construction of the decoupling.* We now use the results of the previous subsections to construct a pseudo-decoupling of (x, y) on a finite orbit $\mathcal{O}$.

For $p = (p_a)_{a\in\mathcal{O}}$ a family of 1-chains, we consider the 0-chains

$$\gamma_x(p) = -\frac{1}{|\mathcal{O}|}\sum_a \partial^y(p_a) \text{ and } \gamma_y(p) = -\frac{1}{|\mathcal{O}|}\sum_a \partial^x(p_a)$$

where all sums run over $\mathcal{O}$. The $\mathbb{C}$-linearity of the boundary homomorphisms implies that γ_x and γ_y are $\mathbb{C}$-linear morphisms from $C_1(\mathcal{O})^{\mathcal{O}}$ to $C_0(\mathcal{O})$. We recall that ω is the 0-chain $\frac{1}{|\mathcal{O}|}\sum_{a\in\mathcal{O}} a$ defined in Lemma 4.4.

Theorem 5.25 (Decoupling theorem). *Let $p^x = (p^x_a)_{a\in\mathcal{O}}$ and $p^y = (p^y_a)_{a\in\mathcal{O}}$ be two families of 1-chains, that are such that for all $a \in \mathcal{O}$ one has*

$$\varepsilon^x(\partial(p^x_a) + (x, y) - a) = 0 \text{ and } \varepsilon^y(\partial(p^y_a) + (x, y) - a) = 0.$$

Then, the pair $(\omega + \gamma_x(p^x), \gamma_y(p^y))$ is a pseudo-decoupling of (x, y).

Proof. Let $(p^x_a)_{a\in\mathcal{O}}$ and $(p^y_a)_{a\in\mathcal{O}}$ be two families that satisfy the conditions of the theorem. For all a in $\mathcal{O}$, we have $\varepsilon^x(\partial(p^x_a)+(x, y)-a) = 0$. By Proposition 5.20 applied to $\Gamma = \mathcal{O}^x$, there exists $c^x_a \in C_1(\mathcal{O}^x)$ such that $\partial(c^x_a) = \partial(p^x_a) - a + (x, y)$, which rewrites as $\partial(p^x_a - c^x_a) = a - (x, y)$. We denote by c^x the family of 1-chains $(c^x_a)_a$. Note that $\partial^y(c^x_a) = 0$ for all a in $\mathcal{O}$ so that $\gamma_x(c^x) = 0$. Similarly, there exists a family $c^y = (c^y_a)_{a\in\mathcal{O}}$ in $C_1(\mathcal{O}^y)^{\mathcal{O}}$ such that we have $\partial(p^y_a - c^y_a) = a-(x, y)$ for all a in $\mathcal{O}$ and $\gamma_y(c^y) = 0$. Therefore, using the linearity of γ_x and γ_y, we find

$$(\omega + \gamma_x(p^x), \gamma_y(p^y)) = (\omega + \gamma_x(p^x - c^x), \gamma_y(p^x - c^x) + \gamma_y((p^y - c^y) - (p^x - c^x))).$$

By construction of c^x and c^y, the 1-chain $(p^y_a - c^y_a) - (p^x_a - c^x_a)$ is a 1-cycle for all a. Hence, by linearity of the boundary homomorphim ∂, the 1-chain $-\frac{1}{|\mathcal{O}|}\sum_a ((p^y_a - c^y_a) - (p^x_a - c^x_a))$ is a 1-cycle. Hence, by (3) of Theorem 5.24, the 0-chain $\gamma_y((p^y - c^y) - (p^x - c^x))$ cancels decoupled fractions.

Therefore, by Lemma 5.13, it only remains to show that the pair $(\omega+\gamma_x(p^x - c^x), \gamma_y(p^x - c^x))$ is a pseudo-decoupling of (x, y). Denote by $q = (q_a)_{a\in\mathcal{O}}$ the family of 1-chains $p^x - c^x$. Then,

$$a - (x, y) = \partial(p^x_a - c^x_a) = \partial(q_a) = \partial^y(q_a) + \partial^x(q_a).$$

Summing this identity over the orbit yields

$$\sum_{a\in\mathcal{O}} a - |\mathcal{O}|(x, y) = \sum_{a\in\mathcal{O}} \partial^y(q_a) + \sum_{a\in\mathcal{O}} \partial^x(q_a),$$

which can be rewritten as

$$(x, y) = (\omega + \gamma_x(q)) + \gamma_y(q).$$

In order to conclude that the pair $(\omega + \gamma_x(q), \gamma_y(q))$ is a pseudo-decoupling, we just need to check that, for all σ_x in G_x, the 0-chain $\sigma_x \cdot (\omega+\gamma_x(q))-(\omega+\gamma_x(q))$ cancels decoupled fractions, and that, for all σ_y in G_y, the 0-chain $\sigma_y \cdot \gamma_y(q) - \gamma_y(q)$ cancels decoupled fractions and apply Lemma 5.14.

Let σ_x be in G_x. Then by compatibility of G with the boundaries (Proposition 5.23), we compute

$$\begin{aligned}\sigma_x \cdot \gamma_x(q) &= -\frac{1}{|\mathcal{O}|}\sum_{a\in\mathcal{O}} \partial^y(\sigma_x \cdot q_a)\\ &= -\frac{1}{|\mathcal{O}|}\sum_{a\in\mathcal{O}} \partial^y(q_{\sigma_x\cdot a}) - \frac{1}{|\mathcal{O}|}\sum_{a\in\mathcal{O}} \partial^y(\sigma_x \cdot q_a - q_{\sigma_x\cdot a}).\end{aligned}$$

The homomorphism σ_x is a bijection on the vertices of $\mathcal{O}$, so the first sum on the right hand-side is equal to $\gamma_x(q)$ so that

$$\sigma_x \cdot \gamma_x(q) - \gamma_x(q) = \partial^y\left(-\frac{1}{|\mathcal{O}|}\sum_{a\in\mathcal{O}}(\sigma_x \cdot q_a - q_{\sigma_x\cdot a})\right). \tag{5.1}$$

Now, that since σ_x fixes x, we have $\sigma_x \cdot (x, y) = (x, v)$ for some v. Thus there exists c in $C_1(\mathcal{O}^x)$ such that $\sigma_x \cdot (x, y) - (x, y) = \partial(c)$. Then, for all $a \in \mathcal{O}$, we have

$$\partial(\sigma_x \cdot q_a - q_{\sigma_x\cdot a} + c) = (\sigma_x \cdot a - \sigma_x \cdot (x, y)) - (\sigma_x \cdot a - (x, y)) + (\sigma_x \cdot (x, y) - (x, y)) = 0.$$

Therefore, the 1-chain $\sigma_x \cdot q_a - q_{\sigma_x\cdot a} + c$ is a 1-cycle for all a so that $-\frac{1}{|\mathcal{O}|}\sum_a (\sigma_x \cdot q_a - q_{\sigma_x\cdot a} + c)$ is also a 1-cycle by linearity of the boundary homorphism ∂. Moreover, since c is in $C_1(\mathcal{O}^x)$, we have $\partial^y(\sigma_x \cdot q_a - q_{\sigma_x\cdot a}) = \partial^y(\sigma_x \cdot q_a - q_{\sigma_x\cdot a} + c)$ for all a in $\mathcal{O}$, so from (5.1), we conclude that $\sigma_x\cdot\gamma_x(q)-\gamma_x(q) = \partial^y\left(-\frac{1}{|\mathcal{O}|}\sum_a(\sigma_x \cdot q_a - q_{\sigma_x\cdot a} + c)\right)$. Theorem 5.24 implies that $\sigma_x\cdot\gamma_x(q)-\gamma_x(q)$ cancels decoupled fractions. Finally, as ω is fixed by σ_x, we deduce that $\sigma_x \cdot (\omega + \gamma_x(q)) - (\omega + \gamma_x(q)) = \sigma_x\cdot\gamma_x(q)-\gamma_x(q)$ cancels decoupled fractions. The proof for $\sigma_y\gamma_y(q)-\gamma_y(q)$ is completely analogous. □

We can now prove the existence of a decoupling of (x, y) for any finite orbit.

Proof of Theorem 5.10. The graph $\mathcal{O}$ is connected. Hence, for every $a \in \mathcal{O}$, there exists a path from (x, y) to a. Denoting by $p^x_a = p^y_a$ the associated 1-chain, we have $\partial(p^x_a) = a - (x, y)$ (see Example 5.19). Therefore, the families $(p^x_a)_{a\in\mathcal{O}}$ and $(p^y_a)_{a\in\mathcal{O}}$ satisfy the assumptions of Theorem 5.25 leading to the existence of a pseudo-decoupling. Theorem 5.12 establishes the existence of a decoupling obtained from a pseudo-decoupling concluding the proof of Theorem 5.10: if the orbit is finite, the pair (x, y) always admits a decoupling in the orbit. □

In [BMPF+22, Definition 6.1], the authors introduce a notion of a multiplicative decoupling of a regular fraction. In our context, we say that a regular fraction $H(X, Y)$ has a multiplicative Galois decoupling if and only if there exists a positive integer m such that

$$H(X, Y)^m = F(X, t)G(Y, t) + \widetilde{K}(X, Y, t)P(X, Y, t),$$

for some rational fractions $F(X, t), G(Y, t)$ and a regular fraction $P(X, Y, t)$.

Theorem 5.10 yields a decoupling of (x, y) with 0-chains $\widetilde{\gamma_x}, \widetilde{\gamma_y}$ and α having rational coefficients. Let d be the common denominator of the rational coefficients of $\widetilde{\gamma_x}, \widetilde{\gamma_y}$ and α which is easily seen to divide the size of the orbit in the proof of Theorem 5.25 when the input 1-chains in p^x and p^y all have integer coefficients. Then, the 0-chains $d\widetilde{\gamma_x}, d\widetilde{\gamma_y}, d\alpha$ have integer coefficients. For such chains, one can define a multiplicative evaluation:

For a 0-chain $c = \sum_{u,v} c_{u,v}(u, v)$ with integer coefficients, define

$$H^{\text{mul}}_c = \prod_{u,v} H(u, v, 1/S(x, y))^{c_{u,v}}.$$

As a direct corollary of the existence of a decoupling in the orbit, the following lemma gives an explicit procedure to test and construct, when it exists, the multiplicative Galois decoupling of a regular fraction H.

Lemma 5.26. *The following statements are equivalent:*

- *$H(X,Y,t)$ has a multiplicative Galois decoupling.*
- *There exists a a positive integer m such that $\left(H^{mul}_{d\alpha}\right)^m = 1$.*

Proof. From Proposition 3.18, the regular fraction $H(X,Y,t)$ admits a multiplicative Galois decoupling if and only if there exist a positive integer m, $f(x) \in k(x)$ and $g(y) \in k(y)$ such that $H^m_{(x,y)} = f(x)g(y)$.
Let us assume that H admits a multiplicative Galois decoupling and let m be a positive integer such that $H(X,Y)^m = F(X)G(Y) + K(X,Y,t)P(X,Y,t)$ for some rationals fractions F, G and a regular fraction P. By multiplicative evaluation of the previous identity on $d\alpha$, we find that $\left(H^{\text{mul}}_{d\alpha}\right)^m = \left(F(X)^{\text{mul}}_{d\alpha}\right)^m \left(G(Y)^{\text{mul}}_{d\alpha}\right)^m$. It is clear that $d\alpha$ is a 0-chain with integer coefficients that cancels decoupled fractions. By Proposition 5.5, the chain $d\alpha$ is a $\mathbb{Z}$-linear combination of 0-chains induced by bicolored loops. One proves easily by a multiplicative analogue of Example 5.4 that if β is a 0-chain induced by a bicolored loop then $F(X)^{\text{mul}}_\beta = G(Y)^{\text{mul}}_\beta = 1$ which concludes the proof of the first implication.
Conversely, if there exists a positive integer m such that $\left(H^{\text{mul}}_{d\alpha}\right)^m = 1$, the decoupling $d\cdot(x,y) = d\widetilde{\gamma_x} + d\widetilde{\gamma_y} + d\alpha$ yields by multiplicative evaluation

$$\left(H^{\text{mul}}_{d(x,y)}\right)^m = H_{(x,y)}{}^{dm} = \left(H^{\text{mul}}_{d\widetilde{\gamma_x}}\right)^m \left(H^{\text{mul}}_{d\widetilde{\gamma_y}}\right)^m.$$

By definition of the decoupling of $(x,y) = \widetilde{\gamma_x} + \widetilde{\gamma_y} + \alpha$, we find that $\sigma \cdot d\widetilde{\gamma_x} = d\sigma \cdot \widetilde{\gamma_x} = d\widetilde{\gamma_x}$ for all $\sigma \in G_x$. A multiplicative analogue of Lemma 3.19 implies easily that $H^{\text{mul}}_{d\widetilde{\gamma_x}}$ is left fixed by G_x so that $H^{\text{mul}}_{d\widetilde{\gamma_x}}$ belongs to $k(x)$. A similar argument shows that $H^{\text{mul}}_{d\widetilde{\gamma_y}}$ belong to $k(y)$ which concludes the proof. □

5.5. **Effective construction.** The evaluation of a regular fraction at a vertex of the orbit, that is, at a pair of algebraic elements in $\mathbb{K}$ might be difficult from an algorithmic point of view since this requires to compute in an algebraic extension of $\mathbb{Q}(x,y)$. This is however the cost we may have to pay in our decoupling procedure if we choose random families of 1-chains satisfying the assumptions of Theorem 5.25.

In this section, we show how, under mild assumption on the *distance transitivity* of the graph of the orbit, one can construct a decoupling in the orbit expressed in terms of specific 0-chains that we call *level lines*. These level lines regroup vertices of the orbit that satisfy the same polynomial relations. Therefore, one can use symmetric functions and efficient methods from computer algebra to evaluate regular fractions on these level lines (see Appendix C).

Definition 5.27. Let a be a vertex of $\mathcal{O}$. We define the x-*distance* of a to be $d_x(a) = \inf\{d(a,a') \,|\, a' \sim^x (x,y)\}$, that is, the length of a shortest path in $\mathcal{O}$ from a to the clique $(x,\cdot)$.

Such a shortest path $(g_0, g_1, \dots, g_r)$, that is, $g_r = a$, $g_0 \sim^x (x,y)$ and $d_x(a) = r$, is called an x-*geodesic* for a. Note that we have $d_x(g_i) = i$ for all $i = 0, \dots, r$. We denote by $\mathcal{P}^x_a$ the set of 1-chains associated with x-geodesics for a as in Example 5.19.

The x-*level lines* $\mathcal{X}_0, \mathcal{X}_1, \dots$ are defined by $\mathcal{X}_i = \{a \in \mathcal{O} \,|\, d_x(a) = i\}$, and we associate to the level line $\mathcal{X}_i$ the 0-chain $X_i = \sum_{a \in \mathcal{X}_i} a$.

Analogously, we define the y-distance d_y, the set $\mathcal{P}_a^y$ of y-geodesics for a, the y-level lines $\mathcal{Y}_0, \mathcal{Y}_1, \dots$, and denote by Y_i the 0-chain associated with the y-level line $\mathcal{Y}_i$.

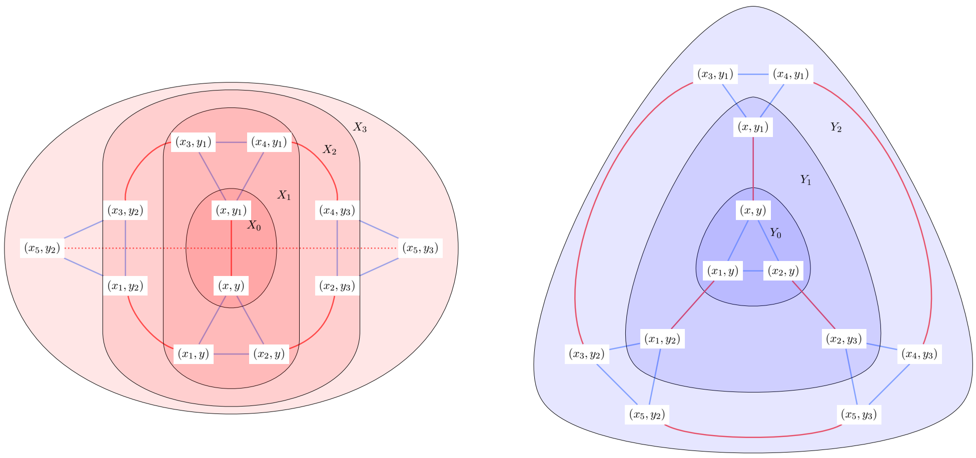


FIGURE 5.3. The level lines for the orbit $\mathcal{O}_{12}$

The level lines can be represented graphically, as in Figure 5.3, or in §5.6 or in Appendix D. The level lines and geodesics are our key tools to construct relevant collections of 1-chains satisfying the conditions of Theorem 5.25. First, the boundaries of a geodesic are easy to express.

Lemma 5.28. *Let a be a vertex of $\mathcal{O}$ and $(g_0, g_1, \dots, g_r)$ an x-geodesic for a. Then $g_i \sim^y g_{i-1}$ if and only if i is odd. Similarly, for $(g_0, g_1, \dots, g_r)$ an y-geodesic for a, then $g_i \sim^x g_{i-1}$ if and only if i is odd.*

Proof. Let $g = (g_0, g_1, \dots, g_r)$ be an x-geodesic of length r. Assume that there exists i such that $g_i \sim^x g_{i+1} \sim^x g_{i+2}$. By transitivity of $\sim^x$, this implies that $g_i \sim^x g_{i+2}$, contradicting the minimality of the geodesic g. Similarly, if there exists i such that $g_i \sim^y g_{i+1} \sim^y g_{i+2}$ then $g_i \sim^y g_{i+2}$, also contradicting the minimality of the geodesic. Therefore, the adjacency types of the edges of the geodesic alternate.

Finally, if $g_0 \sim^x g_1$, then this also contradicts the minimality of the geodesic because then $(x, y) \sim^x g_1$. This fixes the starting parity of the alternating adjacency types of edges of the geodesic, and thus $g_i \sim^y g_{i-1}$ if and only if i is odd. The case of an y-geodesic is symmetric. $\square$

Corollary 5.29. *Let a be a vertex of $\mathcal{O}$, $(g_0, g_1, \dots, g_r)$ an x-geodesic for a and g its associated 1-chain, then $\partial^y(g) = \sum_{\substack{1\leq i\leq r \\ i \text{ odd}}} g_i - g_{i-1}$. Analogously, for $(g_0, g_1, \dots, g_r)$ a y-geodesic for a then $\partial^x(g) = \sum_{\substack{1\leq i\leq r \\ i \text{ odd}}} g_i - g_{i-1}$.*

Recall from Section 5.4.1 that any graph automorphism τ of $\mathcal{O}$ act on the vertex a of $\mathcal{O}$ coordinate-wise and that we denote this action $\tau \cdot a$. We extend the action of τ to any path $(a_1, \dots, a_{n+1})$ as follows

$$\tau \cdot (a_1, \dots, a_{n+1}) = (\tau \cdot a_1, \dots, \tau \cdot a_{n+1}).$$

Note that this action is compatible with the action of graph automorphisms on 1-chains defined in Section 5.4.1. Indeed, if p is the 1-chain associated with the path $(a_1, \dots, a_{n+1})$ as in Example 5.19 then $\tau \cdot p$ is the 1-chain associated with the path $\tau \cdot (a_1, \dots, a_{n+1})$.

The following lemma shows that the geodesics and level lines satisfy some stability properties with respect to the action of elements of G_x and G_y viewed as subgroups of the group of graph automorphisms of $\mathcal{O}$ via their faithful action on the orbit.

Lemma 5.30. *Let σ_x be in G_x and a in $\mathcal{O}$. Then $d_x(\sigma_x \cdot a) = d_x(a)$. Moreover, if $(g_0, \dots, g_r)$ is an x-geodesic for a, then $\sigma_x \cdot (g_0, \dots, g_r)$ an x-geodesic for $\sigma_x \cdot a$. Analogously, if σ_y is in G_y and a in $\mathcal{O}$, then $d_y(\sigma_y \cdot a) = d_y(a)$, and if $(g_0, \dots, g_r)$ an y-geodesic for a, so is $\sigma_y \cdot (g_0, \dots, g_r)$ for $\sigma_y \cdot a$.*

Proof. Assume that $d_x(a) = r$. Then there exists an x-geodesic for a that is $(g_0, g_1, \dots, g_r)$ with $g_r = a$. Apply the graph automorphism σ_x to each of the vertices of this path. Then $(\sigma_x \cdot g_0, \sigma_x \cdot g_1, \dots, \sigma_x \cdot g_r)$ with $\sigma_x \cdot g_r = \sigma_x \cdot a$ is a path of the orbit. By definition, $g_0 \sim^x (x, y)$, thus $\sigma_x \cdot g_0 \sim^x (x, y)$ since x is fixed by G_x. Therefore, $d_x(\sigma_x \cdot a) \leq r = d_x(a)$. Since σ_x is an automorphism, we conclude that $d_x(\sigma_x \cdot a) = d_x(a)$. We finally deduce that $\sigma_x \cdot (g_0, \dots, g_r)$ is a x-geodesic for $\sigma_x \cdot a$. □

This observation leads us to define two subgroups of automorphisms of the graph $\mathcal{O}$. We denote by $\mathrm{Aut}_x(\mathcal{O})$ (resp. $\mathrm{Aut}_y(\mathcal{O})$) the subgroup of graph automorphisms of $\mathcal{O}$ that preserve the x (resp. y)-distance and the adjacency types [‖].

By definition, any element τ in $\mathrm{Aut}_x(\mathcal{O})$ maps an x-geodesic for a onto an x-geodesic for $\tau \cdot a$. Moreover, a graph automorphism preserve the x-distance if and only if it induces a bijective map from $\mathcal{X}_i$ to itself for each i. Analogous results hold for $\mathrm{Aut}_y(\mathcal{O})$.

Lemma 5.30 imply that G_x (resp. G_y) is isomorphic to a subgroup of $\mathrm{Aut}_x(\mathcal{O})$ (resp. $\mathrm{Aut}_y(\mathcal{O})$). The benefit of the groups $\mathrm{Aut}_x(\mathcal{O})$ and $\mathrm{Aut}_y(\mathcal{O})$ is that, unlike G_x and G_y, they only depend on the graph structure of the orbit, and thus are more easily computable. Note however that not all such graph automorphisms come from a Galois automorphism (see for instance the Hadamard example in §5.6.3). We now state an assumption on the *distance transitivity* of the graph of the orbit.

Assumption 5.31. Let a and a' be two pairs of $\mathcal{O}$. If $d_x(a) = d_x(a')$, then there exists a σ_x in $\mathrm{Aut}_x(\mathcal{O})$ such that $\sigma_x \cdot a = a'$. Similarly, if $d_y(a) = d_y(a')$, then there exists a σ_y in $\mathrm{Aut}_y(\mathcal{O})$ such that $\sigma_y(a) = a'$. In other words, $\mathrm{Aut}_x(\mathcal{O})$ (resp. $\mathrm{Aut}_y(\mathcal{O})$) acts transitively on $\mathcal{X}_i$ (resp. $\mathcal{Y}_i$) for all i.

This assumption has been checked for all the finite orbit types appearing for models with steps in $\{-1, 0, 1, 2\}^2$ as well as Hadamard and Fan-models (see the examples in §5.6 or Appendix D). However, Assumption 5.31 does not always hold as illustrated in the following example.

Example 5.32. Consider the weighted model described by the Laurent polynomial $S(X, Y) = \left(X + \frac{1}{X} + Y + \frac{1}{Y}\right)^2$. The kernel polynomial $\widetilde{K}$ is an irreducible polynomial of degree 4 in X and in Y. Therefore, the cardinal of $\mathcal{Y}_0$ is 4 and the only right coordinate of the elements in $\mathcal{Y}_0$ is y. Moreover, each element of $\mathcal{Y}_0$ is x-adjacent to three distinct elements in $\mathcal{Y}_1$ so the cardinality of $\mathcal{Y}_1$ is 12. Now, it is easily seen that the right coordinates of vertices in $\mathcal{Y}_0 \cup \mathcal{Y}_1$ are the roots of the polynomial $\mathrm{Res}(\widetilde{K}(X, y, 1/S(x, y)), \widetilde{K}(X, Y, 1/S(x, y)), X)$. Since x and y are algebraically independent over $\mathbb{Q}$, its irreducible factors in $\mathbb{C}(x, y)[Y]$ are $(Yy - 1)$, $(-y + Y)$,

‖ One can show that this last condition is redundant with the condition on the distance preservation.

$\big(Y^2xy+2Y\,x^2y+Y\,x\,y^2+Y\,x+2Y\,y+xy\big)$ and $\big(Y^2xy-2Y\,x^2y-3Y\,x\,y^2-3Y\,x-2Y\,y+xy\big)$. This proves that the cardinality of the set $\mathcal{V}$ of right-coordinates of elements in $\mathcal{Y}_1$ is 5.

If Assumption 5.31 were true for this model then the transitive action of $\mathrm{Aut}_y(\mathcal{O})$ on $\mathcal{Y}_1$ implies that the sets $K_v=\{(u,w)\mid w=v \text{ and } (u,w)\in\mathcal{Y}_1\}\subset\mathcal{Y}_1$ for v in $\mathcal{V}$ are all isomorphic. Indeed, for v and v' two distinct right coordinates in $\mathcal{Y}_1$, Assumption 5.31 provides σ_y in $\mathrm{Aut}_y(\mathcal{O})$ such that $\sigma_y\cdot(u,v)\subset(u',v')$ because it preserves the y-adjacencies. Its restriction to K_v gives an embedding to $K_{v'}$ because it preserves the y-distance. This proves that K_v and $K_{v'}$ are isomorphic. Since these sets form a partition of $\mathcal{Y}_1$, this would imply that the cardinality of $\mathcal{V}$ divides the cardinality of $\mathcal{Y}_1$. A contradiction.

We now show that Assumption 5.31 is sufficient for (x,y) to admit a decoupling in terms of level lines.

Lemma 5.33 (Under Assumption 5.31)**.** *Let a and a' to be two vertices with $d_x(a)=d_x(a')$. Then there is a bijection between $\mathcal{P}^x_a$ and $\mathcal{P}^x_{a'}$. Analogously, if a and a' satisfy $d_y(a)=d_y(a')$, then there is a bijection between $\mathcal{P}^y_a$ and $\mathcal{P}^y_{a'}$.*

Proof. Use Assumption 5.31 to produce σ_x in $\mathrm{Aut}_x(\mathcal{O})$ such that $\sigma_x(a)=a'$. This σ_x induces a bijection between $\mathcal{P}^x_a$ and $\mathcal{P}^x_{\sigma_x\cdot a}=\mathcal{P}^x_{a'}$ by Lemma 5.30 and the compatibility between the action of σ_x on x-geodesics and its action on the associated 1-chains. □

The following theorem gives a decoupling of (x,y) in terms of level lines.

Theorem 5.34 (Under Assumption 5.31)**.** *Define the following* 0*-chains:*

$$\gamma_x=-\frac{1}{|\mathcal{O}|}\sum_{i\geq1}|\mathcal{X}_i|\sum_{\substack{1\leq j\leq i\\ j\text{ odd}}}\left(\frac{X_j}{|\mathcal{X}_j|}-\frac{X_{j-1}}{|\mathcal{X}_{j-1}|}\right)\quad\text{and}\quad\gamma_y=-\frac{1}{|\mathcal{O}|}\sum_{i\geq1}|\mathcal{Y}_i|\sum_{\substack{1\leq j\leq i\\ j\text{ odd}}}\left(\frac{Y_j}{|\mathcal{Y}_j|}-\frac{Y_{j-1}}{|\mathcal{Y}_{j-1}|}\right).$$

Then $(x,y)=(\omega+\gamma_x)+\gamma_y+\alpha$ is a decoupling of (x,y) in the orbit (with $\omega=\frac{1}{|\mathcal{O}|}\sum_{a\in\mathcal{O}}a$).

Proof. Consider the two families of 1-chains $(p^x_a)_{a\in\mathcal{O}}$ and $(p^y_a)_{a\in\mathcal{O}}$ defined for a in $\mathcal{O}$ as

$$p^x_a=\frac{1}{|\mathcal{P}^x_a|}\sum_{g\in\mathcal{P}^x_a}g\quad\text{and}\quad p^y_a=\frac{1}{|\mathcal{P}^y_a|}\sum_{g\in\mathcal{P}^y_a}g.$$

For all $g=(g_0,\dots,g_r)$ in $\mathcal{P}^x_a$, we have $\partial(g)=a-g_0$ with $g_0\sim^x(x,y)$. Then, $\varepsilon^x(\partial(g)-a+(x,y))=0$. Thus, we find by linearity that $\varepsilon^x(\partial(p^x_a)-a+(x,y))=0$. The same argument shows that $\varepsilon^y(\partial(p^y_a)-a+(x,y))=0$. Therefore, both families of 1-chains $(p^x_a)_{a\in\mathcal{O}}$ and $(p^y_a)_{a\in\mathcal{O}}$ satisfy the conditions of Theorem 5.25, which thus states that if we take

$$\gamma_x=-\frac{1}{|\mathcal{O}|}\sum_{a\in\mathcal{O}}\partial^y(p^x_a)\quad\text{and}\quad\gamma_y=-\frac{1}{|\mathcal{O}|}\sum_{a\in\mathcal{O}}\partial^x(p^y_a),$$

then the pair $(\omega+\gamma_x,\gamma_y)$ is a pseudo decoupling. As the geodesics are stable under the action of their respective Galois groups by Lemma 5.30, it is also a decoupling.

Therefore, we are left to prove that γ_x and γ_y admit the announced (pleasant) expressions. We only treat the case of γ_x, the case of y being totally symmetric.

First, note that, by Lemma 5.33, the cardinality of $\mathcal{P}^x_a$ (resp. $\mathcal{P}^y_a$) depend only on the x-distance (resp. y-distance) of a. For i a non-negative integer, we denote by m^x_i (resp. m^y_i) the cardinality of $\mathcal{P}^x_a$ (resp. $\mathcal{P}^y_a$) for any a such that $d_x(a)=i$ (resp. $d_y(a)=i$).

The expression of the boundary of a geodesic (Lemma 5.29) combined with the partition of $\mathcal{O}$ into x-level lines yields

$$\gamma_x = -\frac{1}{|\mathcal{O}|}\sum_{i\geq 0}\sum_{a\in\mathcal{X}_i}\partial^y(p_a^x) = -\frac{1}{|\mathcal{O}|}\sum_{i\geq 0}\frac{1}{m_i^x}\sum_{a\in\mathcal{X}_i}\sum_{g\in\mathcal{P}_a^x}\sum_{\substack{j \text{ odd}\\ j\leq i}}(g_j - g_{j-1}).$$

If we denote

$$S_j^i = \frac{1}{m_i^x}\sum_{a\in\mathcal{X}_i}\sum_{g\in\mathcal{P}_a^x} g_j,$$

then γ_x rewrites as

$$\gamma_x = -\frac{1}{|\mathcal{O}|}\sum_{i\geq 1}\sum_{\substack{j \text{ odd}\\ j\leq i}} S_j^i - S_{j-1}^i.$$

First, observe that, for any x-geodesic $(g_0,\dots,g_i)$, the j-th component g_j has x-distance j, so the vertices appearing in S_j^i with nonzero coefficients are in $\mathcal{X}_j$. Thus, we can write

$$S_j^i = \sum_{b\in\mathcal{X}_j}\lambda_b^{i,j} b.$$

Let σ_x be in $\mathrm{Aut}_x(\mathcal{O})$. Remind that σ_x induces a bijection on each x-level line and maps bijectively $\mathcal{P}_a^x$ and $\mathcal{P}_{\sigma_x\cdot a}^x$ for all a. Thus, we find

$$\sigma_x\cdot S_j^i = \frac{1}{m_i^x}\sum_{a\in\mathcal{X}_i}\sum_{g\in\mathcal{P}_a^x}\sigma_x\cdot g_j = \frac{1}{m_i^x}\sum_{a\in\mathcal{X}_i}\sum_{g\in\mathcal{P}_a^x}(\sigma_x\cdot g)_j = \frac{1}{m_i^x}\sum_{a\in\mathcal{X}_i}\sum_{g\in\mathcal{P}_{\sigma_x\cdot a}^x} g_j = S_j^i.$$

Under Assumption 5.31, the group $\mathrm{Aut}_x(\mathcal{O})$ acts transitively on $\mathcal{X}_j$. Since S_j^i is fixed by the action of $\mathrm{Aut}_x(\mathcal{O})$, one concludes easily that all the coefficients $\lambda_b^{i,j}$ are equal to some scalar λ_j^i and that $S_j^i = \lambda_j^i X_j \quad (*)$.

To compute the value of λ_j^i, we recall the existence of the augmentation morphism $\varepsilon : C_0(\mathcal{O}) \to \mathbb{C}$ which associates to a 0-chain the sum of its coefficients. We apply ε to each side of $(*)$. On the one hand, $\varepsilon(S_j^i) = \sum_{a\in\mathcal{X}_i}\frac{1}{|\mathcal{P}_a^x|}\sum_{g\in\mathcal{P}_a^x} 1 = \sum_{a\in\mathcal{X}_i} 1 = |\mathcal{X}_i|$. On the other hand, $\varepsilon(\lambda_j^i X_j) = \lambda_j^i|\mathcal{X}_j|$. Therefore, we deduce $\lambda_j^i = \frac{|\mathcal{X}_i|}{|\mathcal{X}_j|}$ and the announced expression for the decoupling follows. □

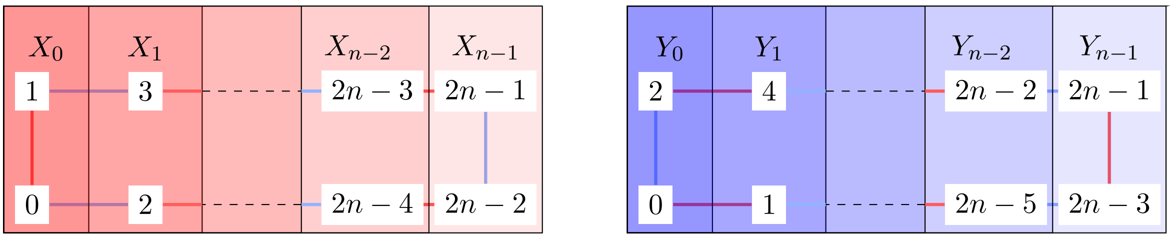
To conclude, we have defined in this section a *distance-transitivity* property that is only graph-theoretic. When this property is satisfied by the orbit-type, it leads to a decoupling expressed in terms of level lines. As described in Appendix C, the evaluation of a regular fraction on a level line is efficient from an algorithmic point of view and so is our procedure for the Galois decoupling of a regular fraction. In the following section and in Appendix D, we easily check Assumption C on various orbit-types and produce the associated decoupling in terms of level-lines.

5.6. **Examples.** In this last subsection and Appendix D, we check Assumption 5.31 and unroll the construction of the decoupling of the previous section for all the finite orbit-types of models with steps in $\{-1,0,1,2\}^2$, namely with orbits $\mathcal{O}_{12}$, $\mathcal{O}_{18}$, $\widetilde{\mathcal{O}_{12}}$ as well as for the cyclic models, the Hadamard models and the fan models.

We summarize the results of this section and Appendix D on the decoupling of XY in the following proposition.

Proposition 5.35. *The regular fraction XY does not decouple for any weighted models with orbit-types Hadamard (see below) and for the family of the fan-models (see Appendix D).*
The regular fraction XY does not decouple for unweighted models with steps in $\{-1,0,1,2\}^2$ with orbit-types $\mathcal{O}_{18}$, $\widetilde{\mathcal{O}_{12}}$.
The fraction XY decouples for the model $\mathcal{G}_3^{\lambda,\mu}$ with $(\lambda,\mu)\in\{(0,1),(1,1)\}$ with orbit-type $\mathcal{O}_{12}$.

5.6.1. *Cyclic orbit.* Assume that the orbit is a cycle of size $2n$, which is the orbit-type of any small-steps model with finite orbit. The graph of the orbit looks as follows, where we have labeled vertices from 0 to $2n-1$. We represent both x-level lines and y-level lines.



Each of the x-level lines has 2 elements, so does any y-level line. The reader can check that the permutation

$$\sigma^x=(0,1)(2,3)\dots(2i,2i+1)\dots(2n-1,2n-2)\quad\text{which corresponds to a horizontal reflection}$$

on the figure on the left-hand side, induces a graph automorphism of $\mathrm{Aut}_x(\mathcal{O})$, that is preserving the x-distance and the type adjacencies. Moreover, σ^x acts transitively on each x-level line. As the situation is completely symmetric for y-level lines, this proves Assumption 5.31 for cyclic orbits.

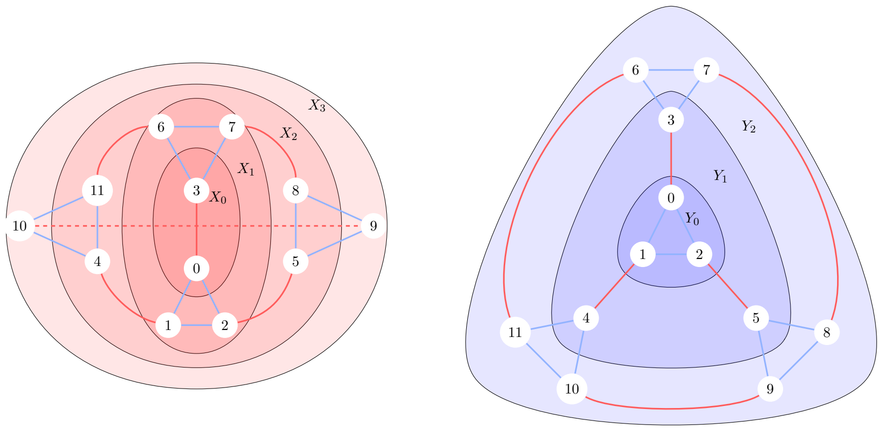
According to Theorem 5.34, we find:

$$(x,y)=\left(\omega-\frac{1}{2n}\sum_{j\text{ odd}}(n-j)\,(X_j-X_{j-1})\right)-\left(\frac{1}{2n}\sum_{j\text{ odd}}(n-j)\,(Y_j-Y_{j-1})\right)+\alpha.$$

In the above equation and in the rest of the section, we only give the explicit expressions of $\widetilde{\gamma_x},\widetilde{\gamma_y}$ and we write them between parenthesis according to their order in the expression $(x,y)=\widetilde{\gamma_x}+\widetilde{\gamma_y}+\alpha$.

The above decoupling equation corresponds to the decoupling construction obtained for small steps walks in [BBMR21, Theorem 4.11].

5.6.2. *The case of $\mathcal{O}_{12}$.* Below are the x and y-level lines for the orbit type $\mathcal{O}_{12}$:



Consider the following permutations of the vertices of the orbit. In this section, we take the convention that the exponents indicate which type of level lines these automorphisms stabilize:

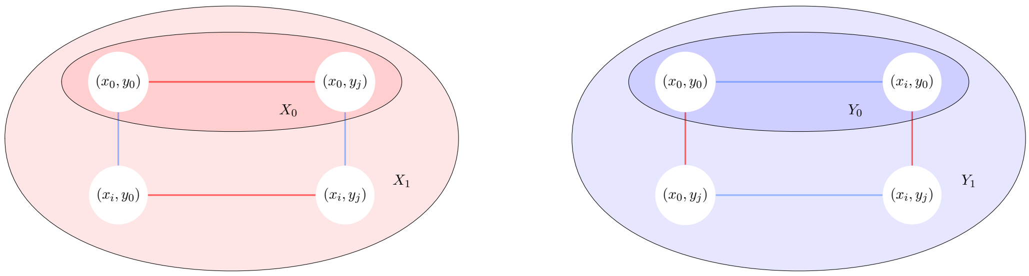
$$\begin{aligned}
\tau^{x,y} &= (1\,2)(4\,5)(6\,7)(9\,10)(8\,11) && \text{the vertical reflection on both sides,}\\
\tau^{x} &= (0\,3)(1\,6)(2\,7)(4\,11)(5\,8) && \text{the horizontal reflection on the left-hand side,}\\
\tau^{y} &= (0\,1\,2)(3\,4\,5)(6\,10\,8)(7\,11\,9) && \text{the } \tfrac{2\pi}{3} \text{ rotation on the right -hand side,}
\end{aligned}$$

The reader can check that these automorphisms are elements of $\mathrm{Aut}_x(\mathcal{O})$ or $\mathrm{Aut}_y(\mathcal{O})$ according to their exponents and that their action on the corresponding level lines is transitive. Therefore Assumption 5.31 holds for the orbit type $\mathcal{O}_{12}$. The cardinality of $\mathcal{O}$ is 12 and one can write $\omega = \frac{1}{12}(X_0 + X_1 + X_2 + X_3)$. Thus, according to Theorem 5.34, the decoupling equation is

$$\begin{aligned}
(x,y) &= \frac{2}{12}\left(\frac{X_2}{4} - \frac{X_3}{2}\right) + \frac{4+4+2}{12}\left(\frac{X_0}{2} - \frac{X_1}{4}\right) + \frac{3+6}{12}\left(\frac{Y_0}{3} - \frac{Y_1}{3}\right) + \omega + \alpha\\
&= \left(\frac{X_0}{2} - \frac{X_1}{8} + \frac{X_2}{8}\right) + \left(\frac{Y_0}{4} - \frac{Y_1}{4}\right) + \alpha.
\end{aligned}$$

5.6.3. *Hadamard models.* The notion of *Hadamard* models has been introduced by Bostan, Bousquet-Mélou and Melczer who proved that these models are always D-finite ([BBMM21, Proposition 21]). Hadamard models are characterized by the shape of their Laurent polynomial: $S(X,Y) = P(X)Q(Y) + R(X)$ for P, Q and R three Laurent polynomials. The Hadamard models form an interesting class because their orbit is always finite and in the form of a cartesian product.

Indeed,[BBMM21, Proposition 22] yields the existence of distinct elements $x_0, \dots, x_{n-1}$ and $y_0, \dots, y_{m-1}$ in $\mathbb{K}$ with $x_0 = x$ and $y_0 = y$ such that $\mathcal{O} = \{x_i \,|\, 1 \leq 0 \leq n-1\} \times \{y_j \,|\, 0 \leq j \leq m-1\}$. As a consequence, the orbits of the Hadamard models, even though their size might be arbitrarily large, are always of diameter two. This means that the distance between any two vertices is at most two as illustrated below:



These orbit-types are very symmetric. The x-level lines $\mathcal{X}_0$ is $\{(x, y_j) \,|\, 0 \leq j \leq m-1\}$ while $\mathcal{X}_1 = \{(x_i, y_j) \,|\, 0 \leq j \leq m-1 \text{ and } 1 \leq i \leq n-1\}$. Thus, $|\mathcal{X}_0| = m$ and $|\mathcal{X}_1| = (n-1)m$. It is easy to prove that any element of $\mathrm{Aut}_x(\mathcal{O})$ is of the form

$$\phi^x_{\sigma,\tau} \colon (x_i, y_j) \mapsto (\sigma(x_i), \tau(y_j)),$$

for τ a permutation of the set $\{y_j \,|\, 0 \leq j \leq m-1\}$ and σ a permutation of $\{x_i \,|\, 0 \leq i \leq n-1\}$ such that $\sigma(x) = x$. An analogous description holds for the y-level lines and $\mathrm{Aut}_y(\mathcal{O})$ proving that the Hadamard models satisfy Assumption 5.31 and that $\mathrm{Aut}_x(\mathcal{O}) \simeq S_{n-1} \times S_m$ and $\mathrm{Aut}_y(\mathcal{O}) \simeq S_n \times S_{m-1}$.

Theorem 5.34 gives the following decoupling:

$$\begin{aligned}(x, y) &= \frac{m(n-1)}{nm}\left(\frac{X_0}{m} - \frac{X_1}{m(n-1)}\right) + \frac{n(m-1)}{nm}\left(\frac{Y_0}{n} - \frac{Y_1}{n(m-1)}\right) + \omega + \alpha \\ &= \left(\frac{1}{m}X_0\right) + \left(\frac{m-1}{nm}Y_0 - \frac{1}{nm}Y_1\right) + \alpha = \left(\frac{1}{m}X_0\right) + \left(\frac{1}{n}Y_0 - \omega\right) + \alpha,\end{aligned}$$

with $\omega = \frac{1}{mn}(Y_0 + Y_1)$. Note that any Hadamard model where $\deg_X \widetilde{K} > 1$ and $\deg_Y \widetilde{K} > 1$ always contains a bicolored square, so the fraction XY never admits a decoupling (see Example 5.6).

The complete description of the groups $\mathrm{Aut}_x(\mathcal{O})$ and $\mathrm{Aut}_y(\mathcal{O})$ obtained above is particularly useful to construct examples of orbits whose graph automorphisms are not necessarily Galois automorphisms as illustrated below.

Example 5.36. Consider the nontrivial unweighted model defined by the $S(X, Y) = \left(X + \frac{1}{X}\right)\left(Y^n + \frac{1}{Y^n}\right)$. Then by Proposition 22 in [BBMM21], the orbit has the form

$$\{x, \frac{1}{x}\} \times \{\zeta^i y, \zeta^i \frac{1}{y} \text{ for } i = 0, \dots, n-1\}$$

where ζ is a primitive n-th root of unit.

Hence, the extension $k(\mathcal{O})$ equals $\mathbb{C}(x, y) = k(x, y)$. Consider the tower of field extension $k(x) \subset k(x, y^n) \subset k(x, y)$. Since $k(x)$ coincides with $\mathbb{C}(x, y^n + \frac{1}{y^n})$ and $k(x, y^n)$, the multiplicativity of the degree of a field extension yields

$$[k(\mathcal{O}) : k(x)] = [\mathbb{C}(x, y) : \mathbb{C}(x, y^n)] \times [\mathbb{C}(x, y^n) : \mathbb{C}(x, y^n + \frac{1}{y^n})] = n \times 2.$$

Indeed, since x and y are algebraically independent over $\mathbb{C}$, the element y^n is not a m-th power in $\mathbb{C}(x, y^n)$ for m dividing n. Thus, the minimal polynomial of y over the field $\mathbb{C}(x, y^n)$ is $Y^n - y^n$ so that $[\mathbb{C}(x, y) : \mathbb{C}(x, y^n)]$ equals n. Moreover, since y^n does not belong to $\mathbb{C}(x, y^n + \frac{1}{y^n})$, its minimal polynomial over the later field is $Y^2 - (y^n + \frac{1}{y^n})Y + 1$.

Thus, $G_x \subsetneq \mathrm{Aut}_x(\mathcal{O})$ because G_x is a dihedral group of size $2n$ and $\mathrm{Aut}_x(\mathcal{O})$ is S_{2n} by the above description.

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Appendix A. The algebraic kernel curve and its covering

In this section, we present an informal discussion on the geometric framework for walks confined in a quadrant. For small steps walks, this approach was developed in [KR12, DHRS18, DHRS20] and allowed these authors to construct analytic weak invariants ([Ras12, BBMR21]), difference equations ([KR12, DHRS20]) as well as efficient algorithms to compute the order of the group or some decoupling in the infinite group case ([HS21]). For small steps models, this geometric framework amounts to interpret the symmetries of the polynomial $\widetilde{K}(X,Y,t)$ as automorphisms of a certain algebraic curve. For large steps models, we shall see that this geometric framework holds in a certain sense if and only if the orbit is finite. Our intention in this section is to introduce a geometric framework and not to give a complete and systematic study of this geometric setting for large steps walks which is a whole subject in its own right.

Though the kernel polynomial $\widetilde{K}(X,Y,t)$ is irreducible over $\mathbb{Q}(t)[X,Y]$, it might be reducible over $\overline{\mathbb{Q}(t)}$. For small steps walks, Proposition 1.2 in [DHRS21] characterizes the models, called *degenerate*, whose associated kernel polynomial is reducible over $\overline{\mathbb{Q}(t)}$. These small steps models correspond to the small steps univariate cases described in §2.2 plus the two cases where the step polynomial $S(X,Y)$ is either a Laurent polynomial in XY or in X/Y. The generating function $Q(X,Y,t)$ of a degenerate model with small steps is always algebraic over $\mathbb{Q}(X,Y,t)$. One could wonder if the degenerate models in the large steps situation still coincide with the univariate ones and are therefore algebraic. The question of the reducibility of the kernel polynomial over $\overline{\mathbb{Q}(t)}$ requires some substantial work and we leave it for further articles and assume from now on that the kernel polynomial is irreducible of degree $d_x = m_x + M_x$ (resp. $d_y = m_y + M_y$) in X (resp. in Y) in the notation of Section 2.2.

Let us fix once for all a complex transcendental value of t so that $\overline{\mathbb{Q}}(t)$ embeds into $\mathbb{C}$. We denote by $\mathbb{P}^1(\mathbb{C})$ the complex projective line, that is, the set of equivalence classes $[\alpha_0 : \alpha_1]$ of elements $(\alpha_0, \alpha_1) \in \mathbb{C}^2$ up to multiplication by a non-zero scalar. The projective line $\mathbb{P}^1(\mathbb{C})$ can be identified to $\mathbb{C} \cup \{\infty\}$ where $\mathbb{C} = \{[\alpha_0 : 1] \text{ with } \alpha_0 \in \mathbb{C}\}$ and ∞ is the point $[1 : 0]$.

We define the kernel curve E_t as follows

$$E_t = \{([x_0 : x_1], [y_0 : y_1]) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \mid \overline{K}(x_0, x_1, y_0, y_1, t) = 0\},$$

where $\overline{K}(x_0, x_1, y_0, y_1, t)$ is the homogeneous polynomial defined by $x_1^{d_x} y_1^{d_y} \widetilde{K}(\frac{x_0}{x_1}, \frac{y_0}{y_1}, t)$ (see [DHRS21, §2] for the small steps case).

The kernel curve $E_t \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ is a projective algebraic curve. It is naturally equipped with two projections $\pi_x : E_t \to \mathbb{P}^1_x, ([x_0 : x_1], [y_0 : y_1]) \mapsto [x_0 : x_1]$ and $\pi_y : E_t \to \mathbb{P}^1_y, ([x_0 : x_1], [y_0 : y_1]) \mapsto [y_0 : y_1]$ where the notation $\mathbb{P}^1_x, \mathbb{P}^1_y$ emphasizes the variable on which one projects. The curve E_t is irreducible by our assumption on $\widetilde{K}$. If we denote by Sing its singular locus, that is, the set of points of E_t at which the tangent is not defined, its genus is given by the formula

$$g(E_t) = 1 + d_x d_y - d_x - d_y - \sum_{P \in \mathrm{Sing}} \sum_i \frac{m_i(P)(m_i(P)-1)}{2}, \tag{A.1}$$

where $m_i(P)$ is a positive integer standing for the multiplicity of a point P, that is, for every $\ell < m_i(P)$, the partial derivatives of $\overline{K}$ of order ℓ vanish at P (see [Har77, Exercise 5.6, Page 231-232 and Example 3.9.2, Page 393]).

Example A.1. The kernel polynomial associated to the model $\mathcal{G}_3^{\lambda,\mu}$ is $\widetilde{K}(X,Y,t) = XY - t(1 + XY^2 + X^2 + X^3Y^2 + \lambda X^2 Y)$ One can easily check that the algebraic curve E_t corresponding to the model $\mathcal{G}_3^{\lambda,\mu}$ is smooth**, so that its genus equals $2 = 1 + 3.2 - 3 - 2$.

If the curve E_t is smooth, it can be endowed with a structure of compact Riemann surface (see [GGD12, Example 1.46]). We recall that the function field of an irreducible projective curve E defined by some bivariate polynomial $F(X,Y)$ is the fraction field of the $\mathbb{C}$-algebra $\mathbb{C}[X,Y]/(F)$ where (F) is the polynomial ideal generated by F. The following categories are equivalent

- the category of smooth projective curves over $\mathbb{C}$ and non-constant morphisms,
- the category of finitely generated field extensions of $\mathbb{C}$ of transcendence degree one and morphisms of field extensions,
- the category of compact Riemann surfaces and their morphisms (see [GGD12, Remark 1.94 and Proposition 1.95]).

When the projective curve E is singular, any automorphism of its function field corresponds to a birational transformation of the curve E but, for simplicity of presentation, we shall only focus on the case where E_t is smooth. Its function field $\mathbb{C}(E_t)$ is the fraction field of the ring $\mathbb{C}[X,Y]/(\widetilde{K})$. It is a field extension of $\mathbb{C}(x) = \mathbb{C}(\mathbb{P}^1_x)$ and $\mathbb{C}(y) = \mathbb{C}(\mathbb{P}^1_y)$.

When the model is with small steps, the curve E_t is of genus one if it is smooth (see [DHRS21, Proposition 2.1]). In that case, the field $\mathbb{C}(E_t)$ is a Galois extension of degree 2 of the fields $\mathbb{C}(x)$ and $\mathbb{C}(y)$. The Galois groups $\mathrm{Gal}(\mathbb{C}(E_t)|\mathbb{C}(x))$ (resp. $\mathrm{Gal}(\mathbb{C}(E_t)|\mathbb{C}(y))$) are cyclic of order two. Their generators correspond via the aforementioned equivalence of categories to two automorphisms Φ, Ψ of E_t which are respectively the deck transformations of the projections from E_t to $\mathbb{P}^1_x$ and to $\mathbb{P}^1_y$. These two automorphisms coincide on a Zariski open set of $E_t \cap \mathbb{C}^2$ with the two birational involutions defined in §3.

When the model has at least one large step and the curve E_t is irreducible and smooth, (A.1) yields that the genus of E_t is strictly greater than one. Hurwitz's Theorem ([GGD12, Theorem 2.41]) implies that the group of automorphisms of E_t, as the group of automorphism of any algebraic curve of genus strictly greater than one, is of finite order bounded by $84(g(E_t)-1)$. The function field $\mathbb{C}(E_t)$ is in general not a Galois extension of $\mathbb{C}(x)$ and of $\mathbb{C}(y)$.

Example A.2. In the notation of Example 2.1, the field $\mathbb{C}(E_t) = \mathbb{C}(x,y)$ associated to the model $\mathcal{G}_3^{\lambda,\mu}$ is not Galois and is a proper subextension of the Galois extension $\mathbb{C}(\mathcal{O}) = \mathbb{C}(x,y,z)$.

If the genus of the curve E_t is strictly greater than one, the same holds for any any covering $\mathcal{M}$ of E_t so that the group of automorphisms of $\mathcal{M}$ is finite (see [GGD12, Theorem 1.76]). Therefore if the curve E_t is smooth, irreducible and the model has at least one large step, Theorem 4.3†† shows that the existence of a Galois extension M of $\mathbb{C}(x)$ and $\mathbb{C}(y)$ is equivalent to the finiteness

**This means that E_t has no singular point. Otherwise, one says that the curve is singular.

††which still holds if one replaces k by $\mathbb{C}$.

of the orbit of the model. Indeed, the condition that $\mathrm{Gal}(M|\mathbb{C}(x))$ and $\mathrm{Gal}(M|\mathbb{C}(y))$ generate a finite group of $\mathrm{Aut}(M)$ is automatic since $\mathrm{Aut}(M)$ which is isomorphic to $\mathrm{Aut}(\mathcal{M})$ is finite. One can thus rewrite Theorem 4.3 in geometric terms as follows.

Theorem A.3. *Assume that the model has at least one large step and that the curve E_t is irreducible and smooth. The following statements are equivalent:*

- *the orbit of the walk is finite,*
- *there exists a covering $\mathcal{M}$ of E_t which is a Galois cover of $\mathbb{P}^1_x$ and $\mathbb{P}^1_y$.*

Under the assumption of Theorem A.3, one can generalize the notion of group of the walk defined by the two birational involutions Φ, Ψ for small steps models (see §3) to the large step framework if and only if the orbit of the walk is finite. If the orbit is finite, the group of the walk is generated by the deck transformations of the two projections of $\mathcal{M}$ onto $\mathbb{P}^1_x, \mathbb{P}^1_y$. It is in general no longer a group of automorphisms of the kernel curve E_t, unless E_t equals $\mathcal{M}$, which happens only in very restricted situations. If the orbit is infinite, there is no hope to realize the symmetries of $\widetilde{K}(X, Y, t)$ as a finite set of birational transformations.

Appendix B. Solving the model $\mathcal{G}_3^{\lambda,\mu}$

In Section 2.2, we illustrate how the construction of Galois invariants and decoupling pairs for the model $\mathcal{G}_3^{\lambda,\mu}$ allows us to construct explicit equations in one catalytic variable satisfied by the sections $Q(0, Y)$ and $Q(X, 0)$. Theorem 3 in [BMJ06] implies that these sections are algebraic which yields the algebraicity of the generating function $Q(X, Y)$. However,[BMJ06] actually gives a general method to obtain explicit polynomial equations for the solutions of equations in one catalytic variable.

In this section, we follow this method to provide an explicit polynomial equation for the excursion generating function $Q(0, 0)$ attached to the model $\mathcal{G}_3^{\lambda,\mu}$. All the computations can be found in the Maple worksheet [Worb] and we give here their guidelines.

We start from the functional equation obtained for $Q(0, Y)$, because it is the simplest of the two and we recall below its canonical form, with $F(Y) = Q(0, Y)$:

$$F(Y) = 1 + t\bigg(\mu^2 t^2 Y F(Y)\left(\Delta^{(1)}F(Y)\right)^2 + \lambda\mu t F(Y)\,\Delta^{(1)}F(Y) + \mu t\left(\Delta^{(1)}F(Y)\right)^2 + 2\mu t F(Y)\,\Delta^{(2)}F(Y) + \mu Y F(Y) + \lambda\Delta^{(2)}F(Y) + 2\Delta^{(3)}F(Y)\bigg),$$

where Δ is the *discrete derivative*: $\Delta G(Y) = \frac{G(Y)-G(0)}{Y}$.

Besides $F(Y)$, there are three unknown functions: $F(0)$ (the excursions series), $F'(0)$ and $F''(0)$. The above equation can hence be rewritten as

$$P(F(Y), F(0), F'(0), F''(0), t, Y) = 0, \tag{B.1}$$

with $P(x_0, x_1, x_2, x_3, t, Y)$ a polynomial with coefficients in $\mathbb{Q}(\lambda, \mu)$.

The method of Bousquet-Mélou and Jehanne consists in constructing more equations from (B.1). For that purpose, we search for fractional power series [‡‡] Y_i's that are solutions of (B.1) and of the following equation

$$(\partial_{x_0} P)\,(F(Y_i), F(0), F'(0), F''(0), t, Y_i) = 0. \tag{B.2}$$

[‡‡] A fractional power series is an element of $\mathbb{C}[[t^{1/d}]]$ for some positive integer d.

Then the paper [BMJ06] points out that any such solution is also a solution of the following equation

$$(\partial_Y P)\,(F(Y_i), F(0), F'(0), F''(0), t, Y_i) = 0. \tag{B.3}$$

Moreover, these solutions are double roots of $D(F(0), F'(0), F''(0), t, Y)$ the discriminant of P with respect to x_0 (see Theorem 14 in [BMJ06]). If there are enough of fractional power series Y_i's (at least the number of unknown functions), then the result of [BMJ06] provides "enough" independant polynomial equations relating the unknown functions (here $F(0)$, $F'(0)$ and $F''(0)$). Where "enough" means that the theory of elimination among these equations yield a polynomial equation for each of the unknown series. In this section, we focus only on the excursion generating function $Q(0, 0)$.

Eliminating $F'(0)$ between (B.1) and (B.3), one finds a first equation between Y_i and $F(Y_i)$:

$$\begin{aligned} -2F(Y_i)\mu t\, {Y_i}^4 + F(0)^2\mu\, t^2 Y_i - 4F(0)F(Y_i)\mu\, t^2 Y_i + 3Y_i\mu\, t^2 F(Y_i)^2 - F(0)\lambda t Y_i \\ + F(Y_i)\lambda t Y_i + F(Y_i)\, {Y_i}^3 - 2F'(0)tY_i - {Y_i}^3 - 4tF(0) + 4F(Y_i)t = 0. \end{aligned} \tag{B.4}$$

Now, eliminating $F(Y_i)$ between (B.4) and (B.1), and removing the trivially nonzero factors, we obtain the following polynomial equation for the Y_i's:

$$2\mu t\, {Y_i}^4 + \lambda t Y_i - {Y_i}^3 + 2t = 0. \tag{B.5}$$

Using Newton polygon's method, we find that, among the four roots of the irreducible polynomial above, exactly three are fractional power series Y_1, Y_2 and Y_3 that are not Laurent series and the last one, denoted Y_0, is a Laurent series with a simple pole at $t = 0$. Moreover, (B.5) yields

$$t = \frac{Y_0}{2\mu {Y_0}^4 + \lambda Y_0 + 2},$$

so that $\mathbb{Q}(\lambda, \mu, t) \subset \mathbb{Q}(\lambda, \mu, Y_0)$. Therefore, we obtain the minimal polynomial $M(Y_0, Y)$ satisfied by the series Y_1, Y_2, Y_3 over $\mathbb{Q}(\lambda, \mu, Y_0)$ as:

$$M(Y_0, Y) = 2{Y_0}^3\mu Y^3 - {Y_0}^2\lambda Y - Y_0\lambda Y^2 - 2{Y_0}^2 - 2Y_0 Y - 2Y^2. \tag{B.6}$$

This polynomial of degree 3 is irreducible over the field $\mathbb{Q}(\lambda, \mu, Y_0) \subset \mathbb{Q}(\lambda, \mu)((t))$ because otherwise one of the series Y_i's would belong to $\mathbb{Q}(\lambda, \mu, Y_0)$ which is impossible since the Y_i's are not Laurent series in t. Since $\mathbb{Q}(\lambda, \mu, Y_0, F(0), F'(0), F''(0)) \subset \mathbb{Q}(\lambda, \mu)((t))$, the same argument shows that $M(Y_0, Y)$ remains irreducible over $\mathbb{Q}(\lambda, \mu, Y_0, F(0), F'(0), F''(0))$.

Now, since the Y_i's are double roots of $D(F(0), F'(0), F''(0), t(Y_0), Y)$, the polynomial $M(Y_0, Y)^2$ must divide $D(F(0), F'(0), F''(0), t(Y_0), Y)$ so that the remainder $R(Y)$ in the euclidian division of $D(F(0), F'(0), F''(0), t(Y_0), Y)$ by $M(Y_0, Y)^2$ should be identically zero. The polynomial $R(Y)$ has degree at most 6 (the discriminant has degree 12 and M^2 has degree 6), and we write it as

$$R(Y) = e_0 + e_1 Y + e_2 Y^2 + e_3 Y^3 + e_4 Y^4 + e_5 Y^5 + e_6 Y^6$$

with e_i a polynomial in Y_0,$F(0)$,$F'(0)$ and $F''(0)$. Hence, each of its coefficient gives an equation $e_i = 0$ on the unknown functions in terms of Y_0. We first eliminate $F''(0)$ between e_0 and e_1 which yields an equation e_6 between Y_0, $F(0)$, $F'(0)$. We get another such equation e_7 by eliminating $F''(0)$ between e_0 and e_2. Finally, eliminating $F'(0)$ between e_6 and e_7 yields an equation e_7 over $\mathbb{Q}(\lambda, \mu)$ between Y_0 and $F(0)$. The polynomial defining the equation e_7 factors into two nontrivial irreducible factors. To decide which of these factors is a polynomial equation for $Q(0, 0, t)$, we compute the first terms of $Q(0, 0, t)$ (which is easy from the functional equation for $Q(x,y)$) and of $Y_0(t)$ (thanks to the Newton method) and we plug these approximations in the two factors of e_7. This gives the following result:

Proposition B.1. *The series $Q(0,0)$ is algebraic of degree 8 over $\mathbb{Q}(\lambda,\mu)(Y_0)$ (for any (λ,μ) with $\mu \neq 0$. Hence, as Y_0 is of degree 4 over $\mathbb{Q}(\lambda,\mu)(t)$, we conclude that $Q(0,0)$ is an algebraic series of degree 32 over $\mathbb{Q}(\lambda,\mu)(t)$.*

We note that any step of our procedure remains valid if one specializes (λ,μ) to $(0,1)$ and $(1,1)$ so that the excursion series $Q(0,0)$ of the models $\mathcal{G}_2$ and $\mathcal{G}_3$ remains algebraic of degree 32 over $\mathbb{Q}(\lambda,\mu)(Y_0)$. This proves the conjecture on the algebraicity degree of the excursion generating functions of the models $\mathcal{G}_2$ and $\mathcal{G}_3$ in [BBMM21].

Appendix C. Formal computation of decoupling with level lines

As explained in Section 5.5, the evaluation of a regular fraction at an arbitrary pair of elements in the orbit is expensive from a computer algebra point of view. We describe below a family of 0-chains called *symmetric chains* which are easy to evaluate on. We will then show that the level lines introduced in §5.5 can be described explicitely in terms of these symmetric chains. Thus, under Assumption 5.31, Theorem 5.34 yields an expression of the decoupling in the orbit in terms of symmetric chains which provides a powerful implementation of the computation of the Galois decoupling of a fraction.

C.1. **Symmetric chains on the orbit.**

Definition C.1. Let $P(X)$ be a square-free polynomial in $\mathbb{C}(x,y)[Z]$. We define two finite subsets of $\mathbb{K}\times\mathbb{K}$ to be $V^1(P) = \{(u,v)\in\mathbb{K}\times\mathbb{K} \,|\, P(u)=0 \wedge S(x,y)=S(u,v)\}$ and $V^2(P) = \{(u,v)\in\mathbb{K}\times\mathbb{K} \,|\, P(v)=0 \wedge S(x,y)=S(u,v)\}$.

We recall here a well known corollary of the theory of symmetric polynomials (see [Lan02, Theorem 6.1]). Let $P(X)$ be a polynomial with coefficients in a field L and let $x_1,\dots,x_n$ be its roots taken with multiplicity in some algebraic closure of L. If $H(X)$ is a rational fraction over L with denominator relatively prime to $P(X)$, then the sum $\sum_i H(x_i)$ is a well defined element of L. There are numerous effective algorithms to compute such a sum based on resultants, trace of a companion matrix, Newton formula... (see for example [BFSS06]).

We extend these methods to the computation of $s=\sum_{(u,v)\in V^1(P)} H(u,v,1/S(x,y))$ for P a square-free polynomial such that $V^1(P)\subset\mathcal{O}$ and $H(X,Y,t)$ a regular fraction as follows.

By definition of $V^1(P)$, we can rewrite s as the double sum

$$s = \sum_{u\,/\,P(u)=0}\ \sum_{v\,/\,\widetilde{K}(u,v,1/S(x,y))=0} H(u,v,1/S(x,y)).$$

Consider the sum $\sum_{v\,/\,\widetilde{K}(x,v,1/S(x,y))=0} H(x,v,1/S(x,y))$. It is a well-defined element of $k(x)$ which can be computed efficiently since it is a symmetric function on the roots of the square-free polynomial $\widetilde{K}(x,Y,1/S(x,y))$. Let $\Sigma(X)$ be fraction in $k(X)$ such that $\Sigma(x)$ equals $\sum_{v\,/\,\widetilde{K}(x,v,1/S(x,y))=0} H(x,v,1/S(x,y))$. Since the group of the orbit G acts transitively on the orbit and preserves the adjacencies, it is easily seen that, for any right coordinate of the orbit u, the sum $\sum_{v\,/\,\widetilde{K}(u,v,1/S(x,y))=0} H(u,v,1/S(x,y))$ coincides with $\Sigma(u)$. Then, $s=\sum_{u\,/\,P(u)=0}\Sigma(u)$ is of the desired form and can also be computed efficiently since it is a symmetric function on the roots of the square-free polynomial P. The process is symmetric for $V^2(P)$.

These observations motivate the following definition.

Definition C.2. A *symmetric chain* is a $\mathbb{C}$-linear combination of 0-chains of the form $\sum_{a\in V^i(P)} a$ with P a square-free polynomial such that $V^i(P) \subset \mathcal{O}$.

From the above discussion, any regular fraction $H(X,Y,t)$ can be evaluated on a symmetric chain in an efficient way.

C.2. **Level lines as symmetric chains.** We now motivate the choice of level lines introduced in §5.5, by showing they are symmetric chains which one can construct efficiently.

We recall that the *square-free part* of a polynomial P in $K[Z]$ is the product of its distinct irreducible factors and can be computed as $P/\gcd(P,P')$.

Now, let P be a polynomial in $\mathbb{C}(x,y)[Z]$. Then we denote by $R_{\widetilde{K},X}(P)$ the square-free part of $\mathrm{Res}(\widetilde{K}(X,Z,1/S(x,y)),P(X),X)$ in $\mathbb{C}(x,y)[Z]$. Similarly, we define $R_{\widetilde{K},Y}(P)$ to be the square-free part of $\mathrm{Res}(\widetilde{K}(Z,Y,1/S(x,y)),P(Y),Y)$ in $\mathbb{C}(x,y)[Z]$. The following lemmas are straightforward so that we omit their proofs.

Lemma C.3. *Let $P(Z)$ be a polynomial in $\mathbb{C}(x,y)[Z]$. Then,*

$$V^2(R_{\widetilde{K},X}(P)) = \{a \in \mathbb{K}\times\mathbb{K} \,|\, \exists a' \in V^1(P),\ a \sim^y a'\}$$

and

$$V^1(R_{\widetilde{K},Y}(P)) = \{a \in \mathbb{K}\times\mathbb{K} \,|\, \exists a' \in V^2(P),\ a \sim^x a'\}.$$

Lemma C.4. *Let i be a positive integer. Any element a of $\mathcal{X}_i$ is adjacent to some element a' of $\mathcal{X}_{i-1}$. Moreover, if i is odd then $a \sim^y a'$ and if i is even then $a \sim^x a'$.*

Now, we construct by induction a sequence of square-free polynomials $(P^x_j(Z))_j \in \mathbb{C}(x,y)[Z]$ which satisfy the equations

$$V^1(P^x_{2i}) = \mathcal{X}_{2i} \cup \mathcal{X}_{2i-1} \text{ and } V^2(P^x_{2i+1}) = \mathcal{X}_{2i+1} \cup \mathcal{X}_{2i} \text{ for all } i.$$

We set $P^x_0(Z) = Z - x$ so that $V^1(P^x_0) = \mathcal{X}_0 \subset \mathcal{O}$.

Now, assume that we have constructed the polynomials $P^x_j(Z)$ for $j = 0,\dots 2i$. By Lemma C.3 and the induction hypothesis, $V^2(R_{\widetilde{K},X}(P^x_{2i}))$ is composed of all the vertices that are y-adjacent to some vertex in $\mathcal{X}_{2i}\cup\mathcal{X}_{2i-1}$. Moreover, by the induction hypothesis, $V^2(P^x_{2i-1}) = \mathcal{X}_{2i-1}\cup\mathcal{X}_{2i}$. Hence, by Lemma C.4 we find that

$$V^2(R_{\widetilde{K},X}(P^x_{2i})) \setminus V^2(P^x_{2i-1}) = \mathcal{X}_{2i+1}\cup\mathcal{X}_{2i}.$$

Hence, if we define P^x_{2i+1} to be $R_{\widetilde{K},X}(P^x_{2i})$ divided by its greatest common divisor with P^x_{2i-1}, then P^x_{2i+1} is square-free, and the above equation ensures that $V^2(P^x_{2i+1}) = \mathcal{X}_{2i+1}\cup\mathcal{X}_{2i}$. We construct P^x_{2i+2} using similar arguments.

Analogously, one can construct a sequence of square-free polynomials $(P^y_j(Z))_j \in \mathbb{C}(x,y)[Z]$ which satisfy

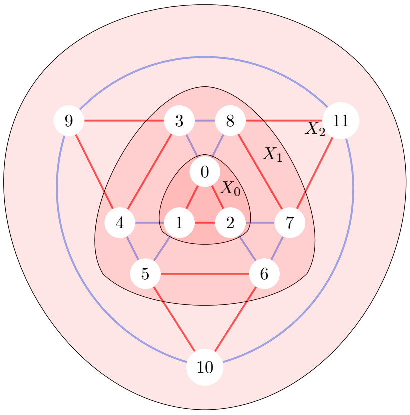
$$V^1(P^y_{2i}) = \mathcal{Y}_{2i}\cup\mathcal{Y}_{2i-1} \text{ and } V^2(P^y_{2i+1}) = \mathcal{Y}_{2i+1}\cup\mathcal{Y}_{2i} \text{ for all } i.$$

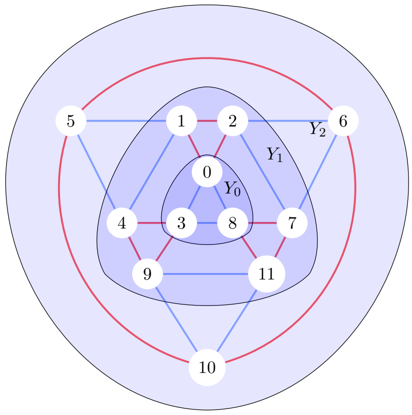
starting from $P^y_0(Z) = Z - y$.

As the x-level lines are disjoint sets of vertices, the 0-chain associated with $\mathcal{X}_{i+1}\cup\mathcal{X}_i$ is just the sum $X_{i+1} + X_i$. Hence, as X_0 and all $X_{i+1}+X_i$ are symmetric chains, all the X_i are symmetric chains as well. The same argument holds for y-level lines. Note that, as expected, the coefficients of the P^x_i are actually in $k(x)$ and the coefficients of the P^y_i are in $k(y)$ (easy from their construction). Hence, when computing the decoupling, the different components are already lifted.

Appendix D. Some more decoupling of orbit types

D.0.1. *The case of* $\widetilde{\mathcal{O}_{12}}$. We represent below the x and y-level lines for the orbit type $\widetilde{\mathcal{O}_{12}}$:





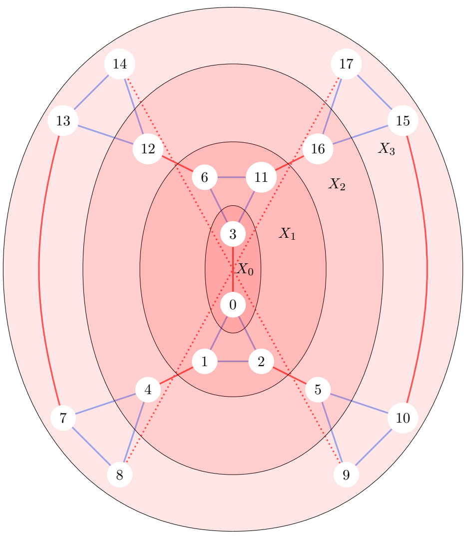
We find the following elements in $\mathrm{Aut}_x(\mathcal{O})$:

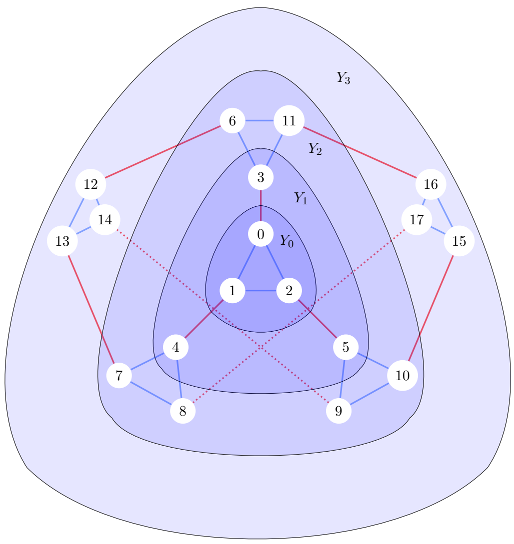
$$\begin{array}{rl|l} \tau^{x,y} &= (1\,2)(3\,8)(4\,7)(5\,6)(9\,11) & \text{vertical reflection} \\ \tau^{x} &= (0\,1\,2)(3\,5\,7)(4\,6\,8)(9\,10\,11) & \frac{2\pi}{3} \text{ rotation} \end{array}$$

One can check that their action is transitive on the x-level lines. As the situation is completely symmetric for y-level lines, Assumption 5.31 holds for this orbit type. Thus, according to Theorem 5.34 and taking $\omega = \frac{1}{12}(X_0 + X_2 + X_3)$, the decoupling equation is

$$\begin{aligned}(x, y) &= \frac{6+3}{12}\left(\frac{X_0}{3} - \frac{X_1}{6}\right) + \frac{6+3}{12}\left(\frac{Y_0}{3} - \frac{Y_1}{6}\right) + \omega + \alpha \\ &= \left(\frac{X_0}{3} - \frac{X_1}{24} + \frac{X_2}{12}\right) + \left(\frac{Y_0}{4} - \frac{Y_1}{8}\right) + \alpha.\end{aligned}$$

D.0.2. *The case of* $\mathcal{O}_{18}$. We represent below the x and y-level lines for the orbit type $\mathcal{O}_{18}$.





We present some elements belonging to the groups $\mathrm{Aut}_x(\mathcal{O})$ and $\mathrm{Aut}_y(\mathcal{O})$:

$\tau^{xy} = (1\,2)(6\,11)(4\,5)(7\,10)(8\,9)(13\,15)(14\,17)(12\,16)$	vertical reflection
$\tau^{y} = (0\,1\,2)(3\,4\,5)(6\,7\,9)(8\,10\,11)(12\,13\,14)(15\,16\,17)$	$\frac{2\pi}{3}$ rotation for $d_y(v) \leq 2$ + rotating each "ear"
$\tau_1^{x} = (0\,3)(1\,6)(2\,11)(4\,12)(5\,16)(7\,13)(8\,14)(9\,17)(10\,15)$	horizontal reflection
$\tau_2^{x} = (15\,17)(8\,10)(4\,5)(7\,9)(13\,14)(1\,2)$	pinching the upper "arms"

The reader can check that these elements act transitively on their respective level lines which proves Assumption 5.31 for $\mathcal{O}_{18}$. Thus, according to Theorem 5.34 and taking $\omega = \frac{1}{18}(X_0 + X_1 + X_2 + X_3)$, the decoupling equation is

$$\begin{aligned}(x,y) &= \frac{8}{18}\left(\frac{X_2}{4} - \frac{X_3}{8}\right) + \frac{4+4+8}{18}\left(\frac{X_0}{2} - \frac{X_1}{4}\right) + \frac{6}{18}\left(\frac{Y_2}{6} - \frac{Y_3}{6}\right) + \frac{3+6+6}{18}\left(\frac{Y_0}{3} - \frac{Y_1}{3}\right) + \omega + \alpha \\ &= \left(\frac{X_0}{2} - \frac{X_1}{6} + \frac{X_2}{6}\right) + \left(\frac{5Y_0}{18} - \frac{5Y_1}{18} + \frac{Y_2}{18} - \frac{Y_3}{18}\right) + \alpha.\end{aligned}$$

D.0.3. *Fan models.* We study a class of models derived from the ones arising in the enumeration of plane bipolar orientations (see [BMFR20]). The *fan* models are derived from those introduced in [BMFR20, Equation (7)] by a horizontal reflection.

Definition D.1. For $i \geq 0$, define $V_i(x,y) = \sum_{0\leq j\leq i} x^j y^{i-j}$. If $z_1, \ldots, z_p$ are complex weights, with z_p being nonzero, we define the *p-fan* to be the model with step polynomial

$$S(x,y) = \frac{1}{xy} + \sum_{i\leq p} z_i V_i(x,y).$$

By [BBMM21, Proposition 3, p.9], the orbits of models related to one another by a reflection are isomorphic so that one can directly use the orbit computations of Proposition 4.4 in [BMFR20] to compute the orbit of a p-fan.

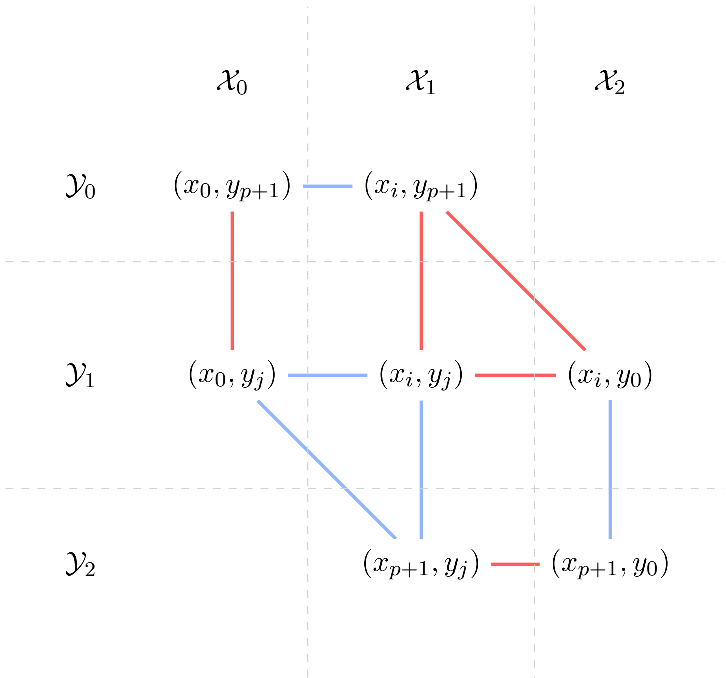
Proposition D.2. *Let $x_0, \ldots, x_p$ be defined as the roots of the equation $S(X,y) = S(x,y)$ with $x_0 = x$ and $x_{p+1} = y$. Moreover, for $0 \leq i \leq p+1$, denote $y_i = x_i$.*

In particular, $y_{p+1} = y$. Then the pairs (x_i, y_j) with $i \neq j$ form the orbit of the walk for the p-fan.

Note that these models all small backwards steps and that they all have an x/y symmetry.

As a result, the orbit is of size $(p+2)(p+1)$, and the cardinalities of the level lines are $|\mathcal{X}_0| = p+1$, $|\mathcal{X}_2| = p+1$ and $|\mathcal{X}_1| = p(p+1)$. The y-level lines are symmetric.

Below is a depiction of this orbit type, with the indices i and j satisfy $0 < i \neq j < p+1$.



Unfortunately, the orbit of the p-fan contains a bicolored square, hence no decoupling of XY is possible (see Example 5.6).

Like the orbit of the Hadamard model, the groups $\mathrm{Aut}_x(\mathcal{O})$ and $\mathrm{Aut}_y(\mathcal{O})$ contain in particular the following family of automorphisms

$$\begin{array}{l|l} \phi^x_{\sigma,\tau} \colon (x_i, y_j) \mapsto (\sigma(x_i), \tau(y_j)) & \sigma \text{ and } \tau \text{ are permutations, with } \sigma(x_0) = x_0 \\ \phi^y_{\sigma,\tau} \colon (x_i, y_j) \mapsto (\sigma(x_i), \tau(y_j)) & \sigma \text{ and } \tau \text{ are permutations, with } \tau(y_{p+1}) = y_{p+1}. \end{array}$$

This family of automorphisms acts transitively on the level lines proving Assumption 5.31.

Thus using Theorem 5.34 we obtain the decoupling equation of (x, y) as

$$\begin{aligned}(x, y) &= \frac{(p+1) + p(p+1)}{(p+1)(p+2)} \left(\frac{X_0}{p+1} - \frac{X_1}{p(p+1)} \right) + \frac{(p+1) + p(p+1)}{(p+1)(p+2)} \left(\frac{Y_0}{p+1} - \frac{Y_1}{p(p+1)} \right) + \omega + \alpha \\ &= \left(\frac{X_0}{p+1} - \frac{X_1}{p\,(p+1)\,(p+2)} + \frac{X_2}{(p+1)\,(p+2)} \right) + \left(\frac{Y_0}{p+2} - \frac{Y_1}{p\,(p+2)} \right) + \alpha.\end{aligned}$$

Appendix E. Computation of a Galois group : Hadamard models

Consider $S(X, Y) = P(X)Q(Y) + R(X)$ a Hadamard model, with PR and Q nonconstant Laurent polynomials over $\mathbb{C}$. We first note that the pair $(\frac{t-R(X)}{P(X)}, Q(Y))$ is a pair of nontrivial Galois invariants, hence the orbit of a Hadamard model is always finite by Theorem 4.3. Proposition 3.22 in [BBMM21] gives a description of the orbit of these models. Its left coordinates are $\mathbf{x} = x_0, \ldots, x_{m-1}$ the m distinct solutions x_i of $P(x)Q(y) + R(x) = P(x_i)Q(y) + R(x_i)$. Its right coordinates are $\mathbf{y} = y_0, \ldots, y_{n-1}$ the n distinct solutions y_i of $Q(y) = Q(y_i)$. Hence, the field $k(\mathcal{O})$ is equal to $\mathbb{C}(\mathbf{x}, \mathbf{y})$.

We now compute the field of Galois invariants, by showing that $k_{\mathrm{inv}} = k(Q(y))$. Writing $Q(Y) = A(Y)/B(Y)$ with A and B relatively prime, we know that the right coordinates of the orbit are the roots of the polynomial $\mu_y(Y) = B(Y) - A(Y)Q(y) \in k(Q(y))[Y] \subset k_{\mathrm{inv}}[Y]$. Thus, by §4.3, the coefficients of this polynomial generate the field of rational invariants, implying that $k(Q(y)) \subset k_{\text{inv}} \subset k(Q(y))$, which shows the claim.

Our goal in the rest of this section is to give an explicit description of the finite group of a Hadamard model in the case where either R is equal to 0 or $P(X)$ is equal to one. In these cases, the field k_{inv} is equal to $\mathbb{C}(P(x), Q(y))$ (resp. $\mathbb{C}(R(x), Q(y))$ in the second case) and the x_i, y_i satisfy $P(x) = P(x_i)$ and $Q(y) = Q(y_i)$. The extensions $\mathbb{C}(\mathbf{x})|\mathbb{C}(P(x))$ and $\mathbb{C}(\mathbf{y})|\mathbb{C}(Q(y))$ are both Galois and we denote their respective Galois groups by H_x and H_y. We shall prove that the group of the walk $G = \text{Gal}(k(\mathcal{O})|k_{\text{inv}})$ is isomorphic to $H_x \times H_y$. We prove this for $S(X, Y) = P(X)Q(Y)$ but the proof is entirely similar for $S = Q(Y) + R(X)$.

First, we recall some terminology. We say that two field extensions $L|K$ and $M|K$, subfields of a common field Ω, are *algebraically independent* if any finite set of elements of L, that are algebraically independent over K, remains algebraically independent over M. We say that $L|K$ and $M|K$ are *linearly disjoint* over K if any finite set of elements of L, that are K-linearly independent, are linearly independent over M. The field compositum of L and M is the smallest subfield of Ω that contains L and M. Finally, we say that $L|K$ is a regular field extension if K is relatively algebraically closed in L and $L|K$ is separable. We recall that K is relatively algebraically closed in L if any element of L that is algebraic over K belongs to K. Note that in our setting, all fields are in characteristic zero so $L|K$ is always separable.

First, let us prove that $\mathbb{C}(\mathbf{x}, Q(y))|\mathbb{C}(P(x), Q(y))$ is Galois with Galois group isomorphic to H_x. We remark that since x and y are algebraically independent over $\mathbb{C}$, the field extension $\mathbb{C}(P(x), Q(y))|\mathbb{C}(P(x))$ is purely transcendental of transcendence degree one, hence regular. Since $\mathbb{C}(\mathbf{x})|\mathbb{C}(P(x))$ is an algebraic extension, the element $Q(y)$ remains transcendental over $\mathbb{C}(\mathbf{x})$. Thus, the field extensions $\mathbb{C}(\mathbf{x})$ and $\mathbb{C}(P(x), Q(y))$ are algebraically independent over $\mathbb{C}(Q(y))$. Thus, by Lemma 2.6.7 in [FJ23], the fields $\mathbb{C}(\mathbf{x})$ and $\mathbb{C}(P(x), Q(y))$ are linearly disjoint over $\mathbb{C}(P(x)$. Then, the field $\mathbb{C}(\mathbf{x}, Q(y))$ that is the compositum of $\mathbb{C}(\mathbf{x})$ and $\mathbb{C}(P(x), Q(y))$, is Galois with Galois group isomorphic to H_x (see page 35 in [FJ23]). Analogously, one can prove that $\mathbb{C}(\mathbf{y}, P(x))|\mathbb{C}(P(x))$ is Galois with Galois group H_y.

To conclude, we note that the field extension $\mathbb{C}(\mathbf{x})|\mathbb{C}$ is regular of transcendence degree 1. Since x is transcendental over $\mathbb{C}(\mathbf{y})$, the fields extensions $\mathbb{C}(\mathbf{x})$ and $\mathbb{C}(\mathbf{y})$ are algebraically independent over $\mathbb{C}$ and therefore linearly disjoint over $\mathbb{C}$ by Lemma 2.6.7 in [FJ23]. By the tower property of the linear disjointness (Lemma 2.5.3 in [FJ23]), we find that $\mathbb{C}(\mathbf{x}, Q(y))$ is linearly disjoint from $\mathbb{C}(\mathbf{y})$ over $\mathbb{C}(Q(y))$. Using once again the tower property, we conclude that $\mathbb{C}(\mathbf{x}, Q(y))$ and $\mathbb{C}(\mathbf{y}, P(x))$ are linearly disjoint over $k_{\text{inv}} = \mathbb{C}(P(x), Q(y))$. Lemma 2.5.6 implies that the following restriction map is a group isomorphism:

$$\begin{array}{rcl} G = \text{Gal}(\mathbb{C}(\mathbf{x}, \mathbf{y})|\mathbb{C}(P(x), Q(y))) & \longrightarrow & \text{Gal}(\mathbb{C}(\mathbf{x}, Q(y))|\mathbb{C}(P(x), Q(y))) \times \text{Gal}(\mathbb{C}(\mathbf{y}, P(x))|\mathbb{C}(P(x), Q(y))) \\ \sigma & \longmapsto & (\sigma|_{\mathbb{C}(\mathbf{x}, Q(y))}, \sigma|_{\mathbb{C}(\mathbf{y}, P(x))}). \end{array}$$

By the above, we conclude that G is isomorphic to $H_x \times H_y$.

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