Lectures on electric fields in composites

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TIFR, Bangalore - February 2013

Contents

Abstract

These lectures are devoted to some properties of periodic composites in electrophysics. On the one hand, the characterization of smooth periodic electric fields among the set of smooth periodic gradients fields is investigated. The laminate fields are also studied in this perspective. On the other hand, the study of the Hall effect in composites shows a gap between the dimension two preserving the bounds of the Hall coefficient, and the dimension three involving unexpected effective properties.

Keywords : Periodic homogenization - Composites - Electric fields - Conductivity - Dynamical systems - Hall effect.

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Notations

- $(e_1, \dots, e_d)$ denotes the canonical basis of $\mathbb{R}^d$.
- I_d denotes the unit matrix of $\mathbb{R}^{d\times d}$, and $R_\perp$ denotes the $90°$ rotation matrix in $\mathbb{R}^{2\times 2}$.
- For $A \in \mathbb{R}^{d\times d}$, A^T denotes the transpose of the matrix A.
- For $\xi, \eta \in \mathbb{R}^d$, $\xi \otimes \eta$ denotes the matrix $[\xi_i\, \eta_j]_{1\le i,j\le d}$.
- Y denotes any closed parallelepiped of $\mathbb{R}^d$, and $Y_d := [-1/2, 1/2]^d$.
- $\langle \cdot \rangle$ denotes the average over Y.
- $C^k_\sharp(Y)$ denotes the space of k-continuously differentiable Y-periodic functions on $\mathbb{R}^d$.
- $L^2_\sharp(Y)$ denotes the space of Y-periodic functions in $L^2(\mathbb{R}^d)$, and $H^1_\sharp(Y)$ denotes the space of functions $\varphi \in L^2_\sharp(Y)$ such that $\nabla\varphi \in L^2_\sharp(Y)^d$.
- For any open set Ω of $\mathbb{R}^d$, $C^\infty_c(\Omega)$ denotes the space of smooth functions with compact support in Ω, and $\mathscr{D}'(\Omega)$ the space of distributions on Ω.
- For $u \in C^1(\mathbb{R}^d)$ and $U = (U_j)_{1\le j\le d} \in C^1(\mathbb{R}^d)^d$,

$$\nabla u := \left(\frac{\partial u}{\partial x_i}\right)_{1\le i\le d} \quad \text{and} \quad DU := \big(\nabla U_1, \dots, \nabla U_d\big) = \left[\frac{\partial U_j}{\partial x_i}\right]_{1\le i,j\le d}. \tag{0.1}$$

  The partial derivative $\dfrac{\partial u}{\partial x_i}$ will be sometimes denoted $\partial_i u$.

- For $\Sigma = [\Sigma_{ij}]_{1\le i,j\le d} \in C^1(\mathbb{R}^d)^{d\times d}$,

$$\mathrm{Div}\,(\Sigma) := \left(\sum_{i=1}^{d} \frac{\partial \Sigma_{ij}}{\partial x_i}\right)_{1\le j\le d} \quad \text{and} \quad \mathrm{Curl}\,(\Sigma) := \left(\frac{\partial \Sigma_{ik}}{\partial x_j} - \frac{\partial \Sigma_{jk}}{\partial x_i}\right)_{1\le i,j,k\le d}. \tag{0.2}$$

- For $\xi^1_1, \dots, \xi^{d-1}$ in $\mathbb{R}^d$, the cross product $\xi^1 \times \cdots \times \xi^{d-1}$ is defined by

$$\xi \cdot \big(\xi^1 \times \cdots \times \xi^{d-1}\big) = \det\big(\xi, \xi^1, \dots, \xi^{d-1}\big), \quad \text{for any } \xi \in \mathbb{R}^d, \tag{0.3}$$

  where det is the determinant with respect to the canonical basis $(e_1, \dots, e_d)$, or equivalently, the k^{th} coordinate of the cross product is given by

$$\big(\xi^1 \times \cdots \times \xi^{d-1}\big) \cdot e_k = (-1)^{k+1} \begin{vmatrix} \xi^1_1 & \cdots & \xi^{d-1}_1 \\ \vdots & \ddots & \vdots \\ \xi^1_{k-1} & \cdots & \xi^{d-1}_{k-1} \\ \xi^1_{k+1} & \cdots & \xi^{d-1}_{k+1} \\ \vdots & \ddots & \vdots \\ \xi^1_d & \cdots & \xi^{d-1}_d \end{vmatrix}. \tag{0.4}$$

1 Composites

1.1 Effective properties in periodic composites

For the sake of simplicity the spatial period will be $Y_d := [-1/2, 1/2]^d$ in the sequel. Consider a periodic conductor in $\mathbb{R}^d$ characterized by a matrix-valued conductivity $\sigma \in L^\infty(\mathbb{R}^d)^{d\times d}$ which is Y_d-periodic, i.e.

$$\sigma(y+k) = \sigma(y), \quad \text{a.e. } y \in \mathbb{R}^d,\ \forall\, k \in \mathbb{Z}^d, \tag{1.1}$$

and which satisfies the following bounds for given $\alpha, \beta > 0$,

$$\sigma(y) \geq \alpha\, I_d \ \text{ and } \ \sigma^{-1}(y) \geq \beta^{-1}\, I_d, \quad \text{a.e. } y \in \mathbb{R}^d. \tag{1.2}$$

The Ohm law linking an electric field ∇u in $L^2(Y_d)^d$ and a divergence free current field j in $L^2(Y_d)^d$ is given by

$$j = \sigma\nabla u, \quad \text{with } \operatorname{div}(j) = 0 \text{ in } \mathbb{R}^d. \tag{1.3}$$

Then, the effective (constant) conductivity σ^* associated with σ is defined by

$$\langle j \rangle = \sigma^* \langle \nabla u \rangle. \tag{1.4}$$

Mathematically, the matrix σ^* is the homogenized limit as $\varepsilon \to 0$ of the sequence $\sigma(\frac{x}{\varepsilon})$ which is εY_d-periodic (see [6] for details). More precisely, by virtue of the Lax-Milgram theorem there exists for any $\lambda \in \mathbb{R}^d$, a unique solution $u_\lambda \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ to the cell problem

$$\begin{cases} \operatorname{div}(\sigma\nabla u_\lambda) = 0 & \text{in } \mathbb{R}^d \\ y \mapsto u_\lambda(y) - \lambda\cdot y & \text{is } Y_d\text{-periodic.} \end{cases} \tag{1.5}$$

Then, σ^* is given by the following periodic limit

$$(\sigma\nabla u_\lambda)\left(\frac{x}{\varepsilon}\right) \rightharpoonup \left\langle \sigma\nabla u_\lambda \right\rangle =: \sigma^*\lambda \quad \text{weakly in } L^2_\sharp(Y_d)^d, \tag{1.6}$$

which corresponds to (1.4) with $j = \sigma\nabla u_\lambda$.

On the other hand, when the matrix-valued conductivity σ is symmetric, the periodic homogenization formula (1.6) combined with the cell problem (1.5) is equivalent to the variational principal

$$\sigma^*\lambda\cdot\lambda = \min\left\{ \left\langle \sigma\,(\lambda + \nabla\varphi)\cdot(\lambda+\nabla\varphi)\, \right\rangle \,:\, \varphi \in H^1_\sharp(Y) \right\} = \left\langle \sigma\nabla u_\lambda \cdot \nabla u_\lambda \right\rangle, \quad \text{for } \lambda \in \mathbb{R}^d. \tag{1.7}$$

1.2 Positivity properties of the electric fields

1.2.1 The two-dimensional case

Let σ be a Y_d-periodic matrix-valued satisfying (1.2). We have the following result:

Theorem 1.1 (Alessandrini, Nesi [1])**.** *Assume that $d = 2$. Let U be the vector-valued defined by $U := (u_{e_1}, u_{e_2})$ according to (1.5). Then, the Y_d-periodic electric field DU satisfies*

$$\langle DU \rangle = I_2 \quad \textit{and} \quad \det(DU) > 0 \quad \textit{in } \mathbb{R}^d. \tag{1.8}$$

Proof. The proof is based on complex analysis and in particular the quasi-conformal mappings, which is out the scope of this lecture. The strict inequality is far to be evident. □

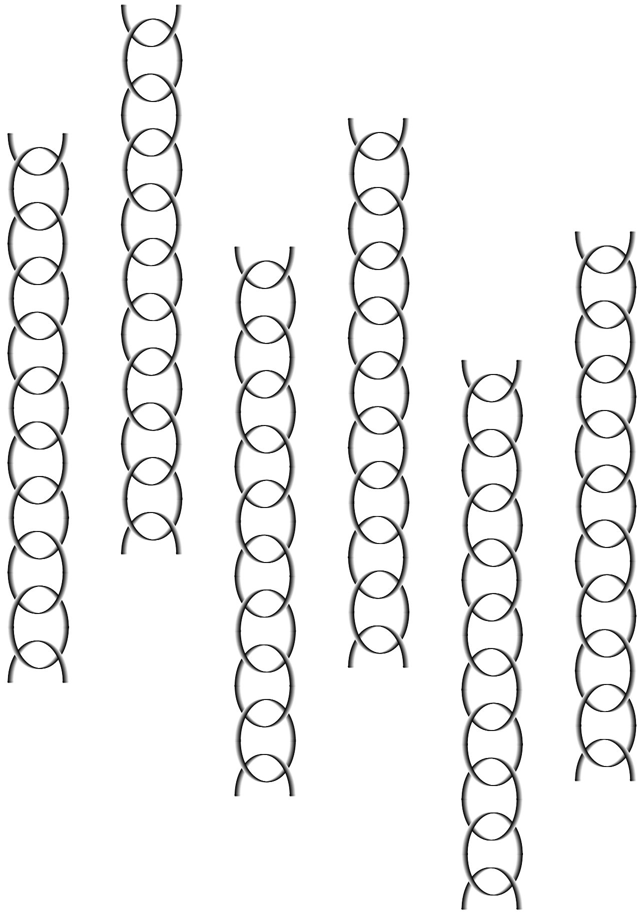

Figure 1: *The periodic chainmail*

1.2.2 Change of sign of the electric field determinant in dimension three

The previous result is specific to dimension 2. Indeed, we have the following result:

Theorem 1.2 (Milton, Nesi, B. [10])**.** *There exists a periodic tridimensional electric field DU in $L^2_\sharp(Y_3)^{3\times 3}$ with $\langle DU\rangle = I_3$, such that* $\det(DU)$ *changes of sign.*

Proof. The proof is based on the periodic chainmail as shown in figure 1 and figure 2. Each high-conductivity chain of the periodic lattice is built from two disjoint and orthogonal links and each link is isometric to a fixed compact torus with a circular section. Let $Q_\sharp$ be the open Y_3-periodic set consisting of all points within the chains, and let Q be the intersection of $Q_\sharp$ with Y_3. The set Q is a union of (see figure 2 and figure 3 below):

- one complete link Q_1 centered in the cube Y_3 orthogonal to the y_1 axis, of inner radius $\rho > 1/4$ and outer radius $R < 1/2$;
- two symmetric (with respect to the plane $y_3 = 0$) half-links Q_2^+, Q_2^- such that the axis of the link which contains Q_2^+, resp.Q_2^-, is the line passing through the point $(0, 0, 1)$, resp. $(0, 0, -1)$, in direction y_2.

Note that the set Q as well as its complementary $Y_3 \setminus Q$ are symmetric with respect to reflection in the three planes $y_i = 0$, $i = 1, 2, 3$.

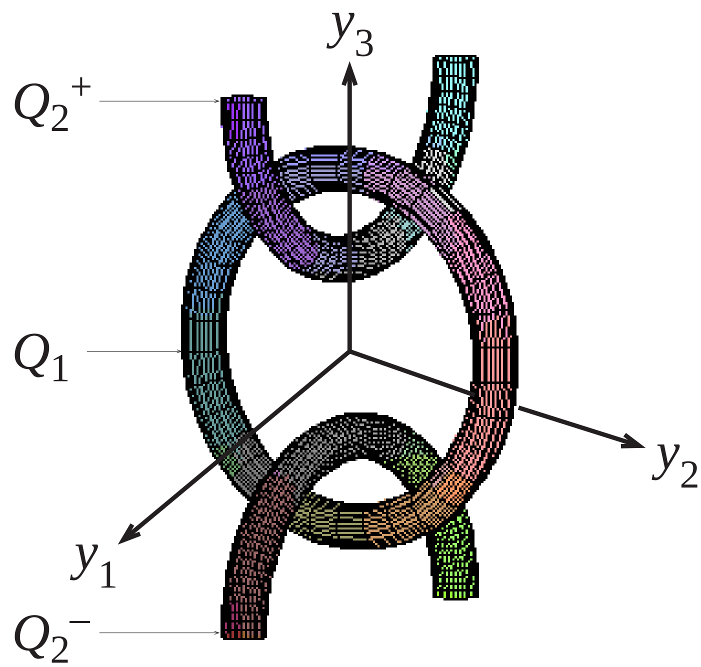

Figure 2: *The period Q of the chainmail with two half links crossing a central one*

Let σ^κ, for $\kappa > 0$, be the Y_3-periodic conductivity function defined by

$$\sigma^\kappa(y) := \begin{cases} \kappa\, I_3 & \text{if } y \in Q_\sharp \\ I_3 & \text{if } y \in Y \setminus Q_\sharp, \end{cases} \tag{1.9}$$

and let U^κ be the solution in $H^1_{\text{loc}}(Y_3)^3$ of

$$\begin{cases} \operatorname{Div}(\sigma^\kappa DU^\kappa) = 0 & \text{in } \mathbb{R}^3 \\ U^\kappa(y) - y & \text{is } Y_3\text{-periodic, with } \langle U^\kappa \rangle = 0. \end{cases} \tag{1.10}$$

By the periodic homogenization formula and the definition (1.12) of U we have

$$\begin{aligned} \int_{Y_3} \sigma^\kappa DU^\kappa : DU^\kappa &= \inf_{V \in H^1_\sharp(Y)^3} \int_{Y_3} \sigma^\kappa\,(I_d + DV) : (I_d + DV) \\ &\leq \int_{Y_3} \sigma^\kappa DU : DU = \int_{Y_3 \setminus Q} DU : DU, \end{aligned} \tag{1.11}$$

hence DU^κ is bounded in $L^2(Y)^9$ and strongly converges to 0 in $L^2(Q)^9$. Since U^κ has a zero Y_3-average, the Poincaré-Wirtinger inequality applied to Y implies that U^κ weakly converges (up to a subsequence) in $H^1_{\text{loc}}(\mathbb{R})^3$ as $\kappa \to \infty$ to a vector-valued function U satisfying

$$\text{for } i = 1, 2, \quad \begin{cases} \Delta u_i = 0 & \text{in } \mathbb{R}^3 \setminus Q_\sharp \\ u_i(y) - y_i & \text{is } Y_3\text{-periodic} \\ u_i = 0 & \text{in } Q, \end{cases} \quad \text{and} \quad \begin{cases} \Delta u_3 = 0 & \text{in } \mathbb{R}^3 \setminus Q_\sharp \\ u_3(y) - y_3 & \text{is } Y_3\text{-periodic} \\ u_3 = 0 & \text{in } Q_1 \\ u_3 = \pm 1/2 & \text{in } Q_2^\pm. \end{cases} \tag{1.12}$$

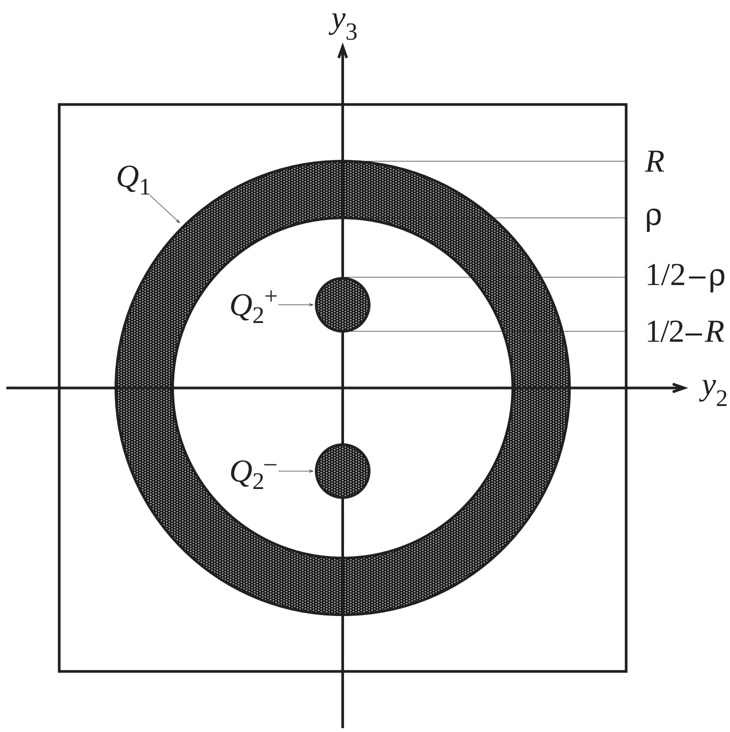

Figure 3: *A cross section of the chainmail period*

Thanks to the strong convergence of U^κ to U, the change of sign of $\det(DU^\kappa)$ then follows from the change of sign of $\det(DU)$ which will be now proved.

On the one hand, by symmetry the matrix DU is diagonal on axis y_3. On the other hand, due to the 1-periodicity and the oddness of the function $y_3 \mapsto u_3(y) - y_3$, we have $u_3(0,0,\rho) = 0$ by link Q_1, and by link Q_2^+,

$$\begin{aligned} u_3(0,0,1/2-\rho) - (1/2+\rho) \quad &= u_3(0,0,1/2+\rho) - (1/2+\rho) \\ &= u_3(0,0,\rho-1/2) - (\rho-1/2) \\ &= -\, u_3(0,0,1/2-\rho) + 1/2 - \rho, \end{aligned} \tag{1.13}$$

hence $u_3(0,0,1/2-\rho) = 1/2$. It follows that

$$\exists\, \tau \in (1/2-\rho, \rho), \quad \frac{\partial u_3}{\partial y_3}(0,0,\tau) = \frac{1}{1-4\rho} < 0 \quad (\text{since } \rho > 1/4). \tag{1.14}$$

Moreover, the strong maximum principle yields

$$\frac{\partial u_i}{\partial y_i}(0,0,y_3) > 0 \quad \text{for any } (0,0,y_3) \in Y_3 \setminus Q \text{ and } i = 1,2. \tag{1.15}$$

Therefore, we obtain that

$$\det(DU)(0,0,\tau) = \frac{\partial u_1}{\partial y_1}(0,0,\tau)\,\frac{\partial u_2}{\partial y_2}(0,0,\tau)\,\frac{\partial u_3}{\partial y_3}(0,0,\tau) < 0, \tag{1.16}$$

which also implies that $\det(DU)$ is negative in the neighborhood $(0,0,\tau)$. The proof is now complete. □

The previous pointwise properties of the electric fields suggest to characterize the electric fields among the smooth periodic (or not) vector (or matrix) fields defined in $\mathbb{R}^d$. This is the subject of the next section.

2 Realizability of electric fields

2.1 The vector field case

Definition 2.1. Let Ω be an (bounded or not) open set of $\mathbb{R}^d$, $d \geq 2$, and let $u \in H^1(\Omega)$. The vector-valued field ∇u is said to be a *realizable* electric field in Ω if there exist a symmetric positive definite matrix-valued $\sigma \in L^\infty_{\text{loc}}(\Omega)^{d\times d}$ such that

$$\operatorname{div}(\sigma \nabla u) = 0 \quad \text{in } \mathscr{D}'(\Omega). \tag{2.1}$$

If σ can be chosen isotropic ($\sigma \to \sigma I_d$), the field ∇u is said to be *isotropically realizable* in Ω.

2.1.1 Isotropic and anisotropic realizability

Theorem 2.2 (Milton, Treibergs, B. [11]). *Let Y be a closed parallelepiped of $\mathbb{R}^d$. Consider $u \in C^1(\mathbb{R}^d)$, $d \geq 2$, such that*

$$\nabla u \text{ is } Y\text{-periodic} \quad \text{and} \quad \langle \nabla u \rangle \neq 0. \tag{2.2}$$

i) Assume that

$$\nabla u \neq 0 \quad \text{everywhere in } \mathbb{R}^d. \tag{2.3}$$

Then, ∇u is an isotropically realizable electric field locally in $\mathbb{R}^d$ associated with a continuous conductivity.

ii) Assume that ∇u satisfies condition (2.2), and is a realizable electric field in $\mathbb{R}^2$ associated with a smooth Y-periodic conductivity. Then, condition (2.3) holds true.

iii) There exists a gradient field ∇u satisfying (2.2), which is a realizable electric field in $\mathbb{R}^3$ associated with a smooth Y_3-periodic conductivity, and which admits a critical point y_0, i.e. $\nabla u(y_0) = 0$.

Remark 2.3. Part *i*) of Theorem 2.2 provides a local result in the smooth case, and still holds without the periodicity assumption on ∇u. It is then natural to ask if the local result remains valid when the potential u is only Lipschitz continuous. The answer is negative as shown in Example 2.4 below. We may also ask if a global realization of a periodic gradient can always be obtained with a periodic isotropic conductivity σ. The answer is still negative as shown in Example 2.6.

The underlying reason for these negative results is that the proof of Theorem 2.2 is based on the rectification theorem which needs at least C^1-regularity and is local.

Example 2.4. Let $\chi : \mathbb{R} \to \mathbb{R}$ be the 1-periodic characteristic function which agrees with the characteristic function of $[0, 1/2]$ on $[0, 1]$. Consider the function u defined in $\mathbb{R}^2$ by

$$u(x) := x_2 - x_1 + \int_0^{x_1} \chi(t)\, dt, \quad \text{for any } x = (x_1, x_2) \in \mathbb{R}^2. \tag{2.4}$$

The function u is Lipschitz continuous, and

$$\nabla u = \chi\, e_2 + (1 - \chi)\,(e_2 - e_1) \quad \text{a.e. in } \in \mathbb{R}^2. \tag{2.5}$$

The discontinuity points of ∇u lie on the lines $\{x_1 = 1/2\,(1 + k)\}$, $k \in \mathbb{Z}$. Let $Q := (-r, r)^2$ for some $r \in (0, 1/2)$.

Assume that there exists a positive function $\sigma \in L^\infty(Q)$ such that $\sigma \nabla u$ is divergence free in Q. Let v be a stream function such that $\sigma \nabla u = R_\perp \nabla v$ a.e. in Q. The function v is unique up to an additive constant, and is Lipschitz continuous. On the one hand, we have

$$0 = \nabla u \cdot \nabla v = (e_2 - e_1) \cdot \nabla v \quad \text{a.e. in } (-r, 0) \times (-r, r), \tag{2.6}$$

hence $v(x) = f(x_1 + x_2)$ for some Lipschitz continuous function f defined in $[-2r, r]$. On the other hand, we have

$$0 = \nabla u \cdot \nabla v = e_2 \cdot \nabla v \quad \text{a.e. in } (0, r) \times (-r, r), \tag{2.7}$$

hence $v(x) = g(x_1)$ for some Lipschitz continuous function g in $[0, r]$. By the continuity of v on the line $\{x_1 = 0\}$, we get that $f(x_2) = g(0)$, hence f is constant in $[-r, r]$. Therefore, we have

$$\nabla v = 0 \;\text{ a.e. in } (-r, 0) \times (0, r) \quad \text{and} \quad \sigma \nabla u = \sigma\,(e_2 - e_1) \neq 0 \;\text{ a.e. in } (-r, 0) \times (0, r), \tag{2.8}$$

which contradicts the equality $\sigma \nabla u = R_\perp \nabla v$ a.e. in Q. Therefore, the field ∇u is non-zero a.e. in $\mathbb{R}^2$, but is not an isotropically realizable electric field in the neighborhood of any point of the lines $\{x_1 = 1/2\,(1 + k)\}$, $k \in \mathbb{Z}$.

Remark 2.5. The singularity of ∇u in Example 2.4 induces a jump of the current at the interface $\{x_1 = 0\}$. To compensate this jump we need to introduce formally an additional current concentrated on this line, which would imply an infinite conductivity there. The assumption of bounded conductivity (in L^∞) leads to the former contradiction. Alternatively, with a smooth approximation of ∇u around the line $\{x_1 = 0\}$, then part *i)* of Theorem 2.2 applies which allows us to construct a suitable conductivity. But this conductivity blows up as the smooth gradient tends to ∇u.

Example 2.6. Consider the function u defined in $\mathbb{R}$ by

$$u(x) := x_1 - \cos(2\pi x_2), \quad \text{for any } x = (x_1, x_2) \in \mathbb{R}^2. \tag{2.9}$$

The function u is smooth, and its gradient ∇u is Y_2-periodic, independent of the variable x_1 and non-zero on $\mathbb{R}^2$.

Assume that there exists a smooth positive function σ defined in $\mathbb{R}^2$, which is a-periodic with respect to x_1 for some $a > 0$, and such that $\sigma \nabla u$ is divergence free in $\mathbb{R}^2$. Set $Q := (0, a) \times (-r, r)$ for some $r \in (0, \frac{1}{2})$. By an integration by parts and taking into account the periodicity of $\sigma \nabla u$ with respect to x_1, we get that

$$\begin{aligned} 0 &= \int_Q \operatorname{div}(\sigma \nabla u)\, dx \\ &= \underbrace{\int_{-r}^{r} \big(\sigma \nabla u(a, x_2) - \sigma \nabla u(0, x_2)\big) \cdot e_1\, dx_2}_{=0} + \int_0^a \big(\sigma \nabla u(x_1, r) - \sigma \nabla u(x_1, -r)\big) \cdot e_2\, dx_1 \\ &= 2\pi \sin(2\pi r) \int_0^a \big(\sigma(x_1, r) + \sigma(x_1, -r)\big)\, dx_1 > 0, \end{aligned} \tag{2.10}$$

which yields a contradiction. Therefore, the Y_2-periodic field ∇u is not an isotropically realizable electric field in the torus.

Proof of Theorem 2.2.

i) Let $x_0 \in \mathbb{R}^d$. First assume that $d > 2$. By the rectification theorem (see, *e.g.*, [4]) there exist an open neighborhood V_0 of x_0, an open set W_0, and a C^1-diffeomorphism $\Phi : V_0 \to W_0$ such that $D\Phi^T \nabla u = e_1$. Define $v_i := \Phi_{i+1}$ for $i \in \{1, \ldots, d-1\}$. Then, we get that $\nabla v_i \cdot \nabla u = 0$ in V_0, and the rank of $(\nabla v_1, \ldots, \nabla v_{d-1})$ is equal to $(d-1)$ in V_0. Consider the continuous function

$$\sigma := \frac{|\nabla v_1 \times \cdots \times \nabla v_{d-1}|}{|\nabla u|} > 0 \quad \text{in } V_0. \tag{2.11}$$

Since by definition, the cross product $\nabla v_1 \times \cdots \times \nabla v_{d-1}$ is orthogonal to each ∇v_i as is ∇u, then due to the condition (2.3) combined with a continuity argument, there exists a fixed $\tau_0 \in \{\pm 1\}$ such that

$$\nabla v_1 \times \cdots \times \nabla v_{d-1} = \tau_0 \, \sigma \nabla u \quad \text{in } V_0. \tag{2.12}$$

Moreover, Theorem 3.2 of [13] implies that $\nabla v_1 \times \cdots \times \nabla v_{d-1}$ is divergence free, and so is $\sigma \nabla u$. Therefore, ∇u is an isotropically realizable electric field in V_0.

When $d = 2$, the equality $\nabla v_1 \cdot \nabla u = 0$ in V_0 yields for some fixed $\tau_0 \in \{\pm 1\}$,

$$\tau_0 \, R_\perp \nabla v_1 = \underbrace{\frac{|\nabla v_1|}{|\nabla u|}}_{\sigma :=} \nabla u \quad \text{in } V_0, \tag{2.13}$$

which also allows us to conclude the proof of (*i*).

ii) It is a straightforward consequence of [1] (Proposition 2, the smooth case).

iii) Ancona [3] first built an example of potential with critical points in dimension $d \geq 3$. The following construction is a regularization of the simpler example of [10] which allows us to derive a change of sign for the determinant of the matrix electric field. Consider the periodic chain-mail $Q_\sharp \subset \mathbb{R}^3$ of [10], and the associated isotropic two-phase conductivity σ^κ which is equal to $\kappa \gg 1$ in $Q_\sharp$ and to 1 elsewhere. Now, let us modify slightly the conductivity σ^κ by considering a smooth Y_3-periodic isotropic conductivity $\tilde{\sigma}^\kappa \in [1, \kappa]$ which agrees with σ^κ, except within a thin boundary layer of each interlocking ring $Q \subset Q_\sharp$, of width κ^{-1} from the boundary of Q. Proceeding as in the proof of Theorem 1.2 it is easy to check that the smooth periodic matrix-valued electric field $D\tilde{U}^\kappa$ solution of

$$\operatorname{Div}\left(\tilde{\sigma}^\kappa D\tilde{U}^\kappa\right) = 0 \ \text{ in } \mathbb{R}^3, \quad \text{with} \quad \langle D\tilde{U}^\kappa \rangle = I_3, \tag{2.14}$$

converges (as $\kappa \to \infty$) strongly in $L^2(Y_3)^{3\times 3}$ to the same limit DU as the electric field DU^κ associated with σ^κ. Then, by virtue of [10] $\det(DU)$ is negative around some point between two interlocking rings, so is $\det\left(D\tilde{U}^\kappa\right)$ for κ large enough. This combined with $\left\langle \det\left(D\tilde{U}^\kappa\right)\right\rangle = 1$ and the continuity of $D\tilde{U}^\kappa$, implies that there exists some point $y_0 \in Y_3$ such that $\det\left(D\tilde{U}^\kappa(y_0)\right) = 0$. Therefore, there exists $\xi \in \mathbb{R}^3 \setminus \{0\}$ such that the potential $u := \tilde{U}^\kappa \cdot \xi$ satisfies $\langle \nabla u \rangle = \xi$ and $\nabla u(y_0) = D\tilde{U}^\kappa(y_0)\,\xi = 0$. Theorem 2.2 is thus proved. ☐

In dimension two we have the following characterization of realizable electric vector fields:

Theorem 2.7 (Milton, Treibergs, B. [11])**.** *Let Y be a closed parallelogram of $\mathbb{R}^2$. Consider a function $u \in C^1(\mathbb{R}^2)$ satisfying (2.2). Assume that there exists a function $v \in C^1(\mathbb{R}^2)$ satisfying (2.2) such that*

$$R_\perp \nabla u \cdot \nabla v = \det(\nabla u, \nabla v) > 0 \quad \textit{everywhere in } \mathbb{R}^2. \tag{2.15}$$

Then, ∇u is realizable in the torus with a symmetric positive definite matrix-valued conductivity σ in $C^0_\sharp(Y)^{2\times 2}$.
Conversely, assume that ∇u is realizable in $\mathbb{R}^2$ with a symmetric positive definite matrix-valued conductivity σ in $C^2_\sharp(Y)^{2\times 2}$. Then, there exists a function $v \in C^1(\mathbb{R}^2)$ satisfying (2.2) and (2.15).

Remark 2.8. The result of Theorem 2.7 still holds under the less regular assumption

$$\nabla u \in L^2_\sharp(Y)^2, \quad \nabla u \neq 0 \text{ everywhere in } \mathbb{R}^2 \quad \text{and} \quad \langle \nabla u \rangle \neq 0. \tag{2.16}$$

Then, the Y-periodic conductivity σ defined by the formula (2.17) below is only defined almost everywhere in $\mathbb{R}^2$, and is not necessarily uniformly bounded from below or above in the cell period Y. However, $\sigma\nabla u$ remains divergence free in the sense of distributions on $\mathbb{R}^2$.

Proof of Theorem 2.7. Let $u, v \in C^1(\mathbb{R}^2)$ be two functions satisfying (2.2) and (2.15). From (2.15) we easily deduce that ∇u does not vanish in $\mathbb{R}^2$. Then, we may define in $\mathbb{R}^2$ the function

$$\sigma := \frac{1}{|\nabla u|^4} \begin{pmatrix} \partial_1 u & \partial_2 u \\ -\partial_2 u & \partial_1 u \end{pmatrix}^T \begin{pmatrix} R_\perp \nabla u \cdot \nabla v & -\nabla u \cdot \nabla v \\ -\nabla u \cdot \nabla v & \frac{|\nabla u \cdot \nabla v|^2 + 1}{R_\perp \nabla u \cdot \nabla v} \end{pmatrix} \begin{pmatrix} \partial_1 u & \partial_2 u \\ -\partial_2 u & \partial_1 u \end{pmatrix}. \tag{2.17}$$

Hence, σ is a symmetric positive definite matrix-valued function in $C^0_\sharp(Y)^{2\times 2}$ with determinant $|\nabla u|^{-4}$. Moreover, a simple computation shows that $\sigma\nabla u = -R_\perp \nabla v$, so that $\sigma\nabla u$ is divergence free in $\mathbb{R}^d$. Therefore, ∇u is a realizable electric field in $\mathbb{R}^d$ associated with the anisotropic conductivity σ.

Conversly, let u be a function in $C^1(\mathbb{R})$ satisfying (2.2) such that ∇u is realizable in $\mathbb{R}^2$ with a symmetric positive definite conductivity σ in $C^2_\sharp(Y)^{2\times 2}$. Consider the unique (up to an additive constant) potential v solution of $\operatorname{div}(\sigma\nabla v) = 0$ in $\mathbb{R}^d$, with $\nabla v \in L^2_\sharp(Y)^d$ and $\langle \nabla v \rangle = R_\perp \langle \nabla u \rangle$. By the classical regularity results on second-order elliptic pde's v belongs to $C^1(\mathbb{R}^2)$. Set $U := (u, v)$, by (2.2) we have

$$\det\big(\langle DU \rangle\big) = R_\perp \langle \nabla u \rangle \cdot \langle \nabla v \rangle = \big|\langle \nabla u \rangle\big|^2 > 0. \tag{2.18}$$

Hence, by Theorem 1.1 we have $\det(DU) > 0$ a.e. in $\mathbb{R}^2$. On the other hand, assume that there exists a point $y_0 \in \mathbb{R}^2$ such that $\det(DU)(y_0) = 0$. Then, there exists $\xi \in \mathbb{R}^2 \setminus \{0\}$ such that the potential $w := U\xi$ satisfies $\nabla w(y_0) = DU(y_0)\,\xi = 0$, which contradicts Proposition 2 of [1] (the smooth case). Therefore, we get that $R_\perp \nabla u \cdot \nabla v = \det(DU) > 0$ everywhere in $\mathbb{R}^2$, that is (2.15). □

Example 2.9. Go back to the Examples 2.4 and 2.6 which provide examples of gradients which are not isotropically realizable electric fields. However, in the context of Theorem 2.7 we can show that the two gradient fields are realizable electric fields associated with anisotropic conductivities:

1. Consider the function u defined by (2.4), and define the function v by

$$v(x) := -x_1 + \int_0^{x_2} \chi(t)\,dt, \quad \text{for any } x = (x_1, x_2) \in \mathbb{R}^2. \tag{2.19}$$

   We have

$$\nabla v = \chi\,(e_2 - e_1) + (1 - \chi)\,(-e_1) \quad \text{a.e. in } \mathbb{R}^2, \tag{2.20}$$

which combined with (2.5) implies that

$$\nabla u \cdot \nabla v = R_\perp \nabla u \cdot \nabla v = 1 \quad \text{a.e. in } \mathbb{R}^2. \tag{2.21}$$

Hence, after a simple computation formula (2.17) yields the rank-1 laminate (see Section 2.2.2) conductivity

$$\sigma = \chi \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} + (1-\chi)\,\frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix} \quad \text{a.e. in } \mathbb{R}^2. \tag{2.22}$$

This combined with (2.5) yields

$$\sigma\nabla u = \chi\,(e_1 + e_2) + (1-\chi)\,e_1 \quad \text{a.e. in } \mathbb{R}^2, \tag{2.23}$$

which is divergence free in $\mathscr{D}'(\mathbb{R}^2)$ since $(e_1 + e_2 - e_1) \perp e_1$.

2. Consider the function u defined by (2.9), and define the function v by $v(x) := x_2$. Then, formula (2.17) yields the smooth conductivity

$$\sigma = \frac{1}{\big(1 + 4\pi^2 \sin^2(2\pi x_2)\big)^2} \begin{pmatrix} \big(1 + 4\pi^2 \sin^2(2\pi x_2)\big)^2 + 4\pi^2 \sin^2(2\pi x_2) & -2\pi \sin(2\pi x_2) \\ -2\pi \sin(2\pi x_2) & 1 \end{pmatrix}, \tag{2.24}$$

This implies that $\sigma\nabla u = e_1$ which is obviously divergence free in $\mathbb{R}^2$.

2.1.2 Isotropic realizability in the whole space

In the previous section we have shown that not all gradients ∇u satisfying (2.2) and (2.3) are isotropically realizable when we assume σ is periodic. In the present section we will prove that the isotropic realizability actually holds in the whole space $\mathbb{R}^d$ when we relax the periodicity assumption on σ. To this end consider for a smooth periodic gradient field $\nabla u \in C^1_\sharp(Y)^d$, the following gradient dynamical system

$$\begin{cases} \dfrac{dX}{dt}(t,x) &= \nabla u\big(X(t,x)\big) \\ X(0,x) &= x, \end{cases} \quad \text{for } t \in \mathbb{R},\ x \in \mathbb{R}^d, \tag{2.25}$$

where t will be referred to as the time. First, we will extend the local rectification result of Theorem 2.2 to the whole space involving a hyperplane. Then, using an alternative approach we will obtain the isotropic realizability in the whole space replacing the hyperplane by an equipotential. Finally, we will give a necessary and sufficient for the isotropic realizability in the torus.

We have the following result:

Proposition 2.10. *Let u be a function in $C^2(\mathbb{R}^d)$ such that ∇u satisfies (2.2) and (2.3). Also assume that there exists an hyperplane $H := \{x \in \mathbb{R}^d : x \cdot \nu = h\}$ such that each trajectory $X(\cdot, x)$ of (2.25), for $x \in \mathbb{R}^d$, intersects H only at one point $z_H(x) = X\big(\tau_H(x), x\big)$ and at a unique time $\tau_H(x) \in \mathbb{R}$, in such a way that ∇u is not tangential to H at $z_H(x)$. Then, the gradient ∇u is an isotropically realizable electric field in $\mathbb{R}^d$.*

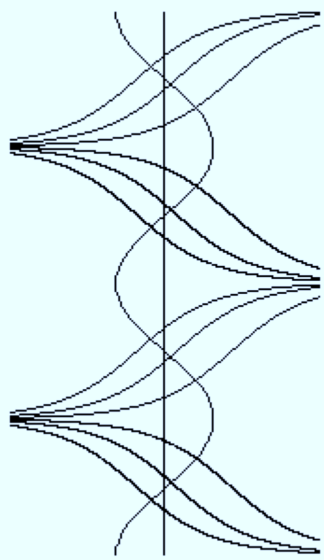

Figure 4: *The trajectories crossing the line* $\{x_1 = 0\}$ *and the equipotential* $\{u = 0\}$

Example 2.11. Go back to Example 2.6 with the function u defined in $\mathbb{R}^2$ by (2.9). The gradient field ∇u is smooth and Y_2-periodic. The solution of the dynamical system (2.25) which reads as

$$\begin{cases} \dfrac{dX_1}{dt}(t,x) = 1, & X_1(0,x) = x_1, \\ \dfrac{dX_2}{dt}(t,x) = 2\pi \sin\big(2\pi X_2(t,x)\big), & X_2(0,x) = x_2, \end{cases} \quad \text{for } t \in \mathbb{R},\ x \in \mathbb{R}^2, \tag{2.26}$$

is given explicitly by (see figure 4)

$$X(t,x) = \begin{cases} (t + x_1)\, e_1 + \Big[n + \frac{1}{\pi} \arctan\big(e^{4\pi^2 t} \tan(\pi x_2)\big)\Big]\, e_2 & \text{if } x_2 \in \big(n - \frac{1}{2}, n + \frac{1}{2}\big) \\ (t + x_1)\, e_1 + \big(n + \frac{1}{2}\big)\, e_2 & \text{if } x_2 = n + \frac{1}{2}, \end{cases} \tag{2.27}$$

where n is an arbitrary integer.

Consider the line $\{x_1 = 0\}$ as the hyperplane H. Then, we have $\tau_H(x) = -\,x_1$. Moreover, using successively the explicit formula (2.27) and the semigroup property

$$X\big(s, X(t,x)\big) = X(s+t,x) \quad \forall\, s,t \in \mathbb{R},\ \forall\, x \in \mathbb{R}^d, \tag{2.28}$$

we have

$$X\big(-X_1(t,x), X(t,x)\big) = X\big(-\,t - x_1, X(t,x)\big) = X(-\,x_1, x), \quad \forall\, t \in \mathbb{R}. \tag{2.29}$$

Hence, the function v defined by $v(x) := X_2(-\,x_1, x)$ satisfies

$$v\big(X(t,x)\big) = X_2\big(-X_1(t,x), X(t,x)\big) = X_2(-\,x_1, x) = v(x), \quad \forall\, t \in \mathbb{R}. \tag{2.30}$$

The function v is thus a first integral of system (2.25). It follows that

$$\frac{d}{dt}\left[v\big(X(t,x)\big)\right] = 0 = \nabla v\big(X(t,x)\big) \cdot \frac{dX}{dt}(t,x) = \nabla v\big(X(t,x)\big) \cdot \nabla u\big(X(t,x)\big), \tag{2.31}$$

which, taking $t = 0$, implies that $\nabla u \cdot \nabla v = 0$ in $\mathbb{R}^2$. Moreover, putting $t = -x_1$ in (2.27), we get that for any $n \in \mathbb{Z}$,

$$v(x) = \begin{cases} n + \frac{1}{\pi} \arctan\left(e^{-4\pi^2 x_1} \tan(\pi x_2)\right) & \text{if } x_2 \in \left(n - \frac{1}{2}, n + \frac{1}{2}\right) \\ n + \frac{1}{2} & \text{if } x_2 = n + \frac{1}{2}. \end{cases} \tag{2.32}$$

Therefore, by (2.13) ∇u is an isotropically realizable electric field in the whole space $\mathbb{R}^2$, with the smooth conductivity

$$\sigma := \frac{|\nabla v|}{|\nabla u|} = \begin{cases} \dfrac{1 + \tan^2(\pi x_2)}{e^{4\pi^2 x_1} + e^{-4\pi^2 x_1} \tan^2(\pi x_2)} & \text{if } x_2 \notin \frac{1}{2} + \mathbb{Z} \\ e^{4\pi^2 x_1} & \text{if } x_2 \in \frac{1}{2} + \mathbb{Z}. \end{cases} \tag{2.33}$$

It may be checked by a direct calculation that $\sigma \nabla u$ is divergence free in $\mathbb{R}^2$.

Remark 2.12. The hyperplane assumption of Theorem 2.10 does not hold in general. Indeed, we have the following heuristic argument:

Let H be a line of $\mathbb{R}^2$, and let Σ be a smooth curve of $\mathbb{R}^2$ having an S-shape across H. Consider a smooth periodic isotropic conductivity σ which is very small in the neighborhood of Σ. Let u be a smooth potential solution of $\operatorname{div}(\sigma \nabla u) = 0$ in $\mathbb{R}^2$ satisfying (2.2), (2.3), and let v be the associated stream function satisfying $\sigma \nabla u = R_\perp \nabla v$ in $\mathbb{R}^2$. The potential v is solution of $\operatorname{div}(\sigma^{-1} \nabla v) = 0$ in $\mathbb{R}^2$. Then, since σ^{-1} is very large in the neighborhood of Σ, the curve Σ is close to an equipotential of v and thus close to a current line of u. Therefore, some trajectory of (2.25) has an S-shape across H. This makes impossible the regularity of the time τ_H which is actually a multi-valued function.

Now, replacing a hyperplane by an equipotential (see figure 1 above) we have the more general result:

Theorem 2.13 (Milton, Treibergs, B. [11])**.** *Let u be a function in $C^3(\mathbb{R}^d)$ such that ∇u satisfies (2.2) and (2.3). Then, the gradient field ∇u is an isotropically realizable electric field in $\mathbb{R}^d$.*

Proof of Theorem 2.13. On the one hand, for a fixed point $x \in \mathbb{R}^d$, define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(t) := u\big(X(t,x)\big)$, for $t \in \mathbb{R}$. The function f is in $C^3(\mathbb{R})$, and

$$f'(t) = \frac{dX}{dt}(t,x) \cdot \nabla u\big(X(t,x)\big) = \left|\nabla u\big(X(t,x)\big)\right|^2, \quad \forall\, t \in \mathbb{R}. \tag{2.34}$$

Since ∇u is periodic, continuous and does not vanish in $\mathbb{R}^d$, there exists a constant $m > 0$ such that $f' \geq m$ in $\mathbb{R}$. It follows that

$$\frac{f(t) - f(0)}{t} \geq m, \quad \forall\, t \in \mathbb{R} \setminus \{0\}, \tag{2.35}$$

which implies that

$$\lim_{t \to \infty} f(t) = \infty \quad \text{and} \quad \lim_{t \to -\infty} f(t) = -\infty. \tag{2.36}$$

This combined with the monotonicity and continuity of f thus shows that there exists a unique $\tau(x) \in \mathbb{R}$ such that

$$u\big(X(\tau(x), x)\big) = 0. \tag{2.37}$$

On the other hand, similar to the hyperplane case, we have that for any $x \in \mathbb{R}^d$,

$$\frac{\partial}{\partial t}\left[u\big(X(t,x)\big)\right]\Big|_{t=\tau(x)} = \left|\nabla u\big(X(\tau(x),x)\big)\right|^2 > 0. \tag{2.38}$$

Hence, from the implicit function theorem combined with the C^2-regularity of $(t,x) \mapsto u\big(X(t,x)\big)$ we deduce that $x \mapsto \tau(x)$ is a function in $C^2(\mathbb{R}^d)$.

Now, define the function w in $\mathbb{R}^d$ by

$$w(x) := \int_0^{\tau(x)} \Delta u\big(X(s,x)\big)\, ds, \quad \text{for } x \in \mathbb{R}^d, \tag{2.39}$$

which belongs to $C^1(\mathbb{R}^d)$ since $u \in C^3(\mathbb{R}^d)$. Note that the semigroup property (2.28) and (2.37) yield

$$u\left[X\big(\tau(x) - t, X(t,x)\big)\right] = u\left[X\big(\tau(x),x\big)\right] = 0, \quad \forall\, t \in \mathbb{R}, \tag{2.40}$$

which by the uniqueness of $\tau(x)$ implies that

$$\tau\big(X(t,x)\big) = \tau(x) - t, \quad \forall\, t \in \mathbb{R}. \tag{2.41}$$

Then, using (2.28), (2.41) and the change of variable $r := s + t$, we get that for any (t,x) in $\mathbb{R} \times \mathbb{R}^d$,

$$w\big(X(t,x)\big) = \int_0^{\tau(x)-t} \Delta u\big(X(s+t,x)\big)\, ds = \int_t^{\tau(x)} \Delta u\big(X(r,x)\big)\, dr, \tag{2.42}$$

which implies that

$$\frac{\partial}{\partial t}\left[w\big(X(t,x)\big)\right] = \nabla w\big(X(t,x)\big) \cdot \nabla u\big(X(t,x)\big) = -\Delta u\big(X(t,x)\big). \tag{2.43}$$

Finally, define the conductivity σ by

$$\sigma(x) := e^{w(x)} = \exp\left(\int_0^{\tau(x)} \Delta u\big(X(s,x)\big)\, ds\right), \quad \text{for } x \in \mathbb{R}^2, \tag{2.44}$$

which belongs to $C^1(\mathbb{R}^d)$. Applying (2.43) with $t = 0$, we obtain that

$$\operatorname{div}(\sigma \nabla u) = e^w \left(\nabla w \cdot \nabla u + \Delta u\right) = 0 \quad \text{in } \mathbb{R}^d, \tag{2.45}$$

which concludes the proof. □

Remark 2.14. In the proof of Theorem 2.13 the condition that ∇u is non-zero everywhere is essential to obtain both:

- the uniqueness of the time $\tau(x)$ for each trajectory to reach the equipotential $\{u = 0\}$,
- the regularity of the function $x \mapsto \tau(x)$.

2.1.3 Isotropic realizability in the torus

We have the following characterization of the isotropic realizability in the torus:

Theorem 2.15 (Milton, Treibergs, B. [11])**.** *Let u be a function in $C^3(\mathbb{R}^d)$ such that ∇u satisfies (2.2) and (2.3). Then, the gradient field ∇u is isotropically realizable with a positive conductivity $\sigma \in L^\infty_\sharp(Y)$, with $\sigma^{-1} \in L^\infty_\sharp(Y)$, if there exists a constant $C > 0$ such that*

$$\forall\, x \in \mathbb{R}^d, \quad \left|\int_0^{\tau(x)} \Delta u\big(X(t,x)\big)\, dt\right| \le C, \tag{2.46}$$

where $X(t,x)$ is defined by (2.25) and $\tau(x)$ by (2.37).
Conversely, if ∇u is isotropically realizable with a positive conductivity $\sigma \in C^1_\sharp(Y)$, then the boundedness (2.46) holds.

Example 2.16. For the function u of Example 2.11 and for $x = (x_1, 0)$, we have by (2.39) and (2.27),

$$w(x) = 4\pi^2 \int_0^{\tau(x)} \cos\big(2\pi X_2(s,x)\big)\, ds = 4\pi^2\, \tau(x),$$

and by (2.37),

$$X_1\big(\tau(x), x\big) = \tau(x) + x_1 = \cos\big(2\pi X_2(\tau(x), x)\big) = 1.$$

Therefore, we get that $\sigma_0(x_1, 0) = 4\pi^2\,(1 - x_1)$, which contradicts the boundedness (2.46). This is consistent with the negative conclusion of Example 2.6.

Proof of Theorem 2.15.
Sufficient condition: Without loss of generality we may assume that the period is $Y = [0,1]^d$. Define the function σ_0 by

$$\sigma_0(x) := \exp\left(\int_0^{\tau(x)} \Delta u\big(X(t,x)\big)\, dt\right), \quad \text{for } x \in \mathbb{R}^d, \tag{2.47}$$

and consider for any integer $n \geq 1$, the conductivity σ_n defined by the average over the $(2n+1)^d$ integer vectors of $[-n,n]^d$:

$$\sigma_n(x) := \frac{1}{(2n+1)^d} \sum_{k \in \mathbb{Z}^d \cap [-n,n]^d} \sigma_0(x+k), \quad \text{for } x \in \mathbb{R}^d. \tag{2.48}$$

On the one hand, by (2.46) σ_n is bounded in $L^\infty(\mathbb{R}^d)$. Hence, there is a subsequence of n, still denoted by n, such that σ_n converges weakly-$*$ to some function σ in $L^\infty(\mathbb{R}^d)$. Moreover, we have for any $x \in \mathbb{R}^d$ and any $k \in \mathbb{Z}^d$ (denoting $|k|_\infty := \max_{1 \leq i \leq d} |k_i|$),

$$\begin{aligned}
\big|\,(2n+1)^d\, \sigma_n(x+k) - (2n+1)^d\, \sigma_n(x)\,\big| &= \Big|\sum_{|j-k|_\infty \leq n} \sigma_0(x+j) - \sum_{|j|_\infty \leq n} \sigma_0(x+j)\Big| \\
&\leq \sum_{\substack{|j-k|_\infty \leq n \\ |j|_\infty > n}} \sigma_0(x+j) + \sum_{\substack{|j-k|_\infty > n \\ |j|_\infty \leq n}} \sigma_0(x+j) \\
&\leq C\, n^{d-1},
\end{aligned} \tag{2.49}$$

where C is a constant independent of n and x. This implies that $\sigma(\cdot + k) = \sigma(\cdot)$ a.e. in $\mathbb{R}^d$, for any $k \in \mathbb{Z}^d$. The function σ is thus Y-periodic and belongs to $L^\infty_\sharp(Y)$. Moreover, since by virtue of (2.46) and (2.47) σ_0 is bounded from below by e^{-C}, so is σ_n and its limit σ. Therefore, σ^{-1} also belongs to $L^\infty_\sharp(Y)$.

On the other hand, by virtue of Theorem 2.13 the gradient field ∇u is realizable in $\mathbb{R}^d$ with the conductivity σ_0. This combined with the Y-periodicity of ∇u yields $\operatorname{div}(\sigma_n \nabla u) = 0$ in $\mathbb{R}^d$. Hence, using the weak-$*$ convergence of σ_n in $L^\infty(\mathbb{R}^d)$ we get that for any $\varphi \in C^\infty_c(\mathbb{R}^d)$, $\nabla u \cdot \nabla \varphi \in L^1(\mathbb{R}^d)$ and

$$0 = \lim_{n \to \infty} \int_{\mathbb{R}^d} \sigma_n \nabla u \cdot \nabla \varphi\, dx = \int_{\mathbb{R}^d} \sigma \nabla u \cdot \nabla \varphi\, dx. \tag{2.50}$$

Therefore, we obtain that $\operatorname{div}(\sigma\nabla u) = 0$ in $\mathscr{D}'(\mathbb{R}^d)$, so that ∇u is isotropically realizable with the Y-periodic bounded conductivity σ.

Necessary condition: Let σ be a positive function in $C^1_\sharp(Y)$ such that $\operatorname{div}(\sigma\nabla u) = 0$ in $\mathbb{R}^d$. Then, the function $w := \ln\sigma$ also belongs to $C^1_\sharp(Y)$, and solves the equation $\nabla w\cdot\nabla u + \Delta u = 0$ in $\mathbb{R}^d$. Therefore, using (2.25) we obtain that for any $x \in \mathbb{R}^d$,

$$\begin{aligned}\int_0^{\tau(x)} \Delta u\big(X(t,x)\big)\,dt &= -\int_0^{\tau(x)} \nabla w\big(X(t,x)\big)\cdot\nabla u\big(X(t,x)\big)\,dt \\ &= -\int_0^{\tau(x)} \nabla w\big(X(t,x)\big)\cdot\frac{dX}{dt}(t,x)\,dt \\ &= w\big(X(0,x)\big) - w\big(X(\tau(x),x)\big) = w(x) - w\big(X(\tau(x),x)\big),\end{aligned} \tag{2.51}$$

which implies (2.46) due to the boundedness of w in $\mathbb{R}^d$. □

Remark 2.17. If we also assume that σ_0 of (2.46) is uniformly continuous in $\mathbb{R}^d$, then the previous proof combined with Ascoli's theorem implies that the conductivity σ is continuous. Indeed, the sequence σ_n defined by (2.48) is then equi-continuous.

2.2 The matrix field case

Definition 2.18. Let Ω be an (bounded or not) open set of $\mathbb{R}^d$, $d \geq 2$, and let $U = (u_1, \ldots, u_d)$ be a function in $H^1(\Omega)^d$. The matrix-valued field DU is said to be a realizable matrix-valued electric field in Ω if there exists a symmetric positive definite matrix-valued $\sigma \in L^\infty_{\rm loc}(\Omega)^{d\times d}$ such that

$$\operatorname{Div}(\sigma DU) = 0 \quad \text{in } \Omega. \tag{2.52}$$

2.2.1 The periodic framework

Theorem 2.19 (Milton, Treibergs, B. [11])**.** *Let Y be a closed parallelepiped of $\mathbb{R}^d$, $d \geq 2$. Consider a function $U \in C^1(\mathbb{R}^d)^d$ such that*

$$DU \text{ is } Y\text{-periodic} \quad \text{and} \quad \det\big(\langle DU\rangle\big) \neq 0. \tag{2.53}$$

i) Assume that (1.8) holds. Then, DU is a realizable electric matrix field in $\mathbb{R}^d$ associated with a continuous conductivity.

ii) Assume that $d = 2$, and that DU is a realizable electric matrix field in $\mathbb{R}^2$, satisfying (2.53) and associated with a smooth conductivity in $\mathbb{R}^2$. Then, condition (1.8) holds true.

iii) In dimension $d = 3$, there exists a smooth matrix field DU satisfying (2.53) and associated with a smooth periodic conductivity, such that $\det(DU)$ *takes positive and negative values in $\mathbb{R}^3$.*

Remark 2.20. Similarly to Remark 2.8 the assertions *i)* and *ii)* of Theorem 2.19 still hold under the less regular assumptions that

$$DU \in L^2_\sharp(Y)^{d\times d} \quad \text{and} \quad \det\big(\langle DU\rangle DU\big) > 0 \ \text{ a.e. in } \mathbb{R}^d. \tag{2.54}$$

Then, the Y-periodic conductivity σ defined by the formula (2.55) below is only defined a.e. in $\mathbb{R}^d$, and is not necessarily uniformly bounded from below or above in the cell period Y. However, σDU remains divergence free in the sense of distributions on $\mathbb{R}^d$.

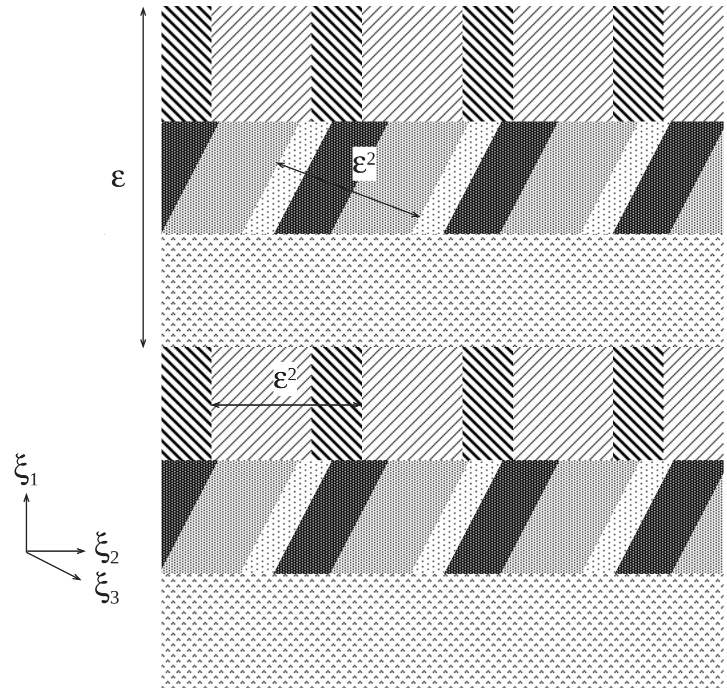

Figure 5: *A rank-2 laminate with* $m_2 = 2$*, and directions* ξ_1*,* $\xi_{1,2} = \xi_2$*,* $\xi_{2,2} = \xi_3$

Proof of Theorem 2.19.

i) Let $U \in C^1(\mathbb{R}^d)^d$ be a vector-valued function satisfying (2.53). Then, we can define the matrix-valued function σ by

$$\sigma := \det\big(\langle DU\rangle DU\big)\,(DU^{-1})^T DU^{-1} = \det\big(\langle DU\rangle\big)\,\mathrm{Cof}\,(DU)\,DU^{-1}, \tag{2.55}$$

where Cof denotes the Cofactors matrix. It is clear that σ is a Y-periodic continuous symmetric positive definite matrix-valued function. Moreover, Piola's identity (see, *e.g.*, [13], Theorem 3.2) implies that

$$\mathrm{Div}\,\big(\mathrm{Cof}\,(DU)\big) = 0 \quad \text{in } \mathbb{R}^d. \tag{2.56}$$

Hence, σDU is Divergence free in $\mathbb{R}^d$. Therefore, DU is a realizable electric matrix field associated with the continuous conductivity σ.

ii) Let DU be an electric matrix field satisfying condition (2.53) and associated with a smooth conductivity in $\mathbb{R}^2$. By the regularity results for second-order elliptic pde's the function U is smooth in $\mathbb{R}^2$. Moreover, by virtue of Theorem 1.2 we have $\det\big(\langle DU\rangle DU\big) > 0$ a.e. in $\mathbb{R}^2$. Therefore, as in the proof of Theorem 2.7 we conclude that (1.8) holds.

iii) This is an immediate consequence of the counter-example of [10] combined with the regularization argument used in the proof of Theorem 2.2 *iii*). □

Remark 2.21. The conductivity σ defined by (2.55) can be derived by applying the coordinate change $x' = U^{-1}(x)$ to the homogeneous conductivity $\big|\det\langle DU\rangle\big|\,I_d$.

2.2.2 The laminate fields

Definition 2.22. Let d, n be two positive integers. A rank-n laminate in $\mathbb{R}^d$ is a multi-scale microstructure defined at n ordered scales $\varepsilon_n \ll \cdots \ll \varepsilon_1$ depending on a small positive parameter $\varepsilon \to 0$, and in multiple directions in $\mathbb{R}^d \setminus \{0\}$, by the following process (see figure 5):

- At the smallest scale ε_n, there is a set of m_n rank-1 laminates, the i^{th} one of which is composed, for $i = 1, \dots, m_n$, of an ε_n-periodic repetition in the direction $\xi_{i,n}$ of homogeneous layers with constant positive definite conductivity matrices $\sigma^h_{i,n}$, $h \in I_{i,n}$.

- At the scale ε_k, there is a set of m_k laminates, the i^{th} one of which is composed, for $i = 1, \dots, m_k$, of an ε_k-periodic repetition in the direction $\xi_{i,k}$ of homogeneous layers and/or a selection of the m_{k+1} laminates which are obtained at stage $(k+1)$ with conductivity matrices $\sigma^h_{i,j}$, for $j = k+1, \dots, n$, $h \in I_{i,j}$.

- At the scale ε_1, there is a single laminate ($m_1 = 1$) which is composed of an ε_1-periodic repetition in the direction $\xi_1 \in \mathbb{R}^d \setminus \{0\}$ of homogeneous layers and/or a selection of the m_2 laminates which are obtained at the scale ε_2 with conductivity matrices $\sigma^h_{i,j}$, for $j = 2, \dots, n$, $h \in I_{i,j}$.

The laminate conductivity at stage $k = 1, \dots, n$, is denoted by $L^\varepsilon_k(\hat\sigma)$, where $\hat\sigma$ is the whole set of the constant laminate conductivities.

Due to the results of [14, 7] there exists a set $\hat P$ of constant matrices in $\mathbb{R}^{d\times d}$, such that the laminate $P_\varepsilon := L^\varepsilon_n(\hat P)$ is a corrector (or a matrix electric field) associated with the conductivity $\sigma_\varepsilon := L^\varepsilon_n(\hat\sigma)$ in the sense of Murat-Tartar [16], *i.e.*

$$\begin{cases} P_\varepsilon \rightharpoonup I_d & \text{weakly in } L^2_{\text{loc}}(\mathbb{R}^d)^{d\times d} \\ \operatorname{Curl}(P_\varepsilon) \to 0 & \text{strongly in } H^{-1}_{\text{loc}}(\mathbb{R}^d)^{d\times d\times d} \\ \operatorname{Div}(\sigma_\varepsilon P_\varepsilon) & \text{is compact in } H^{-1}_{\text{loc}}(\mathbb{R}^d)^d. \end{cases} \tag{2.57}$$

The weak limit of $\sigma_\varepsilon P_\varepsilon$ in $L^2_{\text{loc}}(\mathbb{R}^d)^{d\times d}$ is then the homogenized limit of the laminate. The three conditions of (2.57) satisfied by P_ε extend to the laminate case the three respective conditions

$$\begin{cases} \langle DU\rangle = I_d \\ \operatorname{Curl}(DU) = 0 \\ \operatorname{Div}(\sigma DU) = 0, \end{cases} \tag{2.58}$$

satisfied by any electric matrix field DU in the periodic case.

Remark 2.23. In [7] it is proved that the two differential constraints of (2.57) can be derived by a suitable control of the curl and the divergence jumps between two fields in any neighboring layers of a rank-1 sub-laminate in the direction $\xi = \xi_{i,k}$ and at the scale ε_k of the rank-n laminate. More precisely, consider any pair (P, σ) where P is the (matrix of $\hat P$, resp. average of matrices in $\hat P$) electric field and σ is the (matrix of $\hat\sigma$, resp. the effective) conductivity in any (homogeneous, resp. composite) layer, and the pair (Q, τ) in any neighboring (homogeneous, resp. composite) layer along the direction ξ at the scale ε_k (see figure 6 below). Then, the curl convergence of (2.57) is obtained by the jump conditions

$$P - Q = \xi \otimes \eta, \quad \text{for some } \eta \in \mathbb{R}^d, \tag{2.59}$$

and the divergence convergence of (2.57) is obtained by the jump conditions

$$(\sigma P - \tau Q)^T \xi = 0. \tag{2.60}$$

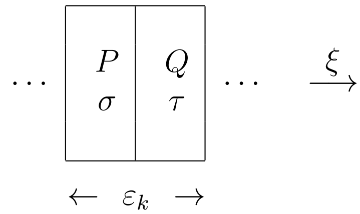

Figure 6: *Two neighboring layers in the direction ξ and at the scale ε_k*

The laminate electric fields and their realizability may be defined as follows:

Definition 2.24. A laminate electric field is any rank-n laminate $L_n^\varepsilon(\hat{P})$ built from a finite set $\hat{P}$ of $\mathbb{R}^{d\times d}$ (according to Definition 2.22), and satisfying the two first conditions of (2.57). Then, the laminate field $L_n^\varepsilon(\hat{P})$ is said to be realizable if there exists a finite set $\hat{\sigma}$ of positive definite matrices such that the rank-n laminate conductivity $L_n^\varepsilon(\hat{\sigma})$ satisfies the third condition of (2.57).

Then, the extension of Theorem 2.19 to laminate fields is the following:

Theorem 2.25 (Milton, Nesi, B. [10] and Milton, Treibergs, B. [11])**.** *Let d, n be two positive integers, and let $L_n^\varepsilon(\hat{P})$ be a rank-n laminate electric field. Then, a necessary and sufficient condition for $L_n^\varepsilon(\hat{P})$ to be a realizable laminate electric field is that* $\det(L_n^\varepsilon(\hat{P})) > 0$ *a.e. in* $\mathbb{R}^d$*, or equivalently that the determinant of each matrix in $\hat{P}$ is positive.*

Proof. The sufficient condition is proved in [11], while the necessary condition is established in [10], Theorem 3.3 (see also [12], Theorem 2.13, for an alternative approach).

Proof of the sufficient condition. Consider a rank-n laminate field $P_\varepsilon = L_n^\varepsilon(\hat{P})$ satisfying the two first convergence of (2.57) and $\det(P_\varepsilon) > 0$ a.e. in $\mathbb{R}^d$, or equivalently $\det(P) > 0$ for any $P \in \hat{P}$. Similarly to (2.55) consider the rank-n laminate conductivity defined by

$$\sigma_\varepsilon := \det\big(P_\varepsilon\big)\,(P_\varepsilon^{-1})^T(P_\varepsilon)^{-1} = L_n^\varepsilon(\hat{\sigma}), \quad \text{where} \quad \hat{\sigma} := \Big\{\det\big(P\big)\,(P^{-1})^T P^{-1} : P \in \hat{P}\Big\}. \tag{2.61}$$

Then, the third condition of (2.57) is equivalent to the condition

$$\operatorname{Div}\big(\operatorname{Cof}(P_\varepsilon)\big) \quad \text{is compact in } H^{-1}_{\rm loc}(\mathbb{R}^d)^d. \tag{2.62}$$

Contrary to the periodic case $\operatorname{Cof}(P_\varepsilon)$ is not divergence free in the sense of distributions. However, following the homogenization procedure for laminates of [7], and using the quasi-affinity of the Cofactors for gradients (see, *e.g.*, [13]), condition (2.62) holds if any matrices P, Q of two neighboring layers in some direction $\xi = \xi_{i,k}$ of the laminate satisfy the jump condition for the divergence (see Remark 2.23 above)

$$\big(\operatorname{Cof}(P) - \operatorname{Cof}(Q)\big)^T\xi = 0. \tag{2.63}$$

More precisely, at a given scale ε_k of the rank-n laminate the matrix P (or Q) is:

1. either a matrix in $\hat{P}$,
2. or the average of rank-1 laminate electric fields obtained at the smallest scales $\varepsilon_{k+1}, \ldots, \varepsilon_n$.

In the first case the matrix P is the constant value of the field in a homogeneous layer of the rank-n laminate. In the second case the average of the Cofactors of the matrices involving in these rank-1 laminates is equal to the Cofactors matrix of the average, that is $\operatorname{Cof}(P)$, by virtue of the quasi-affinity of the Cofactors (see [13], Section 4.1.2) applied iteratively to the rank-1 connected matrices in each rank-1 laminate.

Therefore, it remains to prove equality (2.63) for any matrices P, Q with positive determinant, satisfying the curl jump condition (2.59). Using (2.59) and the multiplicativity of the Cofactors matrix we get that

$$\begin{aligned} \big(\mathrm{Cof}\,(P) - \mathrm{Cof}\,(Q)\,\big)^T &= \mathrm{Cof}\,(Q)^T \left[\mathrm{Cof}\,(I_d + (\xi \otimes \eta)\,Q^{-1})^T - I_d\right] \\ &= \mathrm{Cof}\,(Q)^T \left[\mathrm{Cof}\,(I_d + \xi \otimes \lambda)^T - I_d\right], \quad \text{with } \lambda := (Q^{-1})^T\,\eta. \end{aligned} \tag{2.64}$$

Moreover, if $\xi \cdot \lambda \neq -1$, a simple computation yields

$$\mathrm{Cof}\,(I_d + \xi \otimes \lambda)^T = \det\,(I_d + \xi \otimes \lambda)\,(I_d + \xi \otimes \lambda)^{-1} = (1 + \xi \cdot \lambda)\,I_d - \xi \otimes \lambda, \tag{2.65}$$

which extends to the case $\xi \cdot \lambda = -1$ by a continuity argument. Therefore, it follows that

$$\big(\mathrm{Cof}\,(P) - \mathrm{Cof}\,(Q)\,\big)^T = \mathrm{Cof}\,(Q)^T\,\big((\xi \cdot \lambda)\,I_d - \xi \otimes \lambda\big), \tag{2.66}$$

which implies the desired equality (2.63), since $(\xi \otimes \lambda)\,\xi = (\xi \cdot \lambda)\,\xi$.

Proof of the necessary condition. It is enough to prove that, at any scale ε_k of the lamination and for two neighboring composite (or homogeneous) layers in some direction $\xi = \xi_{i,k}$, associated with the homogenized (or in $\hat{\sigma}$) conductivities σ, τ and the average (or in $\hat{P}$) electric fields P, Q satisfying the jump conditions (2.59) and (2.60), we have

$$\det\,(P)\det\,(Q) > 0. \tag{2.67}$$

Indeed, we deduce from (2.67) and the rank-1 affinity of the determinant that the electric fields in each phase have the same sign. This sign is positive since the global average of the electric fields is equal to I_d according to the first convergence of (2.57).

Now let us prove property (2.67) by contradiction. Namely, assume that there exist P, Q such that $\det\,(P)\det\,(Q) \leq 0$. We have the following alternative:

1. If $\det\,(Q) = 0$, there exists a non-zero vector $\eta \in \mathbb{R}^d$ such that $Q\eta = 0$. Hence, by the jump conditions (2.59) and (2.60) we have $P\eta \parallel \xi$ and $\sigma P\eta \perp \xi$, which implies that $P\eta = 0$ since σ is positive definite. Similarly, we have $P'\eta = 0$ for any average electric field P' at scale ε_k. Then, by considering successively the scales $\varepsilon_{k-1}, \ldots, \varepsilon_1$, we also have $P'\eta = 0$ for any average electric field P at any scale $\gg \varepsilon_k$. Since the global average electric field is equal to I_d at scale ε_1, we finally obtain that $\eta = 0$, which yields the contradiction.

2. Otherwise $\det\,(P) > 0$ and $\det\,(Q) < 0$. Then, introduce a meso-scale ε'_k between ε_{k+1} and ε_k, and replace the two neighboring composites whose homogenized matrices are σ and τ, by a rank-1 laminate at the scale ε'_k, in the same direction ξ and such that the volume fraction of τ is equal to

$$\theta := \frac{\det\,(P)}{\det\,(P) - \det\,(Q)} \in [0, 1]. \tag{2.68}$$

   Since the lamination direction is the same at the scales ε_k and ε'_k, it is easy to deduce from (2.59) and (2.60) that the average electric fields remain unchanged from the smallest scale ε_n to the new scale ε'_k. In particular, the average electric fields associated with σ and τ are still P and Q. By virtue of the reiterated homogenization principle the average electric field at the new scale is equal to the matrix $Q' := \theta\,Q + (1 - \theta)\,P$. However, the determinant of Q' is equal to 0 by (2.68), since P, Q are rank-1 connected and the determinant is rank-1 affine. Therefore, we are led to the first case with Q'.

This concludes the proof of Theorem 2.25. □

3 Hall effect in composites

3.1 The Hall effect and the magneto-resistance

Consider a homogeneous conductor in $\mathbb{R}^d$, with a matrix resistivity ρ_0 which is the inverse of the conductivity σ_0. In the presence of a small magnitude magnetic field $h \in \mathbb{R}^d$ (i.e. $|h| \ll 1$), classical physics claims that the perturbed resistivity ρ_h satisfies

$$\rho_{(-h)} = (\rho_h)^T. \tag{3.1}$$

Hence, the second-order expansion of the perturbed resistivity reads as:

- in dimension $d = 2$ (h is replaced by $h\, e_3 \perp$ to the plane conductor),

$$\rho_h = \rho_0 + h\, r\, R_\perp + h^2\, M + o(h^2), \quad \text{with } R_\perp := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{3.2}$$

  where r is the Hall coefficient and M is magneto-resistance which is a symmetric matrix;

- in dimension $d = 3$,

$$\rho_h = \rho_0 + r\, \mathscr{E}(h) + o(h), \quad \text{where } \mathscr{E}(h) := \begin{pmatrix} 0 & h_3 & -h_2 \\ -h_3 & 0 & h_1 \\ h_2 & -h_1 & 0 \end{pmatrix}. \tag{3.3}$$

For a fixed divergence free current field j, the perturbed electric field $e_h := \rho_h\, j$ satisfies:

- in dimension $d = 2$,

$$e_h = \rho_0\, j + \underbrace{h\, r\, R_\perp\, j}_{\text{Hall field } E_h} + h^2\, M\, j + o(h^2); \tag{3.4}$$

- in dimension $d = 3$,

$$e_h = \rho_0\, j + \underbrace{r\, j \times h}_{\text{Hall field } E_h} + o(h). \tag{3.5}$$

The Hall field which is both perpendicular to j and h is necessary to balance the magnetic force acting on the moving charge carriers as shown in figure 7.

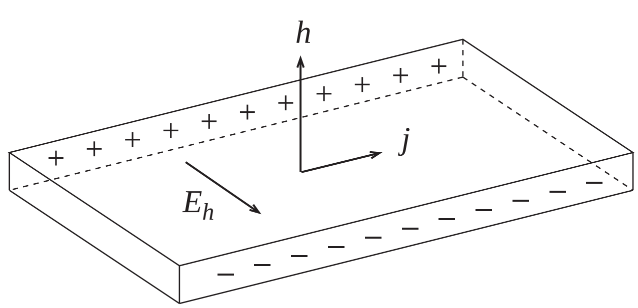

Figure 7: *The Hall field E_h*

3.2 The Hall coefficient in composites

3.2.1 The two-dimensional case

Let $\sigma_0(y) = \rho_0(y)^{-1}$ be a Y_2-periodic matrix-valued conductivity satisfying (1.2). Then, by virtue of (1.5) and (1.6) the unperturbed effective (or homogenized) conductivity $\sigma_0^* = (\rho_0^*)^{-1}$ is given by

$$\sigma_0^* = \langle \sigma_0 DU_0 \rangle, \quad \text{where} \quad \begin{cases} \operatorname{div}(\sigma_0 DU_0) = 0 & \text{in } \mathbb{R}^2 \\ U_0(y) - y & \text{is } Y_2\text{-periodic.} \end{cases} \tag{3.6}$$

Following the Bergman approach [5] we have the following result:

Theorem 3.1 (Milton, Manceau, B. [9])**.** *The effective Hall coefficient is given by*

$$r_* = \frac{\langle r \det(\sigma_0 DU_0) \rangle}{\det(\sigma_0^*)}. \tag{3.7}$$

Moreover, the bounds of the Hall coefficient are preserved, i.e.

$$r_1 \le r(y) \le r_2 \;\; a.e. \;\, in \; Y_2 \quad \Rightarrow \quad r_1 \le r_* \le r_2. \tag{3.8}$$

Proof. On the one hand, we need the periodic div-curl lemma: for any divergence free current field $j \in L^2_\sharp(Y_d)^d$ and any curl free electric field $e \in L^2_\sharp(Y_d)^d$,

$$\langle j \cdot e \rangle = \langle j \rangle \cdot \langle e \rangle. \tag{3.9}$$

Applied to the perturbed energy the div-curl lemma yields

$$\langle (DU_h)^T \sigma_h \, DU_h \rangle = \langle (DU_h)^T \rangle \, \langle \sigma_h DU_h \rangle = \langle DU_h \rangle^T \sigma_h^* \langle DU_h \rangle = \sigma_h^*. \tag{3.10}$$

On the other hand, the expansions of the perturbed quantities σ_h, ρ_h and their homogenized σ_h^*, ρ_h^* read as

$$\begin{cases} \sigma_h = \sigma_0 + s\, h\, R_\perp + o(h) & \rho_h = \rho_0 + r\, h\, R_\perp + o(h) \\ \sigma_h^* = \sigma_0^* + s_*\, h\, R_\perp + o(h) & \rho_h^* = \rho_0^* + r_*\, h\, R_\perp + o(h). \end{cases} \tag{3.11}$$

Moreover, by $\sigma_h \, \rho_h = I_2$ the coefficients s, s_* are given in terms of the Hall coefficients r, r_* by

$$s = -\, r \, \det(\sigma_0) \quad \text{and} \quad s_* = -\, r_* \, \det(\sigma_0^*). \tag{3.12}$$

Hence, putting (3.11) and (3.12) in the energy equalities (3.10) it follows that

$$\begin{aligned} \sigma_h^* &= \sigma_0^* + h \, \langle s\, (DU_0)^T R_\perp \, DU_0 \rangle + o(h) \\ &= \sigma_0^* + h \, \langle s \, \det(DU_0)\, R_\perp \rangle + o(h) \\ &= \sigma_0^* + h\, s_* \, R_\perp + o(h). \end{aligned} \tag{3.13}$$

This combined with (3.12) implies the formula (3.7) for the effective Hall coefficient r_*.

Now, the preservation of the bounds is a consequence of the quasi-affinity of the determinant for divergence free functions in $2d$ (see, e.g., [13]),

$$\langle \det(\sigma_0 DU_0) \rangle = \det\big(\langle \sigma_0 DU_0 \rangle\big) = \det(\sigma_0^*). \tag{3.14}$$

Using the positivity property (1.8) in the previous equality we get that

$$r_1 \, \langle \det(\sigma_0 DU_0) \rangle \le \langle r \, \det(\sigma_0 DU_0) \rangle \le r_2 \, \langle \det(\sigma_0 DU_0) \rangle, \tag{3.15}$$

which combined with (3.7) concludes the proof. □

3.2.2 Reversal of the sign of the Hall coefficient in dimension three

Let $\sigma_0 = \rho_0^{-1}$ be a Y_3-periodic matrix-valued conductivity satisfying (1.2), and for $h \in \mathbb{R}^3$, let $\rho_h = \sigma_h^{-1}$ be a perturbed resistivity satisfying expansion (3.3). In dimension 3, due to the possible anisotropy the first-order term of the effective resistivity $\rho_h^* = (\sigma_h^*)^{-1}$ involves not a single Hall coefficient as in dimension 2, but *a priori* a full Hall matrix $R_* \in \mathbb{R}^{3\times 3}$, i.e.

$$\rho_h^* = \rho_0^* + \mathscr{E}(R^* h) + o(h), \quad \text{where} \quad \mathscr{E}(R^* j) = j \times R^* h, \ \text{ for } j \in \mathbb{R}^3. \tag{3.16}$$

Similarly, the perturbed conductivity σ_h and the effective perturbed conductivity σ_h^* satisfy the first-order expansions

$$\begin{cases} \sigma_h^* = \sigma_0^* + \mathscr{E}(Sh) + o(h), & \text{where } S \in L^\infty_\sharp(Y_3)^{3\times 3} \\ \sigma_h^* = \sigma_0^* + \mathscr{E}(S^* h) + o(h), & \text{where } S^* \in \mathbb{R}^{3\times 3}. \end{cases} \tag{3.17}$$

Note that $S = s\, I_3$ when σ_0 is isotropic. Moreover, expanding the equalities $\sigma_h\, \rho_h = \sigma_h^*\, \rho_h^* = I_3$ and using the algebraic identity

$$P^T\, \mathscr{E}(\xi)\, P = \mathscr{E}\left(\mathrm{Cof}(P)^T \xi\right), \quad \forall\, (\xi, P) \in \mathbb{R}^3 \times \mathbb{R}^{3\times 3}, \tag{3.18}$$

we get the extension of (3.12) to dimension 3,

$$S = -\, r\, \mathrm{Cof}\,(\sigma_0) \quad \text{and} \quad S^* = -\, \mathrm{Cof}\,(\sigma_0^*)\, R^*. \tag{3.19}$$

Contrary to the two-dimensional case the bounds of the Hall coefficients are not necessarily preserved by the homogenization process:

Theorem 3.2 (Milton, B. [8])**.** *There exist a Y_3-periodic conductivity $\sigma_0(y)$, and a Hall coefficient $r(y) > 0$, such that the effective Hall matrix R^* is isotropic and satisfies*

$$R^* = r_*\, I_3 \quad \textit{with} \quad r_* < 0. \tag{3.20}$$

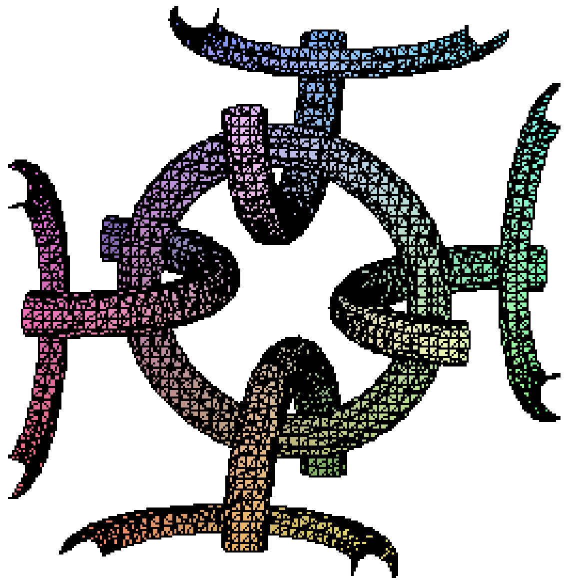

Figure 8: *The period Q of the cubic chainmail with four half links crossing a central one*

Proof. The proof is based on the same type of periodic chainmail $Q_\sharp$ as in Section 1.2.2, but arranged in a cubic way as a Middle-Age armor (see figure 8). For $\kappa > 0$, let σ_0^κ be the unperturbed conductivity defined by (1.9), and let U_0^κ be the associated vector-valued potential solution of (3.6). Also consider the Y_3-periodic isotropic matrix $S = s\, I_3$ associated with σ_h by (3.17), and defined in $\mathbb{R}^3$ by:

- $s = 1$ in the cubic lattice of balls $B_\sharp(\delta)$ of radius $\delta > 0$, centered on points between two interlocking rings as shown in figure 9;
- $s = \gamma > 0$ elsewhere.

Thanks to the cubic symmetry the composite inherits of the isotropy of σ_h, so that the conductivity σ_h^* and the effective matrices R^*, S^* are isotropic. This combined with relation (3.19) implies that we are led to simply prove the change of sign for S^*.

On the one hand, starting from the expansion of the conductivity

$$\sigma_h = \sigma_0^\kappa + s\,\mathscr{E}(h) + o(h), \tag{3.21}$$

then expanding the perturbed energy (3.10) and using the formula (3.18) with $P := DU_0^\kappa$ and $\xi := s\,h$, combined with the expansion (3.17) of σ_h^*, we have

$$\begin{aligned} \sigma_h^* &= \sigma_0^* + \left\langle (DU_0^\kappa)^T \mathscr{E}(Sh) DU_0^\kappa \right\rangle + o(h) \\ &= \sigma_0^* + \left\langle \mathscr{E}\big(\mathrm{Cof}(DU_0^\kappa)^T Sh\big) \right\rangle + o(h) \\ &= \sigma_0^* + \mathscr{E}(S^* h) + o(h). \end{aligned} \tag{3.22}$$

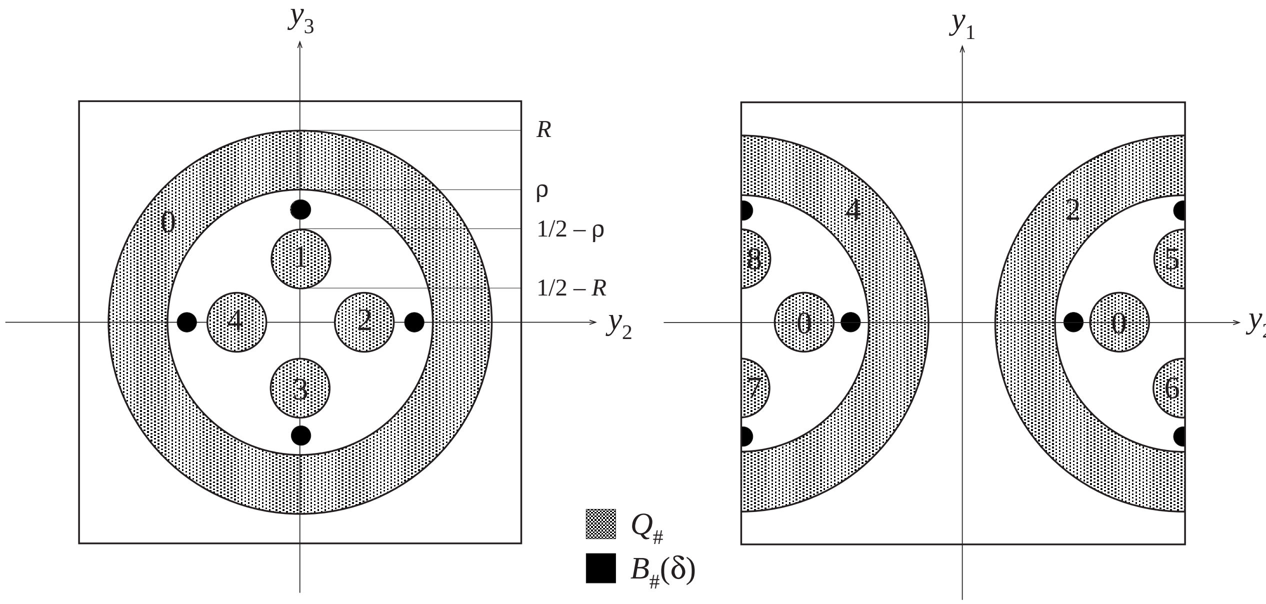

Figure 9: *Two sections of the cubic chainmail period*

Hence, the matrix S^* satisfies

$$S^* = \left\langle \mathrm{Cof}\,(DU_0^\kappa)^T S \right\rangle. \tag{3.23}$$

Since $S^* = s_*\, I_3$, we get that

$$s_* = \frac{1}{3}\left\langle s\,\mathrm{tr}\big(\mathrm{Cof}(DU_0^\kappa)\big)\right\rangle. \tag{3.24}$$

Now, follow the arguments of the proof of Theorem 1.2. By virtue of the cubic symmetry the integral over the balls on which $s = 1$, is reduced to the integral over one ball of radius δ, centered on a point $(0, 0, \tau)$ such that, according to (1.14) and (1.15),

$$\tau \in (1/2 - \rho, \rho) \quad \text{and} \quad DU(0,0,\tau) = \begin{pmatrix} \frac{\partial u_1}{\partial y_1} > 0 & 0 & 0 \\ 0 & \frac{\partial u_2}{\partial y_2} > 0 & 0 \\ 0 & 0 & \frac{\partial u_3}{\partial y_3} = \frac{1}{1-4\rho} < 0 \end{pmatrix}, \tag{3.25}$$

where $U = (u_1, u_2, u_3)$ is the limit of U_0^κ as $\kappa \to \infty$ satisfying (1.12), and $\rho > 1/4$ is the inner radius of the rings. Comparing u_i to the function $y \mapsto u_i(y) - (1/2 - R)^{-1} y_i$, we can also show that

$$0 < \frac{\partial u_i}{\partial x_i}(0,0,\tau) \le \frac{1}{1/2 - R}, \quad \text{for } i = 1, 2. \tag{3.26}$$

Hence, it follows that

$$\mathrm{tr}\big(DU(0,0,\tau)\big) < 0 \quad \text{as } \rho \approx \frac{1}{4}. \tag{3.27}$$

Finally, from (3.24) and the strong convergence of DU_0^κ to DU in $L^2(Y_3)^{3\times 3}$ as $\kappa \to \infty$, we deduce that

$$\begin{aligned} s_* &= \frac{\gamma}{3}\int_{\{s=\gamma\}\cap Y_3} \mathrm{tr}\big(\mathrm{Cof}(DU_0^\kappa)\big)\,dy + \frac{1}{3}\int_{\{s=1\}\cap Y_3} \mathrm{tr}\big(\mathrm{Cof}(DU_0^\kappa)\big)\,dy \\ &= O(\gamma) + \frac{1}{3}\int_{\{s=1\}\cap Y_3} \mathrm{tr}\big(\mathrm{Cof}(DU_0)\big)\,dy + o_\kappa(1) \\ &= O(\gamma) + c\,\delta^3\left[\mathrm{tr}\big(DU(0,0,\tau)\big) + o_\delta(1)\right] + o_\kappa(1), \end{aligned} \tag{3.28}$$

where $c > 0$ is a fixed constant. Therefore, choosing successively

$$\rho \approx \frac{1}{4}, \quad \delta \ll 1, \quad \gamma \ll \delta^3, \quad \kappa \gg 1, \tag{3.29}$$

we obtain that s_* has the same sign as $\operatorname{tr}\big(DU(0,0,\tau)\big)$ which is negative by (3.27). □

3.3 An effective Hall field parallel to the magnetic field

By the expansion (3.5) the Hall field is given by $E_h = r\, j \times h$ in dimension 3, so that E_h is both orthogonal to the magnetic field h and the current field j. Then, taking the average of the perturbed electric field (3.5) and using the expansion (3.16) of the effective resistivity, it follows that the effective Hall field E_h^* satisfies

$$\begin{aligned} \langle e_h \rangle &= \rho_0^* \langle j \rangle + E_h^* + o(h) \\ &= \rho_h^* \langle j \rangle = \rho_0^* \langle j \rangle + \mathscr{E}(R^* h) \langle j \rangle + o(h) \\ &= \rho_0^* \langle j \rangle + \langle j \rangle \times (R^* h) + o(h), \end{aligned} \tag{3.30}$$

which implies that

$$E_h^* = \langle j \rangle \times (R^* h). \tag{3.31}$$

Moreover, by (3.19) and (3.23) the effective Hall matrix is given by

$$R^* = \frac{\sigma_0^*}{\det(\sigma_0^*)} \left\langle r \operatorname{Cof}(\sigma_0 D U_0)^T \right\rangle, \tag{3.32}$$

which is the natural extension to dimension 3 of the formula (3.7) in dimension 2.

Since the effective R^* is not necessarily isotropic, E_h^* may be not orthogonal to h. Actually, we have the following result:

Theorem 3.3 (Milton, B. [8]). *There exists a periodic composite such that the effective Hall field E_h^* is asymptotically parallel to the magnetic field h. It is derived from an effective Hall matrix R^* which is asymptotically antisymmetric.*

Proof. The proof is based on the Y_3-periodic structure the cross section of which is represented in figure 10. For $\kappa > 0$, the unperturbed conductivity σ_0^κ of the composite is defined by

$$\sigma_0^\kappa(y) := \begin{cases} \operatorname{diag}(\kappa, \kappa, 1) & \text{if } y \in Q_s = Q_1 \cup Q_2 \cup Q_3 \cup Q_4 \\ I_3 & \text{if } y \in Y_3 \setminus Q_s, \end{cases} \tag{3.33}$$

Note that the anisotropic conductivity σ^κ can be reduced to an isotropic one first using a suitable lamination in the direction y_3. The Hall coefficient r is defined by

$$r(y) := r_\kappa\, 1_{K_1}(y), \quad \text{for } y \in Y, \tag{3.34}$$

where the constant $r_\kappa > 0$ will be chosen later and 1_{K_1} is the characteristic function of the central square K_1 of side $\ell < \frac{1}{3}$ in figure 10. The regions Q_1, Q_2, Q_3, Q_4 are highly conducting in the plane y_1-y_2, and only the central square K_1 has a non-zero Hall coefficient.

Now, consider the potential $U_0^\kappa = (u_1^\kappa, u_2^\kappa, u_3^\kappa)$ solution of the equation (3.6) with σ_0^κ. Due to the columnar structure in the direction y_3 we have $\nabla u_3^\kappa = e_3$, while u_1^κ and u_2^κ do not depend

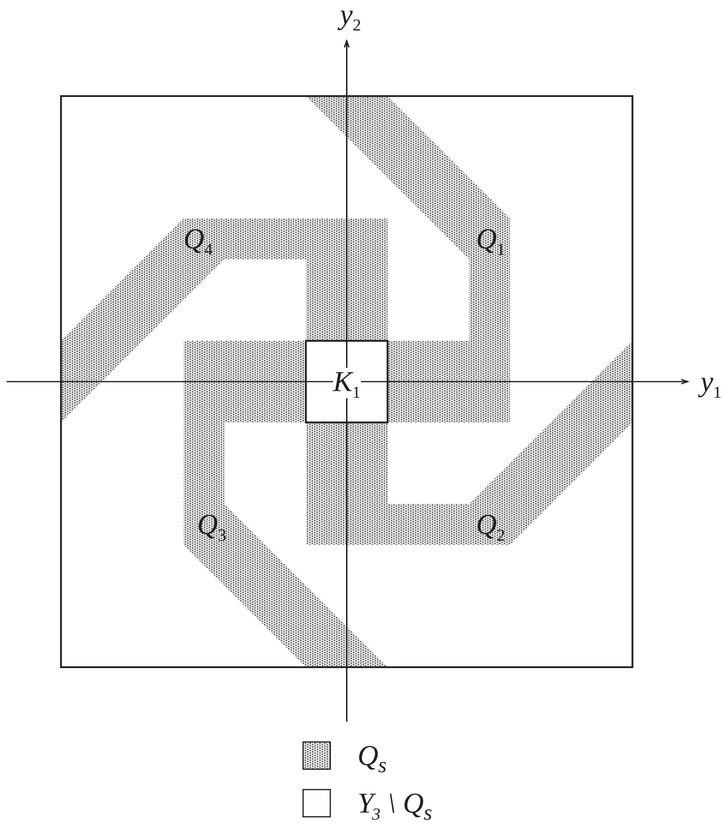

Figure 10: *The cross section of the columnar period cell.*

on the y_3 variable. This combined with the invariance by a rotation of 90° of figure 10 and the definition (3.33) of σ_0^κ, implies that the homogenized conductivity σ_0^* reads as

$$\sigma_0^* = \begin{pmatrix} a_\kappa & 0 & 0 \\ 0 & a_\kappa & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{where } a_\kappa > 0. \tag{3.35}$$

By analogy with a two-phase $(1, \kappa)$ checkerboard (see, e.g., [15], Section 3.3) we can also prove that the effective coefficient a_κ satisfies

$$a_\kappa \approx \sqrt{\kappa}. \tag{3.36}$$

On the other hand, According to the variational principle (1.7) we also have

$$a_\kappa = \int_Y \sigma_0^\kappa \nabla u_1^\kappa \cdot \nabla u_1^\kappa \, dy \geq \kappa \int_{Q_s} |\nabla u_1^\kappa|^2 \, dy, \tag{3.37}$$

which combined with estimate (3.36) implies that ∇u_1^κ strongly converges to zero in $L^2(Q_s)$. That is expected because the electric field should be close to zero in the highly conducting phase, except near the corner contact points. Then, the Poincaré-Wirtinger inequality in the regular connected open set Q_i, for $i = 1, \ldots, 4$, yields

$$⨍_{\partial K_1 \cap \partial Q_i} u_1^\kappa \, dy - ⨍_{\partial Y_2 \cap \partial Q_i} u_1^\kappa \, dy = o(1), \quad \text{as } \kappa \to \infty. \tag{3.38}$$

In other words, the average electric potential along the boundary $\partial K_1 \cap \partial Q_i$ should be close to that along the boundary $\partial Y_2 \cap \partial Q_i$, as expected because the region Q_i is highly conducting in the plane. Moreover, since $y_1 \mapsto u_1^\kappa(y) - y_1$ is Y-periodic, we have (see figure 1)

$$\int_{\partial Y_2 \cap \partial Q_1} u_1^\kappa \, dy = \int_{\partial Y_2 \cap \partial Q_3} u_1^\kappa \, dy \quad \text{and} \quad \int_{\partial Y_2 \cap \partial Q_2} u_1^\kappa \, dy = \int_{\partial Y_2 \cap \partial Q_4} u_1^\kappa \, dy + \ell. \tag{3.39}$$

Hence, it follows from (3.38) and (3.39) that

$$\left\{ \begin{aligned} \int_{K_1} \frac{\partial u_1^\kappa}{\partial y_1} \, dy &= \int_{\partial K_1 \cap \partial Q_1} u_1^\kappa \, dy - \int_{\partial K_1 \cap \partial Q_3} u_1^\kappa \, dy = o(1) \\ \int_{K_1} \frac{\partial u_1^\kappa}{\partial y_2} \, dy &= \int_{\partial K_1 \cap \partial Q_4} u_1^\kappa \, dy - \int_{\partial K_1 \cap \partial Q_2} u_1^\kappa \, dy = -\,\ell + o(1), \end{aligned} \right. \qquad \text{as } \kappa \to \infty. \tag{3.40}$$

Similarly, with a change of sign we get for the function u_2^κ

$$\left\{ \begin{aligned} \int_{K_1} \frac{\partial u_2^\kappa}{\partial y_1} \, dy &= \int_{\partial K_1 \cap \partial Q_1} u_2^\kappa \, dy - \int_{\partial K_1 \cap \partial Q_3} u_2^\kappa \, dy = \ell + o(1) \\ \int_{K_1} \frac{\partial u_2^\kappa}{\partial y_2} \, dy &= \int_{\partial K_1 \cap \partial Q_4} u_2^\kappa \, dy - \int_{\partial K_1 \cap \partial Q_2} u_2^\kappa \, dy = o(1). \end{aligned} \right. \qquad \text{as } \kappa \to \infty. \tag{3.41}$$

Now, putting the Cofactors matrix of DU_0^κ

$$\operatorname{Cof}\left(DU_0^\kappa\right)^T = \begin{pmatrix} \frac{\partial u_2^\kappa}{\partial y_2} & -\frac{\partial u_2^\kappa}{\partial y_1} & 0 \\ -\frac{\partial u_1^\kappa}{\partial y_2} & \frac{\partial u_1^\kappa}{\partial y_1} & 0 \\ 0 & 0 & \frac{\partial u_1^\kappa}{\partial y_1}\frac{\partial u_2^\kappa}{\partial y_2} - \frac{\partial u_1^\kappa}{\partial y_2}\frac{\partial u_2^\kappa}{\partial y_1} \end{pmatrix} \tag{3.42}$$

in (3.32), we deduce from (3.35), (3.40), (3.41) that the effective Hall matrix R_κ^* satisfies

$$R_\kappa^* = \frac{r_\kappa}{a_\kappa^2} \begin{pmatrix} a_\kappa & 0 & 0 \\ 0 & a_\kappa & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} o(1) & -\,\ell + o(1) & 0 \\ \ell + o(1) & o(1) & 0 \\ 0 & 0 & c_\kappa \end{pmatrix}, \tag{3.43}$$

where

$$c_\kappa := \int_{K_1} \left(\frac{\partial u_1^\kappa}{\partial y_1}\frac{\partial u_2^\kappa}{\partial y_2} - \frac{\partial u_1^\kappa}{\partial y_2}\frac{\partial u_2^\kappa}{\partial y_1} \right) dy = \int_{K_1} \det\left(\nabla u_1^\kappa, \nabla u_2^\kappa\right) dy. \tag{3.44}$$

Moreover, by the positivity result (1.8) we have $\det\left(\nabla u_1^\kappa, \nabla u_2^\kappa\right) > 0$ a.e. in Y_2. This combined with the quasi-affinity of the determinant (see [13], Section 4.1.2) yields

$$0 < c_\kappa \le \left\langle \det\left(\nabla u_1^\kappa, \nabla u_2^\kappa\right) \right\rangle = \det\left(\langle \nabla u_1^\kappa, \nabla u_2^\kappa \rangle\right) = 1. \tag{3.45}$$

Then, choosing $r_\kappa := a_\kappa/\ell$, the estimates (3.36) and (3.45) give

$$\lim_{\kappa\to\infty} \frac{r_\kappa\, c_\kappa}{a_\kappa^2} = \lim_{\kappa\to\infty} \frac{c_\kappa}{\ell\, a_\kappa} = 0, \tag{3.46}$$

which finally implies that

$$\lim_{\kappa\to\infty} R_\kappa^* = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.47}$$

Therefore, taking a periodic current field j with $\langle j \rangle \parallel e_3$ and $h \perp e_3$, we obtain that the effective Hall field $E_h = \langle j \rangle \times (R^* h)$ is asymptotically parallel to h, which concludes the proof of Theorem 3.3. □

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