Isomonodromic Deformations

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Abstract

These notes are prepared for a mini-course on isomonodromic deformations (see http://perso.math.univ-toulouse.fr/jisom/) given by the author at the University of Paul Sabatier in Toulouse in June 2012. The main ideas of the theory of isomonodromy deformations and a few references for further study are given. This is a survey paper which is mainly based on [28, 18, 20, 21, 22].

1 Introduction

The discovery of six Painlevé equations by P. Painlevé and B. Gambier was one of the greatest achievements in the theory of analytical functions in XIX–XX centuries.

Although the Painlevé equations were discovered from purely mathematical considerations, they started to play a key role at the end of the XXth century. The theory of the Painlevé equations was elaborated by many scientists (e.g., K. Okamoto [30], Sh. Shimomura, K. Iwasaki, H. Kimura, M. Yoshida [18] and many others).

A new development in the theory of the Painlevé equations was closely related to the theory of isomonodromic deformations of linear systems proposed in papers of the Japanese mathematicians M. Sato, M. Jimbo, T. Miwa, K. Ueno. The development of this theory, which went back to the original papers by L. Schlesinger, R. Fuchs, R. Garnier, was also stimulated by the theory of holonomic quantum fields. Another breakthrough in the theory of the isomonodromic deformations of linear systems was due to A.V. Kitaev, A.R. Its, V.Yu. Novokshonov [17] and many others who studied asymptotic properties of the solutions of the nonlinear ordinary differential equations and, in particular, the Painlevé equations.

2 Painlevé equations

As it has been mentioned above, the discovery of the new six Painlevé equations the solutions of which have no movable critical points is indispensable for the theory of analytic functions in XX century. Such equations are usually called equations of $P-$type. P. Painleve [31] — [34] and B. Gambier [11, 12] classified the second order ordinary differential equations of $P-$type of the form

$$w'' = \mathcal{R}(z, w, w'), \tag{1}$$

where $\mathcal{R}$ is a rational function of w, w' with analytic in $z \in \mathcal{D}$ coefficients.

In the result of the investigation by the modified small parameter method known as α-method, P. Painleve and B. Gambier proved that there are fifty canonical equations of the form (1) with the Painlevé property. Applying linear-fractional transformations of the dependent variable along with the analytical change of the independent variable for each equation, they obtained 50 canonical equations. It appeared after integration that the general solution of 44 equations is expressed either in elementary functions or elliptic functions or in terms of the functions which are the solutions of some linear equations or in terms of the solutions of the Fuchsian equations $\mathcal{P}(z, w, w') = 0$ or, finally, in terms of the solutions of the other 6 equations which are called the Painlevé equations and denoted by (P_1) — (P_6). A lot of useful information about the Painlevé equations and further references are collected at the DLMF project[1]. For instance, the third and the fifth Painlevé equation are respectively given by

$$w'' = \frac{w'^2}{w} - \frac{w'}{z} + \frac{1}{z}\left(\alpha w^2 + \beta\right) + \gamma w^3 + \frac{\delta}{w}, \tag{2}$$

$$w'' = \frac{3w-1}{2w(w-1)}w'^2 - \frac{w'}{z} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1}, \tag{3}$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary complex parameters.

The movable singular points of the solutions (P_1) — (P_6) are poles of the first and second order. The fixed singular points ($z = 0,\, 1, \infty$) are critical.

The Painlevé equations may be regarded as the nonlinear counterparts of the equations for the classical special functions. Their solutions are transcendental in a sense that they cannot generally be expressed in terms of the solutions of either the first order Fuchsian equations or each other. However, there exist integrable cases with certain values of the parameters. In addition, the Painlevé equations have both algebraic solutions and one-parameter families of solutions solvable in terms of special functions. The Painlevé

[1] http://dlmf.nist.gov/32

equations also possess the so-called Bäcklund transformations. Informally, the Bäcklund transformations relate a given solution to another solution either of the same equation possibly with different parameter values or of another equation [16, 4, 30, 8].

In [10] R. Fuchs showed that equation (P_6) described isomonodromy deformations of a linear differential equation of the second order with four regular points. In [13, 37] the other Painlevé equations (P_1) — (P_5) were obtained from (P_6) by a confluence procedure. It was also shown that they described isomonodromy deformations of certain linear differential equations of the second order with both regular and irregular singularities. The isomonodromy deformations technique was developed in [9, 17, 29] and in this sense the Painlevé equations are integrable. Research conducted by R. Fuchs, L. Schlesinger and R. Garnier was continued by M. Jimbo, T. Miwa, K. Ueno and M. Sato [19] — [22] due to the development of the theory of holonomic quantum fields [36]. They proved the Painlevé property of the deformation equations of linear systems. In the framework of their theory they also found new applications of the Painlevé equations to physical problems. There exist connections between the isomonodromic deformations method, differential geometry [6] and the theory of integrable systems [1].

Although the Painlevé equations were obtained from purely mathematical considerations they also appeared in many physical models as reductions of partial differential equations, the so-called soliton equations solvable by the inverse scattering transform [1].

The first study of asymptotic properties of solutions of the Painlevé equations on the complex plane was undertaken by P. Boutroux [3]. In particular he studied the distribution of poles in the neighborhood of infinity, found asymptotic formulas for adjacent poles and curves on the complex plane along which the Painlevé functions were periodic.

N.Yu. Novokshonov used the isomonodromy deformations method for the first time to find asymptotic connection formulas of the solutions of the Painlevé equations. Then his idea was developed in papers by A.R. Its, A.A. Kapaev, A.V. Kitaev [23, 26, 17], B.I. Suleimanov, H. Kimura [25], K. Takano, A.I. Yablonsky and many others.

Starting from the 1950s, and especially in the 1990th, there appeared a number of papers studying different properties of the Painlevé equations including their discrete counterparts. Informally, the continuous Painlevé equations are obtained from their discrete counterparts, i.e., nonlinear difference equations, by a suitable limiting process. The discrete Painlevé equations are extensively applied to many physical problems. An overview of the latest results and open problems in the theory of the discrete Painlevé equations may be found in [5, 15, 35] with a rich bibliography.

3 The method of the isomonodromic deformations in the theory of linear systems

The general solution of a linear ordinary differential equation with rational coefficients is generally multivalued. This property is described by a representation of the fundamental homotopy group of the complex plane deprived of the singular points, the monodromy representation.

The deformation of a linear ordinary differential equation (ODE) while preserving its associated monodromy representation leads to systems of linear partial differential equations, the integrability (compatibility) conditions of which are nonlinear differential equations of P — type [28]. The method of isomonodromic deformations is a powerful tool to associate linear with integrable nonlinear equations and to solve difficult problems such as connection problems of nonlinear differential equations, asymptotic properties of the Painlevé equations and others. It is well-known [29] (see also [18, p. 119] and the references therein) that the Garnier systems in the form of the Hamiltonian systems with a polynomial Hamiltonian appear as a deformation of a second-order Fuchsian equation with $n+3$ singularities. When $n=1$ the Garnier system is equivalent to the sixth Painlevé equation. The isomonodromic deformations method is developed in the papers by H. Flaschka, A.C. Newell [9], M. Jimbo, T. Miwa, K. Ueno [19] — [22], T. Kimura [24, 25], A. Its, V.Yu. Novokshonov [17], M.V. Fedoryuk [7], A.V. Kitaev [27], etc.

Next we introduce the concept of isomonodromic deformations following [28]. Let

$$\frac{d}{dx}y(x) = A(x)y(x) \tag{4}$$

be a homogeneous linear system of order N with an $N\times N$ matrix $A(x)$ rational in x. Let $Y(x)$ be a fundamental $N\times N$ matrix solution of system (4) which satisfies the matrix equation

$$\frac{d}{dx}Y(x) = A(x)Y(x). \tag{5}$$

Let the poles of matrix $A(x)$ be localized at the points $a_\nu,\ \nu\in\{1,...,n\},\ a_\infty=\infty$.

Definition 1. *System* (4) *is called Fuchsian at the point* $\mathbf{a}$ *(and* $\mathbf{a}$ *is a Fuchsian singularity of the system) if* $A(x)$ *has a simple pole at* $\mathbf{a}$. *System* (4) *is called Fuchsian when all its singularities* $a_\nu,\ a_\infty$ *are Fuchsian.*

If system (4) is Fuchsian at points $a_\nu,\ \nu\in\{1,...,n\},\ a_\infty=\infty$, then $A(x)=\sum_{\nu=1}^{n} A_\nu/(x-a_\nu)$. If (4) has no singularity at $x=\infty$, then $\sum_{\nu=1}^{n} A_\nu = 0$.

The matrix $A(x)$ has singular points at which solution $Y(x)$ is generally multivalued. To describe the multivaluedness of $Y(x)$ one considers a monodromy representation of system (5), i.e., a subgroup $GL(N,C)$ related to system (4) [18, p. 75].

We define the projective plane deprived of the singular points by P_a^1 and its universal covering by $\overline{P_a^1}$. Let $\pi : \overline{P_a^1} \to P_a^1$ be a covering map. Then $Y(x)$ is single-valued on $\overline{P_a^1}$.

Let γ be a path in $\overline{P_a^1}$ starting at the point x and ending at x_γ such that $\pi(x_\gamma) = \pi(x)$ (i.e., $\pi(\gamma)$ is a closed path in P_a^1). Matrix $Y(x_\gamma)$ satisfies (5) and, hence, there exists a nonsingular constant matrix $M_\gamma \in GL(N,C)$ such that $Y(x_\gamma) = Y(x)M_\gamma$, where M_γ is a function of the homotopy class $[\gamma]$ of the path γ.

Definition 2. *The mapping $[\gamma] \to M_\gamma$ defines a representation of the fundamental homotopy group of P_a^1 or monodromy representation associated with the differential system* (5).

It is known that any representation of the fundamental homotopy group of P_a^1 is the monodromy representation of a Fuchsian system. The Riemann-Hilbert problem is to prove the existence of the Fuchsian system (4) the monodromy representation of which coincides with a given representation of the fundamental homotopy group. A.A. Bolibruch [2, p. 98], showed that the Riemann-Hilbert problem cannot be solved positively in general case and explicitly constructed linear systems with double poles the monodromy representation of which is not isomorphic to the monodromy of a differential system with simple poles.

Another problem occurring during the study of linear systems is the problem of isomonodromic deformations of a system (5).

Definition 3 [7]. *Let*

$$dy(x)/dx = A(x,t)y(x) \tag{6}$$

be a system of N equations, where $x \in \mathbb{C}$, $t \in {}^m$. System (6) *is called a deformation of a system where $t = t_0$ is fixed.*

Let us choose for each singular point a_ν a path γ_ν with endpoints x and x_ν which encircles a_ν counterclockwise. The monodromy matrices M_ν defined by $Y(x_\nu) = Y(x)M_\nu$, $\nu \in \{1, ..., n, \infty\}$ generate the monodromy group. Since it is always possible to choose paths γ_ν in such a way that their product [18] $\gamma_1 ... \gamma_n \gamma_\infty$ is homotopic to a point, the following monodromy constraint holds $M_1 ... M_n M_\infty = 1$.

Definition 4. *A deformation is isomonodromic if and only if it leaves all M_ν invariant.*

At the beginning of the previous century L. Schlesinger [37] showed that the deformation of the Fuchsian system is isomonodromic if $Y(x)$ as a function of the deformation parameters satisfies a system of linear partial differential equations or $A(x)$ as the function of the same deformation parameters satisfies a completely integrable nonlinear differential system. Later in [9, 20] L. Schlesinger's results were generalized for non-Fuchsian systems. Thus, the isomonodromic deformation establishes a deep connection between linear and nonlinear completely integrable differential equations.

The method of the isomonodromic deformations of linear systems is given in details in [28, 22, 20, 18]. Below we mention only basic facts.

3.1 Isomonodromic deformations of linear systems with Fuchsian singularities

Let $A(x) = \sum_{\nu=1}^{n} A_\nu/(x-a_\nu)$, where matrices A_ν are of order $N\times N$, $A_\infty = -\sum_{\nu=1}^{n} A_\nu$.

Assumption 1. *All matrices A_ν are diagonalizable $A_0^{(\nu)} = G_\nu^{-1} A_\nu G_\nu$, $\nu \in \{1, ..., n, \infty\}$ and without loss of generality $G_\infty = \mathbf{1}$, where $\mathbf{1}$ is an identity matrix, i.e., matrix A_∞ is diagonal.*

In the neighborhood of singular points a_ν we have

$$Y^{(\nu)}(x) = G_\nu \left(\mathbf{1} + \sum_{j=1}^{\infty} Y_j^{(\nu)} (x_\nu)^j \right) (x_\nu)^{A_0^{(\nu)}}, \tag{7}$$

where $(x_\nu) = (x - a_\nu)$, $\nu \in \{1, ..., n\}$, and $(x_\infty) = 1/x$.

Assumption 2. *All eigenvalues of matrices A_ν, $\nu \in \{1, ..., n, \infty\}$, are distinct modulo the nonzero integers.*

Assumption 2 is necessary for the existence of the series in the right-hand side of (7). If assumption 2 is not satisfied, then, generally, the series in the right-hand side of (7) must be supplemented with logarithm. Series in (7) converges inside some circle with the nearest singular point on its boundary. The right-hand side of (7) is multivalued inside the circle of convergence.

Any solution of (5) can be written in the form $Y^{(\nu)}(x)C_\nu$, where $C_\nu \in GL(N, C)$. Let us choose the following fundamental matrix solution $Y^{(\infty)}(x)$ and define the connection matrices by $Y^{(\infty)}(x) = Y^{(\nu)}(x)C_\nu$, $\nu \in \{1, ...n\}$.

According to the expansion (7) after encircling the singular point a_ν along the path γ_ν we get $Y^{(\nu)}(x_\nu) = Y^{(\nu)}(x) e^{2\pi i A_0^{(\nu)}}$. This implies that $Y(x_\nu) = Y(x) M_\nu$, where *the monodromy matrices* M_ν are given by the following formula: $M_\nu = C_\nu^{-1} e^{2\pi i A_0^{(\nu)}} C_\nu$.

Let us denote *the singularity data of the system* by $SD = \{a_\nu,\ A_0^{(\nu)},\ G_\nu,\ \nu \in \{1, ..., n\}\}$, where $\sum_{\nu=1,...,n} G_\nu A_0^{(\nu)} G_\nu^{-1}$ is a diagonal matrix. Let us also denote *the monodromy data* which characterize the monodromy properties of the fundamental matrix solution $Y(x)$ by $MD = \{a_\nu,\ A_0^{(\nu)},\ C_\nu,\ \nu \in \{1, ..., n, \infty\}\}$, where $a_\infty = \infty$, $C_\infty = \mathbf{1}$.

We shall say that $Y(x)$ has *the monodromy properties MD* if $Y(x)$ is holomorphic and invertible in $\overline{P^1_a}$ and $Y(x)C_\nu^{-1}(x_\nu)^{-A_0^{(\nu)}}$ is holomorphic at $x=a_\nu$ for any ν. Note that the matrices in the definition of MD and SD are defined up to some invertible matrix D_ν, i.e., $G_\nu \to G_\nu D_\nu$ and $C_\nu \to D_\nu^{-1}C_\nu$. It is obvious that MD are determined by SD. The converse statement holds as well.

Lemma 1 [28]. *If there exists a matrix $Y(x)$, which has the monodromy properties MD, then it is unique. Moreover, $Y(x)$ satisfies a differential system of the form* (5) *with a rational matrix $A(x)$ with simple poles.*

Hence, SD are uniquely defined by MD.

Under the isomonodromic deformations of the Fuchsian differential system (5) the matrix $A(x)$ is continuously deformed while the monodromy matrices M_ν are not modified. In other words, SD are continuously deformed while *the partial monodromy data* $PMD=\{A_0^{(\nu)},\ C_\nu,\ \nu\in\{1,...,n,\infty\},\ C_\infty=\mathbf{1}\}$ are preserved.

It follows from lemma 1 that we can deform a_ν continuously and independently. Thus, $Y(x)=Y(x,a)$, $G_\nu=G_\nu(a)$. The coefficients of matrices A_ν also depend on $a=\{a_\nu,\ \nu\in\{1,...,n\}\}$.

Theorem 1 [37]. *The deformations of the system of linear differential equations*

$$\frac{\partial}{\partial x}Y(x,a)=\sum_{\nu=1}^{n}\frac{A_\nu(a)}{x-a_\nu}Y(x,a) \tag{8}$$

are isomonodromic if and only if either $Y(x,a)$ satisfies the following set of linear partial differential equations

$$\frac{\partial}{\partial a_\nu}Y(x,a)=-\frac{A_\nu(a)}{x-a_\nu}Y(x,a),\ \nu\in\{1,...,n\}, \tag{9}$$

or matrices $A_\nu(a)$ satisfy the integrability conditions of (8), (9), *i.e., the completely integrable set of nonlinear partial differential equations*

$$\frac{\partial}{\partial a_\mu}A_\nu=\frac{[A_\mu,A_\nu]}{a_\mu-a_\nu},\ (\mu\neq\nu),\ \ \frac{\partial}{\partial a_\nu}A_\nu=-\sum_{\mu\neq\nu,\ \mu=1}^{n}\frac{[A_\mu,A_\nu]}{a_\mu-a_\nu}. \tag{10}$$

A proof can also be found in [18, Lm. 5.2, p. 197].

Observe that by a simple transformation we can always fix three singular points. Hence, when $n<3$ we get trivial cases. We can also fix either the trace of matrices A_ν or one of its eigenvalues.

Let $N=2$, $n=3$. We fix singular points $x=0;\ 1;\ t;\ \infty$. In this case the only deformation parameter is t. Systems (8) and (10) are written in the following way:

$$\frac{\partial}{\partial x}Y(x,t)=\left(\frac{A_0(t)}{x}+\frac{A_1(t)}{x-1}+\frac{A_t(t)}{x-t}\right)Y(x,t),\ \frac{\partial}{\partial t}Y(x,t)=-\frac{A_t(t)}{x-t}Y(x,t); \tag{11}$$

where $A_0+A_1+A_t+A_\infty=0$. Assume that $\pm\theta_0/2,\ \pm\theta_1/2,\ \pm\theta_t/2$ and $\pm\theta_\infty/2$ are eigenvalues of matrices $A_0,\ A_1,\ A_t,\ A_\infty$ and

$$A_\nu=\frac{1}{2}\begin{pmatrix} z_\nu & u_\nu(\theta_\nu-z_\nu)\\ (\theta_\nu+z_\nu)/u_\nu & -z_\nu\end{pmatrix},\quad \nu=0;1;t,\quad A_\infty=\begin{pmatrix}\theta_\infty/2 & 0\\ 0 & -\theta_\infty/2\end{pmatrix},$$

where $z_0,\ z_1,\ z_t,\ u_0,\ u_1,\ u_t$ are the functions of the parameter t. Denote $A(x,t)_{12}=k(x-y)/(2x(x-1)(x-t))$, where $k=tu_0(z_0-\theta_0)-(1-t)u_1(z_1-\theta_1)$, $ky=tu_0(z_0-\theta_0)$. Next we introduce the following notation: $\xi=z_0+z_1$, $\zeta=t(1-y)z_0+(1-t)yz_1$. Then we get the following system:

$$z_0'=-\frac{Z}{2t},\ z_1'=-\frac{Z}{2(1-t)},\ y'=\frac{(1-\theta_\infty)y(1-y)-\zeta}{t(1-t)},\ \frac{k'}{k}=\frac{(1-\theta_\infty)(y-t)}{t(1-t)},$$

where

$$Z=-\left[\frac{1}{t(1-y)}+\frac{1}{(1-t)y}\right]\frac{\zeta^2}{y-t}+\frac{2\zeta\xi}{y-t}-\frac{t(1-y)}{(1-t)y}\theta_0^2+\frac{(1-t)y\theta_1^2}{t(1-y)}.$$

Calculating directly, we get that the function $y(t)$ satisfies the sixth Painlevé equation

$$\begin{aligned}\frac{d^2y}{dt^2} &= \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\right)\frac{dy}{dt}+\\ &+\frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha+\beta\frac{t}{y^2}+\gamma\frac{t-1}{(y-1)^2}+\delta\frac{t(t-1)}{(y-t)^2}\right)\end{aligned} \tag{12}$$

with parameters $\alpha=(1-\theta_\infty)^2/2$, $\beta=-\theta_0^2/2$, $\gamma=\theta_1^2/2$, $\delta=(1-\theta_t^2)/2$. Note that equations (11) are called *the Lax pair.*

It is well-known [14, 28] that the other Painlevé equations can be constructed from the sixth Painlevé equation by successive confluences of singularities. By means of these confluences one can also construct the Lax pairs for the other Painlevé equations. However, in the equations obtained in such a way some poles of matrix $A(x)$ are not simple.

3.2 Isomonodromic deformations of linear systems with irregular singularities

Assume that the rational $N \times N$ matrix $A(x)$ has the following form:

$$A(x) = \sum_{\nu=1}^{n} \sum_{k=0}^{r_\nu} \frac{A_{\nu,k}}{(x-a_\nu)^{k+1}} + \sum_{k=1}^{r_\infty} A_{\infty,k} x^{k-1}.$$

The nonnegative integer r_ν is *the Poncaré rank* at the singular point $x = a_\nu$, $\nu \in \{1, ..., n, \infty\}$.

Assumption 3. *All matrices* A_{ν, r_ν} *are diagonalizable* $A_{\nu,r_\nu} = G_\nu A^{(\nu)}_{-r_\nu} G_\nu^{-1}$, $\nu \in \{1, ..., n, \infty\}$. *Without loss of generality we assume that* A_{∞, r_∞} *is diagonal and* $G_\infty = \mathbf{1}$.

If $r_\nu > 0$ and assumption 3 is fulfilled, then the singularity is called *irregular.* Next we introduce the following notation $\xi_\nu = x - a_\nu$, $\nu \in \{1, ..., n\}$, $\xi_\infty = 1/x$.

The formal expansion of solutions around the irregular singular point is of the following form:

$$\widetilde{Y}^{(\nu)}(x) = G_\nu \left(\mathbf{1} + \sum_{j=1}^{\infty} Y_j^{(\nu)} \xi_\nu^j \right) \exp\left(\sum_{j=-r_\nu}^{-1} \frac{1}{j} T_j^{(\nu)} \xi_\nu^j + T_0^{(\nu)} \log(\xi_\nu) \right), \tag{13}$$

where matrices $T_j^{(\nu)}$ are diagonal. Solution $\widetilde{Y}^{(\nu)}(x)$ is a formal solution of (5), i.e., series is asymptotic and generally divergent. This implies that $\widetilde{Y}^{(\nu)}(x)$ does not define a solution of (5), as in the Fuchsian case. The coefficients of expansion (13) can be determined if the following assumption holds.

Assumption 4. *If* $r_\nu > 0$, *then all eigenvalues of* A_{ν, r_ν} *are distinct, if* $r_\nu = 0$, *then they are distinct modulo the nonzero integers.*

The logarithm in the argument of the exponential is responsible for the multivaluedness of $\widetilde{Y}^{(\nu)}(x)$. The coefficient $T_0^{(\nu)}$ which determines the branching of $\widetilde{Y}^{(\nu)}(x)$ is called *the exponent of formal monodromy.* The polynomial in $1/\xi_\nu$ in the argument of the exponential is responsible for the exponential growth of $\widetilde{Y}^{(\nu)}(x)$ in the neighborhood of the singular point and for *the Stokes phenomenon* [38, p. 50], considered below.

The case $N = 2$ is considered in details in [28]. By analogy in case $N \neq 2$ we construct a set of Stokes sectors $S_{\nu,l}$ on $\overline{P_a^1}$ around each singularity $x = a_\nu$, $\nu \in \{1, ..., n, \infty\}$ with the following properties: the intersection $S_{\nu,l} \bigcap S_{\nu,l'}$ is nonempty if and only if $|l - l'| = 1$; $\pi(S_{\nu,l}) = \pi(S_{\nu,l+2r_\nu})$; $\pi(\bigcup_{l \in \{1,...,2r_\nu\}} S_{\nu,l})$ is a punched neighborhood of the singular point $x = a_\nu$.

Lemma 2 [28]. *There exists a unique holomorphic and invertible in $\overline{P_a^1}$ solution $Y^{(\nu,l)}(x)$ of equation* (5) *such that $Y^{(\nu,l)}(x) \sim \widetilde{Y}^{(\nu)}(x)$ in $S_{\nu,l}$, where $\nu \in \{1,...,n,\infty\}$, l is an integer.*

The Stokes multipliers $S_{\nu,l}$ are defined by $Y^{(\nu,l+1)} = Y^{(\nu,l)} S_{\nu,l}$. By permuting the Stokes lines we can obtain that all $S_{\nu,l}$ are triangular with all diagonal elements equal to 1. Hence, $\det(S_{\nu,l}) = 1$ and $Y^{(\nu,1)}(x_\nu) = Y^{(\nu,1)}(x) e^{2\pi i T_0^{(\nu)}} S_{2r_\nu}^{-1} ... S_1^{-1}$.

To define the connection matrices we have to distinguish one particular solution $Y^{(\nu,l)}(x)$ for every ν. Let us choose $Y^{(\infty,1)}(x)$ as our fundamental matrix solution and write $Y^{(\infty,1)}(x) = Y^{(\nu,l)}(x) C_\nu$, $\nu \in \{1,...,n\}$. Thus, $M_\nu = C_\nu^{-1} e^{2\pi i T_0^{(\nu)}} S_{2r_\nu}^{-1} ... S_1^{-1} C_\nu$. Note that $\det(M_\nu) = e^{2\pi i tr T_0^{(\nu)}}$ and $\det(M_1 ... M_n M_\infty) = 1$.

The singularity data of the system in this case are $SD = \{a_\nu,\ A_{-k_\nu}^{(\nu)},\ G_\nu,\ \nu \in \{1,...,n,\infty\},\ k_\nu \in \{0,1,...,r_\nu\},\ a_\infty = \infty,\ G_\infty = \mathbf{1}\}$, where matrices $A_{-r_\nu}^{(\nu)}$ are diagonal and the following relation holds: $A_0^{(\infty)} = -\sum_{\nu=1}^n G_\nu A_0^{(\nu)} G_\nu^{-1}$. The monodromy data MD, which characterize the monodromy properties of the fundamental matrix solution $Y(x)$, are given by $MD = \{a_\nu,\ T_{-k_\nu}^{(\nu)},\ S_{\nu,l_\nu},\ C_\nu,\ \nu \in \{1,...,n,\infty\},\ k_\nu \in \{0,1,...,r_\nu\},\ l_\nu \in \{1,...,2r_\nu\},\ a_\infty = \infty,\ C_\infty = \mathbf{1}\}$, where matrices $T_{-k_\nu}^{(\nu)}$ are diagonal, matrices S_{ν,l_ν} are triangular with diagonal elements equal to 1.

We shall say that $Y(x)$ has *the monodromy properties MD*, if $Y(x)$ is holomorphic and invertible in $\overline{P_a^1}$ and there exist invertible matrices G_ν, $\nu \in \{1,...,n\}$, and sectors $S_{\nu,l}$ such that

$$Y(x) C_\nu^{-1} S_{\nu,1} ... S_{\nu,l-1} \sim G_\nu (\mathbf{1} + O(\xi)) \exp\left(\sum_{j=-r_\nu}^{-1} \frac{1}{j} T_j^{(\nu)} \xi_\nu^j + T_0^{(\nu)} \log(\xi_\nu) \right)$$

in $S_{\nu,l}$ for $\nu \in \{1,...,n,\infty\}$, $l_\nu \in \{1,...,2r_\nu + 1\}$, $G_\infty = \mathbf{1}$. As in the Fuchsian case matrices G_ν, C_ν and $S_{\nu,l}$ are defined up to some invertible matrix D_ν. Moreover, if assumptions 3 and 4 hold, then the sets SD and MD are homeomorphic.

Let us define the partial monodromy data, which are preserved in isomonodromic deformations, by $PMD = \{T_0^{(\nu)},\ S_{\nu,l_\nu},\ C_\nu,\ \nu \in \{1,...,n,\infty\},\ l_\nu \in \{1,...,2r_\nu\},\ C_\infty = \mathbf{1}\}$. The set of *deformation parameters* is given by $DP = \{a_\nu,\ T_{-k_\nu}^{(\nu)},\ \nu \in \{1,...,n,\infty\},\ k_\nu \in \{1,...,r_\nu\},\ a_\infty = \infty\}$.

Considering SD as independent variables and MD as dependent variables, the isomonodromy problem is formulated as follows: Can we continuously deform SD while preserving PMD? Equivalently, let us consider MD as independent variables and SD as dependent variables. Then the formulation of the problem becomes: what are the deformations of SD under any continuous variation of DP, the PMD being kept fixed?

M. Jimbo et al [22] showed that under assumptions 3 and 4 the deformations of the linear differential system with multiple poles are isomonodromic if the coefficients of the matrix satisfy the nonlinear completely integrable differential system of P — type. Thus, there exists a deep connection between the Painlevé equations and linear systems.

For instance, the fifth Painlevé equation (3) can be obtained as the compatibility condition of the linear system

$$Y_x = A(x)Y, \ \ Y_t = B(x)Y, \tag{14}$$

where

$$\begin{aligned} A(x) &= \frac{t}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} v+\theta_0/2 & -u(v+\theta_0) \\ v/u & -(v+\theta_0/2) \end{pmatrix}\frac{1}{x} + \\ &+ \begin{pmatrix} -w & yu(w-\theta_1/2) \\ -(w+\theta_1/2)/(yu) & w \end{pmatrix}\frac{1}{x-1}, \end{aligned}$$

$$B(x) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x + \frac{1}{t}\begin{pmatrix} 0 & -u(v+\theta_0-y(w-\theta_1/2)) \\ (v-(w+\theta_1/2)/y)/u & 0 \end{pmatrix}$$

and $w = v + (\theta_0 + \theta_\infty)/2$. The compatibility condition $Y_{xt} = Y_{tx}$ of system (14) implies

$$\begin{aligned} t dy/dt &= ty - 2v(y-1)^2 - (y-1)((\theta_0 - \theta_1 + \theta_\infty)y - (3\theta_0 + \theta_1 + \theta_\infty))/2, \\ t dv/dt &= yv(w - \theta_1/2) - (v+\theta_0)(w+\theta_1/2)/y, \\ t du/dt &= u(-2v - \theta_0 + y(w-\theta_1/2) + (w+\theta_1/2)/y), \end{aligned} \tag{15}$$

from which we get that the function $y(t)$ satisfies (3) with parameters $\alpha = (\theta_0 - \theta_1 + \theta_\infty)^2/8$, $\beta = -(\theta_0 - \theta_1 - \theta_\infty)^2/8$, $\gamma = 1 - \theta_0 - \theta_1$, $\delta = -1/2$.

In summary, we have seen that the theory of isomonodromy deformations is a powerful tool to study nonlinear differential equations. There are much more aspects of this theory and there are many recent books and research papers discussing them. The present paper is intended to give a starting point for further study rather than a complete survey of the subject and all modern bibliographical references.

References

[1] *Ablowitz M.J., Clarkson P.A.* Solitons, nonlinear evolution equations and inverse scattering // Lect. Notes Math. — Cambridge: C.U.P., 1991. — Vol. 149. — 350 p.

[2] *Bolibruch A.A.* Fuchsian differential equations and holonomic fibering. — Moscow: MTsNMO, 2000. — 120 p.

[3] *Boutroux P.* Etude asymptotique de transcendantes de M. Painlevé dont est les solutions des équations différentielle de second ordre // Ann. École Norm. Super. — 1914. — Vol. 31. — P. 99 — 159.

[4] *Bassom A., Clarkson P., Hicks C.* The fourth Painlevé equation // Studies in Applied Mathematics. — 1995. — Vol. 5, no. 3. — P. 1 — 71.

[5] *Clarkson P.A., Mansfield E.L., Webster H.N.* On the relationship between the discrete and continuous Painlevé equations // Theoretical and Mathematical Physics. — 2000. — Vol. 122, no. 1. — P. 5 — 22.

[6] *Dubrovin B.* Painlevé transcendents in two-dimensional topological field theory // The Painlevé property, one century later. CRM series in Mathematical Physics / Ed. R. Conte. — New York: Springer-Verlag, 1999. — P. 287 — 412.

[7] *Fedoryuk M.V.* Isomonodromic deformations of equations with irregular singularities // Mathemat. Sbornik. — 1990. — Vol. 181, no. 12. — P. 1623 — 1639.

[8] *G. Filipuk.* Analytic properties of transformations of the Painlevé equations. PhD thesis, Belarusian State University, 2002.

[9] *Flaschka H., Newell A.C.* Monodromy and spectrum-preserving deformations, I // Comm. Math. Phys. — 1980. — Vol. 76. — P. 65 — 116.

[10] *Fuchs R.* Über lineare homogene Differentialgleichungen zweiter Ordnung mit drei im Endlichen gelegene wesentlich singuläre Stellen // Math. Annalen. — 1907. — Bd. 63. — S. 301 — 321.

[11] *Gambier B.* Sur les équations différentielles du second ordre et du premier degré dont l'intégrale générale est à points critiques fixes // C. R. Acad. Sc. Paris. — 1906. — Vol. 143. — P. 741 — 743.

[12] *Gambier B.* Sur les équations différentielles du second ordre et du premier degré dont l'intégrale générale est à points critiques fixes // Acta Math. — 1909. — Vol. 33. — P. 1 — 55.

[13] *Garnier R.* Sur les équations différentielles du troisième ordre dont l'intégrale est uniforme et sur une classe d'équations nouvelles d'ordre supérieur dont l'intégrale générale a ses points critiques fixes // Ann. Sci. de l'École Normale Supérieure. — 1912. — Vol. 29. — P. 1 — 126.

[14] *Garnier R.* Etude de l'intégrale générale de l'équation (VI) de M. Painlevé dans le voisinage de ses singularités transcendantes // Ann. Sci. de l'École Normale Supérieure. — 1917. — Vol. 34, no. 3. — P. 239 — 353.

[15] *Grammaticos B., Nihhoff F.W., Ramani A.* Discrete Painlevé equations // The Painlevé property, one century later. CRM series in Mathematical Physics / Ed. R. Conte. — New York: Springer-Verlag, 1999. — P. 413 — 516.

[16] *Gromak V.I.* To the theory of the Painlevé equations // Differential Equations. — 1975. — Vol. 11, no. 2. — P. 373 — 376.

[17] *Its A.R., Novokshenov V.Yu.* The isomonodromic deformation method in the theory of Painlevé equations // Lecture Notes in Math. — Berlin – Heidelberg – New-York – Tokyo: Springer, 1986. — Vol. 1191. — 313 p.

[18] *K. Iwasaki, H. Kimura, Sh. Shimomura, M. Yoshida.* From Gauss to Painlevé. — Bonn: Vieweg, 1991. — 345 p.

[19] *Jimbo M.* Monodromy problem and the boundary condition for some Painlevé equations // Publ. RIMS, Kyoto Univ. — 1982. — Vol. 18. — P. 1137 — 1161.

[20] *Jimbo M., Miwa T.* Monodromy preserving deformations of linear ordinary differential equations with rational coefficients, II // Physica 2D. — 1981. — Vol. 2. — P. 407 — 448.

[21] *Jimbo M., Miwa T.* Monodromy preserving deformations of linear ordinary differential equations with rational coefficients, III // Physica 4D. — 1981. — Vol. 3. — P. 26 — 46.

[22] *Jimbo M., Miwa T., Ueno K.* Monodromy preserving deformations of linear ordinary differential equations with rational coefficients, I // Physica 2D. — 1981. — Vol. 1. — P. 306 — 352.

[23] *Kapaev A.A.* The Painlevé transcendents as nonlinear special functions. Thesis. — Saint-Petersburg, 1997. — 221 p.

[24] *Kimura T.* On the isomonodromic deformation for linear ordinary differential equations of the second order, I // Proc. Japan Acad. Ser. A. — 1981. — Vol. 57. — P. 285 — 290.

[25] *Kimura T.* On the isomonodromic deformation for linear ordinary differential equations of the second order, II // Proc. Japan Acad. Ser. A. — 1982. — Vol. 58. — P. 294 — 297.

[26] *Kitaev A.V.* The method of isomonodromic deformations and asymptotics of solutions of the "complete" third Painlevé equation // Mathemat. Sbornik. — 1987. — Vol. 134, no. 3. — P. 421 — 444.

[27] *Kitaev A.V.* Non-Schlesinger deformations of ordinary differential equations with rational coefficients // J. Phys. A: Math. Gen. — 2001. — Vol. 34. — P. 2259 — 2272.

[28] *Mahoux G.* Introduction to the theory of isomonodromic deformations of linear ordinary differential equations with rational coefficients // The Painlevé property, one century later. CRM series in Mathematical Physics / Ed. R. Conte. — New York: Springer-Verlag, 1999. — P. 35 — 76.

[29] *Okamoto K.* Isomonodromic deformation and Painlevé equations, and the Garnier system // J. Fac. Sci. Univ. Tokyo. Sect. IA, Math. — 1986. — Vol. 33. — P. 575 — 618.

[30] *Okamoto K.* Studies on the Painlevé equations. I. Sixth Painlevé equation P_{VI} // Ann. Mat. Pura Appl. — 1987. — Vol. 146, no. 4. — P. 337 — 381.

[31] *Painlevé P.* Leçons sur la théorie analytique des équations différentielles. Leçons de Stockholm, delivered in 1895. — Paris: Hermann, 1897. — 589 p.

[32] *Painlevé P.* Mémoire sur les équations différentielles dont l'intégrale générale est uniforme // Bull. de la Soc. Math. de France. — 1900. — Vol. 28. — P. 201 — 261.

[33] *Painlevé P.* Sur les équations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme // Acta Math. — 1902. — Vol. 25. — P. 1 — 85.

[34] *Painlevé P.* Sur les équations différentielles du second ordre à points critiques fixes // C. R. Acad. Sc. Paris. — 1906. — Vol. 143. — P. 1111 — 1117.

[35] *Ramani A., Grammaticos B., Ohta J.* A geometrical description of the discrete Painlevé V and VI equations // Commun. Math. Phys. — 2001. — Vol. 217. — P. 315 — 329.

[36] *Sato M., Jimbo M., Miwa T.* Holonomic quantum fields. — Moscow: Mir, 1983. — 304 p.

[37] *Schlesinger L.* Über eine Klasse von Differentialsystemen beliebiger Ordnung mit festen kritischen Punkten // J. für R. und Angew. Math. — 1912. — Vol. 141. — P. 96 — 145.

[38] *Wasow V.* Asymptotic expansions of solutions of ordinary differential equations. — Moscow: Mir, 1968. — 464 p.

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