Linear and nonlinear ODEs and middle convolution

Galina Filipuk

Institute of Mathematics

University of Warsaw

G.Filipuk@mimuw.edu.pl

Collaborator:

Y. Haraoka (Kumamoto University)

Plan of the Talk:

- Linear special functions and transformations between them:
  - Gauss hypergeometric function
  - Heun function
- Nonlinear special functions:
  - Painlevé transcendents
  - Fuchsian systems of differential equations
  - monodromy, rigidity, monodromy preserving deformations
  - Schlesinger systems
- Middle convolution
- Results for the hypergeometric and Heun equations
- Results for the Painlevé equations and Schlesinger systems:
  - middle convolution is used to derive Okamoto's birational transformation for (P_{VI})
  - deformation equations are invariant under middle convolution for Schlesinger systems.

Linear Special Functions

- Functions defined by linear ordinary differential equations (ODEs) which have many applications in analysis, number theory, mathematical physics and other fields.

- Example. Hypergeometric equation

$$\frac{d^2y(z)}{dz^2} + \left(\frac{c}{z} + \frac{a+b-c+1}{z-1}\right)\frac{dy(z)}{dz} + \frac{a\,b}{z\,(z-1)}\,y(z) = 0,$$

  where $a, b, c \in \mathbb{C},\ y(z) : \mathbb{C} \to \mathbb{C}$.

  Gauss hypergeometric series defined by

$${}_2F_1(a,\, b,\, c)(z) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n\,n!}\,z^n,$$

  where $(a)_n = a(a+1)\ldots(a+n-1),\ n > 0,\ (a)_0 = 1$, is a solution of the hypergeometric equation.

  Singular points of the equation (and, hence, of the solutions since equation is linear) are $z = 0,\ 1,\ \infty$.

There is an integral representation of the solutions of the hypergeometric equation which allows one to calculate a monodromy group (linear representation of the fundamental group of $\mathbb{CP}^1 - \{\text{singular points}\}$ summarizing all analytic continuations of the multi-valued solutions of the equation along closed loops).

- Example. Heun equation

$$\frac{d^2y(x)}{dx^2} + \left(\frac{c}{x} + \frac{d}{x-1} + \frac{a+b-c-d+1}{x-t}\right)\frac{dy(x)}{dx} + \frac{abx-q}{x(x-1)(x-t)}y(x) = 0.$$

Four singularities in the complex plane $x = 0,\ 1,\ t,\ \infty$. Parameter q is called an accessory parameter (in contrast to the hypergeometric equation, it cannot be determined if the monodromy data are given).

Many open questions (e.g., monodromy group, integral representation of solutions). Intriguing applications (e.g., Riemann's zeta function).

- Classification of at least 1 free parameter transformations between GHF ${}_2F_1(a,b,c)(z)$ and Heun function $Hn(t,q;a,b,c,d)(x)$ [R.Vidunas-R.Maier-GF, 2010], e.g.,

$$Hn\left(9, q_1;\ 3a,\ 2a+b, a+b+\frac{1}{3}, 2a-2b+1\right)(x) =$$

$$(1-x)^{-2a}\,{}_2F_1\left(a,\ b,\ a+b+\frac{1}{3}\right)(z_1),$$

$$Hn\left(\frac{8}{9}, q_2;\ 3a,\ 2a+b,\ 2a+2b-\frac{1}{3},\ a+b+\frac{1}{3}\right)(x) =$$

$$= \left(1-\frac{9x}{8}\right)^{-2a}\,{}_2F_1\left(a,\ b, a+b+\frac{1}{3}\right)(z_2),$$

where $q_1 = 18a^2 - 9ab + 6a,\ q_2 = 4a^2 + 4ab - 2a/3,$

$$z_1 = -\frac{x(x-9)^2}{27(x-1)^2}, \quad z_2 = \frac{27x^2(x-1)}{(8-9x)^2}.$$

In general, there are about 50 such transformations. Functions z_j are Belyi functions (branched over 3 points). 38 transformations are related to invariants of elliptic surfaces with 4 singular fibers.

- Integral transformations between Heun functions (e.g., Euler type integral transformations found by Slavyanov, GF).

Nonlinear Equations

- Riccati equation $y' = a(z)y^2 + b(z)y + c(z)$. Linearizable. Solutions have poles in $\mathbb{C}$, hence, meromorphic functions.
- Elliptic function $\wp(z)$ is a solution of $y'^2 = p_3(y)$, where p is a degree 3 polynomial. First order nonlinear equation; $\wp(z)$ has poles in $\mathbb{C}$.

Roughly, differential equation possesses *the Painlevé property* if solutions have only movable poles in $\mathbb{C}$ and *quasi-Painlevé property* if solutions have algebraic branch points.

Painlevé (1888) proved that for the first order ODEs of the form

$$G\left(\frac{dy}{dz}, y, z\right) = 0,$$

where G is a polynomial in dy/dz and y with analytic in z coefficients, the movable singularities (depend on initial conditions) of the solutions are *poles and/or algebraic branch points* (quasi-Painlevé property).

Nonlinear Special Functions

- Works of Painlevé, Picard, Fuchs, Gambier, Bureau: Which equations of the type

$$\frac{d^2y}{dz^2} = F\left(\frac{dy}{dz}, y, z\right),$$

where F is a rational function of dy/dz and y and an analytic function of z, have the *Painlevé property: solutions have no movable critical points*?

- 50 types of equations solutions of which have only movable poles. 44 equations are integrable in terms of linear equations and elliptic functions or *reducible to other six equations which are now known as the Painlevé equations:*

$$y'' = 6y^2 + z, \qquad (P_I)$$

$$y'' = 2y^3 + zy + \alpha, \quad \alpha \in \mathbb{C} \qquad (P_{II})$$

$$\dots$$

$$\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt} + \\ + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right), \qquad (P_{VI})$$

α, β, γ, δ being arbitrary parameters.

- First tedious calculations to find a set of necessary conditions for the absence of movable critical points.

- Next prove that conditions are sufficient (difficult problem, settled only recently by Laine, Shimomura, Steinmetz and others, 1990th; also earlier by Hukuhara in unpublished notes).

- So, it took almost 100 years to rigorously prove that 6 functions (Painlevé transcendents), solutions of the 2nd order nonlinear ODEs, are actually meromorphic functions, i.e., possess the Painlevé property.

- Open problem with higher order equations (situation becomes even more complicated as there are natural barriers for analytic continuation of solutions, e.g., Chazy equation).

- The solutions of the six Painlevé equations (Painlevé transcendents (P_I)–(P_{VI})) are nonlinear special functions. They appear in many areas of modern mathematics (mathematical physics, random matrices, enumerative algebraic geometry, Frobenius manifolds, reductions of integrable PDEs, etc). They are already included in the chapter in the recent update of Abramovich-Stegun's book *Handbook of Mathematical Functions* in the DLMS project.

- It is interesting to note that there is a relation of linear and nonlinear special functions. E.g., elliptic asymptotics of the Painlevé transcendents, special solutions for special values of the parameters expressed in terms of linear special functions, nice determinant representations of solutions, etc. For instance, (P_{VI}) can be regarded as a nonlinear analogue of the hypergeometric equation.
- Painlevé transcendents (P_{II})–(P_{VI}) possess Bäcklund transformations (nonlinear recurrence relations, which map solutions of a given Painlevé equation to the solutions of the same/other Painlevé equation but with different values of the parameters and admit an affine Weyl group formulation) [Okamoto, Mazzocco, GF];
- (P_I)–(P_{VI}) admit a Hamiltonian formulation, can be written in a bilinear form. They are irreducible to classical special functions as proved recently by Umemura using the differential Galois theory.

- Method of isomonodromy deformations (to study asymptotics and connection formulae): the Painlevé equations (and their multivariable generalizations such as the Garnier and Schlesinger systems) are expressed as a compatibility condition

$$\frac{\partial}{\partial t_j}\frac{dY}{dx} = \frac{d}{dx}\frac{\partial Y}{\partial t_j},$$

of two linear systems of equations

$$\frac{dY}{dx} = AY, \quad \frac{\partial Y}{\partial t_j} = BY.$$

Monodromy of the Fuchsian systems

A linear system is called *a Fuchsian system* if it is of the form

$$\frac{dY}{dx} = \sum_{i=1}^{p+1} \frac{C_i}{x - t_i} Y,\ C_k \in \mathbb{C}^{n\times n}.$$

$t_1, t_2, \ldots, t_{p+1} \in \mathbb{CP}^1$: distinct points (simple poles of the Fuchsian system), ∞ is a regular point here. It can be a pole if $C_\infty = -\sum_i C_i \neq 0$.

$X = \mathbb{CP}^1 \setminus \{t_1, t_2, \ldots, t_{p+1}\}$ open, connected set

$x_0 \in X$

$\pi_1(X, x_0)$: fundamental group. $\pi_1(X, x_0) = \langle \gamma_1, \gamma_2, \ldots, \gamma_{p+1} \mid \gamma_1\gamma_2 \cdots \gamma_{p+1} = 1 \rangle$. Analytic continuation along all possible loops in the fundamental group of the (generally multivalued) fundamental solution Y:

$$\gamma_* Y = YM, \quad M \in GL(n, \mathbb{C}), \quad \gamma \in \pi_1(X, x_0).$$

Matrix M is called a monodromy matrix.

Linear differential equations defined on X in the complex plane have a *monodromy group*, which (more precisely) is a *linear representation of the fundamental group* of X, summarizing all the analytic continuations along closed loops within X.

$\pi_1(X, x_0) \to \mathrm{GL}(n, \mathbb{C}), \quad (M_1, M_2, \ldots, M_{p+1}) \in (\mathrm{GL}(n, \mathbb{C}))^{p+1} : \; M_1 M_2 \cdots M_{p+1} = Id_n.$

Direct monodromy problem is to determine the monodromy group of the Fuchsian system. The inverse problem of constructing the equation with given regular singularities and with a given monodromy representation, is called the Riemann-Hilbert problem.

Rigidity

A monodromy representation $(M_1, M_2, \ldots, M_{p+1})$ with $M_1 M_2 \cdots M_{p+1} = Id_n$ is said to be *rigid*, if, for any tuple $(N_1, N_2, \ldots, N_{p+1})$ of matrices in $\mathrm{GL}(n, \mathbb{C})$ satisfying $N_1 N_2 \cdots N_{p+1} = Id_n$ and $N_j = D_j M_j D_j^{-1}$ $(1 \leq j \leq p+1)$, there exists $D \in \mathrm{GL}(n, \mathbb{C})$ such that simultaneously $N_j = D M_j D^{-1}$ for $1 \leq j \leq p+1$.

Example. For system

$$\left(xI_2 - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) \frac{dY}{dx} = \begin{pmatrix} \lambda & 1 \\ -(\lambda - \rho_1)(\lambda - \rho_2) & \rho_1 + \rho_2 - \lambda \end{pmatrix} Y$$

the monodromy group is generated by

$$M_1 = \begin{pmatrix} e(\lambda) & q \\ 0 & 1 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 1 & 0 \\ -(e(\lambda) - e(\rho_1))(e(\lambda) - e(\rho_2))e(\lambda)^{-1} q^{-1} & e(\rho_1 + \rho_2 - \lambda) \end{pmatrix},$$

where $q = e(\lambda) - e(\rho_1) = e(-\gamma) - e(-\alpha)$.

If $\rho_1 = -\alpha,\ \rho_2 = -\beta$ and $\lambda = 1 - \gamma$, and we simultaneously conjugate the matrices by some matrix from $GL(2, \mathbb{C})$, then we have a standard monodromy of the hypergeometric equation:

$$M_1 = \begin{pmatrix} e(-\gamma) & e(-\gamma) - e(-\alpha) \\ 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 \\ e(\gamma-\beta) - 1 & e(\gamma - \alpha - \beta) \end{pmatrix}.$$

Why is this tuple rigid?

M_3 is defined from relation $M_1 M_2 M_3 = Id_2$.

From definition, if $N_1,\ N_2$ and N_3 be any matrices from $GL(2, \mathbb{C})$ with eigenvalues as in $M_1,\ M_2$ and M_3 respectively and $N_1 N_2 N_3 = Id_2$, then we can find explicitly matrix $D \in GL(2, \mathbb{C})$, such that

$$D^{-1} N_1 D = M_1, \quad D^{-1} N_2 D = M_2.$$

Indeed, calculating in *Mathematica* gives

$$D = \begin{pmatrix} -\frac{p_1 e(2(2\alpha+\beta))(-1+e(2\gamma))p_2}{(e(2\alpha)-e(2\gamma))(-e(2(\alpha+\beta))+e(2\gamma))} & -\frac{p_1 e(2(\alpha+\beta))p_2}{e(2(\alpha+\beta))-e(2\gamma)} \\ \frac{p_1 e(2\alpha)(-e(2\beta)+e(2\gamma))}{-e(2(\alpha+\beta)+e(2\gamma))} & p_1 \end{pmatrix},$$

where p_1 and p_2 are arbitrary parameters.

There are several definitions of rigidity (e.g., no accessory parameters in a differential equation). However, the quickest way to check the rigidity for a given tuple of matrices is to calculate the index of rigidity and see whether it is equal to 2 or not.

The index of rigidity

$$\iota = (2-(p+1))n^2 + \sum_{j=1}^{p+1} \dim Z(M_j),$$

where $Z(M_j)$ denotes the centralizer of M_j.

The centralizer of an element z of a group G is the set of elements of G which commute with z.

- ι is even; $\iota \leq 2$ for any irreducible tuple; $\iota = 2$ means rigid (Katz' result).

Example continued. For monodromy of the hypergeometric equation (rank 2 Fuchsian system with 3 singularities $x = 0, 1, \infty$ and $A_\infty = -(A_1 + A_2)$ nonzero) we have

$$n = 2,\ p = 2,\ p+1 = 3, Z(M_j) = 2,\ \iota = (2-3)2^2 + 3 * 2 = -4 + 6 = 2.$$

So, it is a rigid tuple.

- The number $2 - \iota$ can be regarded as the dimension of the moduli space of Fuchsian systems with prescribed local monodromies, i.e., number of accessory parameters. Monodromy determines the residue matrices in Fuchsian systems and vice versa, up to simultaneous conjugation by a matrix from $GL(n, \mathbb{C})$.

Reducibility in Fuchsian systems means that the matrices can be written in the form

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

and so, there are invariant subspaces.

Schlesinger systems

Let

$$\frac{d}{d\lambda}\Psi = A(\lambda)\Psi, \quad A(\lambda) = \sum_{k=1}^{N} \frac{A_k}{\lambda - t_k}, \quad A(\lambda) \in \mathrm{sl}(n, \mathbb{C}). \tag{1}$$

The isomonodromy (or, equivalently, monodromy preserving) condition means that the matrices M_k do not depend on the positions of the poles, i.e.,

$$\frac{d}{dt_i} M_k = 0.$$

Under certain non-resonance assumptions on the eigenvalues θ_k of the matrices A_k and $A_\infty := -\sum_{j=1}^{N} A_j$ one can show that the function Ψ satisfies the following system

$$\frac{\partial}{\partial t_i}\Psi = -\frac{A_i}{\lambda - t_i}\Psi, \quad i = 1, \ldots, N, \tag{2}$$

in the case of the monodromy preserving deformations. The compatibility conditions of

(1) and (2) are known as the Schlesinger equations, or deformation equations,

$$\frac{\partial A_k}{\partial t_i} = \frac{[A_i, A_k]}{t_i - t_k}, \quad k \neq i,$$

$$\frac{\partial A_i}{\partial t_i} = -\sum_{k=1,\, k\neq i}^{N} \frac{[A_i, A_k]}{t_i - t_k}, \quad k = i.$$

Middle Convolution

- By Katz's theorem one can obtain any irreducible rigid local system on the punctured affine line from a local system of rank one by applying a suitable sequence of middle convolutions and scalar multiplications.
- Katz' middle convolution functor MC_μ preserves important properties of local systems such as a number of singularities, the index of rigidity and irreducibility but in general changes the rank and the monodromy group.
- Dettweiler and Reiter's algebraic construction of Katz' middle convolution functor and a relation to the integral transformation for Fuchsian systems).
- The additive version of middle convolution for Fuchsian systems depends on a scalar

$\mu \in \mathbb{C}$ and is denoted by mc_μ. It is a transformation on tuples of matrices

$$(A_1, \dots, A_r) \in \left(\mathbb{C}^{n\times n}\right)^r \to mc_\mu(A_1, \dots, A_r) = (\widetilde{B}_1, \dots, \widetilde{B}_r) \in \left(\mathbb{C}^{m\times m}\right)^r.$$

A construction of mc_μ is as follows.

Let $\mathbf{A} = (A_1, \dots, A_r)$, $A_k \in \mathbb{C}^{n\times n}$. Let us also fix points $t = t_k \in \mathbb{C}$, $k = 1, \dots, r$ and consider a Fuchsian system of rank n given by

$$\frac{dY}{dt} = \sum_{k=1}^{r} \frac{A_k}{t - t_k} Y. \tag{3}$$

First, the operation of *addition* is simply a change of the eigenvalues of the residue matrix: $A_k \to A_k + aI_n$, where $a \in \mathbb{C}$, I_n is the identity matrix.

For $\mu \in \mathbb{C}$ one defines *convolution matrices* $\mathbf{B} = c_\mu(\mathbf{A}) = (B_1, \dots, B_r)$ by

$$B_k = \begin{pmatrix} 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ A_1 & \dots & A_{k-1} & A_k + \mu & A_{k+1} & \dots & A_r \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \in \mathbb{C}^{nr\times nr} \tag{4}$$

such that B_k is zero outside the k-th block row.

The convolution matrices define a new Fuchsian system of rank nr with the same number of singularities as in the original Fuchsian system:

$$\frac{dY_1}{dt} = \sum_{k=1}^{r} \frac{B_k}{t-t_k} Y_1. \tag{5}$$

This system may be reducible. In general, there are the following invariant subspaces of the column vector space $\mathbb{C}^{nr}$:

$$\mathcal{L}_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathrm{Ker(A_k)} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (k\text{−th entry}),\ \mathrm{k} = 1, \ldots, \mathrm{r}, \tag{6}$$

and

$$\mathcal{K} = \bigcap_{k=1}^{r} \mathrm{Ker(B_k)} = \mathrm{Ker(B_1 + \ldots + B_r)}. \tag{7}$$

Let us denote $\mathcal{L} = \oplus_{k=1}^{r} \mathcal{L}_k$ and fix an isomorphism between $\mathbb{C}^{nr}/(\mathcal{K}+\mathcal{L})$ and $\mathbb{C}^m$ for some m. The matrices $\widetilde{\mathbf{B}} = mc_\mu(\mathbf{A}) := (\widetilde{B}_1, \ldots, \widetilde{B}_r) \in \mathbb{C}^{m\times m}$, where $\widetilde{B}_k$ is induced by the action of B_k on $\mathbb{C}^m \simeq \mathbb{C}^{nr}/(\mathcal{K}+\mathcal{L})$ are called *the additive version of the middle convolution of* $\mathbf{A}$ *with parameter* μ. Thus, the resulting irreducible Fuchsian system of rank m is given by

$$\frac{dY_2}{dt} = \sum_{k=1}^{r} \frac{\widetilde{B}_k}{t-t_k} Y_2. \tag{8}$$

The relation between the convolution operation c_μ and integral transformations:

Let $g := (g_{i,j})$ be a matrix with entries $g_{i,j}$ such that they are (multi-valued) holomorphic functions on $X := \mathbb{C} \setminus T,\ T := \{t_1, \ldots, t_r\} \subset \mathbb{C},\ t_i \neq t_j$ for $i \neq j$. Assume that the path α_{r+1} encircles an open neighborhood U of y_0 and the path α_i encircles the point t_i. Then the matrix-valued function

$$I^{\mu}_{[\alpha_{r+1},\alpha_i]}(g)(y) := \int_{[\alpha_{r+1},\alpha_i]} g(x)(y-x)^{\mu-1}dx,\ y \in U,$$

is called the *Euler transform* of g with respect to the Pochhamer contour $[\alpha_{r+1}, \alpha_i] := \alpha_{r+1}^{-1}\alpha_i^{-1}\alpha_{r+1}\alpha_i$ and the parameter $\mu \in \mathbb{C}$.

Let $\mathbf{A} := (A_1, \ldots, A_r),\ A_i \in \mathbb{C}^{n\times n}$ be the residue matrices of the Fuchsian system (3) and $F(t)$ be its fundamental solution. Denote

$$G(t) := \begin{pmatrix} F(t)(t-t_1)^{-1} \\ \vdots \\ F(t)(t-t_r)^{-1} \end{pmatrix}$$

and introduce the period matrix

$$I^\mu(y) := (I^\mu_{[\alpha_{r+1},\alpha_1]}(G)(y), \ldots, I^\mu_{[\alpha_{r+1},\alpha_r]}(G)(y)).$$

Then Dettweiler and Reiter showed that the columns of the period matrix $I^\mu(y)$ are solutions of the Fuchsian system (5) obtained by the convolution with parameter $\mu-1$, i.e., $c_{\mu-1}(\mathbf{A})$, where y is contained in a small open neighborhood U of y_0 (which is encircled by α_{r+1}).

Middle Convolution: Summary

(1) Original Fuchsian system

$$\frac{dY(t)}{dt} = \sum_{i=1}^{p} \frac{A_i}{t - t_i} Y(t), \ \ A_k \in \mathbb{C}^{n \times n}$$

(2) Convolution (Okubo-type system)

$$\frac{dY_1(t)}{dt} = \sum_{i=1}^{p} \frac{B_i}{t - t_i} Y_1(t), \ \ B_k \in \mathbb{C}^{pn \times pn}$$

(3) Middle convolution (irreducible part of the above system)

$$\frac{dY_2(t)}{dt} = \sum_{i=1}^{p} \frac{\widetilde{B}_i}{t - t_i} Y_2(t), \ \ \widetilde{B}_k \in \mathbb{C}^{m \times m}$$

m depends on n, p, and a parameter $\mu \in \mathbb{C}$.

So, number of singularities stay the same whereas matrix dimensions change.

Katz and Dettweiler-Reiter say that if you apply additions and middle convolutions in any order and any finite number of times to a rank one system, one gets all rigid systems with a given number of singularities.

Example. How to get the hypergeometric system from rank one system?

Start with

$$\frac{dy}{dx} = \left(\frac{1+\alpha-\gamma}{x} + \frac{-1-\beta+\gamma}{x-1}\right)y.$$

It is solved by $y(x) = x^{1+\alpha-\gamma}(x-1)^{-1-\beta+\gamma}c$. The parameter μ in middle convolution is given by $\mu = -\alpha$. Get

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \left(\frac{G_1}{x} + \frac{G_2}{x-1}\right)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

where

$$G_1 = \begin{pmatrix} 1-\gamma & -1-\beta+\gamma \\ 0 & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0 & 0 \\ 1+\alpha-\gamma & -1-\alpha-\beta+\gamma \end{pmatrix}.$$

Next, conjugate the system with

$$S = \begin{pmatrix} -1-\beta+\gamma & 0 \\ 0 & 1 \end{pmatrix},$$

i.e., simple transformation $Y \to SY$ and we get

$$\hat{G}_1 = \begin{pmatrix} 1-\gamma & 1 \\ 0 & 0 \end{pmatrix}, \quad \hat{G}_2 = \begin{pmatrix} 0 & 0 \\ (1+\beta-\gamma)(-1-\alpha+\gamma) & -1-\alpha-\beta+\gamma \end{pmatrix}$$

which gives a hypergeometric equation for the first (or second) element of the vector Y. This system is irreducible, there are no nontrivial invariant subspaces and so it is a middle convolution of the original rank 1 system.

Results: Hypergepmetric Equation

The Gauss hypergeometric function is a linear special function of the isomonodromy type (Kitaev):

$$\frac{d}{d\lambda}\Psi = \left(\frac{A_0}{\lambda} + \frac{A_1}{\lambda - 1} + \frac{A_t}{\lambda - t}\right)\Psi, \tag{9}$$

with triangular matrices

$$A_k = \begin{pmatrix} 0 & 0 \\ u_k(t) & 0 \end{pmatrix} + \theta_k \sigma_3.$$

Assume $u_0 + u_1 + u_t = 0$. The Schlesinger equations give the following system for the functions u_0, u_1, u_t:

$$\frac{du_0}{dt} = \frac{2\theta_0 u_t - 2\theta_t u_0}{t}, \quad \frac{du_1}{dt} = \frac{2\theta_1 u_t - 2\theta_t u_1}{t - 1} \tag{10}$$

which is equivalent to the Euler differential equation (hypergeometric equation) for u_0 with

$$a = 2\theta_t, \quad b = 2\theta_0 + 2\theta_1 + 2\theta_t, \quad c = 2\theta_0 + 2\theta_t + 1.$$

One can study the effect of the application of the addition (change of the eigenvalues of the residue matrices) and middle convolution.

Shifting the eigenvalues of the residue matrices in (9) by addition $\Psi = \lambda^{-\theta_0}(\lambda-1)^{-\theta_1}(\lambda-t)^{\theta_t}\Psi_1$ we start with the system (3) with

$$A_1 = \begin{pmatrix} 2\theta_0 & 0 \\ u_0(t) & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2\theta_1 & 0 \\ u_1(t) & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 \\ u_t(t) & -2\theta_t \end{pmatrix}.$$

Theorem. Let $u_0(t)$, $u_1(t)$ and $u_t(t)$ with $u_0(t) + u_1(t) + u_t(t) = 0$ satisfy the hypergeometric equations with

$$\begin{aligned} a_0 &= 2\theta_t, & b_0 &= 2(\theta_0+\theta_1+\theta_t), & c_0 &= 2(\theta_0+\theta_t)+1; \\ a_1 &= 2\theta_t, & b_1 &= 2(\theta_0+\theta_1+\theta_t), & c_1 &= 2(\theta_0+\theta_t); \\ a_t &= 2\theta_t+1, & b_t &= 2(\theta_0+\theta_1+\theta_t), & c_t &= 2(\theta_0+\theta_t)+1. \end{aligned}$$

Then the operations of addition and middle convolution with parameter $\mu = -2(\theta_0+\theta_1)$ applied to system (9) give new solutions of the hypergeometric equations given by

$$\widetilde{u}_0(t) = -\frac{2\theta_1 u_0(t)}{f(t)u_1(t)}, \quad \widetilde{u}_1(t) = \frac{2\theta_0}{f(t)}, \quad \widetilde{u}_t(t) = \frac{2\theta_1 u_0(t) - 2\theta_0 u_1(t)}{f(t)u_1(t)},$$

where

$$f'(t) = 2f(t)\frac{\theta_1 u_0(t) + (\theta_1 + \theta_t)u_1(t)}{u_1(t)(t-1)},$$

and the parameters are given by

$$\begin{aligned}
\widetilde{a}_0 &= 2(\theta_0 + \theta_1 + \theta_t), & \widetilde{b}_0 &= 2\theta_t, & \widetilde{c}_0 &= 2(\theta_0 + \theta_t) + 1; \\
\widetilde{a}_1 &= 2(\theta_0 + \theta_1 + \theta_t), & \widetilde{b}_1 &= 2\theta_t, & \widetilde{c}_1 &= 2(\theta_0 + \theta_t); \\
\widetilde{a}_t &= 2(\theta_0 + \theta_1 + \theta_t) + 1, & \widetilde{b}_t &= 2\theta_t, & \widetilde{c}_t &= 2(\theta_0 + \theta_t) + 1.
\end{aligned}$$

Other cases: connection to Heun equation, Jordan-Pocchammer equation [GF, J. Phys. A (2010)].

Results: Heun Equation

The Heun equation is given by

$$t(t-1)(t-\lambda)y'' + \{\gamma(t-1)(t-\lambda) + \delta t(t-\lambda) + \epsilon t(t-1)\}y' + \alpha\beta(t-a)y = 0,$$

where the Fuchs relation

$$\alpha + \beta - \gamma - \delta - \epsilon + 1 = 0 \tag{11}$$

holds.

The hypergeometric system of the Heun equation is given by

$$\begin{pmatrix} t & 0 & 0 \\ 0 & t-1 & 0 \\ 0 & 0 & t-\lambda \end{pmatrix} \begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} 1-\gamma & 1 & 0 \\ \alpha_{21} & -\delta & 1 \\ \alpha_{31} & \alpha_{32} & -\epsilon-1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \tag{12}$$

where

$$\begin{aligned} \alpha_{21} &= (\gamma-1)\delta - g_0, \\ \alpha_{31} &= \alpha\beta(\gamma-2) + (1-\gamma)\epsilon(\delta+\gamma-2) + (\epsilon-\gamma+2)g_0, \\ \alpha_{32} &= \epsilon(\delta+\gamma-2) - \alpha\beta + g_0, \quad g_0 = \frac{1}{\lambda}\{\epsilon(1-\gamma) + \alpha\beta a\}. \end{aligned}$$

Theorem. The hypergeometric system of the Heun equation (12) is related via middle convolution with parameter μ and a gauge transformation to another hypergeometric system (12) with new values of the parameters given by either

$$\mu = \alpha - 1,\ (2-\alpha-\alpha_1)(1-\alpha+\beta-\alpha_1) = 0,$$

$$\alpha_1\beta_1 = (\alpha-2)(\alpha-\beta-1),$$

$$\gamma_1 = \gamma-\alpha+1,\ \delta_1 = \delta-\alpha+1,\ \epsilon_1 = \epsilon-\alpha+1,$$

$$a_1 = \frac{1+\beta-\delta+\alpha(\delta-1+\beta(a-1))+(\alpha-1)(\alpha-\gamma-\delta)\lambda}{(\alpha-2)(\alpha-\beta-1)}$$

or

$$\mu = \beta - 1,\ (2-\beta-\alpha_1)(1+\alpha-\beta-\alpha_1) = 0,$$

$$\alpha_1\beta_1 = -(\beta-2)(\alpha-\beta+1),$$

$$\gamma_1 = \gamma-\beta+1,\ \delta_1 = \delta-\beta+1,\ \epsilon_1 = \epsilon-\beta+1,$$

$$a_1 = -\frac{\alpha(1+(a-1)\beta)+(\beta-1)(\delta-1+(\beta-\gamma-\delta)\lambda)}{(\beta-2)(\alpha-\beta+1)}.$$

Results: the Painlevé VI Equation and the Schlesinger Systems

The algorithm of Dettweiler and Reiter was introduced for rigid systems, so what happens to general Fuchsian systems?

In the case of the sixth Painlevé equation which describes monodromy preserving deformations of rank 2 Fuchsian system with four singularities on the projective line, the algorithm yields the Okamoto birational transformation.

In general, there is an invariance of the deformation equations (Schlesinger system) under middle convolution.

Monodromy preserving deformations

$$\frac{dY}{dx} = \left(\sum_{j=1}^{p} \frac{A_j}{x - t_j} \right) Y,$$
$$A_{p+1} = -(A_1 + A_2 + \cdots + A_p).$$

Matrices $A_j \in GL(n, \mathbb{C})$ generally depend on t_j. Monodromy preserving deformations is when we keep the monodromy of system above constant in t_j.

Under certain assumptions on eigenvalues of A_j, the Schlesinger system govern monodromy preserving deformations of the Fuchsian system above:

$$\begin{cases} \dfrac{\partial A_i}{\partial t_i} = -\sum_{k \neq i} \dfrac{[A_i, A_k]}{t_i - t_k}, & (\textit{Schlesinger system}) \\ \dfrac{\partial A_j}{\partial t_i} = \dfrac{[A_i, A_j]}{t_i - t_j} & (j \neq i) \ . \end{cases}$$

When we have $p = 4$ singularities $(0, 1, t, \infty)$ and $n = 2$, then the system gives the sixth Painlevé equation:

$$\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt} + \\ + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right),$$

$\alpha, \ \beta, \ \gamma, \ \delta$ being arbitrary parameters and $y(t) : \mathbb{C} \to \mathbb{C}$.

When $n = 2$ but $p > 4$, we have so-called Garnier systems and when $p > 4$ and $n > 2$, we have so-called Schlesinger systems.

- start with the Fuchsian system of rank 2 with 4 singularities $(n = 2, \ \ p = 3)$, deformation of which leads to (P_{VI})

$$\frac{dY}{dx} = \left(\frac{G_0}{x} + \frac{G_1}{x-1} + \frac{G_2}{x-t}\right)Y$$

with $G_i \in GL(n, \mathbb{C})$. Here t is a deformation parameter and if we allow it to vary, the coefficients of the residue matrices of the systems become the functions of t and monodromy matrices do not depend on t iff $y(t)$, which is a function of the coefficients of the residue matrices, satisfies (P_{VI})

- apply middle convolution with parameter equal to one of the eigenvalues of the matrix at infinity
- get a new system of rank 2 ($m = 2$)
- nontrivial result: find explicitly Okamoto transformation for the solutions of (P_{VI}):

$$y_1(t) = mc_{k_1}(y(t)) =$$
$$y - \frac{(\theta_0 + \theta_1 - \theta_\infty + \theta_t)(t-y)(y-1)y}{(\theta_0 + \theta_t - 1 + t(\theta_0 + \theta_1))y - (\theta_0 + \theta_1 + \theta_t - 1)y^2 - t(\theta_0 + (t-1)y')}$$

and parameters

$$
\begin{aligned}
\alpha_1 &= \frac{1}{8}(\theta_0+\theta_1+\theta_\infty+\theta_t-2)^2,\ \beta_1 = -\frac{1}{8}(\theta_0-\theta_1+\theta_\infty-\theta_t)^2,\\
\gamma_1 &= \frac{1}{8}(-\theta_0+\theta_1+\theta_\infty-\theta_t)^2,\ \delta_1 = \frac{1}{2}(1-(\theta_0+\theta_1-\theta_\infty-\theta_t)^2/4),
\end{aligned}
$$

which coincides with Okamoto's birational transformation.

(..., K.Okamoto ('87),

R. Conte ('01): singular manifold method,

M. Mazzocco: Laplace transform to irregular system and easy gauge transformation,

M. Noumi, Y. Yamada ('03): symmetric form,

K. Iwasaki ('03): Riemann-Hilbert correspondence,

...

It is known that the group of birational transformations forms an affine Weyl group of $D_4^{(1)}$ type and it is generated by 5 transformations. The meaning of 4 of them in the context of linear system is that they are obtained by simple gauge transformations. And, surprisingly, the last one is connected with the integral transformation which stems from all this theory of rigid systems.

- we know that the integral transformation of certain type applied to the solution of the Fuchsian system (middle convolution) leads to the transformation of the nonlinear deformation equation (P_{VI}):

$$\begin{array}{ccc} (A_0, A_1, A_2) \in (GL(2,\mathbb{C}))^3 & \xrightarrow{mc_{\kappa_1}} & (\bar{A}_0, \bar{A}_1, \bar{A}_2) \in (GL(2,\mathbb{C}))^3 \\ \updownarrow & & \updownarrow \\ y(t) & \xrightarrow{BTr} & \bar{y}(t) \end{array}$$

(Here the deformation equation is the equation which should be satisfied when we require that the monodromy of the Fuchsian system is independent of the deformation parameters. It is a compatibility condition of 2 linear systems.)

- if we do not impose condition on the parameter of middle convolution, we get certain rank 3 system which was studied by Harnad, Mazzocco and Boalch and which proved useful in searching for new algebraic solutions of (P_{VI})

- we also know (can construct explicitly) any rank Fuchsian systems deformation of which leads to the sixth Painlevé equation. Does it hold in general?

Isomonodromic family

$$\begin{cases} \dfrac{\partial A_i}{\partial t_i} = -\displaystyle\sum_{k\neq i} \frac{[A_i, A_k]}{t_i - t_k}, & (\textit{Schlesinger system}) \\ \dfrac{\partial A_j}{\partial t_i} = \dfrac{[A_i, A_j]}{t_i - t_j} & (j \neq i)\ . \end{cases}$$

is invariant under middle convolution for system

$$\frac{dY}{dx} = \left(\sum_{j=1}^{p} \frac{A_j}{x - t_j} \right) Y, \quad A_{p+1} = -(A_1 + A_2 + \cdots + A_p).$$

Theorem. If for $j = 1, 2, \ldots, p+1$, there is no integral difference between any two distinct eigenvalues of A_j and the Jordan canonical form of A_j is independent of $t_1, t_2, \ldots, t_p$, then the systems

$$\begin{cases} \dfrac{\partial}{\partial t_i} \mathrm{tr}(A_i A_j) = -\displaystyle\sum_{k\neq i,j} \frac{\mathrm{tr}([A_i, A_k]A_j)}{t_i - t_k}, \\ \dfrac{\partial}{\partial t_i} \mathrm{tr}(A_j A_k) = \dfrac{\mathrm{tr}([A_i, A_j]A_k)}{t_i - t_j} + \dfrac{\mathrm{tr}(A_j [A_i, A_k])}{t_i - t_k}. \end{cases} \qquad (*)$$

for the Fuchsian systems obtained by addition and middle convolution with parameters independent of $t_1, t_2, \ldots, t_p$ coincide with the system $(*)$ for the initial Fuchsian system.

Summary

- Linear special functions, defined by linear ODEs, appear in many areas of mathematics. They may have nontrivial transformations.
- Solutions of nonlinear ODEs have complicated singularities. Solutions of the Painlevé equations (the 2nd order equations) have only movable poles and appear in many areas of mathematics, so they are nonlinear special functions. There are very few results on higher-order equations or multivariable generalizations.
- Recent results on integral transformations of the Fuchsian systems: using middle convolution one can get nontrivial relations for linear and nonlinear special functions.

Thank you very much for your attention!