Part 1: Algebraic solutions of Painlevé VI

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Painlevé VI equation:

$$\frac{d^2w}{dt^2} = \frac{1}{2}\left(\frac{1}{w}+\frac{1}{w-1}+\frac{1}{w-t}\right)\left(\frac{dw}{dt}\right)^2 - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{w-t}\right)\frac{dw}{dt} +$$
$$+\frac{w(w-1)(w-t)}{2t^2(t-1)^2}\left((\theta_\infty-1)^2 - \frac{\theta_x^2 t}{w^2} + \frac{\theta_y^2(t-1)}{(w-1)^2} + \frac{(1-\theta_z^2)t(t-1)}{(w-t)^2}\right)$$

Painlevé VI equation:

$$\frac{d^2w}{dt^2} = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-t}\right)\left(\frac{dw}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{w-t}\right)\frac{dw}{dt} +$$
$$+ \frac{w(w-1)(w-t)}{2t^2(t-1)^2}\left((\theta_\infty - 1)^2 - \frac{\theta_x^2 t}{w^2} + \frac{\theta_y^2(t-1)}{(w-1)^2} + \frac{(1-\theta_z^2)t(t-1)}{(w-t)^2}\right)$$

- 4 parameters $\theta_{x,y,z,\infty}$
- most general equation of type $w'' = F(t, w, w')$ without movable critical points (Painlevé property)
- $w(t)$ is meromorphic on the universal cover of $\mathbb{P}^1 \backslash \{0, 1, \infty\}$
- Okamoto affine F_4 Weyl symmetry group
- P_I–P_V are obtained as limiting cases
- applications in nonlinear physics, classical and quantum integrable systems, random matrix theory, differential geometry...

Example: 2D Ising model

(Jimbo, Miwa '81) Diagonal two-point correlation functions are Painlevé VI τ-functions:

$$\tau(t) = (1-t)^{-\frac{N^2}{2}} \langle \sigma(0,0)\sigma(N,N)\rangle_{T<T_c}$$

- temperature parameter $0 < t < 1$
- $(\theta_x, \theta_y, \theta_z, \theta_\infty) = (0, N, N, 1)$
- special case of Riccati solutions
- nontrivial solutions of PV/PIII in the scaling limit (McCoy, Tracy, Wu, Barouch '76)

Transcendental Painlevé VI solutions arise in the study of

- extensions of Sato-Miwa-Jimbo theory of holonomic quantum fields (Palmer, Beatty, Tracy '93; Doyon '03; O.L. '07)
- representation theory of $U(\infty)$ (Borodin, Deift '01)
- quantum cohomology of $\mathbb{P}^2$ (Manin '96)

Solutions of Painlevé VI

According to Watanabe (1998):

$$\text{solutions of } P_{VI} \text{ are either } \begin{cases} \text{Riccati solutions or} \\ \text{`new' transcendental functions or} \\ \text{algebraic functions} \end{cases}$$

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- Lot of examples of algebraic solutions:
  Hitchin (1995); Dubrovin (1995); Dubrovin & Mazzocco (1998); Andreev & Kitaev (2001); Kitaev (2003–2005); Boalch (2003–2007)
- is complete classification possible?

Schwarz list

Question: When does Gauss hypergeometric function ${}_2F_1(a, b, c, \lambda)$ become algebraic? (Schwarz, 1873)

$$\frac{d\Phi}{d\lambda} = \left(\frac{A_x}{\lambda - u_x} + \frac{A_y}{\lambda - u_y} \right) \Phi,$$

- standard choice $u_x = 0$, $u_y = 1$
- $\Phi \in \mathrm{Mat}_{2\times 2}$, $A_{x,y} \in \mathfrak{sl}_2(\mathbb{C})$
- monodromy matrices $M_{x,y} \in SL(2, \mathbb{C})$
- algebraic solutions lead to finite monodromy $\rightarrow$ 15 classes

SCHWARZ'S TABLE

Number	λ	μ	ν
1	1/2	1/2	p/n
2	1/2	1/3	1/3
3	2/3	1/3	1/3
4	1/2	1/3	1/4
5	2/3	1/4	1/4
6	1/2	1/3	1/5
7	2/5	1/3	1/3
8	2/3	1/5	1/5
9	1/2	2/5	1/5
10	3/5	1/3	1/5
11	2/5	2/5	2/5
12	2/3	1/3	1/5
13	4/5	1/5	1/5
14	1/2	2/5	1/3
15	3/5	2/5	1/3

Isomonodromy approach

Painlevé VI describes monodromy preserving deformations of Fuchsian systems

$$\frac{d\Phi}{d\lambda} = \left(\frac{A_x}{\lambda - u_x} + \frac{A_y}{\lambda - u_y} + \frac{A_z}{\lambda - u_z} \right) \Phi, \qquad \Phi \in \mathrm{Mat}_{2\times 2}.$$

- $A_\nu \in \mathfrak{sl}_2(\mathbb{C})$ are independent of λ, with eigenvalues $\pm\theta_\nu/2$
- 4 regular singular points $u_x, u_y, u_z, \infty \in \mathbb{P}^1$
- $A_x + A_y + A_z \stackrel{def}{=} -A_\infty = \begin{pmatrix} -\theta_\infty/2 & 0 \\ 0 & \theta_\infty/2 \end{pmatrix}$
- monodromy matrices $M_x, M_y, M_z \in SL(2,\mathbb{C})$, defined up to overall conjugation ($3 \times 3 - 3 = 6$ parameters)

Painlevé VI $\leftrightarrow$ linear system dictionary:

- P_{VI} independent variable $t = (u_x - u_y)/(u_x - u_z)$; $w(t)$ is a combination of matrix elements of $A_{x,y,z}$
- to each branch of a solution of P_{VI} corresponds a (conjugacy class of) triple of monodromy matrices; eigenvalues of M_x, M_y, M_z, $M_\infty = M_z M_y M_x$ give P_{VI} parameters $\theta_{x,y,z,\infty}$; the other two correspond to integration constants
- analytic continuation induces an action of the pure braid group $\mathcal{P}_3$ on the space $\mathcal{M} = G^3/G$, $G = SL(2, \mathbb{C})$ of conjugacy classes of G-triples

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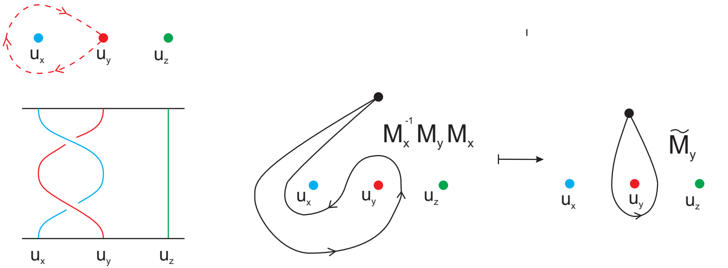

algebraic PVI solutions $\longrightarrow$ finite $\mathcal{P}_3$ orbits

Main question: classify these orbits

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Geometric viewpoint:

- nonlinear action of $\mathrm{Out}\, G$ on $\mathrm{Hom}(G, H)/H$
- here $G = \pi_1(\mathbb{P}^1 \backslash 4 \text{ points})$, $\mathrm{Out}\, G \cong \mathrm{MCG}^*(\mathbb{P}^1 \backslash 4 \text{ points})$, $H = SL(2, \mathbb{C})$

Reconstruction of solutions from monodromy:

- use Jimbo's asymptotic formula to find the leading term $w(t) \sim a_j t^{1-\sigma_j}$ in the Puiseux expansion at 0 of each branch (a_j, σ_j are known functions of $M_{x,y,z}$)
- computing sufficiently many terms, determine the polynomial $P(w,t) = 0$
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Example (finite subgroups of $SL(2,\mathbb{C})$):

Binary tetrahedral, octahedral and icosahedral groups

$$\begin{aligned} 2T &= \langle r,s,t \,|\, r^2 = s^3 = t^3 = rst = 1 \rangle, & |2T| &= 24, \\ 2O &= \langle r,s,t \,|\, r^2 = s^3 = t^4 = rst = 1 \rangle, & |2O| &= 48, \\ 2I &= \langle r,s,t \,|\, r^2 = s^3 = t^5 = rst = 1 \rangle, & |2I| &= 120. \end{aligned}$$

$2T$, $2O$, $2I$ are preimages of T, O, I under the two-fold covering homomorphism

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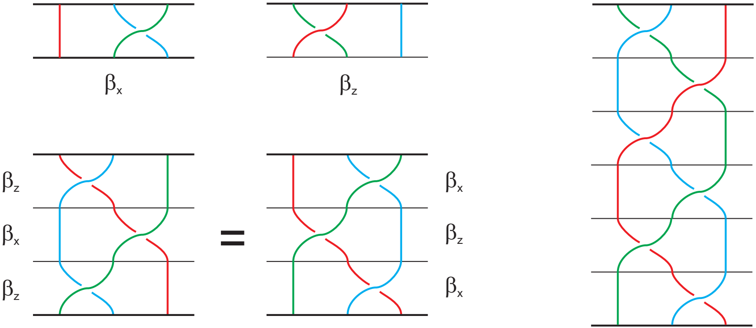
Explicit **counterexamples** with infinite monodromy have been found (e.g. Klein solution)



$\beta_z\ \beta_x\ \beta_z \quad = \quad \beta_x\ \beta_z\ \beta_x$

$(\beta_x\ \beta_z)^3$

braid group defining relations

Remarks:

- center of $\mathcal{B}_3$ acts trivially
- there are isomorphisms

$$\mathcal{B}_3/\mathcal{Z} \cong \Gamma = PSL_2(\mathbb{Z}) = \langle s, t \,|\, s^3 = t^2 = 1 \rangle,$$

$$\mathcal{P}_3/\mathcal{Z} \cong \Lambda = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{Z}) \,|\, a, d \text{ odd}; \; b, c \text{ even} \right\} / \{\pm 1\}.$$

- action of Λ can be extended to that of

$$\bar{\Lambda} = \langle x, y, z \,|\, x^2 = y^2 = z^2 = 1 \rangle \cong C_2 * C_2 * C_2 \,,$$

$$x = \left(\begin{array}{cc} -1 & -2 \\ 0 & 1 \end{array} \right), \quad y = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \quad z = \left(\begin{array}{cc} 1 & 0 \\ -2 & -1 \end{array} \right).$$

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Our problem: find all finite orbits of the $\bar{\Lambda}$ action on $\mathcal{M}$.

$\bar{\Lambda}$ action:

$$x : (M_x, M_y, M_z) \mapsto \left(M_x^{-1}, M_y^{-1}, M_x M_z^{-1} M_x^{-1}\right),$$
$$y : (M_x, M_y, M_z) \mapsto \left(M_y M_x^{-1} M_y^{-1}, M_y^{-1}, M_z^{-1}\right),$$
$$z : (M_x, M_y, M_z) \mapsto \left(M_x^{-1}, M_z M_y^{-1} M_z^{-1}, M_z^{-1}\right).$$

- To a point $(M_x, M_y, M_z) \in \mathcal{M}$ we associate a 7-tuple $(p_x, p_y, p_z, p_\infty, X, Y, Z) \in \mathbb{C}^7$ given by

$$p_x = \mathrm{Tr}\, M_x, \quad p_y = \mathrm{Tr}\, M_y, \quad p_z = \mathrm{Tr}\, M_z, \quad p_\infty = \mathrm{Tr}\,(M_z M_y M_x),$$
$$X = \mathrm{Tr}\,(M_y M_z), \qquad Y = \mathrm{Tr}\,(M_z M_x), \qquad Z = \mathrm{Tr}\,(M_x M_y).$$

- There is a constraint

$$XYZ + X^2 + Y^2 + Z^2 - \omega_X X - \omega_Y Y - \omega_Z Z + \omega_4 = 4,$$

$$\omega_X = p_x p_\infty + p_y p_z, \qquad \omega_Y = p_y p_\infty + p_z p_x, \qquad \omega_Z = p_z p_\infty + p_x p_y,$$
$$\omega_4 = p_x^2 + p_y^2 + p_z^2 + p_\infty^2 + p_x p_y p_z p_\infty.$$

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$$\omega_X = p_x p_\infty + p_y p_z, \qquad \omega_Y = p_y p_\infty + p_z p_x, \qquad \omega_Z = p_z p_\infty + p_x p_y,$$
$$\omega_4 = p_x^2 + p_y^2 + p_z^2 + p_\infty^2 + p_x p_y p_z p_\infty.$$

- **N.B.** p_x, p_y, p_z, p_∞ are fixed by x, y, z !!! (thanks to $\mathrm{Tr}\, M = \mathrm{Tr}\, M^{-1}$, $M \in SL(2, \mathbb{C})$)

Lemma. The induced action of $x, y, z \in \bar{\Lambda}$ on the parameters (X, Y, Z) is

$$x(X,Y,Z) = (\omega_X - X - YZ,\ Y,\ Z)\,,$$
$$y(X,Y,Z) = (X,\ \omega_Y - Y - ZX,\ Z)\,,$$
$$z(X,Y,Z) = (X,\ Y,\ \omega_Z - Z - XY)\,.$$

Proof. Use that $M + M^{-1} = \mathrm{Tr}\, M \cdot \mathbf{1}$ for $M \in SL(2,\mathbb{C})$.

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- our problem reduces to the classification of finite orbits of the $\bar{\Lambda}$ action on $\mathbb{C}^3$
- symmetries: a) permutations b) changes of 2 signs, e.g. $\omega_X \mapsto \omega_X, \omega_Y \mapsto -\omega_Y, \omega_Z \mapsto -\omega_Z, X \mapsto X, Y \mapsto -Y, Z \mapsto -Z$

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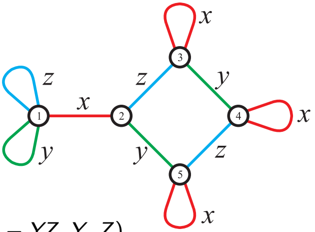
To any orbit O of this action we associate a 3-colored (pseudo)graph $\Sigma(O)$ as follows:

- the vertices of $\Sigma(O)$ represent distinct points $\mathbf{r} = (X, Y, Z) \in O$,
- two vertices $a, b \in \Sigma(O)$ are connected by an undirected edge of color x, y or z if $x(a) = b$ (resp. $y(a) = b$ or $z(a) = b$),
- if a point $a \in \Sigma(O)$ is fixed by the transformation x, y or z, we assign to it a self-loop of the corresponding color.

Example. Set $\boldsymbol{\omega} = (0, 1, 1)$ and consider the orbit of the point $\mathbf{r} = (-1, 1, 1)$. It consists of 5 points:

point	X	Y	Z
1	-1	1	1
2	0	1	1
3	0	1	0
4	0	0	0
5	0	0	1



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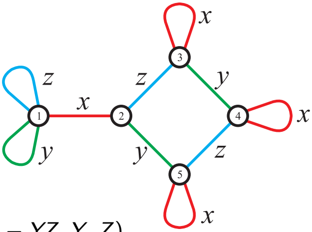

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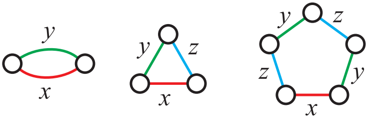
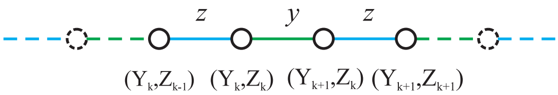
Forbidden subgraphs — examples:





Recursion relations:

$$\begin{aligned} Y_{k+1} &= \omega_Y - Y_k - XZ_k, \\ Z_{k+1} &= \omega_Z - Z_k - XY_{k+1}. \end{aligned}$$

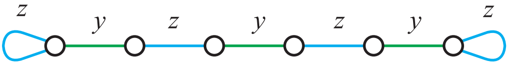
Finite orbit condition implies $Y_{k+N} = Y_k$, $Z_{k+N} = Z_k$, then for $N > 1$

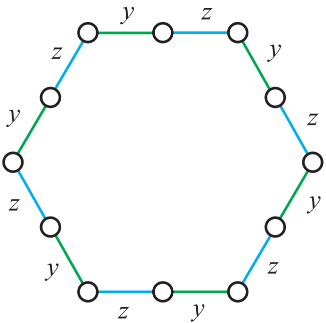
$$X = 2\cos \pi n_X / N, \qquad 0 < n < N,\ n_X \text{ prime to } N$$

Def.1. If $N > 1$ then X is called a good coordinate.

Def.2. A point $(X, Y, Z) \in O$ is called good if it is not fixed by at least two of three transformations x, y, z.

$N{=}6$





Lemma. The coordinates $\{Y_k\}$, $\{Z_k\}$ satisfy

for N even, n_X odd: $\begin{cases} Y_k + Y_{k+N/2} = p_+ + p_- \,, \\ Z_k + Z_{k+N/2} = p_+ - p_- \,, \end{cases}$

for N odd, n_X even: $Y_k + Z_{k+(N-1)/2} = p_+ \,,$

for N odd, n_X odd: $Y_k - Z_{k+(N-1)/2} = p_- \,.$

Here $k = 0, \ldots, N-1$ and $p_\pm = \dfrac{\omega_Y \pm \omega_Z}{2 \pm X}$.

- trigonometric diophantine conditions of type

$$\sum_{j=1}^{4} \cos \pi r_j = 0, \qquad r_{1\ldots 4} \in \mathbb{Q},\, 0 < r_{1\ldots 4} < 1$$

  E.g. for N even, n_X odd:

$$Y_0 + Y_{N/2} = Y_1 + Y_{1+N/2} = \ldots$$

- find rational solutions (algorithmic)
- because of Jimbo-Fricke relation, Y's of distinct suborbit points coincide only if the points are z-neighbors
- if $\omega_Y^2 \neq \omega_Z^2$ it is easy to obtain an upper bound for N !

Upper bounds on N

$$Y_0 + Y_{N/2} = Y_1 + Y_{1+N/2} = Y_2 + Y_{2+N/2} = \ldots (\neq 0)$$

Lemma. Inequivalent irreducible rational n-tuples solving

$$\sum_{j=1}^{n} \cos \pi r_j = 0$$

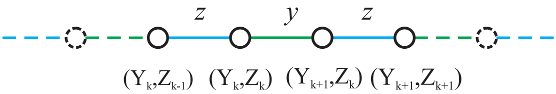
with $1 < n \leq 4$ fall into one of the following classes:

- 4 nontrivial irreducible quadruples

$$\left(0, \frac{1}{5}, \frac{1}{3}, \frac{2}{5}\right), \left(\frac{1}{30}, \frac{1}{6}, \frac{11}{30}, \frac{2}{5}\right), \left(\frac{1}{15}, \frac{4}{15}, \frac{3}{10}, \frac{1}{3}\right), \left(\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{1}{6}\right)$$

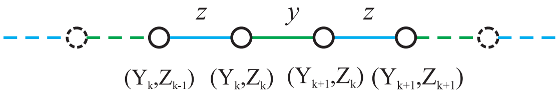
- 1 nontrivial irreducible triple $\left(\frac{1}{10}, \frac{3}{10}, \frac{1}{3}\right)$
- an infinite family of triples of the form $\left(\varphi, \varphi + \frac{1}{3}, \varphi - \frac{1}{3}\right)$, $\varphi \in \mathbb{Q}$
- an infinite family of pairs of the form $\left(\varphi, \frac{1}{2} - \varphi\right)$, $\varphi \in \mathbb{Q}$

Corollary: $N \leq 14$.



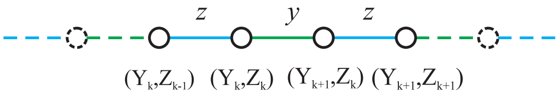
Hint for $\omega_Y^2 = \omega_Z^2$:

$$\omega_Y = Y_k + Y_{k+1} + XZ_k = Y_{k-1} + Y_k + XZ_{k-1}$$



Hint for $\omega_Y^2 = \omega_Z^2$:

$$\omega_Y = \cancel{Y_k} + Y_{k+1} + XZ_k = Y_{k-1} + \cancel{Y_k} + XZ_{k-1}$$



Hint for $\omega_Y^2 = \omega_Z^2$:

$$\omega_Y = \cancel{Y_k} + Y_{k+1} + XZ_k = Y_{k-1} + \cancel{Y_k} + XZ_{k-1}$$

$$\Downarrow$$

$$\cos \pi r_{Y_{k+1}} + \cos \pi (r_X - r_{Z_k}) + \cos \pi (r_X + r_{Z_k}) =$$
$$= \cos \pi r_{Y_{k-1}} + \cos \pi (r_X - r_{Z_{k-1}}) + \cos \pi (r_X + r_{Z_{k-1}})$$

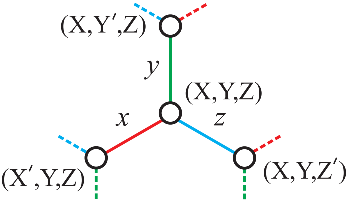
- need rational solutions for 6 cosines

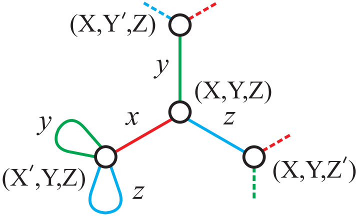
After hard work...

	restrictions on N, n_X	number of possible X
$\omega_Y^2 \neq \omega_Z^2$	$N \leq 10$, n_X odd and even	31
$\omega_Y = \omega_Z \neq 0$	$N \leq 10$, n_X odd and even, $N = 11, 15, 21$, n_X odd	46
$\omega_Y = \omega_Z = 0$ with $\omega_X \neq 0$ or $\omega_4 \neq 0$	$N \leq 15$, n_X odd and even	71

Restrictions on possible values of X for $N > 1$.

- no restrictions iff $\omega_X = \omega_Y = \omega_Z = \omega_4 = 0$





Good generating configurations

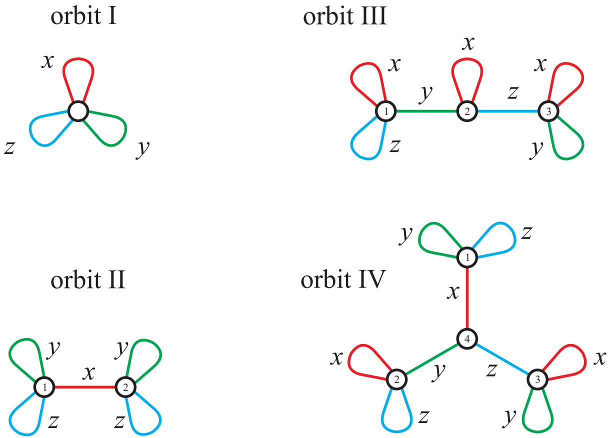
$$\begin{aligned} \omega_X &= X + X' + YZ, \\ \omega_Y &= Y + Y' + XZ, \\ \omega_Z &= Z + Z' + XY, \end{aligned}$$

$$\begin{aligned} \omega_Y &= Y + Y' + XZ, \\ \omega_Z &= Z + Z' + XY, \end{aligned}$$

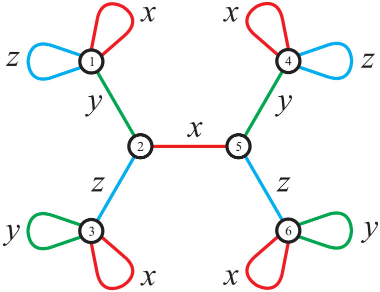
$$\begin{cases} 2Y + X'Z = \omega_Y, \\ 2Z + X'Y = \omega_Z, \end{cases}$$

X', ω_X

$Y(Y - Y') = Z(Z - Z')$



Four orbits without good generating configurations



6-vertex graph without good generating configurations

Summary:

- 4 orbits without GGCs
- all other finite orbits contain GGCs
- if at least one of $\omega_{X,Y,Z,4}$ is non-zero, GGCs belong to an explicitly defined finite set ($\sim 10^8$ elements)
- check which of them do actually lead to finite orbits
- case $\omega_X = \omega_Y = \omega_Z = \omega_4 = 0$ is easy

Nonlinear Schwarz list

Theorem. The list of all nonequivalent finite orbits of the induced $\bar{\Lambda}$ action on $\mathbb{C}^3$ consists of the following:

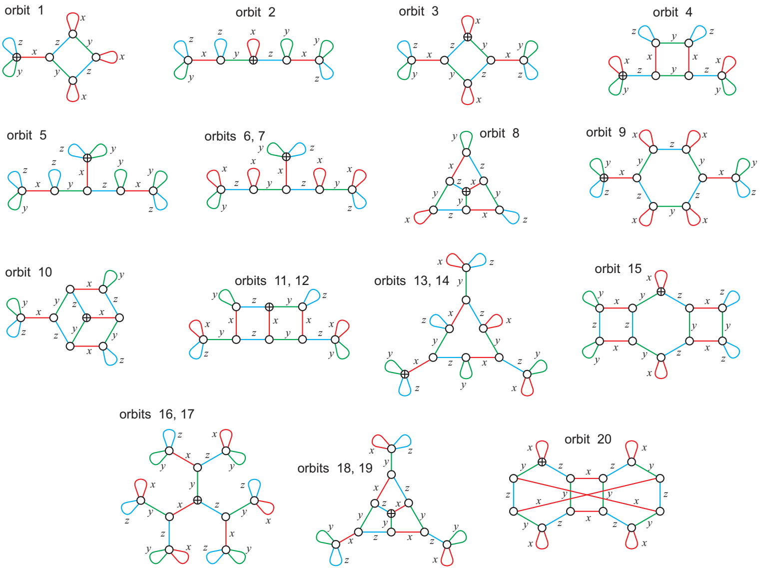
- four orbits I–IV, depending on continuous parameters
- Cayley orbits; all of these can be generated from the points

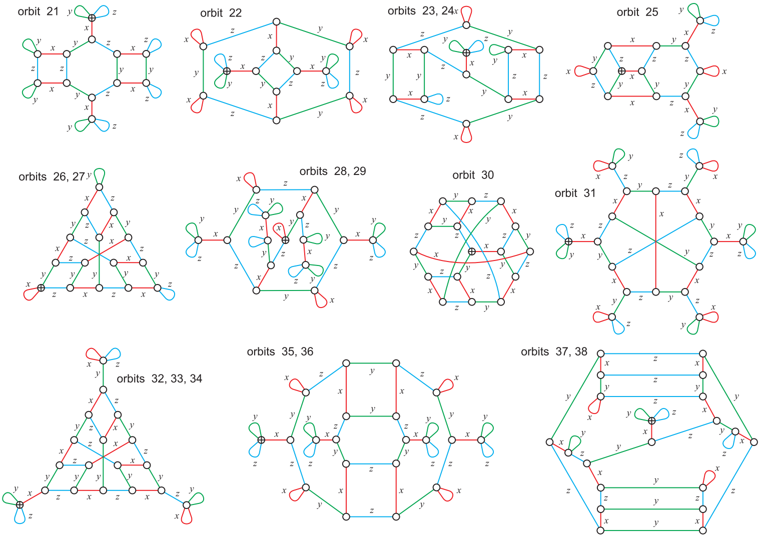
$$\big(-2\cos\pi(r_Y + r_Z), 2\cos\pi r_Y, 2\cos\pi r_Z\big), \qquad r_{Y,Z} \in \mathbb{Q}$$

  with $\omega_X = \omega_Y = \omega_Z = \omega_4 = 0$
- 45 exceptional orbits

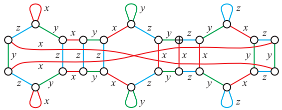
	size	$(\omega_X, \omega_Y, \omega_Z, 4-\omega_4)$	(r_X, r_Y, r_Z)
1	5	$(0, 1, 1, 0)$	$(2/3, 1/3, 1/3)$
2	5	$(3, 2, 2, -3)$	$(1/3, 1/3, 1/3)$
3	6	$(1, 0, 0, 2)$	$(1/2, 1/3, 1/3)$
4	6	$(\sqrt{2}, 0, 0, 1)$	$(1/4, 1/3, 3/4)$
5	6	$(3, 2\sqrt{2}, 2\sqrt{2}, -4)$	$(1/2, 1/4, 1/4)$
6	6	$\left(1-\sqrt{5}, (3-\sqrt{5})/2, (3-\sqrt{5})/2, -2+\sqrt{5}\right)$	$(4/5, 1/3, 1/3)$
7	6	$\left(1+\sqrt{5}, (3+\sqrt{5})/2, (3+\sqrt{5})/2, -2-\sqrt{5}\right)$	$(2/5, 1/3, 1/3)$
8	7	$(1, 1, 1, 0)$	$(1/2, 1/2, 1/2)$
9	8	$(2, 0, 0, 0)$	$(0, 1/3, 2/3)$
10	8	$(1, \sqrt{2}, \sqrt{2}, 0)$	$(1/2, 1/2, 1/2)$
11	8	$\left((3+\sqrt{5})/2, 1, 1, -(\sqrt{5}+1)/2\right)$	$(1/3, 1/2, 1/2)$
12	8	$\left((3-\sqrt{5})/2, 1, 1, (\sqrt{5}-1)/2\right)$	$(1/3, 1/2, 1/2)$
13	9	$\left(2-\sqrt{5}, 2-\sqrt{5}, 2-\sqrt{5}, (5\sqrt{5}-7)/2\right)$	$(4/5, 3/5, 3/5)$
14	9	$\left(2+\sqrt{5}, 2+\sqrt{5}, 2+\sqrt{5}, -(5\sqrt{5}+7)/2\right)$	$(2/5, 1/5, 1/5)$
15	10	$(1, 0, 0, 1)$	$(1/3, 1/3, 2/3)$
16	10	$\left(3-\sqrt{5}, 3-\sqrt{5}, 3-\sqrt{5}, (7\sqrt{5}-11)/2\right)$	$(3/5, 3/5, 3/5)$
17	10	$\left(3+\sqrt{5}, 3+\sqrt{5}, 3+\sqrt{5}, -(7\sqrt{5}+11)/2\right)$	$(1/5, 1/5, 1/5)$
18	10	$\left(-(\sqrt{5}-1)/2, -(\sqrt{5}-1)/2, -(\sqrt{5}-1)/2, 0\right)$	$(1/2, 1/2, 1/2)$
19	10	$\left((\sqrt{5}+1)/2, (\sqrt{5}+1)/2, (\sqrt{5}+1)/2, 0\right)$	$(1/2, 1/2, 1/2)$
20	12	$(0, 0, 0, 3)$	$(2/3, 1/4, 1/4)$
21	12	$(1, 0, 0, 2)$	$(0, 1/4, 3/4)$
22	12	$(2, \sqrt{5}, \sqrt{5}, -2)$	$(1/5, 2/5, 2/5)$
23	12	$\left((3+\sqrt{5})/2, (\sqrt{5}+1)/2, (\sqrt{5}+1)/2, -\sqrt{5}\right)$	$(2/5, 2/5, 2/5)$
24	12	$\left((3-\sqrt{5})/2, -(\sqrt{5}-1)/2, -(\sqrt{5}-1)/2, \sqrt{5}\right)$	$(4/5, 4/5, 4/5)$
25	12	$\left((\sqrt{5}+1)/2, (\sqrt{5}-1)/2, 1, 0\right)$	$(1/2, 1/2, 1/2)$

	size	$(\omega_X, \omega_Y, \omega_Z, 4-\omega_4)$	(r_X, r_Y, r_Z)
26	15	$\left(\frac{3-\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}, \sqrt{5}-1\right)$	$(1/2, 3/5, 3/5)$
27	15	$\left(\frac{3+\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}, -\sqrt{5}-1\right)$	$(1/2, 1/5, 1/5)$
28	15	$\left(\frac{5-\sqrt{5}}{2}, 1-\sqrt{5}, 1-\sqrt{5}, \frac{3\sqrt{5}-5}{2}\right)$	$(3/5, 4/5, 4/5)$
29	15	$\left(\frac{5+\sqrt{5}}{2}, 1+\sqrt{5}, 1+\sqrt{5}, -\frac{3\sqrt{5}+5}{2}\right)$	$(1/5, 2/5, 2/5)$
30	16	$(0, 0, 0, 2)$	$(2/3, 2/3, 2/3)$
31	18	$(2, 2, 2, -1)$	$(0, 1/5, 3/5)$
32	18	$(1-2\cos 2\pi/7, 1-2\cos 2\pi/7, 1-2\cos 2\pi/7, 4\cos 2\pi/7)$	$(6/7, 5/7, 5/7)$
33	18	$(1-2\cos 4\pi/7, 1-2\cos 4\pi/7, 1-2\cos 4\pi/7, 4\cos 4\pi/7)$	$(2/7, 3/7, 3/7)$
34	18	$(1-2\cos 6\pi/7, 1-2\cos 6\pi/7, 1-2\cos 6\pi/7, 4\cos 6\pi/7)$	$(4/7, 1/7, 1/7)$
35	20	$\left(\frac{3-\sqrt{5}}{2}, 0, 0, 1+\sqrt{5}\right)$	$(0, 1/3, 2/3)$
36	20	$\left(\frac{3+\sqrt{5}}{2}, 0, 0, 1-\sqrt{5}\right)$	$(0, 1/3, 2/3)$
37	20	$\left(1, -\frac{\sqrt{5}-1}{2}, -\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}+1}{2}\right)$	$(2/3, 3/5, 3/5)$
38	20	$\left(1, \frac{\sqrt{5}+1}{2}, \frac{\sqrt{5}+1}{2}, -\frac{\sqrt{5}-1}{2}\right)$	$(2/3, 1/5, 1/5)$
39	24	$(1, 1, 1, 1)$	$(1/5, 1/2, 1/2)$
40	30	$\left(-\frac{\sqrt{5}+1}{2}, 0, 0, \frac{3-\sqrt{5}}{2}\right)$	$(2/3, 2/3, 2/3)$
41	30	$\left(\frac{\sqrt{5}-1}{2}, 0, 0, \frac{3+\sqrt{5}}{2}\right)$	$(2/3, 2/3, 2/3)$
42	36	$(1, 0, 0, 2)$	$(0, 1/5, 4/5)$
43	40	$\left(0, 0, 0, \frac{5-\sqrt{5}}{2}\right)$	$(2/5, 2/5, 2/5)$
44	40	$\left(0, 0, 0, \frac{5+\sqrt{5}}{2}\right)$	$(4/5, 4/5, 4/5)$
45	72	$(0, 0, 0, 3)$	$(1/2, 1/5, 2/5)$

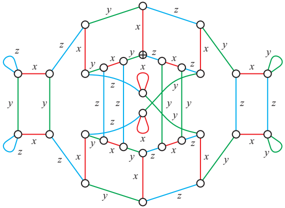

orbit 1
orbit 2
orbit 3
orbit 4
orbit 5
orbits 6, 7
orbit 8
orbit 9
orbit 10
orbits 11, 12
orbits 13, 14
orbit 15
orbits 16, 17
orbits 18, 19
orbit 20


orbit 21
orbit 22
orbits 23, 24
orbit 25
orbits 26, 27
orbits 28, 29
orbit 30
orbit 31
orbits 32, 33, 34
orbits 35, 36
orbits 37, 38

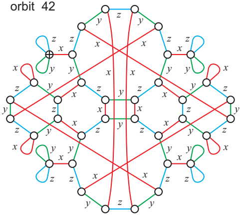
orbit 39



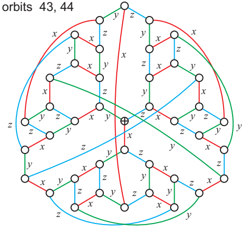
orbits 40, 41



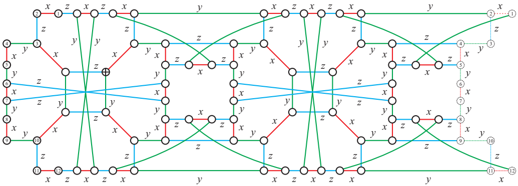
orbit 42



orbits 43, 44



orbit 45



Nonlinear Schwarz list

Theorem. The list of all nonequivalent finite orbits of the induced $\bar{\Lambda}$ action on $\mathbb{C}^3$ consists of the following:

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  with $\omega_X = \omega_Y = \omega_Z = \omega_4 = 0$
- 45 exceptional orbits

Nonlinear Schwarz list

Theorem. The list of all nonequivalent finite orbits of the induced $\bar{\Lambda}$ action on $\mathbb{C}^3$ consists of the following:

- four orbits I–IV, depending on continuous parameters $\Rightarrow$ Riccati & 3 algebraic families
- Cayley orbits; all of these can be generated from the points

$$\left(-2\cos\pi(r_Y+r_Z), 2\cos\pi r_Y, 2\cos\pi r_Z\right), \qquad r_{Y,Z}\in\mathbb{Q}$$

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  with $\omega_X = \omega_Y = \omega_Z = \omega_4 = 0 \Rightarrow$ Picard solutions
- 45 exceptional orbits

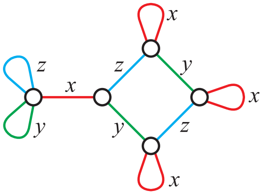
Nonlinear Schwarz list

Theorem. The list of all nonequivalent finite orbits of the induced $\bar{\Lambda}$ action on $\mathbb{C}^3$ consists of the following:

- four orbits I–IV, depending on continuous parameters $\Rightarrow$ Riccati & 3 algebraic families
- Cayley orbits; all of these can be generated from the points

$$\left(-2\cos\pi(r_Y + r_Z), 2\cos\pi r_Y, 2\cos\pi r_Z\right), \qquad r_{Y,Z} \in \mathbb{Q}$$

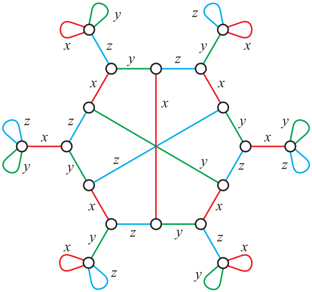
  with $\omega_X = \omega_Y = \omega_Z = \omega_4 = 0 \;\Rightarrow$ Picard solutions
- 45 exceptional orbits $\Rightarrow$ 45 algebraic solutions

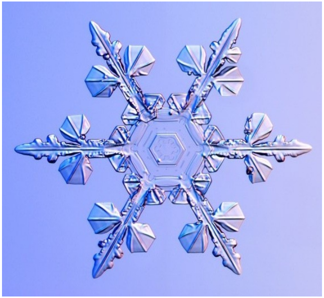


Solution 1, 5 branches, $(\theta_x, \theta_y, \theta_z, \theta_\infty) = (2/5, 1/5, 1/3, 2/3)$:

$$w = \frac{2(s^2+s+7)(5s-2)}{s(s+5)(4s^2-5s+10)},$$

$$t = \frac{27(5s-2)^2}{(s+5)(4s^2-5s+10)^2}.$$





Solution 31 (Dubrovin-Mazzocco great dodecahedron solution), 18 branches, $\boldsymbol{\theta} = (1/3, 1/3, 1/3, 1/3)$:

$$w = \frac{1}{2} - \frac{8s^7 - 28s^6 + 75s^5 + 31s^4 - 269s^3 + 318s^2 - 166s + 56}{18u(s-1)(3s^3 - 4s^2 + 4s + 2)},$$

$$t = \frac{1}{2} + \frac{(s+1)\big(32(s^8+1) - 320(s^7+s) + 1112(s^6+s^2) - 2420(s^5+s^3) + 3167s^4\big)}{54u^3 s(s-1)},$$

$$u^2 = s(8s^2 - 11s + 8). \qquad \text{(elliptic parametrization due to Boalch)}$$