Part 2: Conformal field theory of Painlevé VI

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Standard form of Painlevé VI:

$$\frac{d^2w}{dt^2} = \frac{1}{2}\left(\frac{1}{w}+\frac{1}{w-1}+\frac{1}{w-t}\right)\left(\frac{dw}{dt}\right)^2 - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{w-t}\right)\frac{dw}{dt} + \\ + \frac{2w(w-1)(w-t)}{t^2(t-1)^2}\left((\theta_\infty - 1/2)^2 - \frac{\theta_0^2 t}{w^2} + \frac{\theta_1^2(t-1)}{(w-1)^2} + \frac{(1/4-\theta_t^2)t(t-1)}{(w-t)^2}\right)$$

Sigma form:

$$-\frac{1}{2}\Big(t(t-1)\sigma''\Big)^2 = \\ = \det\begin{pmatrix} 2\theta_0^2 & t\sigma'-\sigma & \sigma'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 \\ t\sigma'-\sigma & 2\theta_t^2 & (t-1)\sigma'-\sigma \\ \sigma'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 & (t-1)\sigma'-\sigma & 2\theta_1^2 \end{pmatrix}$$

- most general 2nd order, 2nd degree ODE without movable critical points [Cosgrove, Scoufis, '93]
- 4 parameters $\boldsymbol{\theta} = (\theta_0, \theta_t, \theta_1, \theta_\infty)$
- tau function: $\sigma = t(t-1)\frac{d}{dt}\ln\tau$

Properties:

- singularities at $0, 1, \infty$ for $\tau(t)$ + movable poles for $w(t)$
- non-autonomous hamiltonian system
- confluence cascade

$$\begin{array}{ccccccc} & & & PIV & & & \\ & & \nearrow & & \searrow & & \\ PVI \to PV & & & & & PII \to PI \\ & & \searrow & & \nearrow & & \\ & & & PIII & & & \end{array}$$

Solutions:

- Riccati/Chazy (hypergeometric)

$$\begin{array}{ccccccc} & & & Hermite & & & \\ & & \nearrow & & \searrow & & \\ {}_2F_1 \to Whittaker & & & & & Airy \to \varnothing \\ & & \searrow & & \nearrow & & \\ & & & Bessel & & & \end{array}$$

- Picard (elliptic)
- algebraic
- transcendental

Back to Ising example

Diagonal two-point correlation function

$$D_N = \langle \sigma(0,0)\sigma(N,N)\rangle_{T<T_c}$$

is a Painlevé VI τ-function [Jimbo, Miwa, '81]:

$$D_N = (1-t)^{\frac{N^2}{2}}\,\tau(t)$$

- temperature parameter $t = \left(\sinh 2\mathcal{K}_x \sinh 2\mathcal{K}_y\right)^{-2}$, $0 < t < 1$
- $\boldsymbol{\theta} = \frac{1}{2}\left(0, N, N, 1\right)$

Toeplitz determinant representation [Montroll, Potts, Ward, '63]:

$$D_N = D_N^{\left(\frac12,-\frac12\right)}, \qquad D_N^{(z,z')} = \det\left[f_{j-k}\right]_{j,k=1,\ldots,N},$$

with the symbol

$$\sum_{\ell\in\mathbb{Z}} f_\ell \zeta^\ell = \left(1-\sqrt{t}\zeta\,\right)^{z}\left(1-\sqrt{t}\zeta^{-1}\right)^{z'}$$

- $D_N^{(z,z')} = (1-t)^{\frac{N(N+z+z')}{2}}\,\tau(t)$ with $\boldsymbol{\theta} = \frac{1}{2}\left(0, N, N+z+z', z-z'\right)$

Example: 2D Ising model (continued)

It is known that [Gessel, '90; Borodin, '01]

$$D_N^{(z,z')} = (1-t)^{-zz'} \sum_{\lambda\in\mathbb{Y},\lambda_1\leq N} P^{(z,z',t)}(\lambda), \tag{1}$$

where $P^{(z,z',t)}(\lambda)$ is the z-measure

$$P^{(z,z',t)}(\lambda) = (1-t)^{zz'} \prod_{(i,j)\in\lambda} \frac{(i-j+z)(i-j+z')}{h_\lambda^2(i,j)}\, t^{|\lambda|} \tag{2}$$

- $\mathbb{Y}$ is the set of all partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell > 0)$, identified with Young diagrams
- $|\lambda| = \lambda_1 + \ldots + \lambda_\ell$ is the total number of boxes in λ
- $h_\lambda(i,j)$ denotes hook length of the box $(i,j)\in\lambda$
- $\sum_{\lambda\in\mathbb{Y}} P^{(z,z',t)}(\lambda) = 1$ is a variant of Cauchy identity for Schur functions [Okounkov, '99]

$$\sum_{\lambda\in\mathbb{Y}} s_\lambda(\mathbf{x})\, s_\lambda(\mathbf{y}) = \prod_{i,j} \left(1 - x_i y_j\right)^{-1}$$

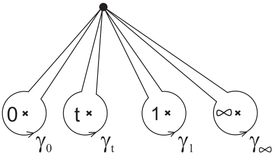
N.B. Relations (1)–(2) provide a "low-temperature" expansion of the Ising-PVI tau function. Our aim is to establish a similar series representation for the **general solution of Painlevé VI**.

Painlevé VI and isomonodromy

PVI describes monodromy preserving deformations of rank 2 linear systems on $\mathbb{P}^1$ with 4 regular singular points $0, t, 1, \infty$:

$$\frac{d\Phi}{dz} = \mathcal{A}(z)\Phi, \qquad \mathcal{A}(z) = \frac{\mathcal{A}_0}{z} + \frac{\mathcal{A}_t}{z-t} + \frac{\mathcal{A}_1}{z-1}$$

- matrices $\mathcal{A}_\nu$ are 2×2, traceless, with eigenvalues $\pm\theta_\nu$
- $\mathcal{A}_0 + \mathcal{A}_t + \mathcal{A}_1 \stackrel{def}{=} -\mathcal{A}_\infty = \operatorname{diag}\{-\theta_\infty, \theta_\infty\}$
- 3 monodromy matrices $\mathcal{M}_{0,t,1} \in G = SL(2,\mathbb{C})$ (note $\mathcal{M}_\infty\mathcal{M}_1\mathcal{M}_t\mathcal{M}_0 = \mathbf{1}$)
- monodromy manifold $\mathcal{M} = G^3/G$, $\dim\mathcal{M} = 6$



Painlevé VI and isomonodromy (continued)

Schlesinger equations:

$$\frac{d\mathcal{A}_0}{dt} = \frac{[\mathcal{A}_t, \mathcal{A}_0]}{t}, \qquad \frac{d\mathcal{A}_1}{dt} = \frac{[\mathcal{A}_t, \mathcal{A}_1]}{t-1}$$

- Lax form $\Rightarrow \theta_{0,t,1,\infty}$ are conserved
- remains 2 degrees of freedom (recall that $\mathcal{A}_0 + \mathcal{A}_t + \mathcal{A}_1 = -\mathcal{A}_\infty$)
- $\left(\frac{\mathcal{A}_0}{z} + \frac{\mathcal{A}_t}{z-t} + \frac{\mathcal{A}_1}{z-1}\right)_{12} = \frac{k(t)(z-w(t))}{z(z-t)(z-1)} \Rightarrow$ standard form of PVI for $w(t)$

Derivation of σPVI [Hitchin, '97]:

- define

$$f = \operatorname{tr}\mathcal{A}_0\mathcal{A}_t, \quad g = \operatorname{tr}\mathcal{A}_1\mathcal{A}_t, \quad h = \operatorname{tr}\mathcal{A}_0[\mathcal{A}_t, \mathcal{A}_1]$$

- then for $\sigma = (t-1)f + tg$ we find $\sigma' = f + g$ and $t(t-1)\sigma'' = -h$
- but for any 2×2 traceless $\mathcal{A}_{0,t,1}$

$$(\operatorname{tr}\mathcal{A}_0[\mathcal{A}_t, \mathcal{A}_1])^2 = -2\det\begin{pmatrix} \operatorname{tr}\mathcal{A}_0^2 & \operatorname{tr}\mathcal{A}_0\mathcal{A}_t & \operatorname{tr}\mathcal{A}_0\mathcal{A}_1 \\ \operatorname{tr}\mathcal{A}_t\mathcal{A}_0 & \operatorname{tr}\mathcal{A}_t^2 & \operatorname{tr}\mathcal{A}_t\mathcal{A}_1 \\ \operatorname{tr}\mathcal{A}_1\mathcal{A}_0 & \operatorname{tr}\mathcal{A}_1\mathcal{A}_t & \operatorname{tr}\mathcal{A}_1^2 \end{pmatrix}$$

- and from $\operatorname{tr}\mathcal{A}_\infty^2 = \operatorname{tr}(\mathcal{A}_0 + \mathcal{A}_t + \mathcal{A}_1)^2$ we can find $\operatorname{tr}\mathcal{A}_0\mathcal{A}_1$

Painlevé VI and isomonodromy (continued)

Sigma Painlevé VI:

$$\left(t(t-1)\sigma''\right)^2 = -2\det\begin{pmatrix} 2\theta_0^2 & t\sigma'-\sigma & \sigma'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 \\ t\sigma'-\sigma & 2\theta_t^2 & (t-1)\sigma'-\sigma \\ \sigma'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 & (t-1)\sigma'-\sigma & 2\theta_1^2 \end{pmatrix}$$

- to any solution corresponds (the conjugacy class of) a triple $(\mathcal{M}_0, \mathcal{M}_t, \mathcal{M}_1)$
- $p_\nu = 2\cos 2\pi\theta_\nu = \mathrm{tr}\,\mathcal{M}_\nu$ (with $\nu = 0, t, 1, \infty$) give four PVI parameters
- remaining two coordinates $\Rightarrow$ integration constants
- introduce $p_{\mu\nu} = 2\cos 2\pi\sigma_{\mu\nu} = \mathrm{tr}\,\mathcal{M}_\mu\mathcal{M}_\nu$, then [Jimbo, '82]

$$p_{0t}p_{1t}p_{01} + p_{0t}^2 + p_{1t}^2 + p_{01}^2 - \omega_{0t}p_{0t} - \omega_{1t}p_{1t} - \omega_{01}p_{01} = 4 - \omega_4, \qquad (3)$$

where $\omega_4 = p_0^2 + p_t^2 + p_1^2 + p_\infty^2 + p_0 p_t p_1 p_\infty$ and

$$\omega_{0t} = p_0 p_t + p_1 p_\infty, \quad \omega_{1t} = p_1 p_t + p_0 p_\infty, \quad \omega_{01} = p_0 p_1 + p_t p_\infty$$

N.B. The triple $\boldsymbol{\sigma}$ satisfying (3) can be interpreted as a pair of PVI integration constants. Our task is: given $\boldsymbol{\sigma}$, to obtain the corresponding solution.

Jimbo's formula ['82]

- expresses the asymptotics of $\tau(t)$ as $t \to 0$, 1, or ∞ in terms of monodromy
- e.g. for $t \to 0$, denote $\sigma = \sigma_{0t}$ and choose $0 < |\mathrm{Re}\,\sigma| < \frac{1}{2}$
- also denote $\Delta_\nu = \theta_\nu^2$ ($\nu = 0, t, 1, \infty$) and $\Delta_\sigma = \sigma^2$; then

$$\tau(t) = \mathrm{const} \cdot \left(t^{\Delta_\sigma - \Delta_0 - \Delta_t} + C_{\pm 1} t^{\Delta_{\sigma \pm 1} - \Delta_0 - \Delta_t} + \text{ smaller terms } \right),$$

with

$$C_{\pm 1} = \frac{\Gamma^2(1 \mp 2\sigma)}{\Gamma^2(1 \pm 2\sigma)} \prod_{\epsilon = \pm} \frac{\Gamma(1 + \epsilon\theta_0 + \theta_t \pm \sigma)\Gamma(1 + \epsilon\theta_\infty + \theta_1 \pm \sigma)}{\Gamma(1 + \epsilon\theta_0 + \theta_t \mp \sigma)\Gamma(1 + \epsilon\theta_\infty + \theta_1 \mp \sigma)} \times$$
$$\times \frac{\left(\theta_0^2 - (\theta_t \mp \sigma)^2\right)\left(\theta_\infty^2 - (\theta_1 \mp \sigma)^2\right)}{4\sigma^2(1 \pm 2\sigma)^2} (-s)^{\pm 1},$$

and

$$s^{\pm 1}\left(\cos 2\pi(\theta_t \mp \sigma) - \cos 2\pi\theta_0\right)\left(\cos 2\pi(\theta_1 \mp \sigma) - \cos 2\pi\theta_\infty\right) =$$
$$= \left(\cos 2\pi\theta_t \cos 2\pi\theta_1 + \cos 2\pi\theta_0 \cos 2\pi\theta_\infty \pm i \sin 2\pi\sigma \cos 2\pi\sigma_{01}\right) -$$
$$- \left(\cos 2\pi\theta_0 \cos 2\pi\theta_1 + \cos 2\pi\theta_t \cos 2\pi\theta_\infty \mp i \sin 2\pi\sigma \cos 2\pi\sigma_{1t}\right) e^{\pm 2\pi i\sigma}.$$

- higher-order corrections can be determined recursively from σPVI (in principle)

Higher order corrections

$$\tau(t) \sim t^{\Delta_\sigma - \Delta_0 - \Delta_t}\Big(1 + \mathcal{B}_1(\theta,\sigma)t + \qquad \ldots\Big)$$

$$+C_{\pm 1}\; t^{\Delta_{\sigma\pm 1} - \Delta_0 - \Delta_t}$$

with

$$\mathcal{B}_1(\theta,\sigma) = \frac{(\Delta_\sigma - \Delta_0 + \Delta_t)(\Delta_\sigma - \Delta_\infty + \Delta_1)}{2\Delta_\sigma},$$

Higher order corrections

$$\tau(t) \sim t^{\Delta_\sigma - \Delta_0 - \Delta_t}\Big(1 + \mathcal{B}_1(\boldsymbol{\theta},\sigma)t + \mathcal{B}_2(\boldsymbol{\theta},\sigma)t^2 \ldots\Big) +$$
$$+ C_{\pm 1}\, t^{\Delta_{\sigma\pm 1} - \Delta_0 - \Delta_t}\Big(1 + \mathcal{B}_1^{(\pm 1)}(\boldsymbol{\theta},\sigma)t + \ldots\Big) + C_{\pm 2} t^{\Delta_{\sigma\pm 2} - \Delta_0 - \Delta_t}\Big(1 + \ldots\Big)$$

with

$$\mathcal{B}_1(\boldsymbol{\theta},\sigma) = \frac{(\Delta_\sigma - \Delta_0 + \Delta_t)(\Delta_\sigma - \Delta_\infty + \Delta_1)}{2\Delta_\sigma},$$
$$\mathcal{B}_2(\boldsymbol{\theta},\sigma) = \frac{(\Delta_\sigma - \Delta_0 + \Delta_t)(\Delta_\sigma - \Delta_0 + \Delta_t + 1)(\Delta_\sigma - \Delta_\infty + \Delta_1)(\Delta_\sigma - \Delta_\infty + \Delta_1 + 1)}{4\Delta_\sigma(2\Delta_\sigma + 1)}$$
$$+ \frac{\left[(1+2\Delta_\sigma)(\Delta_0 + \Delta_t) + \Delta_\sigma(\Delta_\sigma + 1) - 3(\Delta_0 - \Delta_t)^2\right]\left[(1+2\Delta_\sigma)(\Delta_\infty + \Delta_1) + \Delta_\sigma(\Delta_\sigma + 1) - 3(\Delta_\infty - \Delta_1)^2\right]}{2(2\Delta_\sigma + 1)(4\Delta_\sigma - 1)^2},$$
$$\mathcal{B}_1^{(\pm 1)}(\boldsymbol{\theta},\sigma) = \mathcal{B}_1(\boldsymbol{\theta},\sigma \pm 1).$$

Higher order corrections

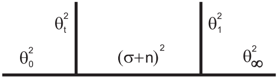
$$\tau(t) \sim t^{\Delta_\sigma-\Delta_0-\Delta_t}\left(1+\mathcal{B}_1(\boldsymbol{\theta},\sigma)t+\mathcal{B}_2(\boldsymbol{\theta},\sigma)t^2\ldots\right)+$$
$$+C_{\pm1}\, t^{\Delta_{\sigma\pm1}-\Delta_0-\Delta_t}\left(1+\mathcal{B}_1^{(\pm1)}(\boldsymbol{\theta},\sigma)t+\ldots\right)+C_{\pm2}t^{\Delta_{\sigma\pm2}-\Delta_0-\Delta_t}\left(1+\ldots\right)$$

with

$$\mathcal{B}_1(\boldsymbol{\theta},\sigma)=\frac{(\Delta_\sigma-\Delta_0+\Delta_t)(\Delta_\sigma-\Delta_\infty+\Delta_1)}{2\Delta_\sigma},$$
$$\mathcal{B}_2(\boldsymbol{\theta},\sigma)=\frac{(\Delta_\sigma-\Delta_0+\Delta_t)(\Delta_\sigma-\Delta_0+\Delta_t+1)(\Delta_\sigma-\Delta_\infty+\Delta_1)(\Delta_\sigma-\Delta_\infty+\Delta_1+1)}{4\Delta_\sigma(2\Delta_\sigma+1)}$$
$$+\frac{\left[(1+2\Delta_\sigma)(\Delta_0+\Delta_t)+\Delta_\sigma(\Delta_\sigma+1)-3(\Delta_0-\Delta_t)^2\right]\left[(1+2\Delta_\sigma)(\Delta_\infty+\Delta_1)+\Delta_\sigma(\Delta_\sigma+1)-3(\Delta_\infty-\Delta_1)^2\right]}{2(2\Delta_\sigma+1)(4\Delta_\sigma-1)^2},$$
$$\mathcal{B}_1^{(\pm1)}(\boldsymbol{\theta},\sigma)=\mathcal{B}_1(\boldsymbol{\theta},\sigma\pm1).$$

Observation. PVI tau function is a linear combination of $c=1$ conformal blocks:

$$\tau(t)=\sum_{n\in\mathbb{Z}} C_n t^{\Delta_{\sigma+n}-\Delta_0-\Delta_t}\mathcal{B}\left(\boldsymbol{\theta},\sigma+n,t\right)$$

Graphical representation of $\mathcal{B}\left(\boldsymbol{\theta},\sigma+n,t\right)$: 

Higher order corrections (continued)

$$\mathcal{B}_3(\boldsymbol{\theta},\sigma)=\frac{(\Delta_\sigma-\Delta_0+\Delta_t+2)(\Delta_\sigma-\Delta_\infty+\Delta_1+2)}{24\Delta_\sigma(4\Delta_\sigma-1)^2(\Delta_\sigma-1)^2}\times$$
$$\times\Big\{(8\Delta_\sigma^2-5\Delta_\sigma+3)(\Delta_\sigma-\Delta_0+\Delta_t)(\Delta_\sigma-\Delta_0+\Delta_t+1)(\Delta_\sigma-\Delta_\infty+\Delta_1)(\Delta_\sigma-\Delta_\infty+\Delta_1+1)$$
$$-4(9\Delta_\sigma^2-4\Delta_\sigma+1)(\Delta_\sigma-\Delta_0+\Delta_t)(\Delta_\sigma-\Delta_0+\Delta_t+1)(\Delta_\sigma-\Delta_\infty+2\Delta_1)$$
$$-4(9\Delta_\sigma^2-4\Delta_\sigma+1)(\Delta_\sigma-\Delta_\infty+\Delta_1)(\Delta_\sigma-\Delta_\infty+\Delta_1+1)(\Delta_\sigma-\Delta_0+2\Delta_t)$$
$$+8(6\Delta_\sigma^3+11\Delta_\sigma^2-6\Delta_\sigma+1)(\Delta_\sigma-\Delta_0+2\Delta_t)(\Delta_\sigma-\Delta_\infty+2\Delta_1)\Big\}$$
$$+\frac{1}{6\Delta_\sigma(\Delta_\sigma-1)^2}\Big\{(\Delta_\sigma^2+3\Delta_\sigma+2)(\Delta_\sigma-\Delta_0+3\Delta_t)(\Delta_\sigma-\Delta_\infty+3\Delta_1)$$
$$+(\Delta_\sigma-\Delta_0+3\Delta_t)(\Delta_\sigma-\Delta_\infty+\Delta_1)(\Delta_\sigma-\Delta_\infty+\Delta_1+1)(\Delta_\sigma-\Delta_\infty+\Delta_1+2)$$
$$+(\Delta_\sigma-\Delta_\infty+3\Delta_1)(\Delta_\sigma-\Delta_0+\Delta_t)(\Delta_\sigma-\Delta_0+\Delta_t+1)(\Delta_\sigma-\Delta_0+\Delta_t+2)$$
$$-2(\Delta_\sigma+1)(\Delta_\sigma-\Delta_0+3\Delta_t)(\Delta_\sigma-\Delta_\infty+2\Delta_1)(\Delta_\sigma-\Delta_\infty+\Delta_1+2)$$
$$-2(\Delta_\sigma+1)(\Delta_\sigma-\Delta_\infty+3\Delta_1)(\Delta_\sigma-\Delta_0+2\Delta_t)(\Delta_\sigma-\Delta_0+\Delta_t+2)\Big\}$$

- more terms can be checked using computer algebra

General isomonodromy problem

Rank N linear system with n regular singular points $a_1, \ldots, a_n$ on $\mathbb{P}^1$:

$$\partial_z \Phi = \mathcal{A}(z)\Phi, \qquad \mathcal{A}(z) = \sum_{\nu=1}^{n} \frac{\mathcal{A}_\nu}{z - a_\nu}$$

- normalization $\Phi(z_0) = \mathbf{1}_N$
- no singularity at $\infty \Rightarrow \sum_{\nu=1}^{n} \mathcal{A}_\nu = 0$
- $\mathcal{A}_\nu$'s assumed to be diagonalizable: $\mathcal{A}_\nu = \mathcal{G}_\nu \mathcal{T}_\nu \mathcal{G}_\nu^{-1}$ with some $\mathcal{T}_\nu = \operatorname{diag}\left\{\lambda_{\nu,1}, \ldots, \lambda_{\nu,N}\right\}$
- introducing $\mathcal{J}(z) = \Phi^{-1}\partial_z \Phi = \Phi^{-1}\mathcal{A}(z)\Phi$, expand $\Phi(z)$ around $z = z_0$:

$$\Phi(z \to z_0) = \mathbf{1}_N + \mathcal{J}(z_0)(z - z_0) + \left(\mathcal{J}^2(z_0) + \partial\mathcal{J}(z_0)\right)\frac{(z - z_0)^2}{2} + \ldots$$

- expansions near singular points:

$$\Phi(z \to a_\nu) = \mathcal{G}_\nu(z)(z - a_\nu)^{\mathcal{T}_\nu}\, \mathcal{C}_\nu$$

  - $\mathcal{G}_\nu(z)$ is holomorphic and invertible in a neighborhood of $z = a_\nu$, and satisfies $\mathcal{G}_\nu(a_\nu) = \mathcal{G}_\nu$
  - $\mathcal{C}_\nu$ are connection matrices; monodromy matrices $\mathcal{M}_\nu = \mathcal{C}_\nu^{-1} e^{2\pi i \mathcal{T}_\nu} \mathcal{C}_\nu$

General isomonodromy problem (continued)

Deformation equations:

$$\partial_{a_\nu}\Phi = -\frac{z_0 - z}{z_0 - a_\nu}\,\frac{\mathcal{A}_\nu}{z - a_\nu}\,\Phi,$$
$$\partial_{z_0}\Phi = -\mathcal{A}(z_0)\,\Phi.$$

- $\mathcal{J}(z)$ remains invariant under isomonodromic variation of z_0 !

Schlesinger equations:

$$\partial_{a_\mu}\mathcal{A}_\nu = \frac{z_0 - a_\nu}{z_0 - a_\mu}\,\frac{[\mathcal{A}_\mu, \mathcal{A}_\nu]}{a_\mu - a_\nu}, \qquad \mu \neq \nu,$$
$$\partial_{a_\nu}\mathcal{A}_\nu = -\sum_{\mu\neq\nu}\frac{[\mathcal{A}_\mu, \mathcal{A}_\nu]}{a_\mu - a_\nu}, \qquad \partial_{z_0}\mathcal{A}_\nu = -\sum_{\mu\neq\nu}\frac{[\mathcal{A}_\mu, \mathcal{A}_\nu]}{z_0 - a_\mu}.$$

Tau function:

$$d\ln\tau = \sum_{\mu<\nu}\mathrm{tr}\,\mathcal{A}_\mu\mathcal{A}_\nu\; d\ln(a_\mu - a_\nu)\,.$$

- τ does not depend on z_0 thanks to

$$\partial_{a_\mu}\ln\tau = \sum_{\nu\neq\mu}\frac{\mathrm{tr}\,\mathcal{A}_\mu\mathcal{A}_\nu}{a_\mu - a_\nu} = \frac{1}{2}\,\mathrm{res}_{\,z=a_\mu}\,\mathrm{tr}\,\mathcal{J}^2(z).$$

Global conformal symmetry

How does τ trasform under $f(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$?

Example (three points): $d \ln \tau$ can be explicitly integrated to

$$\tau(a_1, a_2, a_3) = \text{const} \cdot (a_1 - a_2)^{\Delta_3 - \Delta_2 - \Delta_1} (a_1 - a_3)^{\Delta_2 - \Delta_1 - \Delta_3} (a_2 - a_3)^{\Delta_1 - \Delta_2 - \Delta_3},$$

with $\Delta_\nu = \frac{1}{2} \operatorname{tr} \mathcal{A}_\nu^2$ and $\nu = 1, 2, 3$. Expression for 3-point function of quasiprimary fields with dimensions $\Delta_{1,2,3}$ in 2D CFT !

Proposition: One has

$$\tau(f(a)) = \prod_{\nu=1}^{n} \left[f'(a_\nu) \right]^{-\Delta_\nu} \tau(a)$$

■ It suffices to consider infinitesimal transformations generated by $\left(A + Bz + Cz^2\right) \partial_z$ $\Rightarrow$ check three differential constraints

$$\sum_\nu \partial_{a_\nu} \ln \tau = 0,$$

$$\sum_\nu \left(a_\nu \partial_{a_\nu} \ln \tau + \Delta_\nu\right) = 0,$$

$$\sum_\nu \left(a_\nu^2 \partial_{a_\nu} \ln \tau + 2\Delta_\nu a_\nu\right) = 0.$$

CFT basics

Fields are organized by the representation theory of the Virasoro algebra:

$$[L_n, L_m] = (n-m)\, L_{n+m} + \frac{c}{12}\, n\left(n^2-1\right)\delta_{m+n,0}$$

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}$$

arising from $T(z) = \sum_{n\in\mathbb{Z}} L_n\, z^{-n-2}$.

- primary fields:

$$\begin{cases} L_n|\Delta\rangle = 0 \text{ for } n>0, \\ L_0|\Delta\rangle = \Delta|\Delta\rangle, \end{cases}$$

$$T(z)\phi(w) \sim \frac{\Delta\phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{z-w}$$

- descendants:

$$L_\lambda|\Delta\rangle = L_{-\lambda_\ell}\dots L_{-\lambda_1}|\Delta\rangle, \qquad \lambda\in\mathbb{Y}$$

fields $(L_\lambda\phi)(w)$ obtained by successive OPEs of $T(z)$ with $\phi(w)$

- OPE of two primaries:

$$\phi_0(0)\phi_t(t) = \sum_p C_{0t}^p\, t^{\Delta_p-\Delta_0-\Delta_t} \sum_\lambda \beta_\lambda(\Delta_0,\Delta_t,\Delta_p,c)\, t^{|\lambda|}\,(L_\lambda\phi_p)(0)$$

with β_λ completely fixed by Virasoro symmetry.

Ansatz for Φ

Fundamental matrix solution Φ is completely fixed by its monodromy, normalization and singular behaviour (choice of logarithm branches $\mathcal{L}_\nu = \mathcal{C}_\nu^{-1}\mathcal{T}_\nu\mathcal{C}_\nu = \frac{1}{2\pi i}\ln\mathcal{M}_\nu$).

Starting point (cf [Sato, Miwa, Jimbo, '79]):

$$\Phi_{jk}(z) = (z-z_0)^{2\Delta}\frac{\langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)\bar{\varphi}_j(z_0)\varphi_k(z)\rangle}{\langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)\rangle}, \qquad j,k=1,\ldots,N.$$

Assumptions:

- $\{\mathcal{O}_{\mathcal{L}_\nu}\}$, $\{\bar{\varphi}_j\}$, $\{\varphi_k\}$ are primary fields in a 2D CFT
- OPEs of $\bar{\varphi}$'s with φ's contain $\mathbf{1}$ $\Rightarrow$ same dimensions Δ
- normalization
$$\bar{\varphi}_j(z_0)\varphi_k(z) \sim (z-z_0)^{-2\Delta}\,\delta_{jk}.$$
- dimensions of all other primaries in this OPE are strictly positive integers
- complete OPEs of monodromy fields with auxiliary ones:

$$\mathcal{O}_{\mathcal{L}_\nu}(a_\nu)\varphi_k(z) = \sum_{j=1}^{n}\left((z-a_\nu)^{\mathcal{L}_\nu}\right)_{jk}\sum_{\ell=0}^{\infty}\mathcal{O}_{\mathcal{L}_\nu,j,\ell}(a_\nu)\,(z-a_\nu)^{\ell},$$

If one finds a set of fields with all mentioned properties, the correlator ratio will automatically give Φ.

Tau function

Compute two more orders in the OPE $\bar{\varphi}_j(z_0)\varphi_k(z)$:

$$\bar{\varphi}_j(z_0)\varphi_k(z) = (z-z_0)^{-2\Delta}\Big[\delta_{jk} + J_{jk}(z_0)\,(z-z_0) +$$
$$+\left(\frac{4\Delta}{c}\,T(z_0)\delta_{jk} + (\partial J_{jk})(z_0) + S_{jk}(z_0)\right)\frac{(z-z_0)^2}{2} + O\left((z-z_0)^3\right)\Big].$$

- 1st order: no descendants of 1, new primary J
- 2nd order: level 2 descendant of $\mathbf{1}$, level 1 descendant of J, new primary S

Comparing with

$$\Phi(z\to z_0) = \mathbf{1}_N + \mathcal{J}(z_0)\,(z-z_0) + \left(\mathcal{J}^2(z_0) + \partial\mathcal{J}(z_0)\right)\frac{(z-z_0)^2}{2} + \ldots,$$

one can identify

$$\mathcal{J}(z) = \frac{\langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)J(z)\rangle}{\langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)\rangle},$$
$$\operatorname{tr}\mathcal{J}^2(z) = \frac{\langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)T(z)\rangle}{\langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)\rangle}\,\frac{4N\Delta}{c}.$$

Tau function (continued)

But $\partial_{a_\mu} \ln\tau = \frac{1}{2}\,\mathrm{res}_{\,z=a_\mu}\,\mathrm{tr}\,\mathcal{J}^2(z)$ and

$$\frac{\langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)T(z)\rangle}{\langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)\rangle} = \sum_{\nu=1}^{n}\left\{\frac{\tilde{\Delta}_\nu}{(z-a_\nu)^2}+\frac{1}{z-a_\nu}\,\partial_{a_\nu}\ln\left\langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)\right\rangle\right\}$$

which implies

$$\boxed{\tau(a) = \left\langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)\right\rangle^{\frac{2N\Delta}{c}}}$$

- in particular, for $c = 2N\Delta$ we have $\tilde{\Delta}_\nu = \frac{1}{2}\,\mathrm{tr}\,\mathcal{A}_\nu^2$

Example [SMJ, '79; Moore, '90] (N free complex fermions $\{\bar{\psi}_j\}$, $\{\psi_k\}$)

- $c = 2\cdot N\cdot\frac{1}{2}$, current $J_{jk} = (\bar{\psi}_j\psi_k)$, $T = \frac{1}{2}\sum_k\left[(\bar{\psi}_k\,\partial\psi_k) - (\partial\bar{\psi}_k\,\psi_k)\right]$
- monodromy fields obtained by bosonization

$$\bar{\psi}_k =: e^{-i\phi_k}:, \qquad \psi_k =: e^{i\phi_k}:,$$

$$J_{jk} = \begin{cases} :e^{i(\phi_k-\phi_j)}:, & j\neq k,\\ i\,\partial\phi_k, & j=k,\end{cases} \qquad T = -\frac{1}{2}\sum_k\left(\partial\phi_k\,\partial\phi_k\right),$$

$$\mathcal{O}_{\mathcal{L}_\nu} =: e^{i\sum_k \lambda_{\nu,k}\phi_k^{(\nu)}}:.$$

- need N distinct bosonization schemes

Back to isomonodromy problem

Decompose $\mathcal{A}_\nu$'s as $\mathcal{A}_\nu = \frac{\operatorname{tr}\mathcal{A}_\nu}{N}\,\mathbf{1}_N + \hat{\mathcal{A}}_\nu$, then

$$\Phi_{\mathcal{A}}(z) = \prod_\nu \left(\frac{z-a_\nu}{z_0-a_\nu}\right)^{\frac{\operatorname{tr}\mathcal{A}_\nu}{N}} \Phi_{\hat{\mathcal{A}}}(z),$$

$$\mathcal{J}_{\mathcal{A}}(z) = \frac{1}{N}\sum_\nu \frac{\operatorname{tr}\mathcal{A}_\nu}{z-a_\nu}\,\mathbf{1}_N + \mathcal{J}_{\hat{\mathcal{A}}}(z),$$

$$\tau_{\mathcal{A}}(a) = \prod_{\mu<\nu}(a_\mu - a_\nu)^{\frac{\operatorname{tr}\mathcal{A}_\mu \operatorname{tr}\mathcal{A}_\nu}{N}}\,\tau_{\hat{\mathcal{A}}}(a).$$

Example (continued)

$$N \text{ complex fermions} = \hat{u}(1) \oplus \hat{su}(N)_1$$

Fermion and monodromy fields factorize

$$\bar{\psi}_k = :e^{-i\phi_0/\sqrt{N}}: \otimes\, \hat{\bar{\varphi}}_k, \qquad \psi_k = :e^{i\phi_0/\sqrt{N}}: \otimes\, \hat{\varphi}_k,$$

$$\mathcal{O}_{\mathcal{L}_\nu} = :e^{\frac{i\operatorname{tr}\mathcal{A}_\nu}{\sqrt{N}}\phi_0}: \otimes\, \mathcal{O}_{\hat{\mathcal{L}}_\nu}$$

- dimension $\Delta = \frac{N-1}{2N}$ of $\{\hat{\bar{\varphi}}_k\}$ and $\{\hat{\varphi}_k\}$ agrees with $c_{\hat{su}(N)_1} = N-1$
- dimension of $\mathcal{O}_{\hat{\mathcal{L}}_\nu}$ is equal to $\frac{1}{2}\operatorname{tr}\hat{\mathcal{A}}_\nu^2$
- tracelessness of $\mathcal{A}(z)$ corresponds to factoring out the $\hat{u}(1)$ piece

Conclusion: Isomonodromic tau function can be interpreted as a correlation function of primaries with dimensions $\Delta_\nu = \frac{1}{2}\operatorname{tr}\mathcal{A}_\nu^2$ in a CFT with $c = N - 1$.

Remark. For $N = 2$ the dimension $\Delta = \frac{1}{4}$ of φ's and $\bar{\varphi}$'s corresponds to states degenerate at level 2, and the dimension 1 of $\{J_{jk}\}$ is degenerate at level 3. Hence the correlators

$$\begin{aligned}\mathcal{P}_{jk} &= \langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)\bar{\varphi}_j(z_0)\varphi_k(z)\rangle,\\ \mathcal{Q}_{jk} &= \langle \mathcal{O}_{\mathcal{L}_1}(a_1)\ldots\mathcal{O}_{\mathcal{L}_n}(a_n)J_{jk}(z)\rangle,\end{aligned}$$

have to satisfy linear PDEs of order 2 and 3, fixed by Virasoro symmetry.

Proposition. Under assumption $\operatorname{tr}\mathcal{A}(z) = 0$, the matrices

$$\mathcal{P} = (z - z_0)^{-\frac{1}{2}}\tau\Phi, \qquad \mathcal{Q} = \tau\Phi^{-1}\partial_z\Phi,$$

satisfy

$$\partial_{zz}\mathcal{P} = \left\{\frac{1}{z - z_0}\,\partial_{z_0} + \frac{1}{4\,(z - z_0)^2} + \sum_\nu\left(\frac{1}{z - a_\nu}\,\partial_{a_\nu} + \frac{\Delta_\nu}{(z - a_\nu)^2}\right)\right\}\mathcal{P},$$

$$\partial_{zzz}\mathcal{Q} = \left\{4\sum_\nu\left(\frac{1}{z - a_\nu}\,\partial_{a_\nu z} + \frac{\Delta_\nu}{(z - a_\nu)^2}\,\partial_z\right) + 2\sum_\nu\left(\frac{1}{(z - a_\nu)^2}\,\partial_{a_\nu} + \frac{2\Delta_\nu}{(z - a_\nu)^3}\right)\right\}\mathcal{Q}.$$

Painlevé VI

PVI tau function is a 4-point correlator of monodromy fields,

$$\tau(t) = \langle \mathcal{O}_{\mathcal{L}_0}(0)\mathcal{O}_{\mathcal{L}_t}(t)\mathcal{O}_{\mathcal{L}_1}(1)\mathcal{O}_{\mathcal{L}_\infty}(\infty)\rangle,$$

and these fields are Virasoro primaries with dimensions $\Delta_\nu = \theta_\nu^2$ in a $c = 1$ CFT.

- "conservation of monodromy" $\Rightarrow \{\varphi_k\}$ should have monodromy $\mathcal{M}_t\mathcal{M}_0$ around all fields in the OPE of $\mathcal{O}_{\mathcal{L}_0}$ and $\mathcal{O}_{\mathcal{L}_t}$
- if $\mathcal{M}_t\mathcal{M}_0 = \mathcal{C}_{0t}^{-1}\begin{pmatrix} e^{2\pi i\sigma_{0t}} & 0 \\ 0 & e^{-2\pi i\sigma_{0t}} \end{pmatrix}\mathcal{C}_{0t}$, then it is natural to expect that the set of primaries in the OPE of $\mathcal{O}_{\mathcal{L}_0}$ and $\mathcal{O}_{\mathcal{L}_t}$ consists of an infinite number of monodromy fields $\mathcal{O}_{\mathcal{L}_{0t}^{(n)}}$ with $n \in \mathbb{Z}$ and

$$\mathcal{L}_{0t}^{(n)} = \mathcal{C}_{0t}^{-1}\begin{pmatrix} \sigma_{0t}+n & 0 \\ 0 & -\sigma_{0t}-n \end{pmatrix}\mathcal{C}_{0t}$$

- inserting the OPE $\mathcal{O}_{\mathcal{L}_0}(0)\mathcal{O}_{\mathcal{L}_t}(t)$ into the correlator gives

$$\boxed{\tau(t) = \sum_{n\in\mathbb{Z}} C_n t^{\Delta_{\sigma+n}-\Delta_0-\Delta_t}\mathcal{B}(\boldsymbol{\theta}, \sigma+n, t)}$$

Computation of conformal blocks

1. direct (inversion of the Shapovalov form)
2. recursion relation [Zamolodchikov, '84]
3. combinatorial representations, conjectured in [Alday, Gaiotto, Tachikawa, '09] and proved in [Alba, Fateev, Litvinov, Tarnopolsky, '10]

Structure constants

Jimbo's asymptotic formula can be interpreted as a recurrence relation

$$\frac{C_{n\pm1}}{C_n}=\frac{\Gamma^2\left(1\mp 2(\sigma_{0t}+n)\right)}{\Gamma^2\left(1\pm 2(\sigma_{0t}+n)\right)}\prod_{\epsilon=\pm}\frac{\Gamma\left(1+\epsilon\theta_0+\theta_t\pm(\sigma_{0t}+n)\right)\Gamma\left(1+\epsilon\theta_\infty+\theta_1\pm(\sigma_{0t}+n)\right)}{\Gamma\left(1+\epsilon\theta_0+\theta_t\mp(\sigma_{0t}+n)\right)\Gamma\left(1+\epsilon\theta_\infty+\theta_1\mp(\sigma_{0t}+n)\right)}\times$$

$$\times\frac{\left(\theta_0^2-(\theta_t\mp(\sigma_{0t}+n))^2\right)\left(\theta_\infty^2-(\theta_1\mp(\sigma_{0t}+n))^2\right)}{4\left(\sigma_{0t}+n\right)^2\left(1\pm 2(\sigma_{0t}+n)\right)^2}(-s)^{\pm1}$$

with the solution in terms of Barnes functions

$$\boxed{C_n\left(\boldsymbol{\theta},\boldsymbol{\sigma}\right)=s^n\frac{\prod_{\epsilon,\epsilon'=\pm}G\big(1+\theta_t+\epsilon\theta_0+\epsilon'(\sigma_{0t}+n)\big)\,G\big(1+\theta_1+\epsilon\theta_\infty+\epsilon'(\sigma_{0t}+n)\big)}{G\big(1+2(\sigma_{0t}+n)\big)\,G\big(1-2(\sigma_{0t}+n)\big)}}$$

Structure constants (continued)

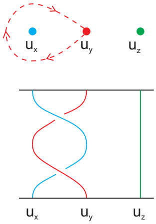
Analytic continuation induces an action of the 3-braid group on monodromy. E.g. for counterclockwise contour around 0

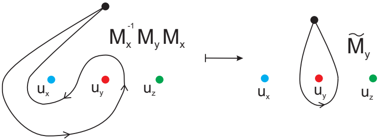
$$\tilde{\mathcal{M}}_0 = \mathcal{M}_t\mathcal{M}_0\mathcal{M}_t^{-1}, \qquad \tilde{\mathcal{M}}_t = \left(\mathcal{M}_t\mathcal{M}_0\right)\mathcal{M}_t\left(\mathcal{M}_t\mathcal{M}_0\right)^{-1}, \qquad \tilde{\mathcal{M}}_1 = \mathcal{M}_1,$$

so that $\tilde{\sigma}_{0t} = \sigma_{0t}$ and

$$\tilde{p}_{01} = \omega_{01} - p_{01} - p_{0t}p_{1t},$$
$$\tilde{p}_{1t} = \omega_{1t} - p_{1t} - p_{0t}\tilde{p}_{01}.$$

- functional relation $C_n(\boldsymbol{\theta}, \tilde{\boldsymbol{\sigma}}) = \kappa \cdot e^{4\pi i n\sigma_{0t}} C_n(\boldsymbol{\theta}, \boldsymbol{\sigma})$, with κ independent on n
- “minimal” solution given by s^n
- remaining G-factor essentially reproduces chiral part of structure constants in time-like Liouville at $c = 1$





Main claim

Complete expansion of Painlevé VI tau function near $t=0$ can be written as

$$\tau(t) = \text{const} \cdot \sum_{n\in\mathbb{Z}} C_n\left(\boldsymbol{\theta},\boldsymbol{\sigma}\right) t^{(\sigma_{0t}+n)^2-\theta_0^2-\theta_t^2}\mathcal{B}(\boldsymbol{\theta},\sigma_{0t}+n;t).$$

The function $\mathcal{B}(\boldsymbol{\theta},\sigma;t)$ is a power series in t which coincides with the general $c=1$ conformal block and is explicitly given by

$$\mathcal{B}\left(\boldsymbol{\theta},\sigma;t\right) = (1-t)^{2\theta_t\theta_1}\sum_{\lambda,\mu\in\mathbb{Y}} \mathcal{B}_{\lambda,\mu}\left(\boldsymbol{\theta},\sigma\right) t^{|\lambda|+|\mu|},$$

$$\mathcal{B}_{\lambda,\mu}\left(\boldsymbol{\theta},\sigma\right) = \prod_{(i,j)\in\lambda} \frac{\left((\theta_t+\sigma+i-j)^2-\theta_0^2\right)\left((\theta_1+\sigma+i-j)^2-\theta_\infty^2\right)}{h_\lambda^2(i,j)\left(\lambda'_j-i+\mu_i-j+1+2\sigma\right)^2}\times$$

$$\times\prod_{(i,j)\in\mu} \frac{\left((\theta_t-\sigma+i-j)^2-\theta_0^2\right)\left((\theta_1-\sigma+i-j)^2-\theta_\infty^2\right)}{h_\mu^2(i,j)\left(\mu'_j-i+\lambda_i-j+1-2\sigma\right)^2}.$$

The structure constants $\{C_n\left(\boldsymbol{\theta},\boldsymbol{\sigma}\right)\}_{n\in\mathbb{Z}}$ can be written in terms of Barnes G-function,

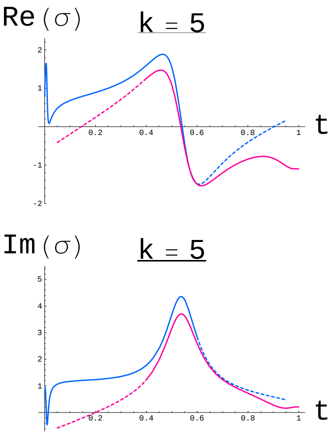
$$C_n\left(\boldsymbol{\theta},\boldsymbol{\sigma}\right) = s^n\frac{\prod_{\epsilon,\epsilon'=\pm} G\left(1+\theta_t+\epsilon\theta_0+\epsilon'(\sigma_{0t}+n)\right) G\left(1+\theta_1+\epsilon\theta_\infty+\epsilon'(\sigma_{0t}+n)\right)}{G\left(1+2(\sigma_{0t}+n)\right)G\left(1-2(\sigma_{0t}+n)\right)}$$

Remarks

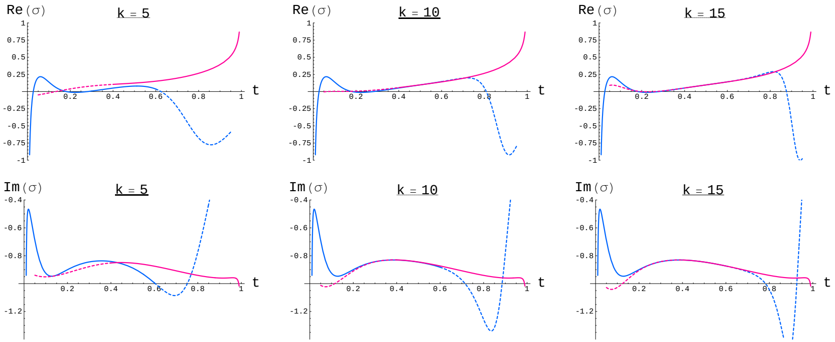
- checked about 30 first terms in the asymptotic expansion of τ (up to level 10) in full generality, more checks to follow
- to prove rigorously, it is sufficient to demonstrate two bilinear relations satisfied by $c = 1$ conformal blocks
- can easily write similar expansions for Painlevé V and III
- expansions at 1, ∞ are obtained by parameter change; for example, near $t = 1$

$$\theta_0 \leftrightarrow \theta_1, \quad \sigma_{0t} \leftrightarrow \sigma_{1t}, \qquad p'_{01} = \omega_{01} - p_{01} - p_{0t}p_{1t}.$$

- series representations suitable for numerical evaluation of PVI functions



$$\begin{pmatrix}\theta_0\\ \theta_t\\ \theta_1\\ \theta_\infty\end{pmatrix}=\begin{pmatrix}0.967351+0.498649i\\ 0.419034+0.178217i\\ 0.483339+0.242348i\\ 0.596821+0.779506i\end{pmatrix},\quad \begin{pmatrix}\sigma_{0t}\\ \sigma_{1t}\end{pmatrix}=\begin{pmatrix}0.367898+0.539999i\\ 0.313642+0.941302i\end{pmatrix}$$



$$\begin{pmatrix} \theta_0 \\ \theta_t \\ \theta_1 \\ \theta_\infty \end{pmatrix} = \begin{pmatrix} 0.501790 + 0.216884i \\ 0.382251 + 0.723641i \\ 0.152700 + 0.358959i \\ 0.158518 + 0.674992i \end{pmatrix}, \quad \begin{pmatrix} \sigma_{0t} \\ \sigma_{1t} \end{pmatrix} = \begin{pmatrix} 0.837497 + 0.943080i \\ 0.411398 + 0.480375i \end{pmatrix}$$

Painlevé III

$$\left(t\sigma''\right)^2=\left(4\left(\sigma'\right)^2-1\right)\left(\sigma-t\sigma'\right)-4\beta\gamma\sigma'+\left(\beta^2+\gamma^2\right).$$

AGT expansion of Painlevé III tau function at $t=0$ can be similarly written as

$$\tau_{\mathrm{PIII}}(t)=\sum_{n\in\mathbb{Z}}C_{\mathrm{PIII}}\left(\beta,\gamma,\sigma+n\right)s_{\mathrm{PIII}}^{n}\,t^{(\sigma+n)^2}\mathcal{B}_{\mathrm{PIII}}\left(\beta,\gamma,\sigma+n;t\right). \qquad (4)$$

Here again σ and s_{PIII} are arbitrary, conformal block $\mathcal{B}_{\mathrm{PIII}}\left(\beta,\gamma,\sigma;t\right)$ is given by

$$\mathcal{B}_{\mathrm{PIII}}\left(\beta,\gamma,\sigma;t\right)=e^{-\frac{t}{2}}\sum_{\lambda,\mu\in\mathbb{Y}}\mathcal{B}_{\lambda,\mu}^{(\mathrm{PIII})}\left(\beta,\gamma,\sigma\right)t^{|\lambda|+|\mu|},$$

$$\mathcal{B}_{\lambda,\mu}^{(\mathrm{PIII})}\left(\beta,\gamma,\sigma\right)=\prod_{(i,j)\in\lambda}\frac{(\beta+\sigma+i-j)(\gamma+\sigma+i-j)}{h_\lambda^2(i,j)\left(\lambda'_j-i+\mu_i-j+1+2\sigma\right)^2}\prod_{(i,j)\in\mu}\frac{(\beta-\sigma+i-j)(\gamma-\sigma+i-j)}{h_\mu^2(i,j)\left(\mu'_j-i+\lambda_i-j+1-2\sigma\right)^2},$$

and the structure constants can be written as

$$C_{\mathrm{PIII}}\left(\beta,\gamma,\sigma\right)=\prod_{\epsilon=\pm}\frac{G\left(1+\beta+\epsilon\sigma\right)G\left(1+\gamma+\epsilon\sigma\right)}{G\left(1+2\epsilon\sigma\right)}. \qquad (5)$$

Bäcklund transformations

	θ_0	θ_t	θ_1	θ_∞	$\mathcal{B}(\boldsymbol{\theta},\sigma;t)$	$\tau(t)$
$\mathbf{s}_0$	$-\theta_0$	θ_t	θ_1	θ_∞	$\mathcal{B}(\boldsymbol{\theta},\sigma;t)$	$\tau(t)$
$\mathbf{s}_t$	θ_0	$-\theta_t$	θ_1	θ_∞	$\mathcal{B}(\boldsymbol{\theta},\sigma;t)$	$\tau(t)$
$\mathbf{s}_1$	θ_0	θ_t	$-\theta_1$	θ_∞	$\mathcal{B}(\boldsymbol{\theta},\sigma;t)$	$\tau(t)$
$\mathbf{s}_\infty$	θ_0	θ_t	θ_1	$-\theta_\infty$	$\mathcal{B}(\boldsymbol{\theta},\sigma;t)$	$\tau(t)$
$\mathbf{s}_\delta$	$\theta_0-\delta$	$\theta_t-\delta$	$\theta_1-\delta$	$\theta_\infty-\delta$	$(1-t)^{\delta_{1t}\delta}\mathcal{B}(\boldsymbol{\theta},\sigma;t)$	$t^{\delta_{0t}\delta}(1-t)^{\delta_{1t}\delta}\tau(t)$
$\mathbf{r}_{0t}$	θ_∞	θ_1	θ_t	θ_0	$\mathcal{B}(\boldsymbol{\theta},\sigma;t)$	$t^{\Delta_{0t}}\tau(t)$
$\mathbf{r}_{1t}$	θ_t	θ_0	θ_∞	θ_1	$(1-t)^{\Delta_{1t}}\mathcal{B}(\boldsymbol{\theta},\sigma;t)$	$(1-t)^{\Delta_{1t}}\tau(t)$
$\mathbf{r}_{01}$	θ_1	θ_∞	θ_0	θ_t	$(1-t)^{\Delta_{1t}}\mathcal{B}(\boldsymbol{\theta},\sigma;t)$	$t^{\Delta_{0t}}(1-t)^{\Delta_{1t}}\tau(t)$
$\mathbf{q}_{01}$	θ_1	θ_t	θ_0	θ_∞		$\tau(1-t)$
$\mathbf{q}_{0\infty}$	θ_∞	θ_t	θ_1	θ_0		$t^{-2\Delta_t}\tau\left(t^{-1}\right)$
$\mathbf{q}_{1\infty}$	θ_0	θ_t	θ_∞	θ_1	$(1-t)^{\Delta_0-\Delta_t-\Delta_\sigma}\mathcal{B}\left(\boldsymbol{\theta},\sigma;\frac{t}{t-1}\right)$	$(1-t)^{-2\Delta_t}\tau\left(\frac{t}{t-1}\right)$

with

$$\delta_{0t}=\theta_0+\theta_t-\theta_1-\theta_\infty,\qquad \Delta_{0t}=\Delta_0+\Delta_t-\Delta_1-\Delta_\infty,$$
$$\delta_{1t}=\theta_1+\theta_t-\theta_0-\theta_\infty,\qquad \Delta_{1t}=\Delta_1+\Delta_t-\Delta_0-\Delta_\infty,$$

Riccati/Chazy solutions

- parameters satisfy

$$\begin{cases} \omega_{0t} = 2p_{0t} + p_{1t}p_{01}, \\ \omega_{1t} = 2p_{1t} + p_{0t}p_{01}, \\ \omega_{01} = 2p_{01} + p_{0t}p_{1t}. \end{cases}$$

- simplest case $\theta_0 + \theta_t + \theta_1 + \theta_\infty = 0$, $\boldsymbol{\sigma} = (\theta_0 + \theta_t, \theta_1 + \theta_t, \theta_0 + \theta_1)$:

$$\tau(t) = \mathrm{const} \cdot t^{2\theta_0\theta_t}(1-t)^{2\theta_t\theta_1}.$$

- four-point correlator $\langle \mathcal{V}_{\theta_0}(0)\mathcal{V}_{\theta_t}(t)\mathcal{V}_{\theta_1}(1)\mathcal{V}_{\theta_\infty}(\infty)\rangle$ of chiral vertex operators $\mathcal{V}_\theta(z) = : e^{i\sqrt{2}\theta\phi(z)} :$ (only one conformal block!)
  - can add screenings
  - transformation s_δ maps CBs of exponential fields (with screening insertions) to CBs with degenerate external dimensions

Riccati/Chazy solutions (continued)

More general situation: for

$$\boldsymbol{\theta}=\frac{1}{2}\left(\eta,N,N-z-z',z'-z+\eta\right),\qquad N\in\mathbb{Z}_{>0},$$

$$\boldsymbol{\sigma}=\frac{1}{2}\left(N+\eta,z+z',z'-z+N\right)\mod\mathbb{Z}$$

Painlevé VI admits a one-parameter family of solutions [Forrester, Witte, '02]

$$\tau(t)=t^{\frac{N\eta}{2}}\left(1-t\right)^{\frac{N(z+z'-N)}{2}}\det\left[f_{j-k}\right]_{j,k=1,\ldots,N}$$

$$f_\ell=\frac{\Gamma\left(1-z'\right)}{\Gamma\left(1-\ell+\eta\right)\Gamma\left(1+\ell-\eta-z'\right)}\,{}_2F_1\left(z,-\ell+\eta+z',1-\ell+\eta,t\right)+$$
$$+\frac{\xi\,\Gamma\left(1-z\right)}{\Gamma\left(1+\ell-\eta\right)\Gamma\left(1-\ell+\eta-z\right)}\,t^{\ell-\eta}\,{}_2F_1\left(z',\ell-\eta+z,1+\ell-\eta,t\right).$$

- dimension $\Delta_t=\frac{N^2}{4}$ is degenerate (level $N+1$)
- $\xi=0\Rightarrow$ single conformal block $\mathcal{B}\left(\frac{\eta}{2},\frac{N}{2},\frac{N-z-z'}{2},\frac{z'-z+\eta}{2},\frac{N+\eta}{2},t\right)$
- subsequently $\eta\to 0\Rightarrow$ Toeplitz determinant $D_N^{(z,z')}$ for z-measures

N.B. Ising lattice diagonal 2-point correlation function coincides with a particular 4-point Virasoro conformal block on the sphere

Riccati/Chazy solutions (continued)

How to recover Gessel from AGT?

- We have $\sigma_{0t} = \theta_0 + \theta_t$. But $\mathcal{B}_{\lambda,\mu}(\boldsymbol{\theta}, \sigma)$ contains the product
$$\prod_{(i,j)\in\mu} (\theta_0 + \theta_t - \sigma + i - j)$$
It vanishes for any non-empty μ as it contains $(i,j) = (1,1)$.
- remaining AGT sum over λ simplifies to
$$\sum_{\lambda\in\mathbb{Y}} t^{|\lambda|} \prod_{(i,j)\in\lambda} \frac{i-j+N}{i-j+N+\eta} \frac{(i-j+z)(i-j+z'+\eta)}{h_\lambda^2(i,j)},$$
and can be restricted to λ with $\lambda_1 \leq N$ thanks to $\theta_t - \theta_0 + \sigma_{0t} = N$
- finally letting $\eta \to 0$ we get
$$D_N^{(z,z')} = \sum_{\lambda\in\mathbb{Y},\lambda_1\leq N} t^{|\lambda|} \prod_{(i,j)\in\lambda} \frac{(i-j+z)(i-j+z')}{h_\lambda^2(i,j)}$$

Transformation $t \leftrightarrow 1-t$:

- $\xi = 0$:
$$\begin{array}{c} 1 \text{ CB at } t=0 \\ (\text{internal dimension } \sigma = \frac{N}{2}) \end{array} \longrightarrow \begin{array}{c} N+1 \text{ CBs at } t=1 \\ (\text{internal dimensions } \sigma_k = \theta_1 - \frac{N}{2} + k,\ k=0,\ldots,N) \end{array}$$
- $\xi \neq 0$: $N+1$ CBs at $t=0$ (internal dimensions $\sigma_k = \theta_0 + \frac{N}{2} - k$, $k = 0,\ldots,N$)

Algebraic solutions

- correspond to finite orbits of braid group action on monodromy [Dubrovin, Mazzocco, '98]
- classification [O.L., Tykhyy, '08]:
  - 3 continuous families
  - 45 exceptional classes ($\boldsymbol{\theta}, \boldsymbol{\sigma}$ rational)

Example: $\boldsymbol{\theta} = \left(2a, a, a, \frac{1}{3}\right)$, $\boldsymbol{\sigma} = \left(3a, \frac{1}{6}, \frac{1}{4}\right)$

$$\tau(t(s)) = \mathrm{const} \cdot (s-2)^{4a^2}(s-1)^{2a^2-\frac{7}{72}}(s+1)^{-10a^2+\frac{1}{8}}(s+2)^{10a^2-\frac{1}{9}},$$

$$t(s) = \frac{(s+1)^2(s-2)}{(s-1)^2(s+2)}$$

$s \in (2, \infty)$ maps to $t \in (0,1)$, and only one conformal block contributes to the corresponding expansion at $t = 0$:

$$\mathcal{B}\left(2a, a, a, \frac{1}{3}, 3a; t\right) = (s_t - 1)^{10a^2-\frac{7}{72}}\left(\frac{s_t+1}{3}\right)^{-18a^2+\frac{1}{8}}\left(\frac{s_t+2}{4}\right)^{14a^2-\frac{1}{9}},$$

with $s_t = \dfrac{(1+\sqrt{t})^{\frac{2}{3}} + (1-\sqrt{t})^{\frac{2}{3}}}{(1-t)^{\frac{1}{3}}}$.

Picard solutions

- $\omega_{0t} = \omega_{1t} = \omega_{01} = \omega_4 = 0$
- parameters can be Backlund transformed to $\boldsymbol{\theta}_{\text{Picard}} = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$
- dimensions $\Delta_\nu = \theta_\nu^2$ correspond to Ashkin-Teller conformal block [Zamolodchikov, '86]

$$\mathcal{B}\left(\boldsymbol{\theta}_{\text{Picard}}, \sigma; t\right) = \frac{\left(16t^{-1}q\right)^{\sigma^2}}{(1-t)^{\frac{1}{8}}\vartheta_3(0|\tau)}$$

where $q = e^{i\pi\tau}$, $\tau = \dfrac{iK'(t)}{K(t)}$ and

$$K(t) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-tx^2)}}, \qquad K'(t) = K(1-t).$$

- structure constants C_n and parameter s simplify to

$$C_n \sim 2^{-4(\sigma_{0t}+n)^2}(-s)^n, \qquad s = -e^{\pm 2\pi i \sigma_{1t}}$$

- conformal expansion $\tau(t) = \sum_{n\in\mathbb{Z}} C_n t^{(\sigma_{0t}+n)^2-\theta_0^2-\theta_t^2}\mathcal{B}(\boldsymbol{\theta}, \sigma_{0t}+n, t)$ then gives theta function series so that finally

$$\tau_{\text{Picard}}(t) = \text{const}\cdot\frac{q^{\sigma_{0t}^2}}{t^{\frac{1}{8}}(1-t)^{\frac{1}{8}}}\frac{\vartheta_3\left(\sigma_{0t}\pi\tau \pm \sigma_{1t}\pi|\tau\right)}{\vartheta_3(0|\tau)}.$$

(this indeed coincides with Picard tau function [Kitaev, Korotkin, '98])

Sine-Gordon twist fields

Two-point correlator $Q(mr) = \langle \mathcal{O}_{\nu}(0)\, \mathcal{O}_{\nu'}(r) \rangle$ is a tau function of Painlevé III

- two singular points $r = 0, \infty$
- form factor expansions as $r \to \infty$
- conformal perturbation theory as $r \to 0$

AGT series:

$$Q(mr) = \sum_{n\in\mathbb{Z}} C_{\text{SG}}(\nu + n, \nu' + n) \left(\frac{m^2 r^2}{4}\right)^{(\nu+n)(\nu'+n)} \sum_{\lambda,\mu\in\mathbb{Y}} \mathcal{B}^{\text{SG}}_{\lambda,\mu}(\nu + n, \nu' + n) \left(\frac{m^2 r^2}{4}\right)^{|\lambda|+|\mu|}$$

with

$$C_{\text{SG}}(\nu, \nu') = G\left[\begin{array}{c} 1+\nu, 1-\nu, 1+\nu', 1-\nu' \\ 1+\nu+\nu', 1-\nu-\nu' \end{array}\right]$$

$$\mathcal{B}^{\text{SG}}_{\lambda,\mu}(\nu, \nu') = \prod_{(i,j)\in\lambda} \frac{(i-j+\nu)(i-j+\nu')}{h^2_{\lambda}(i,j)\left(\lambda'_j - i + \mu_i - j + 1 + \nu + \nu'\right)^2} \prod_{(i,j)\in\mu} \frac{(i-j-\nu)(i-j-\nu')}{h^2_{\mu}(i,j)\left(\mu'_j - i + \lambda_i - j + 1 - \nu - \nu'\right)^2}$$

N.B. Holomorphic conformal blocks describe CPT expansion of massive theory!

GUE scaled gap probability in the bulk

The function

$$\mathcal{G}(t) = \det\left(1 - K_{\text{sine}}\big|_{(0,t)}\right), \qquad t \in (0,\infty)$$

$$K_{\text{sine}}(x,y) = \frac{\sin\frac{x-y}{2}}{\pi(x-y)}$$

is a tau function of Painlevé V.

AGT expansion:

$$\mathcal{G}(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2\pi)^n} \frac{G^6(1+n)}{G^2(1+2n)}\, t^{n^2} \sum_{\lambda,\mu\in\mathbb{Y}\,|\,\lambda_1,\mu'_1\leq n} \mathcal{B}^{\text{sine}}_{\lambda,\mu}(n)\,(it)^{|\lambda|+|\mu|},$$

with

$$\mathcal{B}^{\text{sine}}_{\lambda,\mu}(n) = \prod_{(i,j)\in\lambda} \frac{(i-j+n)^3}{h_\lambda^2(i,j)\left(\lambda'_j - i + \mu_i - j + 1 + 2n\right)^2} \prod_{(i,j)\in\mu} \frac{(i-j-n)^3}{h_\mu^2(i,j)\left(\mu'_j - i + \lambda_i - j + 1 - 2n\right)^2}$$

Conclusions

1. Painlevé VI tau function is a generating function of $c = 1$ conformal blocks
2. AGT combinatorial formulas provide a series representation for general solution of Painlevé VI and an efficient tool of numerical computation
3. special solutions and Bäcklund transformations have natural CFT interpretation

More questions

1. increase rank/genus/number of singular points
2. connection problem for $\tau(t)$/Racah-Wigner coefficients for Virasoro
3. β-deformed Painlevé VI?