Connection problems on higher order linear q-difference equations

Takeshi MORITA*

In this talk, we consider connection problems on higher(r-th) order linear q-difference equations between around the origin and around the infinity. For the sake of simplicity, we deal with the connection problems on the third order (namely, $r = 3$) linear q-difference equations.

We study the following "degenerated" third order q-difference equation:

$$\begin{aligned}&\left[\left(a_1a_2a_3x - b_1/q^2\right)\sigma_q^3 - \left\{(a_1a_2 + a_2a_3 + a_3a_1)x - (b_1/q^2 + 1/q)\right\}\sigma_q^2\right.\\&\left.+\left\{(a_1 + a_2 + a_3)x - 1/q\right\}\sigma_q - x\right]u(x) = 0.\end{aligned} \tag{1}$$

The equation (1) has the formal solution around the origin as follows:

$$u_1(x) = {}_3\varphi_1(a_1, a_2, a_3; b_1; q, x) = \sum_{n\geq 0}\frac{(a_1, a_2, a_3; q)_n}{(b_1; q)_n(q; q)_n}\left\{(-1)q^{\frac{n(n-1)}{2}}\right\}^{-1}x^n.$$

Here, σ_q is the q-shifted operator $\sigma_q f(x) = f(qx)$, the q-shifted factorial $(a; q)_n$ is

$$(a; q)_n := \begin{cases}1, & n = 0,\\(1 - a)(1 - aq)\dots(1 - aq^{n-1}), & n \geq 1,\end{cases}$$

moreover, $(a; q)_\infty := \lim_{n\to\infty}(a; q)_n$ and

$$(a_1, a_2, \dots, a_m; q)_\infty := (a_1; q)_\infty(a_2; q)_\infty \dots (a_m; q)_\infty.$$

The notation ${}_r\varphi_s(a_1, \dots, a_r; b_1, \dots, b_s; q, x)$ is the basic hypergeometric series [2]. The equation (1) also has the fundamental system of solutions around the infinity as follow:

$$\begin{aligned}v_1(x) &= x^{-\alpha_1}{}_3\varphi_2(a_1, a_1q/b_1, 0; a_1q/a_2, a_1q/a_3; q, qb_1/a_1a_2a_3x),\\v_2(x) &= x^{-\alpha_2}{}_3\varphi_2(a_2, a_2q/b_1, 0; a_2q/a_1, a_2q/a_3; q, qb_1/a_1a_2a_3x),\\v_3(x) &= x^{-\alpha_3}{}_3\varphi_2(a_3, a_3q/b_1, 0; a_3q/a_2, a_3q/a_1; q, qb_1/a_1a_2a_3x),\end{aligned}$$

*Graduate School of Information Science and Technology, Osaka University, 1-1 Machikaneyama-machi, Toyonaka, 560-0043, Japan.

where $a_j = q^{\alpha_j}, j = 1, 2, 3$. The relation between the solution around the originu_1 and solutions around the infinity v_1, v_2 and $v_3(x)$ is not so clear. In this talk, we give the relation with the using of the *suitable* resummation method.

We study connection problems on linear q-difference equations with irregular singular points. Connection problems on second order linear q-difference equations between the origin and the infinity are studied by G. D. Birkhoff [1]. The first example of connection formula was given by G. N. Watson (This formula is known as "Watson's formula for the basic hypergeometric series ${}_2\varphi_1(a, b; c; q, x)$") [7] as follows:

$$
\begin{aligned}
{}_2\varphi_1\left(a, b; c; q; x\right) = & \frac{(b, c/a; q)_\infty (ax, q/ax; q)_\infty}{(c, b/a; q)_\infty (x, q/x; q)_\infty} {}_2\varphi_1\left(a, aq/c; aq/b; q, cq/abx\right) \\
& + \frac{(a, c/b; q)_\infty (bx, q/bx; q)_\infty}{(c, a/b; q)_\infty (x, q/x; q)_\infty} {}_2\varphi_1\left(b, bq/c; bq/a; q, cq/abx\right).
\end{aligned}
$$

However, other connection formulae have not known for a long time. At the beginning of 21st century, C. Zhang has shown some connection formulae of solutions of q-difference equations with irregular singular points [8, 9, 10]by the q-Borel-Laplace methods. In connection problems on q-difference equations, two different types of "the q-Borel-Laprace resummation methods" are powerful tools. We review these transformations.

The q-Borel and the q-Laplace transformations of the first kind:

Definition 1. *For any $f(x) = \sum_{n\geq 0} a_n x^n$, the q-Borel transformation $\mathcal{B}_q^+$ is*

$$
\left(\mathcal{B}_q^+ f\right)(\xi) = \varphi(\xi) := \sum_{n\geq 0} a_n q^{\frac{n(n-1)}{2}} \xi^n,
$$

the q-Laplace transformation $\mathcal{L}_{q,\lambda}^+$ is

$$
\left(\mathcal{L}_{q,\lambda}^+ \varphi\right)(x) := \sum_{n\in\mathbb{Z}} \frac{\varphi(q^n\lambda)}{\theta\left(\frac{q^n\lambda}{x}\right)}.
$$

The q-Borel and the q-Laplace transformations of the second kind:

Definition 2. *For $f(x) = \sum_{n\geq 0} a_n x^n$, the q-Borel transformation is defined by*

$$
g(\xi) = \left(\mathcal{B}_q^- f\right)(\xi) := \sum_{n\geq 0} a_n q^{-\frac{n(n-1)}{2}} \xi^n,
$$

and the q-Laplace transformation is given by

$$\left(\mathcal{L}_q^- g\right)(x) := \frac{1}{2\pi i}\int_{|\xi|=r} g(\xi)\theta_q\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi}.$$

Here, $r > 0$ is an enough small number.

These transformations has introduced by J. Sauloy [6]. Recently, I gave connection formulae of the Hahn-Exton q-Bessel function, the q-confluent hypergeometric series, q-Airy function and the divergent series which is related to the Ramanujan function [4, 5, 3] by the using of the q-Borel-Laplace methods. These functions are solutions of second order q-difference equations with special parameters.

But connection formulae of higher order q-difference equations are not clear. In this talk, we give the following theorem with using the q-Borel-Laplace resummation method:

Theorem. For any $x \in \mathbb{C}^* \setminus [-\lambda; q]$, we have

$$\begin{aligned}
&{}_3\tilde{\varphi}_1(a_1, a_2, a_3; b_1; q, \lambda, x) = \mathcal{L}^+_{q,\lambda} \circ \mathcal{B}^+_q\, {}_3\varphi_1(a_1, a_2, a_3; b_1; q, x)\\
&= \frac{(a_2, a_3, b_1/a_1; q)_\infty}{(b_1, a_2/a_1, a_3/a_1; q)_\infty}\frac{\theta(a_1\lambda)}{\theta(\lambda)}\frac{\theta(a_1qx/\lambda)\theta(x)}{\theta(qx/\lambda)\theta(a_1x)}v_1(x)\\
&+ \frac{(a_1, a_3, b_1/a_2; q)_\infty}{(b_1, a_1/a_2, a_3/a_2; q)_\infty}\frac{\theta(a_2\lambda)}{\theta(\lambda)}\frac{\theta(a_2qx/\lambda)\theta(x)}{\theta(qx/\lambda)\theta(a_2x)}v_2(x)\\
&+ \frac{(a_2, a_1, b_1/a_3; q)_\infty}{(b_1, a_2/a_3, a_1/a_3; q)_\infty}\frac{\theta(a_3\lambda)}{\theta(\lambda)}\frac{\theta(a_3qx/\lambda)\theta(x)}{\theta(qx/\lambda)\theta(a_3x)}v_3(x).
\end{aligned}$$

Here, the notation ${}_3\tilde{\varphi}_1(a_1, a_2, a_3; b_1; q, \lambda, x)$ is the q-Borel-Laplace transform of the *divergent* solution $u_1(x)$. The new parameter λ appears in the connection coefficients. We also give the connection matrix for the equation (1). We remark that these coefficients are also new examples of the q-Stokes coefficients.

References

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