Connection problems on higher order linear q-difference equations

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1. Notations

Assumptions: A complex number $q \in \mathbb{C}^*$ is $0 < |q| < 1$.
The q-shifted operator σ_q: $\sigma_q f(x) = f(qx)$.

The basic hypergeometric series with the base q:

$$
{}_r\varphi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q, x) := \sum_{n \geq 0} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s; q)_n (q; q)_n} \left\{ (-1)^n q^{\frac{n(n-1)}{2}} \right\}^{1+s-r} x^n.
$$

The q-shifted factorial $(a; q)_n$

$$
(a; q)_n := \begin{cases} 1, & n = 0, \\ (1-a)(1-aq)\ldots(1-aq^{n-1}), & n \geq 1, \end{cases}
$$

moreover, $(a; q)_\infty := \lim_{n \to \infty} (a; q)_n$ and

$$
(a_1, a_2, \ldots, a_m; q)_\infty := (a_1; q)_\infty (a_2; q)_\infty \ldots (a_m; q)_\infty.
$$

Radius of convergence:
$\infty, 1$ or 0 according to whether $r - s < 1, r - s = 1$ or $r - s > 1$.

The theta function of Jacobi:

$$\theta_q(x) := \sum_{n\in\mathbb{Z}} q^{\frac{n(n-1)}{2}} x^n, \qquad \forall x \in \mathbb{C}^*.$$

Properties of the theta function:

1. Jacobi's triple product identity is

$$\theta_q(x) = \left(q, -x, -\frac{q}{x}; q\right)_\infty .$$

2. The q-difference equation which the theta function satisfies;

$$\theta_q(q^k x) = q^{-\frac{n(n-1)}{2}} x^{-k} \theta_q(x), \quad \forall k \in \mathbb{Z}.$$

3. The inversion formula;

$$\theta_q\left(\frac{1}{x}\right) = \frac{1}{x}\theta_q(x).$$

$[\lambda; q]$-spiral: For any fixed $\lambda \in \mathbb{C}^* \setminus q^{\mathbb{Z}}$, the set $[\lambda; q]$-spiral is

$$[\lambda; q] := \lambda q^{\mathbb{Z}} = \{\lambda q^k; k \in \mathbb{Z}\}.$$

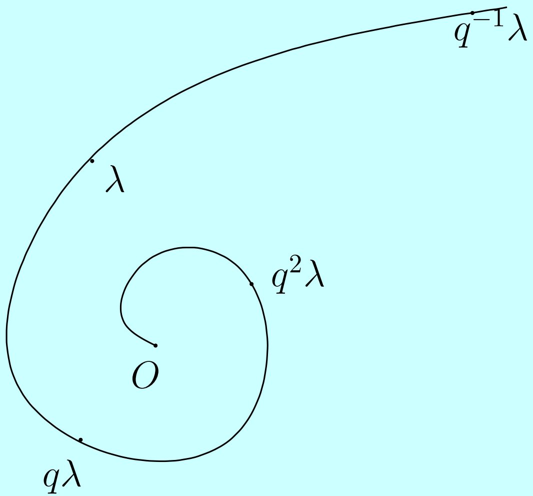


Figure 1. $[\lambda; q]$ – spiral

Relation between the theta function and $[\lambda; q]$-spiral:

Lemma 1. *We have*

$$\theta(\lambda q^k / x) = 0 \overset{iff}{\Longleftrightarrow} x \in [-\lambda; q].$$

2. Linear q-difference equation of the Laplace type

The q-difference equation of the Laplace type:

$$\left\{(a_1x+b_1)\sigma_q^2+(a_2x+b_2)\sigma_q+(a_3x+b_3)\right\}u(x)=0$$

6 parameters: a_1, a_2, a_3, b_1, b_2 and b_3.

By the gauge transformations, we obtain **3 parameters equation:**

$$\left[(c-abqx)\sigma_q^2-\left\{(c+q)-(a+b)qx\right\}\sigma_q+q(1-x)\right]u(x)=0.$$

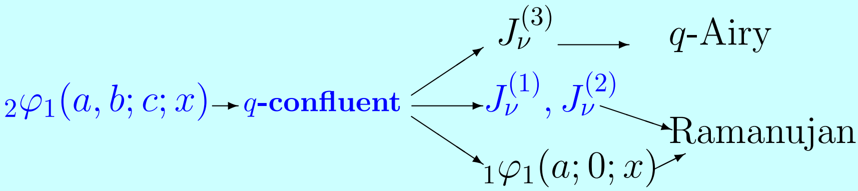
A three parameters solution:

$$u(x)={}_2\varphi_1(a,b;c;q,x)=\sum_{n\geq 0}\frac{(a,b;q)_n}{(c;q)_n(q;q)_n}x^n,$$

which is called **Heine's basic hypergeometric series**.

The degeneration diagram

The **degeneration diagram** for ${}_2\varphi_1(a,b;c;q,x)$[Y. Ohyama, 2011]:



- $J_\nu^{(k)}(k=1,2,3)$ are q-**Bessel functions**.

- The q-**Airy function** and **the Ramanujan function** $A_q(x)$ are q-analogues of the Airy function (Kajiwara, et al., 2004; Ismail, 2005).

- The function ${}_1\varphi_1(a;0;q,x) = \sum_{n\geq 0} \frac{(a;q)_n}{(q;q)_n} \left\{(-1)^n q^{\frac{n(n-1)}{2}}\right\} x^n$ is called **the** q-**Hermite function**. The q-Hermite function ${}_1\varphi_1(a;0;q,x)$ satisfies:

$$\left[-aqx\sigma_q^2 - (q-qx)\sigma_q + q\right] u(x) = 0.$$

Three different q-analogues of the Bessel function:

$$J_\nu^{(1)}(x;q) := \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\left(\frac{x}{2}\right)^\nu {}_2\varphi_1\left(0,0;q^{\nu+1};q,-\frac{x^2}{4}\right), \quad |x|<2,$$

$$J_\nu^{(2)}(x;q) := \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\left(\frac{x}{2}\right)^\nu {}_0\varphi_1\left(-;q^{\nu+1};q,-\frac{q^{\nu-1}x^2}{4}\right), \quad x\in\mathbb{C},$$

$$J_\nu^{(3)}(x;q) := \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} x^\nu {}_1\varphi_1\left(0;q^{\nu+1};q,qx^2\right), \quad x\in\mathbb{C},$$

provided that $\nu \notin \mathbb{Z}$.

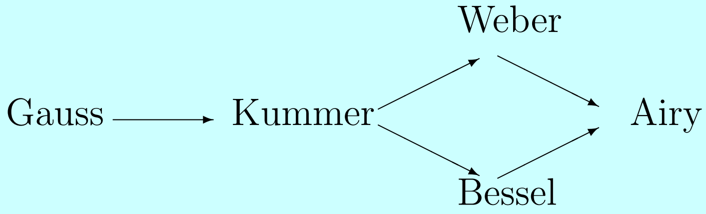
q-difference equations:

$$J_\nu^{(1)}: \quad u(xq) - (q^{\nu/2}+q^{-\nu/2})u(xq^{1/2}) + \left(1+\frac{x^2}{4}\right)u(x) = 0,$$

$$J_\nu^{(2)}: \quad \left(1+\frac{qx^2}{4}\right)u(xq) - (q^{\nu/2}+q^{-\nu/2})u(xq^{1/2}) + u(x) = 0,$$

$$J_\nu^{(3)}: \quad u(xq) - \left\{(q^{\nu/2}+q^{-\nu/2}) - q^{-\nu/2+1}x^2\right\}u(xq^{1/2}) + u(x) = 0.$$

This diagram is a q-analogue of the degeneration diagram for **Gauss' hypergeometric series** $_2F_1$:



Remark. There exist **three different types of q-Bessel functions** $J_\nu^{(j)}, j = 1, 2, 3$ and two **q-analogues of the Airy function**.

Remark. $A_q(x)$ is found by Ramanujan in "**the Lost notebook**"

3.Connection problems on second order linear q-difference equations

G. D. Birkhoff (1914) Connection formulae of second order linear q-difference equations are linear relations in a **matrix form**:

$$\begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} = \begin{pmatrix} C_{11}(x) & C_{12}(x) \\ C_{21}(x) & C_{22}(x) \end{pmatrix} \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix}.$$

$u_1(x)$ **and** $u_2(x)$:solutions **around the origin**
$v_1(x)$ **and** $v_2(x)$:solutions **around the infinity**

Functions C_{ij} $(1 \leq i, j \leq 2)$ are **elliptic functions**:

$$\sigma_q C_{ij}(x) = C_{ij}(x), \quad C_{ij}(e^{2\pi i}x) = C_{ij}(x).$$

4.The first example of the connection matrix

Connection matrix for Heine's ${}_2\varphi_1(a,b;c;q,x)$: Watson's formula

Heine's equation

$$\left[(c-abqx)\sigma_q^2-\{(c+q)-(a+b)qx\}\,\sigma_q+q(1-x)\right]u(x)=0.$$

Local solutions around the origin

$$u_1(x)={}_2\varphi_1(a,b;c;q,x),\quad u_2(x)=\frac{\theta(cx)}{\theta(qx)}{}_2\varphi_1\left(\frac{aq}{c}\frac{bq}{c};\frac{q^2}{c};q,x\right).$$

Local solutions around the infinity

$$y_\infty^{(a,b)}(x)=\frac{\theta(-ax)}{\theta(-x)}{}_2\varphi_1\left(a,\frac{aq}{c};\frac{aq}{b};q,\frac{cq}{abx}\right)$$

and

$$y_\infty^{(b,a)}(x)=\frac{\theta(-bx)}{\theta(-x)}{}_2\varphi_1\left(b,\frac{bq}{c};\frac{bq}{a};q,\frac{cq}{abx}\right).$$

Connection matrix for Heine's equation

$$\begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22}(x) \end{pmatrix} \begin{pmatrix} y_\infty^{(a,b)}(x) \\ y_\infty^{(b,a)}(x) \end{pmatrix},$$

provided that

$$C_{11} = \frac{(b, c/a; q)_\infty}{(c, b/a; q)_\infty}, \quad C_{12} = \frac{(a, c/b; q)_\infty}{(c, a/b; q)_\infty},$$

$$C_{21} = \frac{(bq/c, q/a; q)_\infty}{(q^2/c, b/a; q)_\infty}$$

and

$$C_{22}(x) = \frac{(aq/c, q/b; q)_\infty}{(q^2/c, a/b; q)_\infty} \frac{\theta(-ax)}{\theta\left(-\frac{aq}{c}x\right)} \frac{\theta\left(-\frac{bq}{c}x\right)}{\theta(-bx)}.$$

Remark. C_{11}, C_{12} and C_{21} are constant and $C_{22}(x)$ is a q-elliptic function.

Remark. The first formula has given by **G. N. Watson (1910)**.

But another connection formula have not known for a long time.

We need some *suitable resummation methods* to obtain new connection formulae:

5. The q-Borel-Laplace transformations

We assume that $f(x) = \sum_{n\geq 0} a_n x^n$, $a_0 = 1$.

5.1. The q-Borel-Laplace transformations of the first kind

1. The q-Borel transformation of the first kind is

$$\left(\mathcal{B}_q^+ f\right)(\xi) := \sum_{n\geq 0} a_n q^{\frac{n(n-1)}{2}} \xi^n \left(=: \varphi(\xi)\right).$$

2. The q-Laplace transformation of the first kind is

$$\left(\mathcal{L}_{q,\lambda}^+ \varphi\right)(x) := \frac{1}{1-q} \int_0^{\lambda\infty} \frac{\varphi(\xi)}{\theta_q\left(\frac{\xi}{x}\right)} \frac{d_q \xi}{\xi} = \sum_{n\in\mathbb{Z}} \frac{\varphi(\lambda q^n)}{\theta_q\left(\frac{\lambda q^n}{x}\right)},$$

here, this transformation is given by Jackson's q-integral.

5.2. The q-Borel-Laplace transformations of the second kind

1. The q-Borel transformation of the second kind is

$$(\mathcal{B}_q^- f)(\xi) := \sum_{n\geq 0} a_n q^{-\frac{n(n-1)}{2}} \xi^n \,(=: g(\xi)) \,.$$

2. The q-Laplace transformation of the second kind is

$$\left(\mathcal{L}_q^- g\right)(x) := \frac{1}{2\pi i} \int_{|\xi|=r} g(\xi)\theta_q \left(\frac{x}{\xi}\right) \frac{d\xi}{\xi},$$

   where $r > 0$ is enough small number.

Remark. These resummation methods are introduced by **J. Sauloy** and studied by **C. Zhang**.

Remark. The q-Borel transformation is **the formal inverse** of the q-Laplace transformation:

The q-Borel transformation $\mathcal{B}_q^+$ is formal inverse of the q-Laplace transformation $\mathcal{L}_{q,\lambda}^+$:

Lemma 2. *For any entire function $f(x)$, we have*

$$\mathcal{L}_{q,\lambda}^+ \circ \mathcal{B}_q^+ f = f.$$

The q-Borel transformation $\mathcal{B}_q^-$ also can be considered as a formal inverse of the q-Laplace transformation $\mathcal{L}_q^-$.

Lemma 3. *We assume that the function f can be q-Borel transformed to the analytic function $g(\xi)$ around $\xi = 0$. Then, we have*

$$\mathcal{L}_q^- \circ \mathcal{B}_q^- f = f.$$

6. The q-Stokes phenomenon

In the **differential case**, the resummation of divergent series are convergent on **Sectors** and the Stokes coefficient changes its value on each sector **discretely**.

In **q-difference case**, the resummation converges on the set $\mathbb{C}^* \setminus [-\lambda; q]$. Since $\mathbb{C}^*/q^{\mathbb{Z}} \cong \mathbb{T}$, the q-Stokes coefficient changes its value **continuously**.

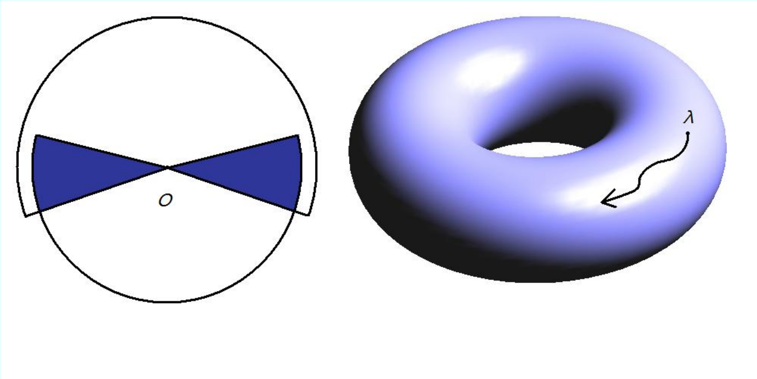


figure: Stokes sectors and q-Stokes regions

7. Examples of connection formulae

7.1. Connection matrix for the q-confluent hypergeometric series

q-confluent hypergeometric equation

$$(1-abqx)u(xq^2)-\{1-(a+b)qx\}\,u(xq)-qxu(x)=0.$$

Local solutions around the origin

$$u_1(x)={}_2\varphi_0(a,b;-;q,x),$$
$$u_2(x)=\frac{(abx;q)_\infty}{\theta(-qx)}{}_2\varphi_1\left(\frac{q}{a},\frac{q}{b};0;q,abx\right)$$

Local solutions around the infinity

$$S_\mu(a,b;q,x)=\frac{\theta(a\mu x)}{\theta(\mu x)}{}_2\varphi_1\left(a,0;\frac{aq}{b};q,\frac{q}{abx}\right),$$
$$S_\mu(b,a;q,x)=\frac{\theta(b\mu x)}{\theta(\mu x)}{}_2\varphi_1\left(b,0;\frac{bq}{a};q,\frac{q}{abx}\right),$$

Connection matrix for q-confluent equation

Theorem. *For any* $\lambda, \mu \in \mathbb{C}^*$, $x \in \mathbb{C}^* \setminus [1;q] \cup [-\mu/a;q] \cup [-\lambda;q]$, *we have*

$$\begin{pmatrix} {}_2f_0(a,b;\lambda,q,x) \\ {}_2f_1(a,b;q,x) \end{pmatrix} = \begin{pmatrix} C_\mu^\lambda(a,b;q,x) & C_\mu^\lambda(b,a;q,x) \\ C_\mu(a,b;q,x) & C_\mu(b,a;q,x) \end{pmatrix} \begin{pmatrix} S_\mu(a,b;q,x) \\ S_\mu(b,a;q,x) \end{pmatrix}.$$

- The set $[\lambda;q]$ is **the q-spiral.**
- ${}_2f_0(a,b;\lambda,q,x)$ is the q-Borel-Laplace transform (of the first kind) of ${}_2\varphi_0(a,b;-;q,x)$ (given by C. Zhang).
- ${}_2f_1(a,b;q,x)$ is the q-Borel-Laplace transform (of the second kind) of ${}_2\varphi_1(a,b;0;q,x)$ (Morita).
- $S_\mu(a,b;q,x)$ is the solution of around the infinity.
- $C_\mu^\lambda(a,b;q,x)$ and $C_\mu(a,b;q,x)$ are elliptic functions.

7.2. Connection matrix of the q-Bessel function $J_\nu^{(1)}(x;q)$

The first q-Bessel equation

$$u(qx) - \left(q^{\nu/2} + q^{-\nu/2}\right) u(q^{1/2}x) + \left(1 + \frac{x^2}{4}\right) u(x) = 0.$$

Local solutions around the origin

$$u_1(x) = J_\nu^{(1)}(x;q) := \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \left(\frac{x}{2}\right)^\nu {}_2\varphi_1\left(0,0;q^{\nu+1};q,-\frac{x^2}{4}\right),$$

$$u_2(x) = J_{-\nu}^{(1)}(q^\nu x;q)$$

where $|x| < 2$.

Local solutions around the infinity

$$v_1(x;\nu) = \frac{\left(\frac{\alpha}{\sqrt{p}x};p\right)_\infty}{\theta_p\left(-\frac{\alpha}{x}\right)} {}_2\varphi_1\left(p^{\nu+\frac{1}{2}}, p^{-\nu+\frac{1}{2}}; -p; p, \frac{\alpha}{\sqrt{p}x}\right)$$

$$v_2(x;\nu) = v_1(q^\nu x; -\nu)$$

Connection matrix of the q-Bessel function

Theorem 1. *(C. Zhang) For any* $\lambda \in \mathbb{C}^*$, $x \in \mathbb{C}^*(0 < |x| < 2)$, *we have*

$$\begin{pmatrix} j^{(1)}_{\nu,\alpha}(t;q) \\ j^{(1)}_{\nu,-\alpha}(t;q) \end{pmatrix} = \begin{pmatrix} C_{\nu,\alpha}(\lambda,t;q) & C_{-\nu,\alpha}(\lambda,t;q) \\ C_{\nu,-\alpha}(\lambda,t;q) & C_{-\nu,-\alpha}(\lambda,t;q) \end{pmatrix} \begin{pmatrix} J^{(1)}_{\nu,\lambda}(x;q) \\ J^{(1)}_{-\nu,\lambda}(x;q) \end{pmatrix}, \qquad (1)$$

where $xt = 1$.

Remark. $j^{(1)}_{\nu,\alpha}(x;q)$ is the solution of the q-Bessel equation around the infinity.

Remark. **Connection formula of the Hahn-Exton q-Bessel function** has given in **SIGMA,7 (2011), 115** (Morita).

7.3 Connection matrix for the Ramanujan equation

The Ramanujan equation:

$$\left(qx\sigma_q^2 - \sigma_q + 1\right) u(x) = 0.$$

A fundamental system of solutions around the origin:

$$u_1(x) = \mathrm{A}_q(x) := \sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n}(-x)^n,$$

$$u_2(x) = \theta(x)\,{}_2\varphi_0(0,0;-;q,-x/q).$$

A fundamental system of solutions around the infinity:

$$v_1(x) = \frac{\theta_q(x)}{\theta_{q^2}(x)}\,{}_1\varphi_1(0;q;q^2,q^2/x),$$

$$v_2(x) = \frac{-q}{1-q}\frac{\theta_q(x/q)}{\theta_{q^2}(x/q)}\frac{1}{x}\,{}_1\varphi_1(0;q^3;q^2,q^3/x).$$

Remark. $u_2(x)$ is **divergent**. $u_1(x)$, $v_1(x)$ nd $v_2(x)$ are **convergent**.
We should define a **resummation** of $u_2(x)$.

The connection formula of the Ramanujan q-difference equation is given by as follows.
For any $x \in \mathbb{C}^*$, we have (Ismail-Zhang, 2007)

$$u_1(x) = \frac{\theta_{q^2}(qx)\theta_{q^2}(x)}{(q,q^2;q^2)_\infty \theta_q(x)} v_1(x) + \frac{\theta_{q^2}(x)\theta_{q^2}\left(\frac{x}{q}\right)}{(q,q^2;q^2)_\infty \theta_q\left(\frac{x}{q}\right)} v_2(x).$$

For any $x \in \mathbb{C}^* \setminus [-\lambda; q]$, we have (Morita, `arXiv:1203.3404`)

$$\tilde{u}_2(x,\lambda) = \frac{(q;q)_\infty \theta_{q^2}\left(-\frac{qx}{\lambda^2}\right)}{\theta_q\left(-\frac{q}{\lambda}\right)\theta_q\left(\frac{x}{\lambda}\right)} \frac{\theta_{q^2}(x)}{\theta_q(x)} v_1(x) + \frac{(q;q)_\infty \theta_{q^2}\left(-\frac{x}{\lambda^2}\right)}{\theta_q\left(-\frac{1}{\lambda}\right)\theta_q\left(\frac{x}{\lambda}\right)} \frac{\theta_{q^2}\left(\frac{x}{q}\right)}{\theta_q\left(\frac{x}{q}\right)} v_2(x).$$

Remark. The connection formula of u_1 is known as an asymptotic formula of the Ramanujan function.