STUDIA MATHEMATICA

Online First version

Intertwinings, second-order Brascamp–Lieb inequalities and spectral estimates

by

Michel Bonnefont (Bordeaux) and Aldéric Joulin (Toulouse)

Abstract. We continue to explore the consequences of the so-called intertwinings between gradients and Markov diffusion operators on $\mathbb{R}^d$ in terms of Brascamp–Lieb type inequalities for log-concave distributions and beyond, extending our inequalities established in a previous paper. First, we identify the extremal functions in the so-called generalized Brascamp–Lieb inequalities, an issue left open in our previous work. Moreover, we derive new generalized Brascamp–Lieb inequalities of second order from which some new lower bounds on the $(d+1)$th positive eigenvalue of the associated Markov diffusion operator are deduced. We apply our spectral results to perturbed product probability measures, freeing us from Helffer's classical method based on uniform spectral estimates for the one-dimensional conditional distributions. In particular we exhibit new examples involving some non-classical nearest-neighbour interactions, for which our spectral estimates turn out to be dimension-free.

1. Introduction. Continuing our previous study [1], the purpose of these notes is to further explore the consequences in terms of spectral-type functional inequalities of the so-called intertwinings between gradients and Markov diffusion operators of the form

$$Lf := \Delta f - (\nabla V)^T \nabla f.$$

Here V is some nice potential on $\mathbb{R}^d$ such that the measure μ with Lebesgue density proportional to e^{-V} is the unique invariant probability measure. Actually, the intertwining approach revealed to be a powerful tool to establish Poincaré and Brascamp–Lieb-type inequalities for such an operator, giving some important information on its spectrum and more precisely on its spectral gap λ_1. Recall that the principle of the intertwining is to write the gradient of a given diffusion operator as a matrix operator of Schrödinger type acting on the gradient, and then to exploit the specific properties of

2020 *Mathematics Subject Classification*: 60J60, 39B62, 47D07, 37A30, 58J50.

Key words and phrases: intertwining, diffusion operator on vector fields, spectral gap, eigenvalues, Brascamp–Lieb type inequalities, log-concave probability measure.

Received 7 April 2020; revised 19 October 2020.

Published online *.

DOI: 10.4064/sm200407-7-11

this alternative operator. In [1] our idea was to consider in the intertwining a weighted gradient instead of the classical Euclidean gradient, the weight being given by multiplication by an invertible matrix A depending on the space variable. As a result, the presence of this weight enabled us to address Brascamp–Lieb type inequalities for many different Markovian dynamics including the log-concave case, i.e. V being convex, and beyond, each situation of interest corresponding more or less to a convenient choice of the weight A.

In the recent years, there have been various extensions of the Brascamp–Lieb inequality taking different forms, which can be established by several methods in connection with optimal transport and celebrated conjectures such as the (B) and KLS conjectures. See for instance the following non-exhaustive list of articles: [18, 14, 17, 6, 24, 8, 27, 25, 7] and mainly the recent paper of Cordero-Erausquin [13] in which, using mass transportation techniques, he established among other things the following improved Brascamp–Lieb inequality: if V is strictly convex in the sense that the Hessian matrix $\nabla^2 V$ is positive-definite, then for every smooth centered function f orthogonal to the coordinate functions in $L^2(\mu)$,

$$\int_{\mathbb{R}^d} f^2\, d\mu \le \int_{\mathbb{R}^d} (\nabla f)^T (\nabla^2 V + \lambda_1 I)^{-1} \nabla f\, d\mu,$$

where I is the identity matrix. The terminology "second-order Brascamp–Lieb inequality" is naturally used to qualify such an inequality since it is related to a higher-order eigenvalue than the spectral gap λ_1 and might be obtained from a Poincaré inequality applied at the level of gradients. Indeed, although not mentioned in [13], a straightforward consequence of this inequality combined with the famous Courant–Fischer Theorem is the following lower bound on the $(d+1)$th positive eigenvalue when V is uniformly convex:

$$\lambda_{d+1} \ge \lambda_1 + \inf_{\mathbb{R}^d} \rho(\nabla^2 V),$$

where $\rho(\nabla^2 V)$ denotes the smallest eigenvalue of the matrix $\nabla^2 V$. The index $d+1$ is relevant since the multiplicity of the spectral gap λ_1, when it is attained, is at most the dimension d, as noticed by Barthe and Cordero-Erausquin [3], following an argument of Klartag [24] (the original result covers the more general strictly convex situation). Although optimal in the standard Gaussian case $V = |\cdot|^2/2$, for which $\lambda_1 = 1$ is of maximal multiplicity d and $\lambda_{d+1} = 2$, there is still room for extension of these results, in particular with a view to relaxing the uniform convexity assumption on the potential V.

Actually, Cordero-Erausquin's paper [13], which is reminiscent of his previous work with Barthe [3], is the starting point of the present paper. Indeed, we intend to make a further step in this direction by addressing these issues in more general situations and to investigate the spectral consequences of these inequalities. Our approach, different from Cordero-Erausquin's but more comparable to that used in [3], is based on the L^2 method of Hörmander [21] and offers, among other things, the following novelty:

- we study the optimality in the generalized Brascamp–Lieb inequality derived in our previous paper [1], where such a problem was left open;
- in the spirit of Cordero-Erausquin's inequality, we establish second-order generalized Brascamp–Lieb inequalities, from which a convenient lower bound for the higher eigenvalue λ_{d+1} is deduced for not necessarily uniformly convex potentials;
- we clarify two centering conditions appearing previously in the literature for uniformly convex potentials;
- we study some perturbation of product probability measures and prove that they satisfy some dimension-free spectral estimates.

For instance, a result illustrating the last point concerns the so-called standard Gaussian product probability measure with quartic interaction. It exhibits a Gaussian part perturbed by a quartic nearest-neighbour interaction.

PROPOSITION 1.1. *Assume that the potential V is of the form*

$$V(x) := \sum_{i=1}^{d} \frac{x_i^2}{2} + J \sum_{i=1}^{d} x_i^2 x_{i+1}^2, \quad x \in \mathbb{R}^d, \quad x_{d+1} := x_1,$$

where $J \geq 0$. Then:

- *there exist explicit constants $J_0 > 0$ and $C > 0$ such that for any $J \in [0, J_0]$ we have $\lambda_1 \geq C$;*
- *there exist explicit constants $\widetilde{J_0} \in (0, J_0)$ and $\widetilde{C} > C$ such that for any $J \in [0, \widetilde{J_0}]$ we have $\lambda_{d+1} \geq \widetilde{C}$.*

All these constants are independent of the dimension.

Let us describe the content of the paper. In Section 2, we recall some basic material on Markov diffusion operators together with the underlying spectral quantities of interest. We also recall the notion of intertwinings between gradients and Markov diffusion operators used in our previous work [1]. Section 3 is then devoted to characterizing the extremal functions in the generalized Brascamp–Lieb inequality, an issue left open in [1]. In Section 4, we derive second-order generalized Brascamp–Lieb inequalities and explore their spectral consequences by proposing a new lower bound on λ_{d+1}. A clarification on the different centering conditions appearing in the literature in the uniformly convex case is also provided. Finally, in Section 5 we apply our spectral estimates and discuss some examples of perturbed product probability measures. The results we obtain for these examples, even for the spectral gap, constitute an important part of the paper. Our approach offers a credible alternative to Helffer's classical method based on uniform spectral estimates for the one-dimensional conditional distributions. In particular, we exhibit two examples involving some strong and non-classical nearest-neighbour interactions, namely the standard Gaussian product probability measure with quartic interaction emphasized above and the Subbotin distribution with convex Lipschitz interaction, for which our spectral estimates reveal to be dimension-free.

2. Preliminaries

2.1. Basic material. Let $\mathcal{C}^\infty(\mathbb{R}^d,\mathbb{R})$ be the space of infinitely differentiable real-valued functions on the Euclidean space $(\mathbb{R}^d, |\cdot|)$, $d \geq 2$, and let $\mathcal{C}_0^\infty(\mathbb{R}^d,\mathbb{R})$ be the subspace of $\mathcal{C}^\infty(\mathbb{R}^d,\mathbb{R})$ of compactly supported functions. Denote by $\|\cdot\|_\infty$ the essential supremum norm with respect to the Lebesgue measure. We consider the Markov diffusion operator defined on $\mathcal{C}^\infty(\mathbb{R}^d,\mathbb{R})$ by

$$Lf := \Delta f - (\nabla V)^T \nabla f,$$

where V is some smooth potential on $\mathbb{R}^d$. Above, Δ and ∇ stand respectively for the Euclidean Laplacian and gradient, and the symbol ^{T} means the transpose of a column vector (or a matrix). If e^{-V} is Lebesgue integrable on $\mathbb{R}^d$, a condition which will be assumed throughout the paper, then we denote by μ the probability measure with Lebesgue density proportional to e^{-V} on $\mathbb{R}^d$. The operator L, which is symmetric with respect to μ, that is, for all $f, g \in \mathcal{C}_0^\infty(\mathbb{R}^d,\mathbb{R})$,

$$\int_{\mathbb{R}^d} f L g \, d\mu = \int_{\mathbb{R}^d} Lf \, g \, d\mu = -\int_{\mathbb{R}^d} (\nabla f)^T \nabla g \, d\mu,$$

is non-positive on $\mathcal{C}_0^\infty(\mathbb{R}^d,\mathbb{R})$. By completeness, the operator is *essentially self-adjoint*, i.e. it admits a unique self-adjoint extension (still denoted L) with domain $\mathcal{D}(L) \subset L^2(\mu)$ in which the space $\mathcal{C}_0^\infty(\mathbb{R}^d,\mathbb{R})$ is dense for the norm induced by L. By the spectral theorem it generates a unique strongly continuous symmetric semigroup $(P_t)_{t\geq 0}$ $(= (e^{tL})_{t\geq 0})$ on $L^2(\mu)$ which is ergodic in the sense that for every $f \in L^2(\mu)$, $P_t f$ converges to $\mu(f)$ in $L^2(\mu)$. Here and hereafter, for a given function $g \in L^1(\mu)$ the notation $\mu(g)$ stands for the integral $\int_{\mathbb{R}^d} g \, d\mu$.

Recall that the spectrum $\sigma(-L)$ of the operator $-L$ is divided into two parts: the *essential spectrum*, that is, the set of limit points in $\sigma(-L)$ and eigenvalues with infinite multiplicity, and the *discrete spectrum*, i.e., the complement of the essential spectrum, consisting of isolated eigenvalues with finite multiplicity. All the elements of the spectrum, called eigenvalues in what follows by abuse of language, are counted with their multiplicities.

Then the Courant–Fischer Theorem [28] gives us variational formulae for the eigenvalues below the bottom of the essential spectrum. More precisely, the first eigenvalue is $\lambda_0 = 0$, possibly embedded in the essential spectrum, the constants being the associated eigenfunctions. Letting $\perp$ denote the orthogonality in $L^2(\mu)$, if for every $n \in \mathbb{N}^*$ we set

$$\lambda_n := \sup_{g_0,\dots,g_{n-1} \in L^2(\mu)} \inf_{\substack{f \in \mathcal{D}(L) \\ f \perp g_i,\, i=0,\dots,n-1}} \frac{-\int_{\mathbb{R}^d} f L f \, d\mu}{\int_{\mathbb{R}^d} f^2 \, d\mu},$$

then either λ_n is located below the bottom of the essential spectrum and thus it is actually the nth positive eigenvalue of the discrete spectrum, or it is itself the bottom of the essential spectrum, all the λ_m coinciding with λ_n when $m \geq n$, and there are at most $n-1$ positive eigenvalues in the discrete spectrum below. The supremum is realized when the g_i are the associated eigenfunctions, and this is the case at least if the spectrum is discrete, for instance when the potential V is uniformly convex, i.e., the smallest eigenvalue $\rho(\nabla^2 V)$ of the Hessian matrix of V is bounded from below (in space) by some positive constant. Note moreover that by a density argument the infimum above might be taken over $\mathcal{C}_0^\infty(\mathbb{R}^d, \mathbb{R})$ instead of $\mathcal{D}(L)$.

The variational formula above leads to the following observation: if the Poincaré-type inequality

$$\lambda \int_{\mathbb{R}^d} f^2 \, d\mu \leq - \int_{\mathbb{R}^d} f \, L f \, d\mu$$

with constant $\lambda > 0$ holds for every function $f \in \mathcal{C}_0^\infty(\mathbb{R}^d, \mathbb{R})$ such that $f \perp g_i$ for some functions g_i in $L^2(\mu)$, $i = 0, \dots, n-1$, g_0 being constant, then we have the lower estimate

$$\lambda_n \geq \lambda.$$

As usual, the case $n = 1$ corresponds to the classical Poincaré inequality and involves the so-called spectral gap λ_1, governing the exponential speed of convergence in $L^2(\mu)$ of the semigroup $(P_t)_{t \geq 0}$. For instance there exists a spectral gap as long as the potential V is convex [23, 5]. This is also the case when V is only convex at infinity, at the price of a perturbation argument [26].

2.2. Intertwinings. Now we turn our attention to the notion of intertwining studied in [1]. Denote by $\mathcal{L}$ the diagonal matrix operator

$$\mathcal{L} := \begin{pmatrix} L & & \\ & \ddots & \\ & & L \end{pmatrix},$$

which acts naturally on the space $\mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$ of smooth vector fields $F : \mathbb{R}^d \to \mathbb{R}^d$. Let us start from the classical intertwining, which is the following: for every $f \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R})$,

$$\nabla Lf = \mathcal{L}^{\nabla^2 V}(\nabla f), \tag{2.1}$$

where $\mathcal{L}^{\nabla^2 V} := \mathcal{L} - \nabla^2 V$ is a Schrödinger-type operator acting on $\mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$, the Hessian matrix $\nabla^2 V$ being understood as a multiplicative operator. Such an intertwining between the gradient and operators is the Euclidean counterpart of the famous Weitzenböck formula for differential forms which appeared a long time ago in Riemannian geometry. Actually, (2.1) is the infinitesimal version of a semigroup identity involving the so-called tangent process arising in probability theory. Such a stochastic interpretation, which is part of the folklore and can be traced back to the 70's, yields a bridge to the famous Bismut formulae and the Malliavin calculus. In a few words, this method consists mainly in differentiating the semigroup $(P_t)_{t\geq 0}$ associated to the Markov process $(X_t^x)_{t\geq 0}$ solution to the stochastic differential equation

$$dX_t^x = \sqrt{2}\, dB_t - \nabla V(X_t^x)dt, \qquad X_0^x = x \in \mathbb{R}^d,$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion on $\mathbb{R}^d$, and writing the result as an alternative semigroup acting on the gradient, which involves the derivative of the mapping $x \mapsto X_t^x$, known as the *tangent process*.

To finish this short introduction to the classical intertwining, let us mention that the operator $\mathcal{L}^{\nabla^2 V}$ appearing in (2.1) is symmetric on the subspace $\mathcal{C}_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$ of smooth compactly supported vector fields on $\mathbb{R}^d$, i.e., for all $F, G \in \mathcal{C}_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} F^T \mathcal{L}^{\nabla^2 V} G\, d\mu = \int_{\mathbb{R}^d} (\mathcal{L}^{\nabla^2 V} F)^T G\, d\mu.$$

Moreover, as noticed by Johnsen in [22], when rewriting it as (the negative of) a Hodge Laplacian, the operator $\mathcal{L}^{\nabla^2 V}$ is also non-positive on $\mathcal{C}_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$.

As announced, we now focus on the intertwining relation emphasized in [1], which is a generalization of (2.1) in the sense that we introduce a matrix weight in this formula to obtain more flexibility in the forthcoming analysis: given $A : \mathbb{R}^d \to \mathcal{M}_{d\times d}(\mathbb{R})$ a smooth invertible matrix, we have the following intertwining between weighted gradient and operators: for every $f \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R})$,

$$A\,\nabla Lf = \mathcal{L}_A^{M_A}(A\,\nabla f), \tag{2.2}$$

where $\mathcal{L}_A^{M_A}(\cdot) := A\mathcal{L}^{\nabla^2 V}(A^{-1}\,\cdot)$, i.e., the operator $\mathcal{L}_A^{M_A}$ acting on $\mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$ is obtained from $\mathcal{L}^{\nabla^2 V}$ by conjugating with the matrix A (the matrix version of the so-called Doob h-transform for probabilists, with $h = A^{-1}$). Rewriting it as a Schrödinger-type operator, we have $\mathcal{L}_A^{M_A} = \mathcal{L}_A - M_A$ (justifying *a posteriori* the presence of M_A in the definition of $\mathcal{L}_A^{M_A}$), where the not necessarily diagonal operator $\mathcal{L}_A$ is defined on $\mathcal{C}^\infty(\mathbb{R}^d,\mathbb{R}^d)$ as

$$\mathcal{L}_A F := \mathcal{L}F + 2A\,\nabla A^{-1}\,\nabla F,$$

the quantities ∇A^{-1} and ∇F being defined respectively as the matrix and vector of gradients $(\nabla a^{i,j})_{i,j=1,\dots,d}$ and $(\nabla F_i)_{i=1,\dots,d}$ if $A^{-1} = (a^{i,j})_{i,j=1,\dots,d}$, and $\nabla A^{-1}\,\nabla F$ is a vector field defined by contraction as

$$(\nabla A^{-1}\,\nabla F)_i := \sum_{j=1}^{d} (\nabla a^{i,j})^T\,\nabla F_j, \qquad i = 1,\dots,d.$$

The matrix M_A, seen as a multiplicative operator, is given by

$$M_A := A\,\nabla^2 V A^{-1} - A\,\mathcal{L}A^{-1},$$

where $\mathcal{L}A^{-1}$ stands for the matrix $(La^{i,j})_{i,j=1,\dots,d}$ if $A^{-1} := (a^{i,j})_{i,j=1,\dots,d}$. The identity (2.2) is a generalization of the classical intertwining in the sense that the choice $A = I$ reduces it to (2.1).

Let us turn to the properties of symmetry and non-positivity of the operators $\mathcal{L}_A^{M_A}$ and $\mathcal{L}_A$. Let S be the positive-definite matrix $S := (AA^T)^{-1}$ and denote by $L^2(S,\mu)$ the space consisting of vector fields F such that

$$\|F\|_{L^2(S,\mu)} := \sqrt{\int_{\mathbb{R}^d} F^T S F\,d\mu} < \infty.$$

First, since $\mathcal{L}^{\nabla^2 V}$ and $\mathcal{L}_A^{M_A}$ are conjugate operators, the second one inherits from the first one the symmetry and non-positivity on $\mathcal{C}_0^\infty(\mathbb{R}^d,\mathbb{R}^d)$, but with respect to the scalar product of $L^2(S,\mu)$, i.e., for all $F, G \in \mathcal{C}_0^\infty(\mathbb{R}^d,\mathbb{R}^d)$,

$$\begin{aligned}\int_{\mathbb{R}^d} F^T S\mathcal{L}_A^{M_A} G\,d\mu &= \int_{\mathbb{R}^d} F^T (AA^T)^{-1} A\mathcal{L}^{\nabla^2 V}(A^{-1}G)\,d\mu \\ &= \int_{\mathbb{R}^d} (A^{-1}F)^T \mathcal{L}^{\nabla^2 V}(A^{-1}G)\,d\mu.\end{aligned}$$

As a result, the operator is essentially self-adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^d,\mathbb{R}^d)$ and admits a unique extension (denoted $\mathcal{L}_A^{M_A}$) with domain $\mathcal{D}(\mathcal{L}_A^{M_A})$. Dealing now with the operator $\mathcal{L}_A$, we deduce from the symmetry of $\mathcal{L}_A^{M_A}$ that it is symmetric on $\mathcal{C}_0^\infty(\mathbb{R}^d,\mathbb{R}^d)$ in $L^2(S,\mu)$ if and only if the multiplicative operator M_A is, i.e., if and only if one of the following equivalent assertions is satisfied:

- the matrix $(A^{-1})^T\mathcal{L}A^{-1}$ is symmetric;
- the matrix SM_A is symmetric;
- the matrix $\nabla^2 V - \mathcal{L}A^{-1}A$ is symmetric.

However it is not clear that such a set of equivalent assumptions implies the non-positivity of $\mathcal{L}_A$ (in contrast to what we claimed too quickly in [1]). In order for the operator $\mathcal{L}_A$ to be both symmetric and non-positive on

$\mathcal{C}_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$, one makes the following slightly stronger assumption:

$(\mathcal{S})$ the matrix $(A^{-1})^T \nabla A^{-1}$ is symmetric.

Indeed, in this case, as shown in [1], for all $F, G \in \mathcal{C}_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} (\mathcal{L}_A F)^T S G \, d\mu = - \int_{\mathbb{R}^d} (\nabla F)^T S \nabla G \, d\mu,$$

where

$$(\nabla F)^T S \nabla G := \sum_{i,j=1}^{d} (\nabla F_i)^T S_{i,j} \nabla G_j.$$

Note that a direct way to observe that $(\mathcal{S})$ is a stronger assumption than the three equivalent ones mentioned above is to take divergence on both sides of the identity characterizing $(\mathcal{S})$ to get the symmetry of the matrix $(A^{-1})^T \Delta A^{-1}$, and thus that of $(A^{-1})^T \mathcal{L} A^{-1}$ by adding the drift term involving the gradient of V.

Finally, under $(\mathcal{S})$ the operator $\mathcal{L}_A$ is essentially self-adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$ and admits a unique extension (still denoted $\mathcal{L}_A$) with domain $\mathcal{D}(\mathcal{L}_A) \subset L^2(S, \mu)$.

To close this section, we mention that although requiring $(\mathcal{S})$ seems to be a bit strange, it is satisfied for instance when the matrix A is chosen to be diagonal, allowing one to cover many interesting situations as we will see in Section 5.

3. Optimality in the generalized Brascamp–Lieb inequality. As observed in our previous paper [1], the intertwining approach has many interesting consequences in terms of functional inequalities and among them Brascamp–Lieb-type inequalities. This section is devoted to characterizing the extremal functions in the generalized Brascamp–Lieb inequality established in [1], where that issue was left open.

Recall first that the *classical Brascamp–Lieb inequality* [11] stands as follows: if the potential V is strictly convex in the sense that the Hessian matrix $\nabla^2 V$ is positive-definite, then for every sufficiently smooth function f which is *centered*, that is, orthogonal to the constants in $L^2(\mu)$, we have

$$\int_{\mathbb{R}^d} f^2 \, d\mu \le \int_{\mathbb{R}^d} (\nabla f)^T (\nabla^2 V)^{-1} \nabla f \, d\mu.$$

In contrast to the classical Poincaré inequality, such an inequality always admits extremal functions given by $f = c^T \nabla V$ for some constant vector $c \in \mathbb{R}^d$.

Based on the intertwining approach, the generalized Brascamp–Lieb inequality established in [1] is stated as follows.

THEOREM 3.1. *Assume* $(\mathcal{S})$ *and that the symmetric matrix* $\nabla^2 V - \mathcal{L}A^{-1}A$ *is positive-definite* (*at each point of* $\mathbb{R}^d$). *Then for any sufficiently smooth centered function* f,

$$\int_{\mathbb{R}^d} f^2 \, d\mu \le \int_{\mathbb{R}^d} (\nabla f)^T (\nabla^2 V - \mathcal{L}A^{-1}A)^{-1} \nabla f \, d\mu. \tag{3.1}$$

As expected, the classical Brascamp–Lieb inequality is recovered by choosing the identity matrix $A = I$. One of the motivations of the generalized Brascamp–Lieb inequality above resides in its connection with the following spectral estimate, also appearing in [1]. Below, the quantity $\rho(M)$ denotes the smallest eigenvalue of a given positive-definite matrix M.

THEOREM 3.2. *Assume* $(\mathcal{S})$ *and that the symmetric matrix* $\nabla^2 V - \mathcal{L}A^{-1}A$ *is uniformly* (*in space*) *bounded from below, in the sense of symmetric matrices, by some positive constant. Then we have the following lower bound on the spectral gap:*

$$\lambda_1 \ge \inf_{\mathbb{R}^d} \rho(\nabla^2 V - \mathcal{L}A^{-1}A). \tag{3.2}$$

Hence the aim is to find a convenient matrix A satisfying the assumption $(\mathcal{S})$ so that the right-hand side in (3.2) is positive and gives relevant information on the spectral gap. In particular, such an investigation is related to the question of optimality in the generalized Brascamp–Lieb inequality (3.1), a problem to which we now turn. It corresponds to the first new result of this paper.

For a given diffeomorphism $H \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$, we denote by J_H its (invertible) Jacobian matrix. Our key observation is the following: if the matrix A is of the form $A := (J_H^T)^{-1}$ for some diffeomorphism $H \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$, then a straightforward computation entails the identity

$$\nabla^2 V - \mathcal{L}A^{-1}A = -J_{\mathcal{L}H}^T (J_H^T)^{-1}. \tag{3.3}$$

Such a formula, which is nothing but a version of the classical intertwining (2.1) applied at the level of matrices,

$$J_{\mathcal{L}H}^T = \mathcal{L}J_H^T - \nabla^2 V \, J_H^T,$$

is the multi-dimensional version of the practical criterion emphasized in the one-dimensional case by the authors in [8, 10]. Such an identity will be used many times below. As announced, our first main result is the following.

THEOREM 3.3. *Let* A *be a matrix of the form* $A := (J_H^T)^{-1}$ *for some diffeomorphism* $H \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$. *Assume* $(\mathcal{S})$ *and that the symmetric matrix in* (3.3) *is positive-definite. Then equality holds in* (3.1) *if and only if the smooth centered function* f *is of the form*

$$f = v^T \mathcal{L}H$$

for some constant vector $v \in \mathbb{R}^d$.

Proof. The proof is based on a convenient representation of the L^2 norm enabled by the intertwining (2.2), in the spirit of the L^2 method of Hörmander. For simplicity, let us keep first the notation A. At the price of an approximation procedure (cf. [1]), we might assume that the smallest eigenvalue of the symmetric matrix $\nabla^2 V - \mathcal{L}A^{-1}A$ is bounded from below by some positive constant, so that by Theorem 3.2 the operator $-L$ has a spectral gap, i.e., $\lambda_1 > 0$. In that case, for every sufficiently smooth centered function f, there exists a unique smooth centered solution $g := -\int_0^\infty P_t f\, dt \in \mathcal{D}(L)$ to the Poisson equation $f = Lg$ (recall that the spectral gap encodes the exponential decay in $L^2(\mu)$ of the semigroup $(P_t)_{t\geq 0}$).

Fix such a function f. Using then integration by parts and the intertwining relation (2.2), we have

$$\begin{aligned}
\int_{\mathbb{R}^d} f^2\, d\mu &= \int_{\mathbb{R}^d} f^2\, d\mu - \int_{\mathbb{R}^d} (f - Lg)^2\, d\mu = 2\int_{\mathbb{R}^d} f Lg\, d\mu - \int_{\mathbb{R}^d} (Lg)^2\, d\mu \\
&= -2\int_{\mathbb{R}^d} (\nabla f)^T \nabla g\, d\mu + \int_{\mathbb{R}^d} (\nabla g)^T \nabla Lg\, d\mu \\
&= -2\int_{\mathbb{R}^d} (A\nabla f)^T SA\nabla g\, d\mu + \int_{\mathbb{R}^d} (A\nabla g)^T SA\nabla Lg\, d\mu \\
&= -2\int_{\mathbb{R}^d} (A\nabla f)^T SA\nabla g\, d\mu + \int_{\mathbb{R}^d} (A\nabla g)^T S\mathcal{L}_A^{M_A}(A\nabla g)\, d\mu \\
&= \int_{\mathbb{R}^d} (A\nabla g)^T S\mathcal{L}_A(A\nabla g)\, d\mu + \int_{\mathbb{R}^d} (A\nabla f)^T SM_A^{-1} A\nabla f\, d\mu \\
&\quad - \int_{\mathbb{R}^d} (A\nabla f + M_A A\nabla g)^T SM_A^{-1}(A\nabla f + M_A A\nabla g)\, d\mu,
\end{aligned}$$

where we have used the symmetry of the matrix SM_A to get the last equality (recall that it is a consequence of the assumption $(\mathcal{S})$). Now, rewriting the intertwining (2.2) gives

$$A\nabla f + M_A A\nabla g = \mathcal{L}_A(A\nabla g),$$

and a straightforward computation leads to the identity

$$A^T SM_A^{-1} A = (\nabla^2 V - \mathcal{L}A^{-1}A)^{-1},$$

so that we obtain the following representation of the L^2 norm:

$$\begin{aligned}
\int_{\mathbb{R}^d} f^2\, d\mu = &\int_{\mathbb{R}^d} (A\nabla g)^T S\mathcal{L}_A(A\nabla g)\, d\mu + \int_{\mathbb{R}^d} (\nabla f)^T (\nabla^2 V - \mathcal{L}A^{-1}A)^{-1} \nabla f\, d\mu \\
&- \int_{\mathbb{R}^d} (A^{-1}\mathcal{L}_A(A\nabla g))^T (\nabla^2 V - \mathcal{L}A^{-1}A)^{-1} A^{-1}\mathcal{L}_A(A\nabla g)\, d\mu.
\end{aligned}$$

Since under $(\mathcal{S})$ the operator $\mathcal{L}_A$ is non-positive, equality holds in (3.1) if and only if the first and third terms on the RHS vanish, i.e., $\mathcal{L}_A(A\nabla g)$

$= 0$. In other words, the vector $A\nabla g$ is an eigenfunction associated to the eigenvalue 0 for the operator $\mathcal{L}_A$, meaning that $A\nabla g = v$ for some constant vector $v \in \mathbb{R}^d$, or equivalently $\nabla g = A^{-1}v$. Finally, since $A = (J_H^T)^{-1}$ for some diffeomorphism $H \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$, the centered function g is given by

$$g = v^T H - \int_{\mathbb{R}^d} v^T H \, d\mu,$$

which implies the following identity for the centered function f:

$$f = v^T \mathcal{L} H.$$

The proof of the optimality result is now complete. ∎

Looking carefully at the proof, we observe that if the matrix A is not taken as the inverse of the Jacobian matrix of a diffeomorphism, then it may happen that there does not exist a function f saturating (3.1) since the vector $A^{-1}v$ has no reason *a priori* to be a gradient. Moreover, note that $(\mathcal{S})$ might not be satisfied for every choice of diffeomorphism H, hence in practical situations one has to choose it carefully to ensure this property, and an idea will be to choose it diagonal, as we will see in Section 5.

Coming back to Theorem 3.2, we may wonder when the spectral gap is attained, if equality can hold in the spectral estimate (3.2) for convenient choice of $A = (J_H^T)^{-1}$ with $H \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$ some diffeomorphism, similarly to the optimality result in Theorem 3.3. Actually, the answer to this question is not clear. However the representation (3.3) suggests that equality in (3.2) might hold at the price of some strong assumptions. More precisely, assume for simplicity that the potential V is strictly convex. If the spectral gap λ_1 is of (maximal) multiplicity d and the corresponding d eigenfunctions define some diffeomorphism $H \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$, then

$$-J_{\mathcal{L}H}^T (J_H^T)^{-1} = J_H^T \lambda_1 I (J_H^T)^{-1} = \lambda_1 I,$$

so that the assumption $(\mathcal{S})$ is satisfied and equality in (3.2) is realized. Certainly, this observation is only theoretical since the eigenfunctions associated to λ_1 are not known in general and thus neither is the desired vector field H. We mention that a first attempt in this direction has been done very recently by Barthe and Klartag [4], who addressed among other things the multiplicity problem of the spectral gap. For instance, when λ_1 is attained, a non-trivial situation for which the multiplicity is maximal is given by some strictly convex potential V sharing the symmetries of the hypercube, that is, invariant with respect to the coordinate hyperplanes (i.e., unconditional) and the coordinate permutations.

4. Second-order inequalities and spectral estimates. Recall that the second-order Brascamp–Lieb inequality established by Cordero-Erausquin [13], which mainly motivates the present work, is stated as follows: if V is strictly convex, then for every smooth centered function f orthogonal to the coordinate functions, i.e., $\int_{\mathbb{R}^d} f \,\mathrm{id}\, d\mu = 0$ with id the identity vector field on $\mathbb{R}^d$, we have

$$\int_{\mathbb{R}^d} f^2\, d\mu \leq \int_{\mathbb{R}^d} (\nabla f)^T (\nabla^2 V + \lambda_1 I)^{-1} \nabla f \, d\mu. \tag{4.1}$$

Combining this with the Courant–Fischer Theorem yields the following lower bound on the $(d+1)$th positive eigenvalue when V is uniformly convex:

$$\lambda_{d+1} \geq \lambda_1 + \inf_{\mathbb{R}^d} \rho(\nabla^2 V). \tag{4.2}$$

In this part, we extend via the intertwining approach the inequality (4.1) by providing generalized Brascamp–Lieb inequalities of second order, together with a lower bound on λ_{d+1}. In particular, this allows us to consider situations beyond the case of a uniformly convex potential V, generalizing the spectral estimate (4.2).

4.1. Second-order generalized Brascamp–Lieb inequalities and spectral bounds. To obtain Brascamp–Lieb-type inequalities of second order, the idea is to analyze more carefully the non-positive term

$$\int_{\mathbb{R}^d} (A\nabla g)^T S \mathcal{L}_A (A\nabla g)\, d\mu,$$

which was ignored in the proof of Theorem 3.3. Since it can be interpreted as an energy term, this might be done by considering some adapted notion of spectral gap for the operator $-\mathcal{L}_A$. Before stating the desired result, contained in Theorem 4.1, we need to introduce some preliminaries.

First let us consider the notion of mean in the space $L^2(S,\mu)$. We assume in what follows that the matrix $\int_{\mathbb{R}^d} S\, d\mu$ given by

$$\Big(\int_{\mathbb{R}^d} S\, d\mu\Big)_{i,j} := \int_{\mathbb{R}^d} S_{i,j}\, d\mu, \qquad i,j = 1,\ldots,d,$$

is well-defined, i.e., all its entries are finite, and the matrix is invertible. Now, for every $F \in L^2(S,\mu)$, the mean $m_S(F)$ of the vector field F is defined as

$$m_S(F) := \Big(\int_{\mathbb{R}^d} S\, d\mu\Big)^{-1} \int_{\mathbb{R}^d} SF\, d\mu,$$

and is the unique constant vector of $\mathbb{R}^d$ minimizing the functional

$$\mathbb{R}^d \ni c \mapsto \int_{\mathbb{R}^d} (F-c)^T S (F-c)\, d\mu.$$

In particular, every constant vector $c \in \mathbb{R}^d$ belongs to $L^2(S,\mu)$ and we have

$m_S(c) = c$. Moreover, for all $F, G \in L^2(S, \mu)$,

$$\int_{\mathbb{R}^d} (F - m_S(F))^T S(G - m_S(G))\, d\mu = \int_{\mathbb{R}^d} F^T SG\, d\mu - m_S(F)^T \Big(\int_{\mathbb{R}^d} S\, d\mu\Big) m_S(G).$$

In what follows, for a given vector field $F \in L^2(S, \mu)$, we set

$$\widetilde{F} := F - m_S(F),$$

which is thus centered in $L^2(S, \mu)$, i.e.,

$$\int_{\mathbb{R}^d} S\widetilde{F}\, d\mu = 0.$$

We are now in a position to define the spectral gap in $L^2(S, \mu)$ of the operator $-\mathcal{L}_A$ under the assumption $(\mathcal{S})$:

$$\lambda_1^A := \inf\bigg\{ \frac{-\int_{\mathbb{R}^d} \widetilde{F}^T S \mathcal{L}_A \widetilde{F}\, d\mu}{\int_{\mathbb{R}^d} \widetilde{F}^T S \widetilde{F}\, d\mu} : F \in \mathcal{D}(\mathcal{L}_A) \bigg\}.$$

In particular, λ_1^A is the best constant $\lambda > 0$ such that the following Poincaré inequality holds: for every $F \in \mathcal{C}_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$,

$$\lambda \int_{\mathbb{R}^d} \widetilde{F}^T S \widetilde{F}\, d\mu \le -\int_{\mathbb{R}^d} \widetilde{F}^T S \mathcal{L}_A \widetilde{F}\, d\mu. \tag{4.3}$$

Although such a notion of spectral gap is somewhat theoretical, it is possible to get estimates for some interesting examples, as we will see in Section 5.

As announced, we are now in a position to state our second main result of the paper, corresponding to second-order generalized Brascamp–Lieb inequalities.

THEOREM 4.1. *Assume* $(\mathcal{S})$ *and that the invertible matrix* $\int_{\mathbb{R}^d} S\, d\mu$ *is well-defined. Assume moreover that the symmetric matrix* $\lambda_1^A I + \nabla^2 V - \mathcal{L}A^{-1}A$ *is positive-definite. Let* $f \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ *be centered and assume that there exists a unique smooth centered solution* $g \in \mathcal{D}(L)$ *to the Poisson equation* $f = Lg$. *Then the following second-order generalized Brascamp–Lieb inequalities hold:*

$$\begin{aligned}\int_{\mathbb{R}^d} f^2\, d\mu \le &\int_{\mathbb{R}^d} (\nabla f)^T (\lambda_1^A I + \nabla^2 V - \mathcal{L}A^{-1}A)^{-1} \nabla f\, d\mu \\ &- \int_{\mathbb{R}^d} \Theta^T S(\lambda_1^A I + M_A)^{-1} \Theta\, d\mu \\ &+ m_S(A\nabla g)^T \Big(\int_{\mathbb{R}^d} SM_A\, d\mu\Big) m_S(A\nabla g),\end{aligned}$$

where $\Theta := (\mathcal{L}_A + \lambda_1^A I)(\widetilde{A\nabla g}) - M_A m_S(A\nabla g)$, *and*

$$\int_{\mathbb{R}^d} f^2\, d\mu \le \int_{\mathbb{R}^d} (\nabla f)^T (\lambda_1^A I + \nabla^2 V - \mathcal{L}A^{-1}A)^{-1} \nabla f\, d\mu - \int_{\mathbb{R}^d} \Upsilon^T S(\lambda_1^A I + M_A)^{-1} \Upsilon\, d\mu + \lambda_1^A m_S(A\nabla g)^T \Big(\int_{\mathbb{R}^d} S\, d\mu \Big) m_S(A\nabla g),$$

where $\Upsilon := (\mathcal{L}_A + \lambda_1^A I)(A\nabla g)$. *In particular, under the centering condition* $m_S(A\nabla g) = 0$, *we have*

$$\int_{\mathbb{R}^d} f^2\, d\mu \le \int_{\mathbb{R}^d} (\nabla f)^T (\lambda_1^A I + \nabla^2 V - \mathcal{L}A^{-1}A)^{-1} \nabla f\, d\mu. \tag{4.4}$$

Proof. It is sufficient to prove only the first two inequalities, since (4.4) is then a straightforward consequence. Indeed, under the centering condition $m_S(A\nabla g) = 0$ we have $\Theta = \Upsilon$ so that the first two inequalities coincide, and moreover

$$\int_{\mathbb{R}^d} \Theta^T S(\lambda_1^A I + M_A)^{-1} \Theta\, d\mu = \int_{\mathbb{R}^d} (A^{-1}\Theta)^T (\lambda_1^A I + \nabla^2 V - \mathcal{L}A^{-1}A)^{-1} A^{-1}\Theta\, d\mu,$$

which is non-negative.

We start by proving the first inequality. Since the intertwining (2.2) can be rewritten as

$$A\nabla f + M_A A\nabla g = \mathcal{L}_A(A\nabla g),$$

which is centered in $L^2(S,\mu)$, one deduces that

$$\int_{\mathbb{R}^d} S(A\nabla f + M_A A\nabla g)\, d\mu = 0.$$

Writing then the L^2 norm similarly to that in the proof of Theorem 3.3 and centering, from the Poincaré inequality (4.3) we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} f^2\, d\mu &= -2 \int_{\mathbb{R}^d} (A\nabla f)^T S A\nabla g\, d\mu + \int_{\mathbb{R}^d} (A\nabla g)^T S \mathcal{L}_A^{M_A}(A\nabla g)\, d\mu \\ &= -2 \int_{\mathbb{R}^d} (A\nabla f)^T S \widetilde{A\nabla g}\, d\mu + \int_{\mathbb{R}^d} \widetilde{A\nabla g}^T S(\mathcal{L}_A - M_A)(\widetilde{A\nabla g})\, d\mu \\ &\quad - 2 m_S(A\nabla g)^T \int_{\mathbb{R}^d} S(A\nabla f + M_A A\nabla g)\, d\mu \\ &\quad + m_S(A\nabla g)^T \Big(\int_{\mathbb{R}^d} S M_A\, d\mu \Big) m_S(A\nabla g) \end{aligned} \tag{4.5}$$

$$\begin{aligned}
&\le -2 \int_{\mathbb{R}^d} (A\nabla f)^T S \widetilde{A\nabla g}\, d\mu - \int_{\mathbb{R}^d} \widetilde{A\nabla g}^T S(\lambda_1^A I + M_A)\widetilde{A\nabla g}\, d\mu \\
&\quad + m_S(A\nabla g)^T \Big(\int_{\mathbb{R}^d} S M_A \, d\mu \Big) m_S(A\nabla g) \\
&= \int_{\mathbb{R}^d} (A\nabla f)^T S(\lambda_1^A I + M_A)^{-1}(A\nabla f)\, d\mu - \int_{\mathbb{R}^d} \Theta^T S(\lambda_1^A I + M_A)^{-1}\Theta\, d\mu \\
&\quad + m_S(A\nabla g)^T \Big(\int_{\mathbb{R}^d} S M_A \, d\mu \Big) m_S(A\nabla g),
\end{aligned}$$

since once again by the intertwining (2.2),

$$\Theta = A\nabla f + (\lambda_1^A I + M_A)\, \widetilde{A\nabla g}.$$

Now, we have the identity

$$A^T S(\lambda_1^A I + M_A)^{-1} A = \left(\lambda_1^A I + \nabla^2 V - \mathcal{L}A^{-1}A\right)^{-1},$$

thus achieving the proof of the first second-order generalized Brascamp–Lieb inequality.

To establish the second one, we observe that the RHS in (4.5) might be written in a slightly different way when ignoring the centering. Indeed, since

$$\Upsilon = A\nabla f + (\lambda_1^A I + M_A)A\nabla g,$$

the RHS can be rewritten after some computations as

$$\begin{aligned}
&\int_{\mathbb{R}^d} (A\nabla f)^T S(\lambda_1^A I + M_A)^{-1} A\nabla f\, d\mu - \int_{\mathbb{R}^d} \Upsilon^T S(\lambda_1^A I + M_A)^{-1}\Upsilon\, d\mu \\
&\qquad\qquad + \lambda_1^A m_S(A\nabla g)^T \Big(\int_{\mathbb{R}^d} S\, d\mu \Big) m_S(A\nabla g),
\end{aligned}$$

from which the desired inequality follows. ■

Let us mention that the two second-order generalized Brascamp–Lieb inequalities above are very close to each other in spirit and imply the same key inequality (4.4). Actually, they differ only because we rearrange in two ways the terms arising in the same quadratic form. If we forget the non-positive terms involving Θ and Υ, it is not clear whether one is better than the other in general, although it seems that the second one is less tractable because the spectral gap λ_1^A in it can be difficult to estimate.

As we may observe from the proof above, optimality in the second-order generalized Brascamp–Lieb inequality (4.4) under the centering condition $m_S(A\nabla g) = 0$ is obtained if and only if the Poincaré inequality (4.3) is saturated and if Θ $(= \Upsilon)$ vanishes. Both conditions are actually the same and mean that $A\nabla g$ is an eigenfunction associated to the eigenvalue λ_1^A for the operator $-\mathcal{L}_A$. However, if it exists, we do not know its expression even for the particular choice $A = (J_H^T)^{-1}$ emphasized in Section 3, where $H \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$ is some diffeomorphism.

Note that the centering condition $m_S(A\nabla g) = 0$ essentially focuses on the function g, so that we cannot obtain directly from (4.4) an estimate on λ_{d+1} since the orthogonality conditions should be required only on f. However once again the choice $A = (J_H^T)^{-1}$, where $H \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$ is a diffeomorphism, enables us to solve this problem. Recall that in this case we have

$$\nabla^2 V - \mathcal{L}A^{-1}A = -J_{\mathcal{L}H}^T (J_H^T)^{-1}.$$

Our third main result is stated as follows (for a better readability, we sometimes keep at the same time the notation A and H).

THEOREM 4.2. *Let $A := (J_H^T)^{-1}$ for some diffeomorphism $H \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$ which satisfies $(\mathcal{S})$. Moreover, assume that the invertible matrix $\int_{\mathbb{R}^d} S\, d\mu$ is well-defined. If the symmetric matrix $-J_{\mathcal{L}H}^T (J_H^T)^{-1}$ has its smallest eigenvalue bounded from below by some positive constant, then*

$$\lambda_{d+1} \geq \lambda_1^A + \inf_{\mathbb{R}^d} \rho(-J_{\mathcal{L}H}^T (J_H^T)^{-1}). \tag{4.6}$$

Proof. The proof is straightforward: by Theorem 3.2 we have $\lambda_1 > 0$ so that the assumptions of Theorem 4.1 are satisfied. Hence the second-order generalized Brascamp–Lieb inequality (4.4) holds under the centering condition $m_S(A\nabla g) = 0$, which can be rewritten as

$$0 = \int_{\mathbb{R}^d} (A^T)^{-1} \nabla g\, d\mu = \int_{\mathbb{R}^d} J_H \nabla g\, d\mu = -\int_{\mathbb{R}^d} HLg\, d\mu = -\int_{\mathbb{R}^d} Hf\, d\mu. \tag{4.7}$$

Finally, the Courant–Fischer Theorem concludes the proof. ∎

According to our assumptions above, it might happen that $\lambda_1^A = 0$. In this case, (4.6) does not provide any additional information since we always have

$$\lambda_{d+1} \geq \lambda_1 \geq \inf_{\mathbb{R}^d} \rho(-J_{\mathcal{L}H}^T (J_H^T)^{-1}).$$

Hence to observe in practice the relevance of Theorem 4.2, we need to carefully estimate the spectral gap λ_1^A.

Similarly to the optimality problem studied for the spectral gap in Section 3, let us analyze the equality case in (4.6). To do so, a bit of spectral analysis is required. Let A be an invertible matrix satisfying the assumption $(\mathcal{S})$ and such that the symmetric matrix $\nabla^2 V - \mathcal{L}A^{-1}A$ is uniformly bounded from below by some positive constant. Hence by Theorem 3.2 we have $\lambda_1 > 0$. Denote the subspaces of (weighted) gradients by $\nabla_I := \{\nabla f : f \in \mathcal{C}_0^\infty(\mathbb{R}^d, \mathbb{R})\} \subset L^2(I, \mu)$ and $\nabla_A := \{A\nabla f : f \in \mathcal{C}_0^\infty(\mathbb{R}^d, \mathbb{R})\} \subset L^2(S, \mu)$. According to the intertwinings (2.1) and (2.2), we observe that ∇_I and ∇_A

are stable subspaces for the operators $\mathcal{L}^{\nabla^2 V}$ and $\mathcal{L}_A^{M_A}$ of Schrödinger type, respectively, i.e., $\mathcal{L}^{\nabla^2 V}(\nabla_I) \subset \nabla_I$ and $\mathcal{L}_A^{M_A}(\nabla_A) \subset \nabla_A$. By construction, since the (self-adjoint extensions of the) restricted operators $\mathcal{L}^{\nabla^2 V}|_{\nabla_I}$ and $\mathcal{L}_A^{M_A}|_{\nabla_A}$ are unitarily equivalent, multiplication by A^{-1} being a unitary transformation from $L^2(S,\mu)$ to $L^2(I,\mu)$, their spectra coincide. Such a property has already been noticed in [8, 1], dealing with the bottom of the spectra. However the relation to the original operator L is more subtle.

Indeed, if we restrict L to the space orthogonal to the constant functions, i.e. to $\mathbb{1}^\perp := \mathcal{D}(L) \cap \{f \in L^2(\mu) : f \perp 1\}$, then there exists a unitary transformation between $L|_{\mathbb{1}^\perp}$ and $\mathcal{L}^{\nabla^2 V}|_{\nabla_I}$, a property emphasized by Johnsen [22] under the key assumption $\lambda_1 > 0$. More precisely, if U stands for the Riesz transform $U := \nabla(-L)^{-1/2}$ acting on $\mathbb{1}^\perp$, with values in $\mathcal{D}(\mathcal{L}^{\nabla^2 V}|_{\nabla_I})$ and which is well-defined thanks to the Riesz-type potential representation

$$(-L)^{-1/2} := \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{t}} P_t \, dt$$

(cf. e.g. [2]), then U is a unitary mapping and

$$\mathcal{L}^{\nabla^2 V}|_{\nabla_I} = U L|_{\mathbb{1}^\perp} U^*.$$

Note that U might also be written as $(-\mathcal{L}^{\nabla^2 V})^{-1/2}\nabla$. As a result, their spectra coincide. Summarizing, we have

$$\sigma(-L|_{\mathbb{1}^\perp}) = \sigma(-\mathcal{L}^{\nabla^2 V}|_{\nabla_I}) = \sigma(-\mathcal{L}_A^{M_A}|_{\nabla_A}). \tag{4.8}$$

Now, under the notation and assumptions of Theorem 4.2, we wonder if equality can hold in (4.6). As we have seen at the end of Section 3, if V is strictly convex and the spectral gap λ_1 is attained, is of (maximal) multiplicity d and the corresponding d eigenfunctions define some diffeomorphism $H \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$, then the choice $A = (J_H^T)^{-1}$ entails $M_A = \lambda_1 I$ so that from (4.8) we obtain, in terms of spectra,

$$\sigma(-L)\backslash\{0\} = \{\lambda + \lambda_1 : \lambda \in \sigma(-\mathcal{L}_A|_{\nabla_A})\},$$

where $\mathcal{L}_A|_{\nabla_A}$ is the (self-adjoint extension of the) restriction of the operator $\mathcal{L}_A$ to the subspace of weighted gradients ∇_A, which is stable in this very specific situation because it is for the operator $\mathcal{L}_A^{M_A} = \mathcal{L}_A - \lambda_1 I$. Since λ_1 has multiplicity d, we get equality in (4.6) as long as the spectral gap λ_1^A coincides with the first positive eigenvalue of $-\mathcal{L}_A|_{\nabla_A}$. Note however that this spectral gap seems difficult to estimate in this case since it depends on the vector field H whose coordinates are the eigenfunctions associated to λ_1, which are unknown in general, except in some very particular cases.

4.2. The case $A = I$. This short subsection is devoted to presenting the results obtained by considering the classical intertwining (2.1), corresponding to the choice $A = I$ in (2.2). In some sense, we are reduced to the situation analyzed by Cordero-Erausquin in [13]. An important point will be to clarify and discuss two different centering conditions appearing in these second-order Brascamp–Lieb or Poincaré inequalities in the uniformly convex case.

When A is written as $A = (J_H^T)^{-1}$, the case $A = I$ corresponds to $H = \mathrm{id}$ and the assumption $(\mathcal{S})$ is trivially satisfied. Hence Theorem 4.1 leads to the following corollary.

Corollary 4.3. *Assume that the symmetric matrix $\lambda_1 I + \nabla^2 V$ is positive-definite. Let $f \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ be centered and assume that there exists a unique smooth centered solution $g \in \mathcal{D}(L)$ to the Poisson equation $f = Lg$. Then the following second-order Brascamp–Lieb inequalities hold:*

$$\begin{aligned}\int_{\mathbb{R}^d} f^2\, d\mu \le & \int_{\mathbb{R}^d} (\nabla f)^T (\nabla^2 V + \lambda_1 I)^{-1} \nabla f\, d\mu \\ & - \int_{\mathbb{R}^d} \Theta^T (\nabla^2 V + \lambda_1 I)^{-1} \Theta\, d\mu \\ & + \Big(\int_{\mathbb{R}^d} \nabla g\, d\mu\Big)^T \Big(\int_{\mathbb{R}^d} \nabla^2 V\, d\mu\Big)\Big(\int_{\mathbb{R}^d} \nabla g\, d\mu\Big),\end{aligned} \tag{4.9}$$

where $\Theta := (\mathcal{L} + \lambda_1 I)(\nabla g - \int_{\mathbb{R}^d} \nabla g\, d\mu) - \nabla^2 V \int_{\mathbb{R}^d} \nabla g\, d\mu$, and

$$\begin{aligned}\int_{\mathbb{R}^d} f^2\, d\mu \le & \int_{\mathbb{R}^d} (\nabla f)^T (\nabla^2 V + \lambda_1 I)^{-1} \nabla f\, d\mu \\ & - \int_{\mathbb{R}^d} \Upsilon^T (\nabla^2 V + \lambda_1 I)^{-1} \Upsilon\, d\mu + \lambda_1 \Big|\int_{\mathbb{R}^d} \nabla g\, d\mu\Big|^2,\end{aligned} \tag{4.10}$$

where $\Upsilon := (\mathcal{L} + \lambda_1 I)(\nabla g)$. In particular, under the centering condition $\int_{\mathbb{R}^d} \nabla g\, d\mu = 0$, we have

$$\begin{aligned}\int_{\mathbb{R}^d} f^2\, d\mu \le & \int_{\mathbb{R}^d} (\nabla f)^T (\nabla^2 V + \lambda_1 I)^{-1} \nabla f\, d\mu \\ & - \int_{\mathbb{R}^d} ((\mathcal{L} + \lambda_1 I)(\nabla g))^T (\nabla^2 V + \lambda_1 I)^{-1} (\mathcal{L} + \lambda_1 I)(\nabla g)\, d\mu.\end{aligned}$$

Since by (4.7) one has

$$\int_{\mathbb{R}^d} \nabla g\, d\mu = - \int_{\mathbb{R}^d} f\, \mathrm{id}\, d\mu,$$

the latter inequality slightly improves (with a remainder term) Cordero-Erausquin's inequality (4.1) under the centering condition

$$\int_{\mathbb{R}^d} f\, \mathrm{id}\, d\mu = 0. \tag{4.11}$$

In particular, in the uniformly convex case $\rho(\nabla^2 V) \ge \rho > 0$, one has $\lambda_1 \ge \rho$ so that the following second-order Poincaré inequality holds: for any centered function $f \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ satisfying the centering condition (4.11),

$$\int_{\mathbb{R}^d} f^2\, d\mu \le \frac{1}{2\rho} \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu,$$

an inequality established first in a paper of Cordero-Erausquin and his coauthors [14] and improved by Hargé [17] as follows:

$$\int_{\mathbb{R}^d} f^2\, d\mu \le \frac{1}{2\rho} \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu - \rho \Big| \int_{\mathbb{R}^d} f \,\mathrm{id}\, d\mu \Big|^2.$$

However a notable difference appears since both [14, 17] exhibit another centering condition on the function f:

$$\int_{\mathbb{R}^d} \nabla f\, d\mu = 0. \tag{4.12}$$

Actually, our approach also allows us to slightly improve Hargé's inequality under the same centering condition (4.12):

Corollary 4.4. *Assume that the potential V is uniformly convex, i.e., $\rho(\nabla^2 V) \ge \rho$ for some $\rho > 0$. Let $f \in \mathcal{C}_0^\infty(\mathbb{R}^d, \mathbb{R})$ be centered and denote $g \in \mathcal{D}(L)$ the unique smooth centered solution to the Poisson equation $f = Lg$. Then the following Poincaré inequality with remainder terms holds:*

$$\begin{aligned}\int_{\mathbb{R}^d} f^2\, d\mu \le{}& \frac{1}{\rho+\lambda_1} \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu - \rho |c|^2 - 2c^T \int_{\mathbb{R}^d} \nabla f\, d\mu \\ &- \frac{1}{\rho+\lambda_1} \int_{\mathbb{R}^d} |(\mathcal{L} - \nabla^2 V + (\lambda_1+\rho) I)(\nabla g - c) - \nabla^2 V c|^2\, d\mu,\end{aligned} \tag{4.13}$$

where $c := \int_{\mathbb{R}^d} \nabla g\, d\mu = -\int_{\mathbb{R}^d} f\, \mathrm{id}\, d\mu$. In particular, for any centered function $f \in \mathcal{C}_0^\infty(\mathbb{R}^d, \mathbb{R})$ satisfying either (4.11) *or* (4.12), *we have*

$$\int_{\mathbb{R}^d} f^2\, d\mu \le \frac{1}{\rho+\lambda_1} \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu.$$

Proof. The proof is somewhat similar to that of Theorem 4.1, the only difference being the use of the inequality $\nabla^2 V \ge \rho I$ at the very beginning:

$$\begin{aligned}\int_{\mathbb{R}^d} f^2\, d\mu &= -2 \int_{\mathbb{R}^d} (\nabla f)^T \nabla g\, d\mu + \int_{\mathbb{R}^d} (\nabla g)^T \mathcal{L}^{\nabla^2 V} (\nabla g)\, d\mu \\ &\le -2 \int_{\mathbb{R}^d} (\nabla f)^T (\nabla g - c)\, d\mu - (\lambda_1+\rho) \int_{\mathbb{R}^d} |\nabla g - c|^2\, d\mu - \rho |c|^2 - 2c^T \int_{\mathbb{R}^d} \nabla f\, d\mu \\ &= \frac{1}{\lambda_1+\rho} \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu - \frac{1}{\rho+\lambda_1} \int_{\mathbb{R}^d} |\nabla f + (\lambda_1+\rho)(\nabla g - c)|^2\, d\mu \\ &\quad - \rho |c|^2 - 2c^T \int_{\mathbb{R}^d} \nabla f d\mu,\end{aligned}$$

and since the intertwining (2.2) with $A = I$ can be rewritten in the present context as $\nabla f = (\mathcal{L} - \nabla^2 V)(\nabla g)$, the proof is complete. ■

Of course, the question of the centering condition is justified since in general the conditions (4.11) and (4.12) differ. Indeed, the gradients ∇f and ∇g have no reason to be centered simultaneously, except in the standard Gaussian case since we always have the identity

$$\int_{\mathbb{R}^d} \nabla f \, d\mu = - \int_{\mathbb{R}^d} \nabla^2 V \nabla g \, d\mu.$$

Note that this is also the case when the potential V is unconditional as also is the function f, as noticed by Barthe and Cordero-Erausquin [3].

To go one step further in the standard Gaussian case, the inequalities (4.9), (4.10) and (4.13), which all coincide in this specific Gaussian setting, give

$$\begin{aligned} \int_{\mathbb{R}^d} f^2 \, d\mu \leq \frac{1}{2} \int_{\mathbb{R}^d} |\nabla f|^2 \, d\mu + \frac{1}{2} \Big| \int_{\mathbb{R}^d} \nabla f \, d\mu \Big|^2 \\ - \frac{1}{2} \int_{\mathbb{R}^d} \Big| (\mathcal{L} + I) \Big(\nabla g - \int_{\mathbb{R}^d} \nabla g \, d\mu \Big) \Big|^2 \, d\mu, \end{aligned} \tag{4.14}$$

which slightly improves the inequality of Goldstein–Nourdin–Peccati [16] obtained directly by a simple spectral decomposition using Hermite polynomials. Note that the inequality (4.14) might also be obtained by spectral decomposition, with equality if and only if f is an Hermite polynomial of degree one, two or three. In particular, it would be interesting to compare (4.14) with the dimension-dependent inequalities appearing in the literature, namely with the following two inequalities obtained through the so-called Borrell-Brascamp–Lieb approach:

- Bobkov–Ledoux's inequality [6]:

$$\int_{\mathbb{R}^d} f^2 \, d\mu \leq 6 \int_{\mathbb{R}^d} |\nabla f|^2 \, d\mu - 6 \int_{\mathbb{R}^d} \frac{|(\nabla f)^T x|^2}{d + |x|^2} \, d\mu;$$

- Bolley–Gentil–Guillin's inequality [7]:

$$\int_{\mathbb{R}^d} f^2 \, d\mu \leq \int_{\mathbb{R}^d} |\nabla f|^2 \, d\mu - \int_{\mathbb{R}^d} \frac{|f - (\nabla f)^T x|^2}{d + |x|^2} \, d\mu;$$

and also with an inequality we established in [9] by exploiting the spherical invariance of the standard Gaussian distribution:

$$\int_{\mathbb{R}^d} f^2 \, d\mu \leq \frac{d(d+3)}{d-1} \int_{\mathbb{R}^d} \frac{|\nabla f|^2}{1 + |x|^2} \, d\mu.$$

5. Application to perturbed product measures. In this section, we present examples of perturbed product probability measures for which the spectral estimates of Theorems 3.2 and 4.2 apply. The results we obtain for these examples, even for the spectral gap, constitute an important part of the paper. Let us consider the potential

$$V(x) := \sum_{i=1}^{d} U_i(x_i) + \varphi(x), \qquad x \in \mathbb{R}^d,$$

where the smooth one-dimensional functions $U_i : \mathbb{R} \to \mathbb{R}$ are such that the e^{-U_i} are Lebesgue integrable on $\mathbb{R}$ and the interaction function $\varphi : \mathbb{R}^d \to \mathbb{R}$ is smooth and such that e^{-V} is Lebesgue integrable on $\mathbb{R}^d$. Hence the probability measure μ is nothing but a perturbation of the product probability measure with density proportional to $\exp(-\sum_{i=1}^{d} U_i)$ on $\mathbb{R}^d$.

Recall first the basic result for estimating the spectral gap of these models with interaction. If the potentials U_i are uniformly convex at infinity and the function φ is convex on $\mathbb{R}^d$ (the latter assumption might be weakened to a sufficiently small non-positive lower bound on its Hessian matrix), then the whole potential V itself is uniformly convex at infinity so that the famous Holley–Stroock perturbation principle applies (cf. [20]), the lower bound obtained on the spectral gap depending poorly on the dimension (except in the case $\varphi = 0$, for which μ is a product probability measure, meaning that λ_1 is dimension-free according to the well-known tensorization property of the Poincaré inequality).

Another method which turned out to be more convenient in terms of the dependence on the parameters of interest was developed by Helffer at the end of the 90's for some models such that the potentials U_i are uniformly convex at infinity and the nearest-neighbour interaction has quadratic growth. It mainly focuses on the use of the one-dimensional conditional distributions and is based on a uniform spectral gap assumption for these one-dimensional measures. In our language, the final estimate obtained by Helffer [18] is the following: if $\overline{\nabla^2 V}$ stands for the matrix $\nabla^2 V$ with null diagonal, i.e.,

$$\overline{\nabla^2 V} := \nabla^2 V - \sum_{i=1}^{d} \partial_{i,i}^2 V \, E_{i,i},$$

where $E_{i,j}$ is the matrix with all entries equal to 0 except the one in the ith line and jth column which is 1, then the spectral gap satisfies the lower bound

$$\lambda_1 \geq \inf_{\mathbb{R}^d} \rho(\overline{\nabla^2 V}) + \min_{i=1,\ldots,d} \inf_{\hat{x}_i \in \mathbb{R}^{d-1}} \lambda_1^{\hat{x}_i},$$

where for any fixed $x \in \mathbb{R}^d$, $\hat{x}_i := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d) \in \mathbb{R}^{d-1}$ and $\lambda_1^{\hat{x}_i}$ denotes the spectral gap of the one-dimensional conditional probability measure corresponding to the ith coordinate under μ, the other being fixed equal to $\hat{x_i}$. Such an approach has been further explored and extended in [15, 26], and more recently in [3], in which the principle of the method is nicely and briefly summarized. In particular the Brascamp–Lieb approach used in [3] suggests that the latter estimate might be refined as

$$\lambda_1 \geq \inf_{x\in\mathbb{R}^d} \rho\Big(\overline{\nabla^2 V}(x) + \sum_{i=1}^{d} \lambda_1^{\hat{x_i}} E_{i,i}\Big), \tag{5.1}$$

a lower bound also emphasized by Chen [12].

Actually, our intertwining approach is of different nature than Helffer's since it is more global in space and avoids the use of these one-dimensional conditional distributions. We will see below that on the one hand it leads to a different set of assumptions ensuring some convenient estimates on the spectral gap, allowing us to consider some interesting examples which do not enter into the previous framework, and on the other hand we go beyond the spectral gap by estimating the $(d+1)$th positive eigenvalue λ_{d+1}. To illustrate the relevance of this approach, let us start by considering two non-classical examples, the standard Gaussian product probability measure with quartic interaction and the Subbotin distribution with a convex Lipschitz interaction term. As far as we know, the results (even for the spectral gap) for these two models are new.

5.1. Two non-classical examples. The first model deals with a standard Gaussian product probability measure perturbed by a quartic nearest-neighbour interaction. The potential V is of the following form:

$$V(x) = \sum_{i=1}^{d} \frac{x_i^2}{2} + J\sum_{i=1}^{d} x_i^2\, x_{i+1}^2, \qquad x \in \mathbb{R}^d,$$

with J some non-negative parameter controlling the interaction term. Since $J \geq 0$, the function e^{-V} is Lebesgue integrable on $\mathbb{R}^d$. Here and in the remainder of the paper, for any given element $x \in \mathbb{R}^d$, we use the conventions $x_{d+1} := x_1$ and $x_{-1} := x_d$. For this model we have

$$U_i(x_i) := x_i^2/2 \quad \text{and} \quad \varphi(x) := J\sum_{i=1}^{d} x_i^2 x_{i+1}^2.$$

Actually, a result of Helffer and Nier [19] tells us that the spectrum of the non-negative operator $-L$ is discrete. In particular they noticed that the quartic interaction term makes the potential V far from being convex. Indeed, for any $x \in \mathbb{R}^d$ we have

$$\nabla^2\varphi(x) = 2J\begin{pmatrix} x_2^2+x_d^2 & 2x_1x_2 & 0 & \dots & 0 & 2x_1x_d \\ 2x_1x_2 & x_1^2+x_3^2 & 2x_2x_3 & 0 & & 0 \\ 0 & & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & & 0 \\ 0 & & 0 & 2x_{d-2}x_{d-1} & x_{d-2}^2+x_d^2 & 2x_{d-1}x_d \\ 2x_1x_d & 0 & \dots & 0 & 2x_{d-1}x_d & x_{d-1}^2+x_1^2 \end{pmatrix},$$

and it is not difficult to see that $\nabla^2 V$ is not uniformly bounded from below. The present setting is the prototype example for which quantitative spectral estimates are difficult to address with Helffer's method. Indeed, careful attention paid to the particular form of V shows that the ith one-dimensional conditional distributions knowing $\hat{x_i}$ are centered Gaussian with variance $1/(1+2J(x_{i-1}^2+x_{i+1}^2))$, so that the matrix involved in (5.1) coincides surprisingly with $\nabla^2 V$ since in this Gaussian setting we know that the spectral gap is the inverse of the variance, i.e., $\lambda_1^{\hat{x_i}} = 1+2J(x_{i-1}^2+x_{i+1}^2)$.

Hence it is a challenging question to propose a convenient lower bound on the spectral gap for this model, together with an estimate on the higher eigenvalue λ_{d+1}: this is the matter of Proposition 1.1 appearing in the Introduction. The more precise result covering Proposition 1.1 is the following (since it is part of a strategy that might be applied to more general models, we delay the proof to Section 5.3).

Proposition 5.1. *Assume that*

$$V(x) := \sum_{i=1}^d \frac{x_i^2}{2} + J\sum_{i=1}^d x_i^2\, x_{i+1}^2, \qquad x\in\mathbb{R}^d,$$

where $J\geq 0$. Then we have the following dimension-free spectral estimates:

- $\lambda_1 \geq \frac{1+\sqrt{1-16J}}{2}$ *for any* $0\leq J\leq \frac{1}{16}$;
- $\lambda_{d+1} \geq \frac{1+\sqrt{1-16J}}{2} + \frac{\sqrt{1-16J}+\sqrt{1-32J}}{2}$ *for any* $0\leq J\leq\frac{1}{32}$.

Now, let us focus on the second example, the Subbotin distribution with a convex Lipschitz nearest-neighbour interaction. We consider the convex potential

$$V(x) := \sum_{i=1}^d \frac{|x_i|^a}{a} + J\sum_{i=1}^d |x_{i+1}-x_i|, \quad x\in\mathbb{R}^d,$$

where J is non-negative. Here the one-dimensional potentials are $U_i(x_i) := |x_i|^a/a$ with $a\in[1,2]$ (the underlying probability measure with Lebesgue density proportional to e^{-U_i} is called the *Subbotin*, or *exponential power, distribution*) and the interaction is convex and of Lipschitz type, i.e.,

$$\varphi(x) := J \sum_{i=1}^{d} |x_{i+1} - x_i|, \qquad x \in \mathbb{R}^d.$$

Once again a result of [19] tells us that the spectrum $\sigma(-L)$ is discrete in the case $a \in (1,2]$, whereas for $a = 1$, we only know that $\lambda_1 > 0$. For this example, it seems difficult to use the estimate (5.1) since the spectral gaps $\lambda_1^{\hat{x}_i}$, $i = 1, \dots, d$, of the one-dimensional conditional probability measures are hard to estimate in terms of the remainder coordinates $\hat{x}_i$ (such an observation was already noted in [15]). The result we obtain for this model is the following (again the proof is postponed to Section 5.3).

Proposition 5.2. *Let*

$$V(x) := \sum_{i=1}^{d} \frac{|x_i|^a}{a} + J \sum_{i=1}^{d} |x_i - x_{i+1}|, \qquad x \in \mathbb{R}^d,$$

where $a \in [1,2]$ and $J \geq 0$. Then we have the following dimension-free spectral estimates:

- $\lambda_1 \geq a(3a-2)/8 - 2J^2$ *for* $J \in [0, \sqrt{a(3a-2)}/4)$;
- $\lambda_{d+1} \geq a(3a-2)(1+(a-1)^{2/a})/8 - 4J^2$ *for* $J \in [0, \sqrt{a(3a-2)(a-1)^{2/a}}/4)$, *provided* $a \in (1,2]$.

Let us conclude this short subsection devoted to the two examples by some comments. In the case of the standard Gaussian product probability measure with quartic interaction or with a convex Lipschitz interaction term (i.e., the second example with $a = 2$), we observe that the optimal values $\lambda_1 = 1$ and $\lambda_{d+1} = 2$, available in the non-interacting case $J = 0$, are recovered. However this is not the case for general Subbotin distributions (that is, $a \neq 2$) since we know that in the non-interacting case $J = 0$ we have $\lambda_1 \geq a^2/4$ (cf. [10]). Certainly, all our estimates above have no reason to be sharp in full generality, meaning that there is still room for improvement. However, they offer a dimension-free behaviour somewhat similar to the non-interacting case, meaning that both situations are comparable as long as the interaction parameter J is sufficiently small. As a final remark, we do not know what happens when J is larger, and in particular if there exists some phase transition. Such a question could be the matter of forthcoming research.

5.2. The general case. Even if the two models above are quite different in essence, Propositions 5.1 and 5.2 share a large part of a common proof, as we will see in Section 5.3, and are consequences of the following results, which might be interesting in themselves to treat other examples. The first one is devoted to the spectral gap whereas the second one deals with the higher eigenvalue λ_{d+1}. In particular, λ_1 and λ_{d+1} can belong to the essential spectrum. Moreover, although the forthcoming analysis might be adapted to the non-convex case, for simplicity we only consider one-dimensional convex potentials U_i.

Below, we denote by ∂_i (resp. $\partial^2_{i,j}$) the first- (resp. second-) order partial derivative with respect to the ith coordinate (resp. to the ith and jth coordinates).

Proposition 5.3. *Assume that*

$$V(x) := \sum_{i=1}^{d} U_i(x_i) + \varphi(x), \qquad x \in \mathbb{R}^d,$$

where the one-dimensional potentials U_i are smooth, convex and such that the functions e^{-U_i} are Lebesgue integrable on $\mathbb{R}$, and the interaction φ is smooth on $\mathbb{R}^d$ and such that e^{-V} is Lebesgue integrable on $\mathbb{R}^d$. Moreover, assume that there exist $\varepsilon_1, \ldots, \varepsilon_d \in [0,1)$ and $\alpha > 0$ such that

$$\nabla^2\varphi(x) + \sum_{i=1}^{d} \{(1-\varepsilon_i)(U_i''(x_i) + \varepsilon_i U_i'(x_i)^2) + \varepsilon_i \partial_i \varphi(x) U_i'(x_i)\} E_{i,i} \geq \alpha I.$$

Then the spectral gap satisfies the bound

$$\lambda_1 \geq \alpha.$$

Proposition 5.4. *Under the same notation and assumptions as in Proposition* 5.3, *assume now that the $\varepsilon_1, \ldots, \varepsilon_d$ belong to $[0, 1/2)$. Assume moreover that for every fixed $i = 1, \ldots, d$ there exist $\widetilde{\varepsilon_i} \in [0,1)$ and $\beta_i > 0$ such that*

$$\begin{aligned}\nabla^2\varphi(x) + \sum_{j\neq i} \{(1-\varepsilon_j)(U_j''(x_j) + \varepsilon_j U_j'(x_j)^2) + \varepsilon_j \partial_j \varphi(x) U_j'(x_j)\} E_{j,j} \\ + \{(1-\widetilde{\varepsilon_i})[(1-2\varepsilon_i)U_i''(x_i) + \widetilde{\varepsilon_i}(1-2\varepsilon_i)^2 U_i'(x_i)^2] \\ + \widetilde{\varepsilon_i}(1-2\varepsilon_i)\partial_i\varphi(x)U_i'(x_i)\} E_{i,i} \geq \beta_i I.\end{aligned}$$

Then

$$\lambda_{d+1} \geq \alpha + \min_{i=1,\ldots,d} \beta_i.$$

Proof of Proposition 5.3. Plugging (3.3) into Theorem 3.2 through the identity $A = (J_H^T)^{-1}$, we have to find some convenient diffeomorphism $H \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$ such that the assumption $(\mathcal{S})$ is satisfied together with the fact that the smallest eigenvalue of the symmetric matrix $-J^T_{\mathcal{L}H}(J^T_H)^{-1}$ is bounded from below by some positive constant. The idea is to consider a diagonal diffeomorphism, i.e. $H(x) = (h_1(x_1), \ldots, h(x_d))^T$, $x \in \mathbb{R}^d$, where the h_i are some increasing functions on $\mathbb{R}$. In this case the matrix A is diagonal, with $a_{i,i} = 1/h_i'$, so that the assumption $(\mathcal{S})$ is trivially satisfied. Given a matrix $M \in \mathcal{M}_{d\times d}(\mathbb{R})$, recall that $\overline{M}$ stands for the matrix M with null diagonal. Since $\overline{\nabla^2 V} = \overline{\nabla^2 \varphi}$, we have

$$
\begin{aligned}
-J_{\mathcal{L}H}^T(x)(J_H^T)^{-1}(x) = \left(-\frac{\partial_i L h_j(x)}{h_j'(x_j)}\right)_{i,j=1,\ldots,d} &= \overline{\nabla^2 V}(x) - \sum_{i=1}^d \frac{\partial_i L h_i(x)}{h_i'(x_i)} E_{i,i} \\
&= \overline{\nabla^2 \varphi}(x) + \sum_{i=1}^d \left(\frac{(-L_i h_i)'(x_i)}{h_i'(x_i)} + \partial_{i,i}^2 \varphi(x) + \frac{\partial_i \varphi(x) h_i''(x_i)}{h_i'(x_i)}\right) E_{i,i} \\
&= \nabla^2 \varphi(x) + \sum_{i=1}^d \left(\frac{(-L_i h_i)'(x_i)}{h_i'(x_i)} + \frac{\partial_i \varphi(x) h_i''(x_i)}{h_i'(x_i)}\right) E_{i,i},
\end{aligned}
$$

where for each $i = 1, \ldots, d$, L_i denotes the one-dimensional dynamics defined as

$$
L_i h(y) := h''(y) - U_i'(y) h'(y), \qquad y \in \mathbb{R},
$$

having an invariant probability measure whose Lebesgue density on $\mathbb{R}$ is proportional to e^{-U_i}. Choosing then the one-dimensional functions

$$
h_i' = e^{\varepsilon_i U_i},
$$

where the parameters ε_i belong to $[0, 1)$, we obtain

$$
\begin{aligned}
&-J_{\mathcal{L}H}^T(x)(J_H^T)^{-1}(x) \\
&\quad = \nabla^2 \varphi(x) + \sum_{i=1}^d \{(1-\varepsilon_i)(U_i''(x_i) + \varepsilon_i U_i'(x_i)^2) + \varepsilon_i \partial_i \varphi(x) U_i'(x_i)\} E_{i,i},
\end{aligned}
$$

from which the desired conclusion follows. ■

As expected, the choice $\varepsilon_i = 0$ for all $i = 1, \ldots, d$ is well-adapted for uniformly convex potentials U_i but becomes irrelevant in the present setting of only convex U_i.

Proof of Proposition 5.4. Using the notation of the proof of Proposition 5.3, we have $h_i' \in L^2(\mu)$ as long as $\varepsilon_i \in [0, 1/2)$ for all $i = 1, \ldots, d$. The choice of a diagonal diffeomorphism H entails that the matrix S is diagonal, with $S_{i,i} = 1/a_{i,i}^2 = (h_i')^2$. In particular the assumptions of Theorem 4.2 are satisfied and it leads to the bound

$$
\lambda_{d+1} \geq \lambda_1^A + \alpha.
$$

Now it remains to estimate the spectral gap λ_1^A of the operator $-\mathcal{L}_A$. The choice of the diagonal matrix $A = (J_H^T)^{-1}$ entails that $\mathcal{L}_A$ is diagonal, i.e.,

$$
\mathcal{L}_A = \begin{pmatrix} L_A^1 & & \\ & \ddots & \\ & & L_A^d \end{pmatrix},
$$

where for each $i = 1, \dots, d$, the operator L_A^i is given for any $f \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R})$ by

$$L_A^i f = Lf + 2a_{i,i}(\nabla a_{i,i}^{-1})^T \nabla f = Lf - (\nabla \log a_{i,i}^2)^T \nabla f = \Delta f - (\nabla V_A^i)^T \nabla f.$$

Above, the potential $V_A^i := V + \log a_{i,i}^2$ can be rewritten explicitly as

$$V_A^i(x) = V(x) - \log h_i'(x_i)^2 = \sum_{j \neq i} U_j(x_j) + (1 - 2\varepsilon_i) U_i(x_i) + \varphi(x).$$

Denote by $\lambda_1^{A,i}$ the spectral gap for the (self-adjoint entension of the) symmetric and non-negative operator $-L_A^i$ whose invariant probability measure μ_A^i on $\mathbb{R}^d$ has Lebesgue density proportional to $e^{-V_A^i}$. The spectral gap λ_1^A is the optimal constant $\lambda > 0$ in the Poincaré inequality (4.3) which can be rewritten in the present diagonal context as follows: for any $F \in \mathcal{C}_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$,

$$\lambda \sum_{i=1}^d \int_{\mathbb{R}^d} \widetilde{F_i}^2 \, d\mu_A^i \leq - \sum_{i=1}^d \int_{\mathbb{R}^d} \widetilde{F_i} L_A^i \widetilde{F_i} \, d\mu_A^i,$$

with

$$\widetilde{F_i} = F_i - m_S(F)_i = F_i - \int_{\mathbb{R}^d} F_i \, d\mu_A^i.$$

Therefore we immediately have the equality

$$\lambda_1^A = \min_{i=1,\dots,d} \lambda_1^{A,i}$$

(to see this, we apply the Poincaré inequality for each probability measure μ_A^i to get $\geq$, whereas the reverse inequality is obtained by considering vector fields F with all coordinates vanishing except one). Hence we are reduced to estimating from below the spectral gap $\lambda_1^{A,i}$ for each $i = 1, \dots, d$, and this is done by using Proposition 5.3 adapted to the present situation, i.e., for the potential V_A^i in place of V, the only difference between them being that the one-dimensional potential U_i in V is replaced by $(1 - 2\varepsilon_i)U_i$ to obtain V_A^i. ∎

5.3. The proofs for the two examples. We are now ready to provide the proofs of the dimension-free spectral estimates for the two examples of interest, contained in Propositions 5.1 and 5.2. As announced, both are based on Propositions 5.3 and 5.4. Let us start with the standard Gaussian product probability measure with quartic interaction.

Proof of Proposition 5.1. Recall that

$$U_i(x_i) := x_i^2/2 \quad \text{and} \quad \varphi(x) := J \sum_{i=1}^d x_i^2 x_{i+1}^2, \qquad x \in \mathbb{R}^d,$$

where $J \geq 0$. First, let us see how Proposition 5.3 can be used in the present context. For any $y \in \mathbb{R}^d$ we have

$$\begin{aligned} y^T \nabla^2 \varphi(x) y &= 8J \sum_{i=1}^d y_i x_i y_{i+1} x_{i+1} + 2J \sum_{i=1}^d y_i^2 (x_{i-1}^2 + x_{i+1}^2) \\ &\geq -2J \Big(\sum_{i=1}^d y_i^2 x_i^2 + \sum_{i=1}^d y_{i+1}^2 x_{i+1}^2 \Big) = -4J \sum_{i=1}^d y_i^2 x_i^2, \end{aligned}$$

where we used the trivial inequality $uv \geq -u^2/2 - v^2/2$, $u, v \in \mathbb{R}$. Hence we obtain the following inequality between matrices (the RHS below is diagonal):

$$\nabla^2 \varphi(x) \geq -4J \begin{pmatrix} x_1^2 & & \\ & \ddots & \\ & & x_d^2 \end{pmatrix}, \quad x \in \mathbb{R}^d. \tag{5.2}$$

Moreover, for all $x \in \mathbb{R}^d$ and $i = 1, \ldots, d$ we have

$$\partial_i \varphi(x) U_i'(x_i) = 2J(x_{i-1}^2 + x_{i+1}^2) x_i^2 \geq 0. \tag{5.3}$$

The inequalities (5.2) and (5.3) tell us that in order to apply Proposition 5.3 to obtain an estimate on λ_1, we have to find $\varepsilon_i \in [0, 1)$, $i = 1, \ldots, d$, such that the smallest eigenvalue of the diagonal matrix

$$\begin{pmatrix} (1-\varepsilon_1)(1+\varepsilon_1 x_1^2) - 4Jx_1^2 & & \\ & \ddots & \\ & & (1-\varepsilon_d)(1+\varepsilon_1 x_d^2) - 4Jx_d^2 \end{pmatrix}$$

is bounded from below by some $\alpha > 0$. Observing that the latter matrix is uniformly bounded from below by $(\min_{i=1,\ldots,d} 1 - \varepsilon_i)$ under the constraint that $0 \leq J \leq \min_{i=1,\ldots,d} \varepsilon_i (1-\varepsilon_i)/4$, by a simple optimization in the $\varepsilon_i \in [0, 1)$ yields

$$\alpha := \frac{1 + \sqrt{1 - 16J}}{2}, \quad 0 \leq J \leq \frac{1}{16},$$

the optimal parameters ε_i being all equal to $\varepsilon := (1 - \sqrt{1-16J})/2 \in [0, 1/2]$.

Dealing now with λ_{d+1}, we fix $i = 1, \ldots, d$ and we need to find some $\widetilde{\varepsilon_i} \in [0, 1)$ such that the smallest eigenvalue of the matrix appearing in Proposition 5.4 is bounded from below by some $\beta_i > 0$, all the ε_j being fixed equal to ε, with this time $J \in [0, 1/16)$ so that all the ε_j are in $[0, 1/2)$, a requirement of Proposition 5.4. Thanks to (5.2) and (5.3), we are led to study the diagonal matrix

$$\sum_{j \neq i} ((1-\varepsilon)(1+\varepsilon x_j^2) - 4Jx_j^2) E_{j,j} + ((1-\widetilde{\varepsilon_i})(1 - 2\varepsilon + \widetilde{\varepsilon_i}(1-2\varepsilon)^2 x_i^2) - 4Jx_i^2) E_{i,i},$$

which is uniformly bounded from below by the constant $(1-\widetilde{\varepsilon_i})(1-2\varepsilon)$ under the constraint $0 \leq J \leq \min_{i=1,\ldots,d}(1-\widetilde{\varepsilon_i})\widetilde{\varepsilon_i}(1-2\varepsilon)^2/4$ (the latter is $< 1/16$ when J is positive). Optimizing in $\widetilde{\varepsilon_i} \in [0,1)$ leads to the desired lower bound: for any $J \in [0,1/32]$,

$$\beta_i := (\sqrt{1-16J} + \sqrt{1-32J})/2,$$

the optimal parameter being $\widetilde{\varepsilon_i} = (1-\sqrt{(1-32J)/(1-16J)})/2$. ∎

Let us now turn to the case of the Subbotin distribution with a convex Lipschitz interaction term.

Proof of Proposition 5.2. Given the parameter $a \in [1,2]$ of the Subbotin distribution, the quantities of interest are

$$U_i(x_i) := \frac{|x_i|^a}{a} \quad \text{and} \quad \varphi(x) := J\sum_{i=1}^{d} |x_{i+1}-x_i|, \quad x \in \mathbb{R}^d,$$

with $J \geq 0$. Once again let us see how Propositions 5.3 and 5.4 might be applied. Although the U_i are not of class $\mathcal{C}^2$ at the origin, this does not play an important role in our study and thus can be ignored, at the price of an unessential regularization procedure. However we need to regularize the interaction function φ by considering the following smooth convex version:

$$\varphi_\tau(x) := J\sum_{i=1}^{d} \sqrt{\tau^2 + (x_{i+1}-x_i)^2},$$

for which the parameter τ will tend to 0.

First we focus our attention on the spectral gap λ_1. According to Proposition 5.3, we aim at finding $\varepsilon_i \in [0,1)$ such that the matrix

$$\nabla^2\varphi_\tau(x) + \sum_{i=1}^{d}\{(1-\varepsilon_i)(U_i''(x_i)+\varepsilon_i U_i'(x_i)^2) + \varepsilon_i\partial_i\varphi_\tau(x)U_i'(x_i)\}E_{i,i}$$

is uniformly bounded from below by some $\alpha > 0$. Since the function $x \mapsto \partial_i\varphi_\tau(x)U_i'(x_i)$ does not have a constant sign, we have to proceed differently from the first example and the idea is thus to decorrelate the terms $\varphi_\tau(x)$ and $U_i'(x_i)$ by using the trivial inequality $uv \geq -u^2/2 - v^2/2$, $u,v \in \mathbb{R}$, applied to $u = \partial_i\varphi(x)$ and $v = \varepsilon_i U_i'(x_i)$. Hence the latter matrix is bounded from below by the matrix

$$\nabla^2\varphi_\tau(x) + \sum_{i=1}^{d}\left\{-\frac{\partial_i\varphi_\tau(x)^2}{2} + \left(1-\frac{3\varepsilon_i}{2}\right)(U_i''(x_i)+\varepsilon_i U_i'(x_i)^2)\right\}E_{i,i},$$

where we used the convexity of U_i and the fact that $1-\varepsilon_i \geq 1-3\varepsilon_i/2$ to simplify the forthcoming optimization. Now we have

$$\inf_{x_i\in\mathbb{R}}\{U_i''(x_i)+\varepsilon_i U_i'(x_i)^2\} = \frac{a}{2}\left(1-\frac{a}{2}\right)^{1-2/a}\varepsilon_i^{2/a-1},$$

and choosing all the ε_i to be $\varepsilon = 1 - a/2 \in [0, 1/2]$ (thus $1 - 3\varepsilon_i/2 > 0$), we see that this matrix is bounded from below by

$$\left(\inf_{\mathbb{R}^d} \rho(\nabla^2 \varphi_\tau) - \max_{i=1,\dots,d} \frac{\|\partial_i \varphi_\tau\|_\infty^2}{2} + \frac{a(3a-2)}{8} \right) I.$$

Since ϕ_τ is convex and $\|\partial_i \varphi_\tau\|_\infty \le 2J$, letting $\tau \to 0$ yields the lower bound

$$\alpha := \frac{a(3a-2)}{8} - 2J^2,$$

which is positive for J small enough, i.e., for any $J \in [0, \sqrt{a(3a-2)}/4)$.

Let us now consider the higher eigenvalue λ_{d+1}, with $a \in (1, 2]$, i.e., a is now different from 1. Let $i = 1, \dots, d$ be fixed together with all the $\varepsilon_j = \varepsilon$ which belong this time to $[0, 1/2)$ since $a \neq 1$. In order to use Proposition 5.4, we are looking for some $\widetilde{\varepsilon_i} \in [0, 1)$ such that the smallest eigenvalue of the matrix appearing in the proposition is bounded from below by some $\beta_i > 0$. A somewhat similar analysis tells us that it is sufficient to bound from below by some $\beta_i > 0$ the smallest eigenvalue of the matrix

$$\nabla^2 \varphi_\tau(x) + \sum_{j \neq i} \left\{ -\frac{\partial_j \varphi_\tau(x)^2}{2} + \left(1 - \frac{3\varepsilon}{2}\right) (U_j''(x_j) + \varepsilon U_j'(x_j)^2) \right\} E_{j,j}$$
$$+ \left\{ -\frac{\partial_i \varphi_\tau(x)^2}{2} + \left(1 - \frac{3\widetilde{\varepsilon_i}}{2}\right) \left((1-2\varepsilon) U_i''(x_i) + \widetilde{\varepsilon_i}(1-2\varepsilon)^2 U_i'(x_i)^2\right) \right\} E_{i,i}.$$

We have

$$\inf_{x_i \in \mathbb{R}} \{(1-2\varepsilon) U_i''(x_i) + \widetilde{\varepsilon_i}(1-2\varepsilon)^2 U_i'(x_i)^2\}$$
$$= \frac{a}{2} \left(1 - \frac{a}{2}\right)^{1-2/a} (1-2\varepsilon)^{2/a} \widetilde{\varepsilon_i}^{2/a-1},$$

and choosing once again $\widetilde{\varepsilon_i} = \varepsilon$ (so that $1 - 3\widetilde{\varepsilon_i}/2 > 0$) and letting $\tau \to 0$ entails that β_i might be chosen as

$$\beta_i := \frac{a(3a-2)(a-1)^{2/a}}{8} - 2J^2,$$

which is positive provided $J \in [0, \sqrt{a(3a-2)(a-1)^{2/a}}/4)$. ∎

Acknowledgments. The authors are grateful to the anonymous referees for various helpful suggestions.

MB is partially supported by the French ANR-18-CE40-0012 RAGE and ANR-19-CE40-0010-01 QUAMPROCS projects.

AJ is partially supported by the French ANR-18-CE40-006 MESA project.

References

[1] M. Arnaudon, M. Bonnefont and A. Joulin, *Intertwinings and generalized Brascamp–Lieb inequalities*, Rev. Math. Iberoamer. 34 (2018), 1021–1054.

[2] D. Bakry, I. Gentil and M. Ledoux, *Analysis and Geometry of Markov Diffusion Operators*, Grundlehren Math. Wiss. 348, Springer, Cham, 2014.

[3] F. Barthe and D. Cordero-Erausquin, *Invariances in variance estimates*, Proc. London Math. Soc. 106 (2013), 33–64.

[4] F. Barthe and B. Klartag, *Spectral gaps, symmetries and log-concave perturbations*, Bull. Hellenic Math. Soc. 64 (2020), 1–31.

[5] S. G. Bobkov, *Isoperimetric and analytic inequalities for log-concave probability measures*, Ann. Probab. 27 (1999), 1903–1921.

[6] S. G. Bobkov and M. Ledoux, *Weighted Poincaré-type inequalities for Cauchy and other convex measures*, Ann. Probab. 37 (2009), 403–427.

[7] F. Bolley, I. Gentil and A. Guillin, *Dimensional improvements of the logarithmic Sobolev, Talagrand and Brascamp–Lieb inequalities*, Ann. Probab. 46 (2018), 261–301.

[8] M. Bonnefont and A. Joulin, *Intertwining relations for one-dimensional diffusions and application to functional inequalities*, Potential Anal. 41 (2014), 1005–1031.

[9] M. Bonnefont, A. Joulin and Y. Ma, *Spectral gap for spherically symmetric log-concave probability measures, and beyond*, J. Funct. Anal. 270 (2016), 2456–2482.

[10] M. Bonnefont, A. Joulin and Y. Ma, *A note on spectral gap and weighted Poincaré inequalities for some one-dimensional diffusions*, ESAIM Probab. Statist. 20 (2016), 18–29.

[11] H. J. Brascamp and E. H. Lieb, *On extensions of the Brunn–Minkowski and Prékopa–Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation*, J. Funct. Anal. 22 (1976), 366–389.

[12] M. F. Chen, *Spectral gap and logarithmic Sobolev constant for continuous spin systems*, Acta Math. Sinica (Engl. Ser.) 24 (2008), 705–736.

[13] D. Cordero-Erausquin, *Transport inequalities for log-concave measures, quantitative forms and applications*, Canad. J. Math. 69 (2017), 481–501.

[14] D. Cordero-Erausquin, F. Fradelizi and B. Maurey, *The (B) conjecture for the Gaussian measure of dilates of symmetric convex sets and related problems*, J. Funct. Anal. 214 (2004), 410–427.

[15] I. Gentil and C. Roberto, *Spectral gaps for spin systems: some non-convex phase examples*, J. Funct. Anal. 180 (2001), 66–84.

[16] L. Goldstein, I. Nourdin and G. Peccati, *Gaussian phase transitions and conic intrinsic volumes: Steining the Steiner formula*, Ann. Appl. Probab. 27 (2017), 1–47.

[17] G. Hargé, *Reinforcement of an inequality due to Brascamp and Lieb*, J. Funct. Anal. 254 (2008), 267–300.

[18] B. Helffer, *Remarks on decay of correlations and Witten Laplacians, Brascamp–Lieb inequalities and semiclassical limit*, J. Funct. Anal. 155 (1998), 571–586.

[19] B. Helffer and F. Nier, *Criteria to the Poincaré inequality associated with Dirichlet forms in* $\mathbb{R}^d$, $d \geq 2$, Int. Math. Res. Notices 2003, 1199–1223.

[20] R. Holley and D. Stroock, *Logarithmic Sobolev inequalities and stochastic Ising models*, J. Statist. Phys. 46 (1987), 1159–1194.

[21] L. Hörmander, L^2 *estimates and existence theorems for the* $\bar{\partial}$ *operator*, Acta Math. 113 (1965), 89–152.

[22] J. Johnsen, *On the spectral properties of Witten-Laplacians, their range projections and Brascamp–Lieb's inequality*, Integral Equations Operator Theory 36 (2000), 288–324.

[23] R. Kannan, L. Lovász and M. Simonovits, *Isoperimetric problems for convex bodies and a localization lemma*, Discrete Comput. Geom. 13 (1995), 541–559.
[24] B. Klartag, *A Berry–Esseen type inequality for convex bodies with an unconditional basis*, Probab. Theory Related Fields 145 (2009), 1–33.
[25] A. V. Kolesnikov and E. Milman, *Brascamp–Lieb-type inequalities on weighted Riemannian manifolds with boundary*, J. Geom. Anal. 27 (2017), 1680–1702.
[26] M. Ledoux, *Logarithmic Sobolev inequalities for unbounded spin systems revisited*, in: Séminaire de Probabilités XXXV, Lecture Notes in Math. 1755, Springer, Berlin, 2001, 167–194.
[27] V. H. Nguyen, *Dimensional variance inequalities of Brascamp–Lieb type and a local approach to dimensional Prékopa's theorem*, J. Funct. Anal. 266 (2014), 931–955.
[28] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. IV. Analysis of Operators*, Academic Press, New York, 1978.

Michel Bonnefont
UMR CNRS 5251
Institut de Mathématiques de Bordeaux
Université Bordeaux 1
Bordeaux, France
E-mail: michel.bonnefont@math.u-bordeaux1.fr
URL: http://www.math.u-bordeaux1.fr/~mibonnef/

Aldéric Joulin
UMR CNRS 5219
Institut de Mathématiques de Toulouse
Université de Toulouse
Toulouse, France
E-mail: ajoulin@insa-toulouse.fr
URL: http://perso.math.univ-toulouse.fr/joulin/