

# A NOTE ON EIGENVALUES ESTIMATES FOR ONE-DIMENSIONAL DIFFUSION OPERATORS

MICHEL BONNEFONT AND ALDÉRIC JOULIN

ABSTRACT. Dealing with one-dimensional diffusion operators, we obtain upper and lower variational formulae on the eigenvalues given by the max-min principle, generalizing the celebrated result of Chen and Wang on the spectral gap. Our inequalities reveal to be sharp at least when the eigenvalues considered belong to the discrete spectrum of the operator, since in this case both lower and upper bounds coincide and involve the associated eigenfunctions. Based on the intertwining between diffusion operators and some convenient gradients with weights, our approach also allows to estimate the gap between the two first positive eigenvalues when the spectral gap belongs to the discrete spectrum.

## 1. INTRODUCTION

Given a probability measure  $\mu$  on  $\mathbb{R}^n$  whose density with respect to the Lebesgue measure is (proportional to)  $e^{-V}$ , where  $V$  is some smooth potential, one can associate a canonical self-adjoint diffusion operator,

$$Lf = \Delta f - \nabla V \nabla f,$$

where  $\Delta$  is the Laplacian and  $\nabla$  stands for the Euclidean gradient on  $\mathbb{R}^n$ . Under some reasonable assumptions on  $V$ , it is well-known that the underlying Markov process converges in distribution to the invariant and reversible probability measure  $\mu$ . Moreover the speed of convergence in  $L^2(\mu)$  is given by the so-called spectral gap  $\lambda_1(-L)$  of the operator, that is, its first positive eigenvalue. We refer to Section 2 for precise definitions together with more details about this large body of work.

Except in some very particular situations, the exact value of the spectral gap is unknown in general and it is thus of particular interest to obtain some estimates on this quantity. In the one-dimensional case, things are a little simpler and it reveals to be of crucial importance since the study of the convergence to equilibrium of many high-dimensional Markovian models might be reduced to a careful analysis of an associated one-dimensional diffusion. Two decades ago, a lower variational

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formula for  $\lambda_1(-L)$  was proposed by Chen and Wang [10] through a coupling approach. It can be written as follows:

$$\lambda_1(-L) \geq \sup_b \inf_{x \in \mathbb{R}} V''(x) - \frac{Lb(x)}{b(x)},$$

where the supremum runs over all smooth positive function  $b$ . Moreover, if the spectral gap belongs to the discrete spectrum of the operator  $-L$ , they proved that the equality holds by choosing  $b$  as the derivative of an associated eigenfunction. Recently, such a formula has been revisited in [6, 8] by a totally different method, namely the use of the so-called intertwining between gradients and operators. We mention that Chen-Wang's formula is not a special feature of the one-dimensional case since it holds in higher dimension, as it has been established in [1]. However, the analysis is rather delicate and involves some tedious technicalities since the objects of interest act on differential forms rather than on functions, cf. also [7], an instance which does not really occur in the one-dimensional case.

Dealing with one-dimensional diffusion operators, the purpose of this note is to extend Chen-Wang's variational formula to higher eigenvalues given by the max-min principle. We provide lower and upper variational formulae on these eigenvalues and study the case of optimality. Let us briefly describe the main ideas of the proof of the two-sided estimates which is divided into several steps. Roughly speaking, the intertwining method consists in differentiating with some weight a diffusion operator and trying to write the result as a Schrödinger type operator acting on this derivative with weight. At this step, a key argument is to adapt to the weighted case a result established by Johnsen [12] who noticed that the intertwining is a unitary transformation between these two operators, so that their spectra coincide. According to this observation, the point then is to bound the associated multiplicative, or 0-order, potential appearing in this Schrödinger operator, yielding to a new diffusion operator. Then the preceding procedure has to be iterated for this new diffusion operator. At each step of the analysis, the weight appearing in the gradient used for the intertwining has to be chosen conveniently, since all the objects of interest strongly depend on it (together with the previous ones) in their very definition, in particular when studying the case of optimality in the estimates.

The content of the paper is the following. In Section 2 we introduce the notation and the main ideas of the intertwining approach. Moreover we recall some basic facts on spectral analysis for self-adjoint operators, including the diffusion and Schrödinger operators emphasized by the intertwining. Section 3 is devoted to our main result, contained in Theorem 3.4, in which some lower and upper variational formulae for eigenvalues given by the max-min principle are provided, together with a study of the possible case of optimality involving the underlying eigenfunctions. As mentioned above, the proof is quite technical and requires some preparation and intermediate results. In Section 4, we illustrate our main

result by some classical and less classical examples. On the one hand we consider the standard case of a uniformly convex potential, for which we recover the one-dimensional version of a result due to Milman [16], comparing the eigenvalues of a diffusion operator with the ones of the Ornstein-Uhlenbeck operator associated to the standard Gaussian distribution. On the other hand, the intertwining approach is sufficiently robust to cover more interesting examples involving only convex potentials, e.g. the so-called Subbotin distribution, and even non-convex ones such as a double-well or an oscillating Gaussian potentials. Up to some universal constants, the examples under review exhibit sharp estimates with respect to the parameters of interest, in accordance with the celebrated Weyl law describing the asymptotic behavior of these eigenvalues. Finally, we propose in Section 5 a first attempt to study the gap between consecutive eigenvalues of a diffusion operator. The proof, which also uses the intertwining method, is based on a celebrated result of Brascamp and Lieb [9] about log-concavity properties of the ground state of a standard Schrödinger operator consisting of a Laplacian plus some potential. For the moment we are only able to estimate the gap between the two first positive eigenvalues, but the intertwining approach seems to be relevant to address this problem in full generality.

## 2. PRELIMINARIES AND BASIC MATERIAL

**2.1. Intertwinings.** Let  $\mathcal{C}^\infty(\mathbb{R})$  be the space of infinitely differentiable real-valued functions on the real line and let  $\mathcal{C}_0^\infty(\mathbb{R})$  be the subspace of  $\mathcal{C}^\infty(\mathbb{R})$  consisting of smooth compactly supported functions. The original diffusion operator we consider in this paper is defined on  $\mathcal{C}^\infty(\mathbb{R})$  by

$$Lf = f'' - V' f',$$

where  $V \in \mathcal{C}^\infty(\mathbb{R})$  is some smooth potential. The underlying invariant (and reversible) measure  $\mu$  has Lebesgue density proportional to  $e^{-V}$  and is assumed, throughout the paper, to be finite and renormalized into a probability measure.

More generally if  $\mathcal{F}$  stands for the set of functions in  $\mathcal{C}^\infty(\mathbb{R})$  which do not vanish, then we introduce for a given function  $a \in \mathcal{F}$  the following diffusion operator

$$L_a f = f'' - V_a' f', \tag{2.1}$$

where

$$V_a = V + \log(a^2).$$

The potential  $V_a$  is then in  $\mathcal{C}^\infty(\mathbb{R})$  and the invariant (and reversible) measure  $\mu_a$ , which is non-necessarily finite, has its Lebesgue-density proportional to  $e^{-V_a}$ . In particular choosing  $a = 1$  we have  $V_1 = V$  and thus  $L_1 = L$  and  $\mu_1 = \mu$ . In other words when  $a$  is chosen to be constant then the dynamics remains the same (up to some constant factors).

Let us recall a result which is the cornerstone of the forthcoming analysis. It deals with the notion of intertwining between diffusion operators and weighted gradients initiated in [6] in the one-dimensional case. Denote  $\partial_a$  the usual gradient multiplied by a function  $a \in \mathcal{F}$ , seen as a weight in the sequel, i.e.  $\partial_a f = af'$  (we will sometimes use the notation  $\partial$  for the usual derivative). In the present language the result stands as follows (the proof is straightforward after some basic algebra).

**Lemma 2.1.** *Letting  $a \in \mathcal{F}$ , if  $b \in \mathcal{F}$  is a given weight then we have the following intertwining relation between operators and weighted gradient: for every  $f \in C^\infty(\mathbb{R})$ ,*

$$\partial_b L_a f = (L_{ab} - M_a^b) \partial_b f, \quad (2.2)$$

where the operator  $L_{ab}$  is obtained as in (2.1), the function  $a$  being replaced by the product  $ab$ , and the smooth potential  $M_a^b$  is given by

$$\begin{aligned} M_a^b &= V_a'' - b L_a(1/b) \\ &= V_a'' + \frac{L_{ab}(b)}{b} \\ &= \frac{(-L_a h)'}{h'}, \end{aligned} \quad (2.3)$$

the last equality being obtained by rewriting  $b$  as  $b = 1/h'$  with  $h' \in \mathcal{F}$ .

Above the operator  $L_{ab} - M_a^b$  is a Schrödinger type operator because of the presence of the multiplicative, or 0-order, potential  $M_a^b$ . It is worthwhile noticing that multiplying the function  $b$  by a non-zero constant does not modify neither the operator  $L_{ab}$  nor the potential  $M_a^b$ .

Actually, the intertwining (2.2) might be decomposed as the composition of two transformations (given naturally in this order but they actually commute):

◦ first we consider the classical intertwining for the operator  $L_a$ , i.e., applying (2.2) without weight (that is,  $b = 1$ ) entails

$$(L_a f)' = (L_a - M_a^1) f'.$$

Observing that the potential  $M_a^1$  rewrites as  $V_a''$ , the latter identity is at the heart of the famous Bakry-Émery theory about the so-called  $\Gamma_2$  calculus, cf. for instance [3] for a nice introduction to the topic, with precise references. Let us mention that we cannot iterate in general the classical intertwining involving second derivatives, unless the potential  $M_a^1$  is constant (however we will see later in some convenient cases that we are able to iterate the more general intertwining (2.2) by choosing the weight  $b$  so that the potential  $M_a^b$  is constant).

◦ then the Schrödinger operator  $L_{ab} - M_a^b$  is the so-called (Doob)  $h$ -transform with  $h = 1/b$  of the operator  $L_a - M_a^1$  obtained at the previous step, that is,

$$b(L_a - M_a^1)(f/b) = (L_{ab} - M_a^b)f,$$

with the potential  $M_a^b$  rewritten as  $M_a^b = M_a^1 - bL_a(1/b)$ . Recall that the Doob transform exhibits an underlying group structure: if  $h, k \in \mathcal{F}$  are two given functions, then the  $hk$ -transform is nothing but the  $h$ -transform of the  $k$ -transform (it is also the  $k$ -transform of the  $h$ -transform). In particular the  $h$ -transform and the original operator are the same if and only if  $h$  is constant.

According to these observations, we understand why the operator  $L_{ab}$  does not see the different roles of  $a$  and  $b$  and why the potentials  $M_a^b$ ,  $M_1^{ab}$ ,  $M_{ab}^1$  and  $M_b^a$  all differ.

**2.2. Spectral analysis.** Let us recall some basic material on the spectrum of self-adjoint operators. By an essentially self-adjoint operator  $T$ , we mean that the operator  $T$  initially defined on  $\mathcal{C}_0^\infty(\mathbb{R})$  admits a unique self-adjoint extension (still denoted  $T$ ) with domain  $\mathcal{D}(T) \subset L^2(\nu)$ , where  $\nu$  is some non-negative measure on  $\mathbb{R}$ , possibly infinite, in which  $\mathcal{C}_0^\infty(\mathbb{R})$  is dense for the norm induced by  $T$ , i.e.,

$$\|f\|_{\mathcal{D}(T)} = \sqrt{\|f\|_{L^2(\nu)}^2 + \|Tf\|_{L^2(\nu)}^2}.$$

Above and in the sequel we use a general notation  $T$  in order to unify the two cases we are interested in:

- (i) diffusion operators:  $T = -L_a$  is associated to the measure  $\nu = \mu_a$ , assumed to be finite;
- (ii) Schrödinger operators:  $T = -L_{ab} + M_a^b$  with potential  $M_a^b \in \mathcal{C}^\infty(\mathbb{R})$ , is endowed with the measure  $\nu = \mu_{ab}$ .

The operators of the type (i) and (ii) are symmetric and non-negative on  $\mathcal{C}_0^\infty(\mathbb{R})$ . Indeed this is a classical observation for diffusion operators, together with the symmetry of the Schrödinger operators of (ii). However the non-negativity of the latter is a bit more subtle since in the one-dimensional case it is a consequence of the intertwining of Lemma 2.1: for all  $f, g \in \mathcal{C}_0^\infty(\mathbb{R})$ ,

$$\begin{aligned} \int_{\mathbb{R}} f(-L_{ab} + M_a^b)g d\mu_{ab} &= \int_{\mathbb{R}} \partial_b h_1 (-L_{ab} + M_a^b) \partial_b h_2 d\mu_{ab} \\ &= \int_{\mathbb{R}} \partial_b h_1 \partial_b (-L_a h_2) d\mu_{ab} \\ &= \int_{\mathbb{R}} h_1' (-L_a h_2)' d\mu_a \\ &= \int_{\mathbb{R}} L_a h_1 L_a h_2 d\mu_a, \end{aligned}$$

where  $h_1, h_2 \in \mathcal{C}^\infty(\mathbb{R})$  are some primitive functions of  $f/b$  and  $g/b$ , respectively. Hence they are essentially-self-adjoint, cf. [11].

The spectrum of the self-adjoint operator  $T$ , denoted  $\sigma(T)$ , corresponds to the complement of the set of  $\lambda \in \mathbb{R}$  such that the operator  $T - \lambda I$  is invertible from  $\mathcal{D}(T)$  to  $L^2(\mu)$  and has a continuous inverse. As  $T$  is non-negative,  $\sigma(T)$  is a subset of  $[0, +\infty)$  and is divided into two disjoint parts: the discrete spectrum  $\sigma_{disc}(T)$ ,

that is, the set of isolated eigenvalues with finite multiplicity (= 1 in our one-dimensional setting; by an eigenfunction associated to an eigenvalue  $\lambda$  we mean some non identically null  $g \in \mathcal{D}(T)$  such that  $Tg = \lambda g$ ) and the essential spectrum  $\sigma_{ess}(T)$ , i.e., the complement of the discrete spectrum consisting of limit points in  $\sigma(T)$  (the case of eigenvalues with infinite multiplicity cannot occur in the one-dimensional case, the dimension of the associated eigenspace being at most 2 according to the general theory of second order linear differential equations). To identify whether or not a given real number belongs to the spectrum, the renowned Weyl criterion states that  $\lambda \in \sigma(T)$  if and only if there exists a sequence  $(g_n)_{n \in \mathbb{N}} \subset \mathcal{D}(T)$  with  $\|g_n\|_{L^2(\nu)} = 1$  such that

$$\lim_{n \rightarrow +\infty} \|Tg_n - \lambda g_n\|_{L^2(\nu)} = 0.$$

Moreover,  $\lambda \in \sigma_{ess}(T)$  if and only if, in addition to the above properties, the sequence  $(g_n)_{n \in \mathbb{N}}$  has no convergent subsequence.

The bottom of the spectrum is given by the following variational formula

$$\lambda_0(T) = \inf_{f \in \mathcal{D}(T)} \frac{\int_{\mathbb{R}} f T f d\nu}{\int_{\mathbb{R}} f^2 d\nu}.$$

For instance in the diffusion case (i) we have  $\lambda_0(-L_a) = 0$  and the constants are the associated eigenfunctions whereas in the Schrödinger setting (ii) we only have  $\lambda_0(-L_{ab} + M_a^b) \geq 0$ . Coming back to the general case, the higher eigenvalues below the bottom of the essential spectrum are given by the well-known max-min principle, cf. [11]. Denote  $\perp_{\nu}$  the orthogonality induced by the scalar product in  $L^2(\nu)$ .

**Theorem 2.2** (Max-min principle). *Consider the following variational formulae: for every  $n \in \mathbb{N}^*$ ,*

$$\lambda_n(T) = \sup_{g_0, g_1, \dots, g_{n-1} \in L^2(\nu)} \inf_{\substack{f \in \mathcal{D}(T) \\ f \perp_{\nu} g_i, i=0, \dots, n-1}} \frac{\int_{\mathbb{R}} f T f d\nu}{\int_{\mathbb{R}} f^2 d\nu}.$$

Then either,

(a)  $\lambda_n(T)$  is the  $(n+1)^{th}$  eigenvalue when ordering the eigenvalues in increasing order and  $T$  has a discrete spectrum in  $[0, \lambda_n(T)]$ , i.e.,  $\sigma_{ess}(T) \cap [0, \lambda_n(T)] = \emptyset$ ;  
or,

(b) it is itself the bottom of the essential spectrum  $\sigma_{ess}(T)$ , all the  $\lambda_m(T)$  coinciding with  $\lambda_n(T)$  when  $m \geq n$ , and there are at most  $n$  eigenvalues in  $\sigma_{disc}(T) \cap [0, \lambda_n(T)]$ .

By abuse of language, all the elements  $\lambda_n(T)$  arising in the max-min principle will be called eigenvalues in the remainder of the paper.

As usual a density argument allows to take the infimum above over  $\mathcal{C}_0^\infty(\mathbb{R})$  instead of  $\mathcal{D}(T)$ . Note also that the supremum is realized when the  $g_i$  are the associated eigenfunctions and it holds at least if the spectrum is discrete, i.e.

$\sigma_{ess}(T) = \emptyset$  - in this case (b) does not occur and the supremum (resp. infimum) is a maximum (resp. minimum), justifying the standard terminology “max-min principle”. In particular the sequence of eigenvalues tends to infinity as  $n$  goes to infinity. In the diffusion framework (i),  $\sigma_{ess}(T) = \emptyset$  if  $V_a$  is uniformly convex or when

$$\lim_{|x| \rightarrow +\infty} \frac{1}{2} V_a'(x)^2 - V_a''(x) = +\infty, \quad (2.4)$$

cf. [11]. See also the necessary and sufficient condition of Muckenhoupt type provided by Wang in [17, Corollary 3.4.14]. For instance a classical framework involving a discrete spectrum is the standard Gaussian case, i.e. the original potential  $V$  is uniformly convex and given by  $V = |\cdot|^2/2$  and  $\mu$  is the standard Gaussian probability distribution. We have  $\sigma(-L) = \mathbb{N}$  and the associated eigenfunctions are the Hermite polynomials. Another example of interest satisfying this time (2.4) is the so-called Subbotin (or exponential-power) distribution with potential of the type  $V = |\cdot|^\alpha/\alpha$  with  $\alpha > 1$  (for  $\alpha \in (1, 2)$  its regularized version at the origin has to be considered), for which the eigenvalues are not known explicitly. We will come back to these basic examples later.

The first positive eigenvalue is of crucial importance since it governs the exponential speed of convergence in  $L^2(\nu)$  of the semigroup  $(e^{-tT})_{t \geq 0}$ . In the Schrödinger case (ii), if  $\lambda_0(-L_{ab} + M_a^b) > 0$  then we have for every  $f \in \bar{L}^2(\mu_{ab})$ ,

$$\|e^{t(L_{ab} - M_a^b)} f\|_{L^2(\mu_{ab})} \leq e^{-\lambda_0(-L_{ab} + M_a^b)t} \|f\|_{L^2(\mu_{ab})}, \quad (2.5)$$

whereas in the diffusion setting (i) the case  $n = 1$  is concerned and corresponds to the existence of the so-called spectral gap of the operator  $-L_a$ , i.e.,  $\lambda_1(-L_a) > 0$ . Indeed for every function  $f \in L_0^2(\mu_a) = \{f \in L^2(\mu_a) : f \perp_{\mu_a} 1\}$ , we have

$$\|e^{tL_a} f\|_{L^2(\mu_a)} \leq e^{-\lambda_1(-L_a)t} \|f\|_{L^2(\mu_a)}. \quad (2.6)$$

The existence of a spectral gap means that the first eigenvalue 0 is isolated:  $0 \in \sigma_{disc}(-L_a)$ . It is the case as soon as the potential  $V_a$  is uniformly convex, an instance of the celebrated Bakry-Émery criterion, or only convex [13, 5, 2] (such an assumption might be weakened to require only the convexity at infinity, at the price of a perturbation argument). Certainly, a stronger condition ensuring the existence of the spectral gap is  $\lambda_1(-L_a) \in \sigma_{disc}(-L_a)$ . More generally, as mentioned in the Introduction, a relevant criterion ensuring its existence is the formula of Chen and Wang, cf. [10] and also [6, 8] where it has been revisited: if there exists a weight  $b \in \mathcal{F}$  such that  $\inf_{x \in \mathbb{R}} M_a^b(x) > 0$ , then

$$\lambda_1(-L_a) \geq \inf_{x \in \mathbb{R}} M_a^b(x). \quad (2.7)$$

Recently such an inequality has been successfully used to estimate the spectral gap in a large variety of examples, cf. [6, 8].

Another feature of the spectral gap in the diffusion case (i) resides in the relationship with the regularity of the solution to the Poisson equation. More precisely

if  $\lambda_1(-L_a) > 0$  then for every centered function  $f \in \mathcal{C}_0^\infty(\mathbb{R})$  the Poisson equation

$$-L_a g = f,$$

admits a unique solution  $g = (-L_a)^{-1} f \in \mathcal{D}_0(-L_a)$  which is in  $\mathcal{C}^\infty(\mathbb{R})$ , where  $\mathcal{D}_0(-L_a) = \{f \in \mathcal{D}(-L_a) : f \perp_{\mu_a} 1\}$ . Here, the operator  $(-L_a)^{-1}$  or more generally  $(-L_a)^{-\alpha}$  for  $\alpha > 0$  is well-defined on  $L_0^2(\mu_a)$  as a Riesz-type potential:

$$(-L_a)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} e^{tL_a} dt,$$

where  $\Gamma$  is the Gamma function  $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt$ , cf. e.g. [3], since the inequality (2.6) leads to the boundedness in  $L_0^2(\mu_a)$  of the operator  $(-L_a)^{-\alpha}$ ,  $\alpha > 0$ . In the Schrödinger case (ii) we use (2.5) and the same analysis as above remains valid provided the first eigenvalue  $\lambda_0(-L_{ab} + M_a^b)$  is positive, at the price of removing the orthogonality condition  $f \perp_{\mu_a} 1$  and replacing the spectral gap by  $\lambda_0(-L_{ab} + M_a^b)$ . Such Riesz-type representation will be used when necessary in the sequel.

### 3. MAIN RESULT

Before stating our main result Theorem 3.4 about eigenvalues estimates, let us present first some important ingredients that will be used in its proof.

**3.1. Johnsen's theorem revisited.** The first result that will be used in the sequel is Johnsen's theorem [12] which identifies the spectrum of the two operators involved in the (multidimensional version of the) intertwining of Lemma 2.1. Considering the original operator  $L$  in the one-dimensional case, his result can be simplified (the restriction of the operators to the subspace generated by gradients is no longer required in dimension 1, since every smooth function is the gradient of its primitive functions; such an observation has already been used above when considering the essential self-adjointness property) and applies under a set of equivalent assumptions, among them  $\lambda_1(-L) > 0$ . His result is then the following.

**Theorem 3.1** (Johnsen). *If the operator  $-L$  has a spectral gap, i.e.,  $\lambda_1(-L) > 0$ , then we have equality of the spectra*

$$\sigma(-L) \setminus \{0\} = \sigma(-L + V''),$$

*with the analogous relation for the essential spectra. In terms of the eigenvalues given by the max-min principle, we have for every  $n \in \mathbb{N}^*$ ,*

$$\lambda_n(-L) = \lambda_{n-1}(-L + V'').$$

Roughly speaking, the main idea of Johnsen is to observe that we have

$$-L + V'' = U (-L)|_{\mathcal{D}_0(-L)} U^*,$$

where  $U$  is the so-called Riesz transform

$$U = \partial(-L)^{-1/2} = (-L + V'')^{-1/2} \partial,$$



correctly defined on  $\mathcal{D}_0(-L)$  (since it is assumed that  $\lambda_1(-L) > 0$ ) and with values in  $\mathcal{D}(-L + V'')$ , which is a unitary transformation (a surjective isometry). In other words the operators  $(-L)|_{\mathcal{D}_0(-L)}$  and  $-L + V''$  are unitarily equivalent. Then for every  $\lambda \in \mathbb{R}$  and every sequence  $(f_n)_{n \in \mathbb{N}} \subset \mathcal{D}_0(-L)$  we have

$$\|(-L + V'' - \lambda I) g_n\|_{L^2(\mu)} = \|(-L - \lambda I) f_n\|_{L^2(\mu)}, \quad \text{with } g_n = U f_n,$$

and Weyl's criterion allows to conclude.

In the present work, we need an analogue of Johnsen's theorem 3.1 adapted to the presence of a weight in the gradient appearing in the intertwining of Lemma 2.1. The exact computation of the spectral gap  $\lambda_1(-L_a)$  in the assumption below being of difficult access in general, we rather use for applications Chen-Wang's formula (2.7), i.e. if there exists  $b \in \mathcal{F}$  such that  $\inf_{x \in \mathbb{R}} M_a^b(x) > 0$ , then

$$\lambda_1(-L_a) \geq \inf_{x \in \mathbb{R}} M_a^b(x).$$

The result is stated as follows.

**Theorem 3.2** (Johnsen's theorem revisited). *Let  $a \in \mathcal{F}$  be such that the measure  $\mu_a$  is finite. Assume moreover that  $\lambda_1(-L_a) > 0$ . Then for all weight  $b \in \mathcal{F}$  we have*

$$\sigma(-L_a) \setminus \{0\} = \sigma(-L_{ab} + M_a^b),$$

*with the analogous relation for the essential spectra. In terms of the eigenvalues given by the max-min principle, we have for every  $n \in \mathbb{N}^*$ ,*

$$\lambda_n(-L_a) = \lambda_{n-1}(-L_{ab} + M_a^b).$$

*Proof.* To get the result we proceed similarly to Johnsen, where this time we have

$$-L_{ab} + M_a^b = U (-L_a)|_{\mathcal{D}_0(-L_a)} U^*,$$

the operator  $U$  being the weighted Riesz transform

$$U = \partial_b (-L_a)^{-1/2} = (-L_{ab} + M_a^b)^{-1/2} \partial_b,$$

defined from  $\mathcal{D}_0(-L_a)$  to  $\mathcal{D}(-L_{ab} + M_a^b)$ , which is also a unitary mapping.  $\square$

The form of the previous unitary mapping is not innocent, since it is nothing but the composition of the two unitary transformations already describes above: the classical intertwining and the  $h$ -transform.

**3.2. A key lemma.** Another result of interest is the following key lemma, emphasizing a particular property of the underlying eigenfunction associated to the first positive eigenvalue. Although it is part of the folklore, cf. for instance Sections 6 and 7 in [10] and also the more recent articles [15, 6, 4], let us give a proof of this property for completeness since it will be used many times in the sequel.

**Lemma 3.3.** *We have the following characterizations:*

- *In the diffusion case (i):*

- if  $g_1^a$  is an eigenfunction associated to the spectral gap  $\lambda_1(-L_a)$ , then up to a change of sign we have  $(g_1^a)' > 0$ ;
- reciprocally if  $g^a$  is an eigenfunction of the diffusion operator  $-L_a$  such that  $(g^a)' > 0$  (up to a change of sign) then the corresponding eigenvalue is the spectral gap  $\lambda_1(-L_a)$ .
- In the Schrödinger case (ii):
  - if  $\tilde{g}_0^{a,b}$  is an eigenfunction associated to the eigenvalue  $\lambda_0(-L_{ab} + M_a^b)$  (this eigenfunction is called the ground state), then up to a change of sign we have  $\tilde{g}_0^{a,b} > 0$ ;
  - reciprocally if  $\tilde{g}^{a,b}$  is an eigenfunction of the Schrödinger operator  $-L_{ab} + M_a^b$  such that  $\tilde{g}^{a,b} > 0$  (up to a change of sign) then the corresponding eigenvalue is  $\lambda_0(-L_{ab} + M_a^b)$ , the bottom of the spectrum of the operator  $-L_{ab} + M_a^b$ .

*Proof.* First note that according to the intertwining of Lemma 2.1, both problems for cases (i) and (ii) are actually equivalent, the relationship between  $g_1^a$  and  $\tilde{g}_0^{a,b}$  being that  $\tilde{g}_0^{a,b}$  and  $\partial_b g_1^a$  are proportional to each other. To see this, we have on the one hand

$$(-L_{ab} + M_a^b)(\partial_b g_1^a) = -\partial_b L_a g_1^a = \lambda_1(-L_a) \partial_b g_1^a = \lambda_0(-L_{ab} + M_a^b) \partial_b g_1^a,$$

where to get the last equality we used Theorem 3.2 with  $n = 1$ . On the other hand, we have

$$\int_{\mathbb{R}} (\partial_b g_1^a)^2 d\mu_{ab} = \int_{\mathbb{R}} ((g_1^a)')^2 d\mu_a = \int_{\mathbb{R}} g_1^a (-L_a g_1^a) d\mu_a = \lambda_1(-L_a) \int_{\mathbb{R}} (g_1^a)^2 d\mu_a,$$

so that  $\partial_b g_1^a \in L^2(\mu_{ab})$  if and only if  $g_1^a \in L^2(\mu_a)$ . Hence  $\partial_b g_1^a$  is an eigenfunction associated to the eigenvalue  $\lambda_0(-L_{ab} + M_a^b)$  which has multiplicity one, therefore  $\partial_b g_1^a$  is proportional to  $\tilde{g}_0^{a,b}$ .

Hence let us prove the desired conclusion only in the diffusion case (i). Our attention is concentrated on the first item. We claim that in the variational formula of the spectral gap,

$$\begin{aligned} \lambda_1(-L_a) &= \inf_{f \in \mathcal{D}_0(-L_a)} \frac{-\int_{\mathbb{R}} f L_a f d\mu_a}{\int_{\mathbb{R}} f^2 d\mu_a} \\ &= \inf_{f \in \mathcal{D}(-L_a)} \frac{\int_{\mathbb{R}} f'^2 d\mu_a}{\text{Var}_{\mu_a}(f)}, \end{aligned}$$

where  $\text{Var}_{\mu_a}(f)$  denotes the variance of  $f$  under the measure  $\mu_a$ , i.e.,

$$\text{Var}_{\mu_a}(f) = \int_{\mathbb{R}} f^2 d\mu_a - \left( \int_{\mathbb{R}} f d\mu_a \right)^2,$$

the infimum might be taken over monotonic functions. Indeed if  $f \in \mathcal{D}(-L_a)$  is a given function, then a primitive function  $g$  of the function  $|f'|$  which belongs to

the space  $\mathcal{D}(-L_a)$  satisfies

$$\int_{\mathbb{R}} g'^2 d\mu_a = \int_{\mathbb{R}} f'^2 d\mu_a,$$

and also

$$\mathrm{Var}_{\mu_a}(f) \leq \mathrm{Var}_{\mu_a}(g),$$

since the variance trivially rewrites as

$$\mathrm{Var}_{\mu_a}(f) = \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \left( \int_x^y f'(t) dt \right)^2 d\mu_a(x) d\mu_a(y).$$

Therefore the first eigenfunction is monotonic. Now assume by contradiction that there exists some  $x_0 \in \mathbb{R}$  such that  $(g_1^a)'(x_0) = 0$ . Hence we have  $g_1^a(x_0) \neq 0$  (otherwise  $g_1^a$  would be identically 0 by the Cauchy-Lipschitz theorem). Since

$$-L_a g_1^a = \lambda_1(-L_a) g_1^a,$$

we obtain  $(g_1^a)''(x_0) \neq 0$  and thus  $g_1^a$  admits a local extrema in  $x_0$ , which contradicts its monotonicity.

Let us now briefly establish the converse. Since we have

$$-L_a g^a = \lambda(-L_a) g^a,$$

with  $\lambda(-L_a) \neq 0$ , the eigenfunction  $g^a$  being non-constant, it yields

$$\lambda_1(-L_a) \leq \lambda(-L_a),$$

by the definition of the spectral gap. Then the reverse inequality is obtained by choosing in Chen-Wang's formula (2.7) the function  $b = 1/(g^a)'$ . The proof is complete.  $\square$

**3.3. Main result.** Now we are in position to state the main result of the present paper, contained in Theorem 3.4, corresponding to lower and upper variational formulae for the eigenvalues of the original operator  $-L$ , generalizing to higher eigenvalues the famous Chen-Wang formula (2.7) on the spectral gap. In particular we exhibit the conditions under which the optimality holds. Let us briefly introduce some notation. If for some  $i \in \mathbb{N}^*$  we have  $\lambda_i(-L) \in \sigma_{disc}(-L)$  (resp.  $\lambda_1(-L_a) \in \sigma_{disc}(-L_a)$  for some  $a \in \mathcal{F}$ ), we denote  $g_i$  (resp.  $g_1^a$ ) an eigenfunction (recall that the associated eigenspaces are one-dimensional and moreover by ellipticity these eigenfunctions are in  $C^\infty(\mathbb{R})$  since the potentials  $V$  and  $V_a$  are in  $C^\infty(\mathbb{R})$ ). By Lemma 3.3 we know that the derivative of the eigenfunction associated to the spectral gap is of constant sign and does not vanish. To let the forthcoming recursive argument start consistently, we introduce below the artificial weight  $a_0 = 1$  so that  $L_{a_0} = L$ ,  $\mu_{a_0} = \mu$  and  $g_1^{a_0} = g_1$ . Finally, anticipating on the notation of Theorem 3.4 below, we recall that by Chen-Wang's formula (2.7) the existence of some  $a_i \in \mathcal{F}$  such that  $\inf_{x \in \mathbb{R}} M_{a_0 \dots a_{i-1}}^{a_i}(x) > 0$  is a sufficient condition ensuring that  $\lambda_1(-L_{a_0 \dots a_{i-1}}) > 0$ , provided the measure  $\mu_{a_0 \dots a_{i-1}}$  is finite.

**Theorem 3.4. Two-sided estimates.** *Given  $n \in \mathbb{N}^*$ , we have the following estimates:*

$$\sup_{a_1, \dots, a_n} \sum_{i=1}^n \inf_{x \in \mathbb{R}} M_{a_0 \dots a_{i-1}}^{a_i}(x) \leq \lambda_n(-L) \leq \inf_{a_1, \dots, a_n} \sum_{i=1}^n \sup_{x \in \mathbb{R}} M_{a_0 \dots a_{i-1}}^{a_i}(x), \quad (3.1)$$

where the supremum in the left-hand-side (resp. the infimum in the right-hand-side) runs over all functions  $a_1, \dots, a_n \in \mathcal{F}$  such that for all  $i \in \{1, \dots, n\}$ ,

- the measure  $\mu_{a_0 \dots a_{i-1}}$  is finite;
- the spectral gap exists, i.e.,  $\lambda_1(-L_{a_0 \dots a_{i-1}}) > 0$ .

Above we adopt the convention that the left-hand-side (resp. right-hand-side) is infinite if at least one of the  $M_{a_0 \dots a_{i-1}}^{a_i}$  is not bounded from below (resp. above).

**Optimality.** *Given  $n \in \mathbb{N}^*$ , if  $\lambda_i(-L) \in \sigma_{disc}(-L)$  for every  $i \in \{1, \dots, n\}$  then  $\lambda_1(-L_{a_0 \dots a_{i-1}}) \in \sigma_{disc}(-L_{a_0 \dots a_{i-1}})$  for every  $i \in \{1, \dots, n\}$ , the  $a_i$  being given recursively by*

$$a_i = \frac{1}{(g_1^{a_0 \dots a_{i-1}})'},$$

meaning that  $\partial_{a_i} \dots \partial_{a_1} g_i$  is a non-zero constant. In this case the equalities hold in (3.1) and we have

$$\lambda_n(-L) = \sum_{i=1}^n \lambda_1(-L_{a_0 \dots a_{i-1}}). \quad (3.2)$$

*Proof.* First let us establish the inequality in the left-hand-side of (3.1), the other sense being somewhat similar by reversing the sign in the forthcoming inequalities. Pick some functions  $a_1, \dots, a_n \in \mathcal{F}$  such that for all  $i \in \{1, \dots, n\}$  the measure  $\mu_{a_0 \dots a_{i-1}}$  is finite together with  $\lambda_1(-L_{a_0 \dots a_{i-1}}) > 0$ . The strategy of the proof of the desired lower bound is to proceed recursively on  $i \in \{1, \dots, n\}$ . The assumptions under consideration for  $i = 1$  are  $\lambda_1(-L_{a_0}) > 0$  and the finiteness of the measure  $\mu_{a_0}$  (the latter is an hypothesis assumed through the whole paper), so that by Theorem 3.2 applied with  $(a, b) = (a_0, a_1)$  we get

$$\lambda_n(-L_{a_0}) = \lambda_{n-1}(-L_{a_0 a_1} + M_{a_0}^{a_1}) \geq \lambda_{n-1}(-L_{a_0 a_1}) + \inf_{x \in \mathbb{R}} M_{a_0}^{a_1}(x),$$

the inequality being a direct consequence of the max-min principle. Next we iterate the procedure: since  $\mu_{a_0 a_1}$  is assumed to be finite and  $\lambda_1(-L_{a_0 a_1}) > 0$ , Theorem 3.2 applied now with  $(a, b) = (a_0 a_1, a_2)$  gives

$$\lambda_{n-1}(-L_{a_0 a_1}) = \lambda_{n-2}(-L_{a_0 a_1 a_2} + M_{a_0 a_1}^{a_2}) \geq \lambda_{n-2}(-L_{a_0 a_1 a_2}) + \inf_{x \in \mathbb{R}} M_{a_0 a_1}^{a_2}(x),$$

so that combining both estimates leads to the inequality

$$\lambda_n(-L_{a_0}) \geq \lambda_{n-2}(-L_{a_0 a_1 a_2}) + \inf_{x \in \mathbb{R}} M_{a_0}^{a_1}(x) + \inf_{x \in \mathbb{R}} M_{a_0 a_1}^{a_2}(x).$$

The mechanism of a recursive procedure is now clear and entails the desired lower bound after taking the supremum over such functions  $a_i$ , the final step of the

recursion being

$$\begin{aligned}\lambda_n(-L_{a_0}) &\geq \lambda_0(-L_{a_0\dots a_n}) + \sum_{i=1}^n \inf_{x \in \mathbb{R}} M_{a_0\dots a_{i-1}}^{a_i}(x) \\ &\geq \sum_{i=1}^n \inf_{x \in \mathbb{R}} M_{a_0\dots a_{i-1}}^{a_i}(x).\end{aligned}$$

Note that  $\lambda_0(-L_{a_0\dots a_n})$  is non-negative but has no reason a priori to vanish since we do not assume the finiteness of the measure  $\mu_{a_0\dots a_n}$ . Finally taking the suprema over the admissible functions  $a_1, \dots, a_n$  yields the desired result.

From now on, let us focus our attention on the possible optimality in the inequalities (3.1). According to the identity (2.3), if we define the functions  $h_i$  such that  $h'_i = 1/a_i \in \mathcal{F}$ , then the inequalities (3.1) might be rewritten as

$$\sup_{i=1}^n \inf_{x \in \mathbb{R}} \frac{(-L_{a_0\dots a_{i-1}} h_i)'(x)}{h'_i(x)} \leq \lambda_n(-L) \leq \inf_{i=1}^n \sup_{x \in \mathbb{R}} \frac{(-L_{a_0\dots a_{i-1}} h_i)'(x)}{h'_i(x)}. \quad (3.3)$$

Therefore the desired optimality result holds if we show that  $\lambda_1(-L_{a_0\dots a_{i-1}}) \in \sigma_{disc}(-L_{a_0\dots a_{i-1}})$  for every  $i \in \{1, \dots, n\}$ , so that the extrema in the inequalities (3.3) are realized for the functions  $h_i$  chosen as the eigenfunctions  $g_1^{a_0\dots a_{i-1}}$  associated to the  $\lambda_1(-L_{a_0\dots a_{i-1}})$ . These functions verify  $h'_i \in \mathcal{F}$  by Lemma 3.3.

Let us start by choosing the function  $h_1$  as an eigenfunction  $g_1^{a_0}$  associated to  $\lambda_1(-L_{a_0})$ . We apply Theorem 3.2 with  $(a, b) = (a_0, a_1)$  to get for all  $k \in \{1, \dots, n\}$ ,

$$\lambda_k(-L_{a_0}) = \lambda_{k-1}(-L_{a_0 a_1} + M_{a_0}^{a_1}) = \lambda_{k-1}(-L_{a_0 a_1}) + \lambda_1(-L_{a_0}),$$

because with this choice we have  $M_{a_0}^{a_1} = \lambda_1(-L_{a_0})$  according to (2.3). Since we assumed  $\lambda_k(-L) \in \sigma_{disc}(-L_{a_0})$  for every  $k \in \{1, \dots, n\}$ , one deduces that  $\lambda_{k-1}(-L_{a_0 a_1}) \in \sigma_{disc}(-L_{a_0 a_1})$  for every  $k \in \{1, \dots, n\}$ . In particular it contains the eigenvalue  $\lambda_1(-L_{a_0 a_1})$  which reveals to be the spectral gap of the operator  $-L_{a_0 a_1}$  since the measure  $\mu_{a_0 a_1}$  is finite, i.e.,

$$\mu_{a_0 a_1}(\mathbb{R}) = \int_{\mathbb{R}} ((g_1^{a_0})')^2 d\mu_{a_0} < +\infty,$$

the function  $g_1^{a_0}$  being an eigenfunction of the operator  $-L_{a_0}$ .

Now we go one step beyond. The preceding argument allows us to choose for  $h_2$  an eigenfunction  $g_1^{a_0 a_1}$  associated to  $\lambda_1(-L_{a_0 a_1})$ . The measure  $\mu_{a_0 a_1}$  being finite, we can apply Theorem 3.2 with  $(a, b) = (a_0 a_1, a_2)$  to get for every  $k \in \{1, \dots, n-1\}$ ,

$$\lambda_k(-L_{a_0 a_1}) = \lambda_{k-1}(-L_{a_0 a_1 a_2} + M_{a_0 a_1}^{a_2}) = \lambda_{k-1}(-L_{a_0 a_1 a_2}) + \lambda_1(-L_{a_0 a_1}),$$

where we have  $M_{a_0 a_1}^{a_2} = \lambda_1(-L_{a_0 a_1})$  by the identity (2.3). As above, the fact that  $\lambda_k(-L_{a_0 a_1}) \in \sigma_{disc}(-L_{a_0 a_1})$  for every  $k \in \{1, \dots, n-1\}$  implies that  $\lambda_{k-1}(-L_{a_0 a_1 a_2}) \in \sigma_{disc}(-L_{a_0 a_1 a_2})$  for every  $k \in \{1, \dots, n-1\}$ , and so  $\lambda_1(-L_{a_0 a_1 a_2})$  is the spectral gap of the operator  $-L_{a_0 a_1 a_2}$  since the measure  $\mu_{a_0 a_1 a_2}$  is also finite:

$$\mu_{a_0 a_1 a_2}(\mathbb{R}) = \int_{\mathbb{R}} ((g_1^{a_0 a_1})')^2 d\mu_{a_0 a_1} < +\infty,$$

the function  $g_1^{a_0 a_1}$  being an eigenfunction of the operator  $-L_{a_0 a_1}$ .

Following the previous analysis, we iterate the argument by choosing for each function  $h_i$  an eigenfunction  $g_1^{a_0 \dots a_{i-1}}$  associated to the eigenvalue  $\lambda_1(-L_{a_0 \dots a_{i-1}})$ , which lies in the set  $\sigma_{disc}(-L_{a_0 \dots a_{i-1}})$  according to the preceding step of the recursion, the measure  $\mu_{a_0 \dots a_{i-1}}$  being also finite. Hence we can apply Theorem 3.2 with  $(a, b) = (a_0 \dots a_{i-1}, a_i)$  to get for every  $k \in \{1, \dots, n - (i - 1)\}$ ,

$$\lambda_k(-L_{a_0 \dots a_{i-1}}) = \lambda_{k-1}(-L_{a_0 \dots a_i} + M_{a_0 \dots a_{i-1}}^{a_i}) = \lambda_{k-1}(-L_{a_0 \dots a_i}) + \lambda_1(-L_{a_0 \dots a_{i-1}}),$$

thanks to the identity (2.3) giving us  $M_{a_0 \dots a_{i-1}}^{a_i} = \lambda_1(-L_{a_0 \dots a_{i-1}})$ . Since the previous step in the iterative procedure tells us that  $\lambda_k(-L_{a_0 \dots a_{i-1}}) \in \sigma_{disc}(-L_{a_0 \dots a_{i-1}})$  for every  $k \in \{1, \dots, n - (i - 1)\}$ , we get that  $\lambda_{k-1}(-L_{a_0 \dots a_i}) \in \sigma_{disc}(-L_{a_0 \dots a_i})$  for every  $k \in \{1, \dots, n - (i - 1)\}$  and in particular the eigenvalue  $\lambda_1(-L_{a_0 \dots a_i})$  belongs to this set and is the spectral gap of the operator  $-L_{a_0 \dots a_i}$ , the measure  $\mu_{a_0 \dots a_i}$  being finite. As such, we arrive at the end at

$$\begin{aligned} \lambda_n(-L_{a_0}) &= \lambda_{n-1}(-L_{a_0 a_1}) + \lambda_1(-L_{a_0}) \\ &= \lambda_{n-2}(-L_{a_0 a_1 a_2}) + \lambda_1(-L_{a_0 a_1}) + \lambda_1(-L_{a_0}) \\ &= \dots \\ &= \lambda_1(-L_{a_0 \dots a_{n-1}}) + \dots + \lambda_1(-L_{a_0 a_1}) + \lambda_1(-L_{a_0}), \end{aligned}$$

i.e., the identity (3.2) holds.

To conclude the proof of the optimality result, we show that for all  $i \in \{1, \dots, n\}$  the identity  $a_i = 1/(g_1^{a_0 \dots a_{i-1}})'$  means that  $\partial_{a_i} \dots \partial_{a_1} g_i$  is a non-zero constant, and this might be established through a recursive argument again. First this is true for  $i = 1$  by definition of  $a_1$ . For  $i = 2$ , the intertwining of Lemma 2.1 applied with  $(a, b) = (a_0, a_1)$  entails

$$\begin{aligned} L_{a_0 a_1}(\partial_{a_1} g_2) &= \partial_{a_1} L_{a_0} g_2 + M_{a_0}^{a_1} \partial_{a_1} g_2 \\ &= -(\lambda_2(-L_{a_0}) - \lambda_1(-L_{a_0})) \partial_{a_1} g_2 \\ &= -\lambda_1(-L_{a_0 a_1}) \partial_{a_1} g_2, \end{aligned}$$

by the identity (3.2) applied at rank  $n = 2$ . Now we have

$$\int_{\mathbb{R}} (\partial_{a_1} g_2)^2 d\mu_{a_0 a_1} = \int_{\mathbb{R}} (g_2')^2 d\mu_{a_0} < +\infty,$$

and thus  $\partial_{a_1} g_2$  is an eigenfunction associated to the spectral gap  $\lambda_1(-L_{a_0 a_1})$ . Since its multiplicity is one, we conclude that the eigenfunctions  $g_1^{a_0 a_1}$  and  $\partial_{a_1} g_2$  are proportional to each other so that by Lemma 3.3 the derivative of  $\partial_{a_1} g_2$  is of constant sign and does not vanish. Therefore the identity  $a_2 = 1/(g_1^{a_0 a_1})'$  means nothing but the fact that  $\partial_{a_2} \partial_{a_1} g_2$  is a non-zero constant.

Now the key point of the recursive argument is to notice that with this choice of the  $a_i$ , we obtain an intertwining of higher order by successive differentiations

since the  $M_{a_0 \dots a_{j-1}}^{a_j}$  are constant and equal  $\lambda_1(-L_{a_0 \dots a_{j-1}})$ , as previously observed: for all  $i \in \{2, \dots, n\}$ ,

$$\begin{aligned} \partial_{a_{i-1}} \dots \partial_{a_1} L_{a_0} f &= L_{a_0 \dots a_{i-1}} (\partial_{a_{i-1}} \dots \partial_{a_1} f) - \left( \sum_{j=1}^{i-1} M_{a_0 \dots a_{j-1}}^{a_j} \right) \partial_{a_{i-1}} \dots \partial_{a_1} f \\ &= L_{a_0 \dots a_{i-1}} (\partial_{a_{i-1}} \dots \partial_{a_1} f) - \left( \sum_{j=1}^{i-1} \lambda_1(-L_{a_0 \dots a_{j-1}}) \right) \partial_{a_{i-1}} \dots \partial_{a_1} f \\ &= L_{a_0 \dots a_{i-1}} (\partial_{a_{i-1}} \dots \partial_{a_1} f) - \lambda_{i-1}(-L_{a_0}) \partial_{a_{i-1}} \dots \partial_{a_1} f, \end{aligned}$$

where in the last equality we used (3.2) at rank  $i-1$ . In other words the intertwining of order  $i-1$  is related to the eigenvalue  $\lambda_{i-1}(-L_{a_0})$ . Taking for instance  $f$  as an eigenfunction associated to a non-zero eigenvalue  $\lambda(-L_{a_0})$  of the operator  $-L_{a_0}$  yields

$$L_{a_0 \dots a_{i-1}} (\partial_{a_{i-1}} \dots \partial_{a_1} f) = -(\lambda(-L_{a_0}) - \lambda_{i-1}(-L_{a_0})) \partial_{a_{i-1}} \dots \partial_{a_1} f, \quad (3.4)$$

and in particular for  $f = g_i$  we get thanks to (3.2) applied at rank  $i$ ,

$$\begin{aligned} L_{a_0 \dots a_{i-1}} (\partial_{a_{i-1}} \dots \partial_{a_1} g_i) &= -(\lambda_i(-L_{a_0}) - \lambda_{i-1}(-L_{a_0})) \partial_{a_{i-1}} \dots \partial_{a_1} g_i \\ &= -\lambda_1(-L_{a_0 \dots a_{i-1}}) \partial_{a_{i-1}} \dots \partial_{a_1} g_i. \end{aligned}$$

A somewhat similar argument as in the previous step of the recursion leads us to prove that

$$\int_{\mathbb{R}} (\partial_{a_{i-1}} \dots \partial_{a_1} g_i)^2 d\mu_{a_0 \dots a_{i-1}} < +\infty, \quad (3.5)$$

therefore  $\partial_{a_{i-1}} \dots \partial_{a_1} g_i$  is an eigenfunction associated to the spectral gap  $\lambda_1(-L_{a_0 \dots a_{i-1}})$  which has multiplicity one, hence it is proportional to  $g_1^{a_0 \dots a_{i-1}}$ . Finally by Lemma 3.3 its derivative is of constant sign and does not vanish so that the identity  $a_i = 1/(g_1^{a_0 \dots a_{i-1}})'$  means that  $\partial_{a_i} \partial_{a_{i-1}} \dots \partial_{a_1} g_i$  is a non-zero constant. The proof of Theorem 3.4 is now achieved.  $\square$

Actually, the argument used to prove (3.5) is based on the identity (3.4) as follows: if  $f$  stands for some eigenfunction associated to a non-zero eigenvalue  $\lambda(-L_{a_0})$  of the operator  $-L_{a_0}$ , then for all  $k \in \mathbb{N}^*$ ,

$$\begin{aligned} \int_{\mathbb{R}} (\partial_{a_k} \dots \partial_{a_1} f)^2 d\mu_{a_0 \dots a_k} &= \int_{\mathbb{R}} \left( (\partial_{a_{k-1}} \dots \partial_{a_1} f)' \right)^2 d\mu_{a_0 \dots a_{k-1}} \\ &= \int_{\mathbb{R}} \partial_{a_{k-1}} \dots \partial_{a_1} f (-L_{a_0 \dots a_{k-1}} (\partial_{a_{k-1}} \dots \partial_{a_1} f)) d\mu_{a_0 \dots a_{k-1}} \\ &= (\lambda(-L_{a_0}) - \lambda_{k-1}(-L_{a_0})) \int_{\mathbb{R}} (\partial_{a_{k-1}} \dots \partial_{a_1} f)^2 d\mu_{a_0 \dots a_{k-1}}, \end{aligned}$$

where to obtain the last equality we used (3.4) for  $i = k$  (in particular if  $f$  is one of the  $g_l$  for  $l \in \{0, \dots, k-1\}$ ) then the smooth function  $\partial_{a_k} \dots \partial_{a_1} f$  is identically

null). Then a straightforward recursive argument leads to the following formula, which is interesting in its own right:

$$\int_{\mathbb{R}} (\partial_{a_k} \dots \partial_{a_1} f)^2 d\mu_{a_0 \dots a_k} = \prod_{l=0}^{k-1} (\lambda(-L_{a_0}) - \lambda_l(-L_{a_0})) \int_{\mathbb{R}} f^2 d\mu_{a_0}.$$

Choosing then  $k = i - 1$  and  $f = g_i$  entail (3.5).

Let us comment Theorem 3.4. To the authors knowledge, the two-sided estimates (3.1) together with the optimality result consisting of decomposing the  $n$ th positive eigenvalue as the sum of  $n$  spectral gaps of different operators, are new. Behind the optimality result is hidden an interesting property of the eigenfunctions  $g_i$  associated to the operator  $-L$ , known in the famous Sturm-Liouville theory as a Chebyshev system in the sense of Karlin and Studden [14]. In other words, the proof above gives some interesting information on the oscillations of the  $g_i$  in the sense that, beyond the strict monotonicity of  $g_1$  (an instance of Lemma 3.3), the following functions are also strictly monotone:

$$\frac{g'_2}{g'_1}, \quad \left(\frac{g'_3}{g'_1}\right)' / \left(\frac{g'_2}{g'_1}\right)', \quad \left(\frac{\left(\frac{g'_4}{g'_1}\right)'}{\left(\frac{g'_2}{g'_1}\right)'}\right)' / \left(\frac{\left(\frac{g'_3}{g'_1}\right)'}{\left(\frac{g'_2}{g'_1}\right)'}\right)', \quad \text{etc.}$$

Certainly, such a property is trivial in the standard Gaussian case introduced previously since it is obviously satisfied by the Hermite polynomials. However the proof of Theorem 3.4 reveals that it concerns actually a much more general framework through the one-dimensional diffusion operators we investigate in the present paper.

As a final remark of Theorem 3.4, we mention that the constraint on the last eigenvalue  $\lambda_n(-L)$  might be relaxed in the optimality result. Indeed, let us only assume that  $\lambda_i(-L) \in \sigma_{disc}(-L)$ , for  $i \in \{1, \dots, n-1\}$ , so that  $\lambda_n(-L)$  is allowed to be the bottom of  $\sigma_{ess}(-L)$ . Then we arrive *a priori* at the following result (once again we use the notation involving the artificial weight  $a_0 = 1$ ):

$$\sup_{a_n \in \mathcal{F}} \inf_{x \in \mathbb{R}} M_{a_0 \dots a_{n-1}}^{a_n}(x) \leq \lambda_n(-L) - \lambda_{n-1}(-L) \leq \inf_{a_n \in \mathcal{F}} \sup_{x \in \mathbb{R}} M_{a_0 \dots a_{n-1}}^{a_n}(x), \quad (3.6)$$

where the  $a_i$  are given recursively by the well-defined system (each  $a_i$  might be defined up to some multiplicative constant)

$$\partial_{a_i} \dots \partial_{a_1} g_i = 1, \quad i \in \{1, \dots, n-1\}.$$

Such a result, written in this form, will be exploited in Section 5 to address the problem of estimating the gap between consecutive eigenvalues. In this situation, we claim that the equality holds (at least) in the left-hand-side of (3.6), which might be rewritten, since the identity (3.2) still holds under the present assumptions, as

$$\lambda_1(-L_{a_0 \dots a_{n-1}}) = \sup_{a_n \in \mathcal{F}} \inf_{x \in \mathbb{R}} M_{a_0 \dots a_{n-1}}^{a_n}(x).$$



To establish the identity (??), it is sufficient to show the inequality  $\leq$ , the other being a direct consequence of Chen-Wang's formula (2.7) applied with  $a = a_0 \dots a_{n-1}$ . According to the choice of the  $a_i$ ,  $i \in \{1, \dots, n-1\}$ , the measure  $\mu_{a_0 \dots a_{n-1}}$  is finite and by (3.2) we know that  $\lambda_1(-L_{a_0 \dots a_{n-1}}) > 0$  since  $\lambda_{n-1}(-L)$  is assumed to belong to the set  $\sigma_{disc}(-L)$  and thus is isolated. Hence applying Theorem 3.2 gives with  $(a, b) = (a_0 \dots a_{n-1}, 1)$ ,

$$\lambda_1(-L_{a_0 \dots a_{n-1}}) = \lambda_0(-L_{a_0 \dots a_{n-1}} + V''_{a_0 \dots a_{n-1}}),$$

since  $M_{a_0 \dots a_{n-1}}^1 = V''_{a_0 \dots a_{n-1}}$ . Now choosing  $\lambda < \lambda_0(-L_{a_0 \dots a_{n-1}} + V''_{a_0 \dots a_{n-1}})$  so that the operator  $-L_{a_0 \dots a_{n-1}} + V''_{a_0 \dots a_{n-1}} - \lambda I$  is invertible from  $\mathcal{D}(-L_{a_0 \dots a_{n-1}} + V''_{a_0 \dots a_{n-1}})$  to  $L^2(\mu_{a_0 \dots a_{n-1}})$  and taking some non identically null  $g \in \mathcal{C}_0^\infty(\mathbb{R})$  with  $g \geq 0$ , there exists a unique smooth function  $f \in \mathcal{D}(-L_{a_0 \dots a_{n-1}} + V''_{a_0 \dots a_{n-1}})$  which is in  $\mathcal{C}^\infty(\mathbb{R})$ , such that

$$(-L_{a_0 \dots a_{n-1}} + V''_{a_0 \dots a_{n-1}} - \lambda I) f = g.$$

Using its Riesz-type representation somewhat similar to (??), the operator  $(-L_{a_0 \dots a_{n-1}} + V''_{a_0 \dots a_{n-1}} - \lambda I)^{-1}$  is positivity improving, i.e.,  $f > 0$ . Now the latter expression rewrites as  $(-L_{a_0 \dots a_{n-1}} + V''_{a_0 \dots a_{n-1}}) f = \lambda f + g$  and since  $g \geq 0$  we get

$$\inf_{x \in \mathbb{R}} V''_{a_0 \dots a_{n-1}}(x) - \frac{L_{a_0 \dots a_{n-1}} f(x)}{f(x)} \geq \lambda,$$

i.e.,  $\inf_{x \in \mathbb{R}} M_{a_0 \dots a_{n-1}}^{a_n}(x) \geq \lambda$  when letting  $a_n = 1/f \in \mathcal{F}$ . Finally taking the supremum over  $a_n \in \mathcal{F}$  and letting  $\lambda$  tends to  $\lambda_0(-L_{a_0 \dots a_{n-1}} + V''_{a_0 \dots a_{n-1}}) = \lambda_1(-L_{a_0 \dots a_{n-1}})$  yields the desired inequality. Note that it is an interesting question to investigate whether the supremum in the above theorem is attained for some function  $a_n \in \mathcal{F}$ .

#### 4. EXAMPLES

In this section we investigate some concrete situations where Theorem 3.4 might be illustrated.

**4.1. The uniformly convex case.** Let us briefly consider the case of a uniformly convex potential,  $\inf_{x \in \mathbb{R}} V''(x) > 0$ , for which we know that the spectrum of the operator  $-L$  is discrete (so that the spectral gap exists). Choosing in Theorem 3.4 all the  $a_i = 1$  entails that  $M_{a_0 \dots a_{i-1}}^{a_i} = V''$  for all  $i \geq 1$ , so that we get the following result.

**Theorem 4.1.** *Assuming that  $\inf_{x \in \mathbb{R}} V''(x) > 0$ , we have for every  $n \in \mathbb{N}^*$ ,*

$$n \inf_{x \in \mathbb{R}} V''(x) \leq \lambda_n(-L) \leq n \sup_{x \in \mathbb{R}} V''(x),$$

*with the convention that the right-hand-side is infinite if  $V''$  is not bounded from above.*

In other words, if we denote  $L^\rho$  the operator with potential  $V = \rho|\cdot|^2/2$ , where  $\rho > 0$ , we have in this Gaussian setting  $\lambda_n(-L^\rho) = n\rho$  and the previous result means that the eigenvalues might be compared as follows:

$$\lambda_n(-L^{\rho_1}) \leq \lambda_n(-L) \leq \lambda_n(-L^{\rho_2}), \quad (4.1)$$

where  $\rho_1 = \inf_{x \in \mathbb{R}} V''(x)$  et  $\rho_2 = \sup_{x \in \mathbb{R}} V''(x)$  are assumed to be positive and finite. Such an observation has been put forward in a recent paper of Milman [16] in which he proceeds to a systematic comparison of eigenvalues for the Riemannian version of the operator  $L$  (called a weighted Laplacian in his article) under some uniform convexity assumptions on the potential.

Actually, the uniformly convex case is special since by Johnsen's theorem 3.1, we get for all  $n \in \mathbb{N}^*$ ,

$$\lambda_n(-L) = \lambda_{n-1}(-L + V''),$$

from which the following result about the gap between consecutive eigenvalues is straightforward (we will see in Section 5 that estimating the gap between consecutive eigenvalues is much more complicated when dealing with non uniformly convex potentials).

**Theorem 4.2.** *Assuming that  $\inf_{x \in \mathbb{R}} V''(x) > 0$ , then we have for every  $n \in \mathbb{N}^*$ ,*

$$\inf_{x \in \mathbb{R}} V''(x) \leq \lambda_n(-L) - \lambda_{n-1}(-L) \leq \sup_{x \in \mathbb{R}} V''(x),$$

*with the convention that the right-hand-side is infinite if  $V''$  is not bounded from above.*

Hence in the uniformly convex case Theorem 4.1 is nothing but a direct consequence of Theorem 4.2, meaning that the two-sided estimates of Theorem 3.4 are somewhat useless in this context (however the optimality result is still interesting). The next part proposes to address more interesting examples beyond the uniformly convex case, for which Theorem 3.4 is really required to estimate  $\lambda_n(-L)$ .

**4.2. Subbotin distributions.** Now let us investigate how Theorem 3.4 might be applied to a non uniformly convex potential, through the example of  $V = |\cdot|^\alpha/\alpha$  where  $\alpha \geq 1$ . This is the so-called Subbotin (or exponential power) distribution (for  $\alpha = 2$  we are reduced to the standard Gaussian case). The potential  $V$  is strictly convex on  $\mathbb{R}$  but its second derivative vanishes at infinity for  $\alpha \in (1, 2)$  and at zero for  $\alpha > 2$  (although  $V$  is not smooth at the origin for  $\alpha \in (1, 2)$ , this point might be ignored at the price of an unessential regularizing procedure). According to (2.4) we have  $\sigma_{ess}(-L) = \emptyset$  for  $\alpha > 1$  and thus the spectrum is discrete. The case of the Laplace distribution, i.e.  $\alpha = 1$ , is somewhat particular since the essential spectrum is not empty (actually we have  $\sigma_{disc}(-L) = \{0\}$  and  $\sigma_{ess}(-L) = [1/4, +\infty)$ ) and therefore is not concerned with the forthcoming analysis. Let us concentrate our attention on the range  $\alpha \in (1, 2)$  only, the framework  $\alpha > 2$  being tackled differently in the next part. Our aim is to apply Theorem 3.4 to this

example, or more precisely to obtain at least a relevant lower bound on  $\lambda_n(-L)$  for  $n \geq 1$ . We set  $a_1 = e^{-\varepsilon_1 V}$  and for all  $i \geq 2$ ,

$$a_i = \exp\left(-\varepsilon_i \prod_{k=1}^{i-1} (1 - 2\varepsilon_k) V\right),$$

where the  $\varepsilon_i \in (0, 1/2)$  are some constants to be chosen conveniently later. Then for all  $i \geq 2$  we have  $a_i = e^{-\varepsilon_i V_{a_1 \dots a_{i-1}}}$  where

$$V_{a_1 \dots a_{i-1}} = \prod_{k=1}^{i-1} (1 - 2\varepsilon_k) V.$$

Moreover we have for all  $x \in \mathbb{R}$ ,

$$\begin{aligned} M_{a_1 \dots a_{i-1}}^{a_i}(x) &= (1 - \varepsilon_i) \left( V_{a_1 \dots a_{i-1}}''(x) + \varepsilon_i (V_{a_1 \dots a_{i-1}}'(x))^2 \right) \\ &= (1 - \varepsilon_i) \left( \prod_{k=1}^{i-1} (1 - 2\varepsilon_k) V''(x) + \varepsilon_i \prod_{k=1}^{i-1} (1 - 2\varepsilon_k)^2 (V'(x))^2 \right) \\ &= C_i |x|^{\alpha-2} + D_i |x|^{2(\alpha-1)}, \end{aligned}$$

with the constants  $C_i$  and  $D_i$  defined by

$$C_i = (\alpha - 1)(1 - \varepsilon_i) \prod_{k=1}^{i-1} (1 - 2\varepsilon_k) \quad \text{and} \quad D_i = \varepsilon_i (1 - \varepsilon_i) \prod_{k=1}^{i-1} (1 - 2\varepsilon_k)^2.$$

The minimum of  $M_{a_1 \dots a_{i-1}}^{a_i}$  is attained at point

$$x = \left( \frac{(2 - \alpha)C_i}{2(\alpha - 1)D_i} \right)^{1/\alpha},$$

so that

$$\begin{aligned} \inf_{x \in \mathbb{R}} M_{a_1 \dots a_{i-1}}^{a_i}(x) &= \left( \frac{2 - \alpha}{2(\alpha - 1)} \right)^{1-2/\alpha} \frac{\alpha}{2(\alpha - 1)} C_i \left( \frac{C_i}{D_i} \right)^{1-2/\alpha} \\ &= \left( \frac{2 - \alpha}{2} \right)^{1-2/\alpha} \frac{\alpha}{2} (1 - \varepsilon_i) \varepsilon_i^{(2-\alpha)/\alpha} \prod_{k=1}^{i-1} (1 - 2\varepsilon_k)^{2/\alpha}. \end{aligned}$$

Hence by Theorem 3.4 we have the following estimate: for all  $n \geq 1$ ,

$$\lambda_n(-L) \geq \left( \frac{2 - \alpha}{2} \right)^{1-2/\alpha} \frac{\alpha}{2} \sum_{i=1}^n (1 - \varepsilon_i) \varepsilon_i^{(2-\alpha)/\alpha} \prod_{k=1}^{i-1} (1 - 2\varepsilon_k)^{2/\alpha}.$$

Now we need to find some  $\varepsilon_i$  such that the series diverges (recall that  $\lambda_n(-L)$  tends to infinity as  $n$  tends to infinity) and, to that aim, assuming for instance  $\sum_k \varepsilon_k < +\infty$  enforces the infinite product to be positive and thus the constraint  $\sum_i \varepsilon_i^{(2-\alpha)/\alpha} = +\infty$ . Finally the choice  $\varepsilon_k = (k + 1)^{-\beta}/2 \in (0, 1/2)$  for some  $\beta$  in the range  $(1, \alpha/(2 - \alpha))$  allows us to get the estimate

$$\lambda_n(-L) \geq C_{\alpha, \beta} n^{1-\beta(2/\alpha-1)},$$

with  $C_{\alpha,\beta} > 0$  some explicit constant only depending on  $\alpha$  and  $\beta$ . In particular choosing  $\beta$  in a neighborhood of 1 (choosing directly  $\beta = 1$  in the very definition of  $\varepsilon_i$  above is allowed but not relevant since it leads to a lower bound on  $\lambda_n(-L)$  converging to 0 as  $n$  is large) entails a lower bound that is close to optimality for large  $n$  given by the rate  $n^{2-2/\alpha}$ . Indeed such a result, which is available for all  $\alpha > 1$ , might be deduced from Weyl's asymptotic law adapted to the present setting, cf. for instance Milman's formula (2.9) in [16]: as  $\lambda \rightarrow +\infty$ , the eigenvalue counting function behaves as

$$\#\{k \geq 1 : \lambda_k(-L) \leq \lambda\} = C_\alpha \lambda^{\frac{\alpha}{2(\alpha-1)}} (1 + o(1)), \quad (4.2)$$

with  $C_\alpha > 0$  some explicit constant. We suspect however that the correct order of magnitude might be recovered by choosing more cleverly the  $\varepsilon_i$ .

**4.3. Lower bounding the eigenvalues through only one intertwining.** If we analyze in detail the proof of the lower bound of Theorem 3.4, we observe that at each step of the recursive argument we invoke Theorem 3.2 which is based on the intertwining of Lemma 2.1. In other words, Theorem 3.4 is obtained by proceeding to successive intertwinings of the type (2.2). In this short part, we propose an alternative method, still based on this intertwining formula, but which uses it only once. More precisely, the idea is to use some convenient weight such that the operator  $-L_{ab}$  in the right-hand-side of (2.2) has its spectrum which might be easier to analyze. Certainly, such an approach has no reason to entail optimality in general since  $-L_{ab}$  is prescribed a priori but it has the advantage to simplify considerably the intertwining approach, in particular when  $-L_{ab}$  is the Gaussian operator  $-L^\rho$  for some convenient  $\rho > 0$ , for which we have seen that  $\lambda_n(-L^\rho)$  behaves linearly in  $n$ . Hence let us start from the original operator  $-L$  and consider the weight  $a_1 = e^{-(V-Z)/2}$ , where  $Z$  is some smooth potential to be determined later. Then the intertwining of Lemma 2.1 applied with the couple  $(a, b) = (1, a_1)$  gives

$$L_{a_1} f = f'' - Z' f', \quad \text{and} \quad M_1^{a_1} = \frac{V''}{2} + \frac{V'^2}{4} + \frac{Z''}{2} - \frac{Z'^2}{4}.$$

Hence if we know a priori that  $\lambda_1(-L) > 0$  then Theorem 3.2 applies. Assuming that there exists  $\kappa \in \mathbb{R}$  such that  $\inf_{x \in \mathbb{R}} M_1^{a_1}(x) \geq \kappa$ , then we have for all  $n \in \mathbb{N}^*$ ,

$$\lambda_n(-L) \geq \lambda_{n-1}(-L_{a_1}) + \kappa.$$

In particular if we choose a Gaussian potential  $Z = \rho |\cdot|^2/2$  with  $\rho > 0$ , then the following result holds.

**Theorem 4.3.** *Assume that  $\lambda_1(-L) > 0$  and the existence of some constant  $\kappa \in \mathbb{R}$  such that*

$$M_1^{a_1} = \frac{V''}{2} + \frac{V'^2}{4} + \frac{\rho}{2} - \frac{\rho^2 x^2}{4} \geq \kappa. \quad (4.3)$$

Then we have for all  $n \in \mathbb{N}^*$ ,

$$\lambda_n(-L) \geq \lambda_{n-1}(-L^\rho) + \kappa = \rho(n-1) + \kappa.$$

Certainly, the lower bound above is meaningful only if the right-hand-side is positive. However an important point is to allow the constant  $\kappa$  to be non-positive, as we will see below in the examples. Note that if  $\kappa > 0$  then the assumption  $\lambda_1(-L) > 0$  is redundant, according to Chen-Wang's formula (2.7) applied with  $(a, b) = (1, a_1)$  (we suspect however that this redundancy holds in full generality).

In the case of a uniformly convex potential  $V$ , this result allows to recover the lower bound in (4.1) when comparing with the Gaussian setting. Assume that  $V'' \geq \rho$  with  $\rho > 0$ . At the price of a translation in space causing no trouble, let us assume without loss of generality that the unique minimum of  $V$  is attained at the origin, so that we have

$$|V'(x)| \geq \rho|x|, \quad x \in \mathbb{R}.$$

Then the criterion (4.3) indicates that  $M_1^{a_1} \geq \rho$  and therefore we obtain for all  $n \in \mathbb{N}^*$ ,

$$\lambda_n(-L) \geq \lambda_{n-1}(-L^\rho) + \rho = \lambda_n(-L^\rho).$$

Actually, the criterion (4.3) is sufficiently robust to involve many various situations of interest that might be difficult to cover otherwise. Let us start by considering once again the Subbotin distribution through the strictly convex potential  $V = |\cdot|^\alpha/\alpha$  but this time with  $\alpha > 2$ . As mentioned earlier, we know that  $\lambda_1(-L) > 0$ . We assume that  $Z = \rho|\cdot|^2/2$  with  $\rho > 0$  to be determined later. Since at infinity we have  $(V')^2 \gg V''$ , we neglect the contribution of  $V''$  in the inequality (4.3), so that we are looking for the minimum of the function

$$x \mapsto \frac{|x|^{2(\alpha-1)}}{4} - \frac{\rho^2 x^2}{4} + \frac{\rho}{2}, \quad x \in \mathbb{R},$$

which is negative and equals  $\kappa := -C_\alpha \rho^{\frac{2(\alpha-1)}{\alpha-2}} + \rho/2$ , where  $C_\alpha > 0$  is some explicit constant only depending on  $\alpha$ . Therefore we obtain by Theorem 4.3 the following estimate: for all  $n \in \mathbb{N}^*$ ,

$$\lambda_n(-L) \geq \rho(n-1) - C_\alpha \rho^{\frac{2(\alpha-1)}{\alpha-2}} + \frac{\rho}{2}.$$

Finally optimizing in  $\rho > 0$  yields for all  $n \in \mathbb{N}^*$  the lower bound

$$\lambda_n(-L) \geq \tilde{C}_\alpha n^{\frac{2(\alpha-1)}{\alpha}},$$

which is sharp as  $n$  is large, according to Weyl's asymptotic law (4.2). Above  $\tilde{C}_\alpha > 0$  is some explicit constant that only depends on  $\alpha$ .

Another interesting situation is the case of a somewhat degenerated potential of the type

$$V(x) = \frac{x^2}{2} + \alpha \sin(\beta x^2), \quad x \in \mathbb{R},$$

where  $\alpha, \beta > 0$  are some small parameters to be calibrated so that the criterion (2.4) ensuring a discrete spectrum is satisfied. In other words,  $V$  is some Gaussian potential perturbed by an oscillating (and bounded) function whose oscillation intensity depends on  $\alpha$  and  $\beta$ . We have for all  $x \in \mathbb{R}$ ,

$$V'(x) = x \left(1 + 2\alpha\beta \cos(\beta x^2)\right) \quad \text{and} \quad V''(x) = 1 + 2\alpha\beta \cos(\beta x^2) - 4\alpha\beta^2 x^2 \sin(\beta x^2),$$

and therefore the degeneracy resides in the fact that  $\liminf_{|x| \rightarrow +\infty} V''(x) = -\infty$ . However provided at least  $\alpha$  or  $\beta$  is sufficiently small, there exists positive constants  $\rho$  and  $\kappa$  both depending on  $\alpha$  or  $\beta$  such that (4.3) is satisfied. Hence Theorem 4.3 can be applied to this example.

## 5. ESTIMATING THE GAP BETWEEN THE TWO FIRST POSITIVE EIGENVALUES BEYOND THE UNIFORMLY CONVEX CASE

As mentioned previously in Section 4, Johnsen's theorem 3.1 allows us to obtain in the uniformly convex case some estimates on the gap between consecutive eigenvalues. However the situation is much more complicated when dealing with non uniformly convex potentials. Letting  $n \in \mathbb{N}^*$  be fixed, if we assume that  $\lambda_i(-L) \in \sigma_{disc}(-L)$  for every  $i \in \{1, \dots, n-1\}$ , then we saw in Section 3 that (??) (combined with (3.2)) leads to the following identity: with  $a_0 = 1$ ,

$$\lambda_n(-L) - \lambda_{n-1}(-L) = \sup_{a_n \in \mathcal{F}} \inf_{x \in \mathbb{R}} M_{a_0 \dots a_{n-1}}^{a_n}(x),$$

the  $a_i$  being given recursively, up to some multiplicative constant, by

$$\partial_{a_i} \dots \partial_{a_1} g_i = 1, \quad i \in \{1, \dots, n-1\},$$

where each eigenfunction  $g_i$  is associated to the eigenvalue  $\lambda_i(-L)$ ,  $i \in \{1, \dots, n-1\}$ . Hence the gap  $\lambda_n(-L) - \lambda_{n-1}(-L)$  in the spectrum of the operator  $-L$  can be estimated as soon as we are able to control the potential  $M_{a_0 \dots a_{n-1}}^{a_n}$  for some well-chosen weight  $a_n \in \mathcal{F}$ . As expected, this problem reveals to be more difficult than only estimating the eigenvalues  $\lambda_n(-L)$  themselves as proposed in Section 3. Indeed in this case the weights  $a_i$  are unknown since the eigenfunctions  $g_i$  of the operator  $-L$  are unknown in general. In this final part of the paper, we introduce a strategy to cover the case  $n = 2$ , i.e. to estimate from below the gap between the two first positive eigenvalues.

Letting  $n = 2$ , we assume that  $\lambda_1(-L) \in \sigma_{disc}(-L)$  and thus we focus our attention on the control of a potential of the type  $M_{a_1}^{a_2}$  for some weight  $a_2 \in \mathcal{F}$

(the artificial weight  $a_0$  has been removed to lighten the notation) and with  $a_1$  given by

$$\partial_{a_1} g_1 = 1, \quad \text{i.e.,} \quad a_1 = \frac{1}{g_1'}$$

Hence the question is to find such a relevant function  $a_2 \in \mathcal{F}$  such that we have  $\inf_{x \in \mathbb{R}} M_{a_1}^{a_2}(x) > 0$  from which would follow the desired estimate

$$\lambda_2(-L) - \lambda_1(-L) \geq \inf_{x \in \mathbb{R}} M_{a_1}^{a_2}(x).$$

For instance the choice  $a_2 = 1$  entails that  $M_{a_1}^{a_2} = V_{a_1}''$  and therefore the requirement  $\inf_{x \in \mathbb{R}} M_{a_1}^{a_2}(x) > 0$  reduces to the Bakry-Émery criterion, that is, the uniform convexity of the potential  $V_{a_1}$ , which has no explicit expression since  $g_1$  is unknown in general. To overcome this difficulty, the idea is to use Theorem 3.2 with some convenient weight so that the devoted Schrödinger operator is nothing but the Laplacian perturbed by a certain potential, and then to use a celebrated result of Brascamp and Lieb about the log-concavity of the associated ground state. More precisely when choosing the couple  $(a, b) = (1, e^{-V/2})$ , we have by Theorem 3.2 for the first positive eigenvalues,

$$\lambda_1(-L) = \lambda_0(-\Delta + M_1^b).$$

Above  $\Delta$  stands for the Laplacian  $\Delta f = f''$  and the potential  $M_1^b$  is

$$\begin{aligned} M_1^b &= V'' - e^{-V/2} L(e^{V/2}) \\ &= V'' + e^{V/2} \Delta(e^{-V/2}) \\ &= \frac{V''}{2} + \frac{(V')^2}{4}. \end{aligned}$$

Since  $\lambda_1(-L) \in \sigma_{disc}(-L)$  we have  $\lambda_0(-\Delta + M_1^b) \in \sigma_{disc}(-\Delta + M_1^b)$  and therefore the ground state  $\tilde{g}_0^b$  exists and is positive (up to a change of sign), cf. Lemma 3.3. Moreover, as already mentioned in the proof of Lemma 3.3, the intertwining of Lemma 2.1 allows us to write

$$(-\Delta + M_1^b)(\partial_b g_1) = -\partial_b L g_1 = \lambda_1(-L) \partial_b g_1 = \lambda_0(-\Delta + M_1^b) \partial_b g_1,$$

with moreover, the measure  $\mathbb{L}$  standing for the Lebesgue measure on  $\mathbb{R}$ ,

$$\int_{\mathbb{R}} (\partial_b g_1)^2 d\mathbb{L} = \int_{\mathbb{R}} (g_1')^2 d\mu < +\infty,$$

so that  $\partial_b g_1$  is an eigenfunction associated to the eigenvalue  $\lambda_0(-\Delta + M_1^b)$ . Since it has multiplicity one,  $\partial_b g_1$  is proportional to the ground state  $\tilde{g}_0^b$ . At the price of a convenient dilation on  $g_1$ , we can assume that both coincide. Then the famous result of Brascamp and Lieb [9] adapted to the present framework (in particular they consider  $\Delta/2$  instead of  $\Delta$ , requiring in our framework a slight modification of the constants they obtain) states that the ground state  $\tilde{g}_0^b$  is uniformly log-concave

as soon as  $M_1^b$  is uniformly convex, that is, if  $M_1^b$  satisfies  $(M_1^b)'' \geq \kappa$  for some  $\kappa > 0$ , then the ground state satisfies  $(-\log \tilde{g}_0^b)'' \geq \sqrt{\kappa/2}$ . However we also have

$$-\log \tilde{g}_0^b = -\log \partial_b g_1 = -\log \left( e^{-V/2} g_1' \right) = \frac{V}{2} - \log(g_1') = \frac{V_{a_1}}{2},$$

thus leading to  $V_{a_1}'' \geq \sqrt{2\kappa}$ , i.e., the measure  $\mu_{a_1}$  is uniformly log-concave. Finally combining all the preceding computations, we get the following result.

**Theorem 5.1.** *Assume that  $\lambda_1(-L) \in \sigma_{disc}(-L)$ . If the potential  $V''/2 + (V')^2/4$  has its second derivative bounded from below by some  $\kappa > 0$ , then the following estimate between the two first positive eigenvalues holds:*

$$\lambda_2(-L) - \lambda_1(-L) \geq \sqrt{2\kappa}.$$

Beyond the standard Gaussian case for which Theorem 5.1 is sharp, the assumptions allow us to consider non uniformly convex potentials and even non-convex situations. For instance with  $V = |\cdot|^4/4$  (recall that in this case we have  $\lambda_1(-L) \in \sigma_{disc}(-L)$ , the spectrum being discrete according to (2.4)), we have

$$\frac{V''}{2} + \frac{(V')^2}{4} = \frac{3x^2}{2} + \frac{x^6}{4},$$

and we get  $\kappa = 3$  so that

$$\lambda_2(-L) - \lambda_1(-L) \geq \sqrt{6}.$$

For the double-well potential  $V = |\cdot|^4/4 - \beta|\cdot|^2/2$ , exhibiting a concave region in a neighbourhood of the origin controlled by the size of the parameter  $\beta > 0$  (once again (2.4) holds for this example and thus  $\lambda_1(-L) \in \sigma_{disc}(-L)$ ), we have

$$\frac{V''}{2} + \frac{(V')^2}{4} = \frac{3x^2 - \beta}{2} + \frac{x^2(x^2 - \beta)^2}{4},$$

and provided  $\beta \in (0, \sqrt{30/7})$  we get  $\kappa = 3 - 7\beta^2/10 > 0$ . Finally we obtain

$$\lambda_2(-L) - \lambda_1(-L) \geq \sqrt{6 - 7\beta^2/5}.$$

As a concluding remark of this work, we mention that the procedure above might be iterated for all  $n \in \mathbb{N}^*$ , the main assumption being the uniform convexity of the potential  $V_{a_1 \dots a_{n-1}}''/2 + (V_{a_1 \dots a_{n-1}}')^2/4$ , which is not satisfactory for the moment since the  $a_i$ , which are defined through the eigenfunctions  $g_i$ , are unknown in general. However we believe that the current approach through the intertwining is promising for the future and will allow to address this problem in full generality, for instance by choosing more conveniently the weight  $a_2$ .



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(M. Bonnefont) UMR CNRS 5251, INSTITUT DE MATHÉMATIQUES DE BORDEAUX, UNIVERSITÉ BORDEAUX 1, FRANCE

*Email address:* <mailto:michel.bonnefont@math.u-bordeaux.fr>

*URL:* <http://www.math.u-bordeaux.fr/~mibonnef/>

(A. Joulin) UMR CNRS 5219, INSTITUT DE MATHÉMATIQUES DE TOULOUSE, UNIVERSITÉ DE TOULOUSE, FRANCE

*Email address:* <mailto:ajoulin@insa-toulouse.fr>

*URL:* <http://perso.math.univ-toulouse.fr/joulin/>