ON THE SPECTRAL GAP FOR PUNCTURED CONVEX DOMAINS

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ABSTRACT. This short note is dedicated to our colleague Patrick Cattiaux who brought in the last thirty years and among many other things an important contribution to the study of functional inequalities and especially the estimation of the spectral gap of diffusion operators. Inspired by one of his paper dealing with this problem for the Gaussian distribution on a punctured domain, we propose in this short note to study this issue for a class of log-concave probability measures on some simple punctured convex domains, namely, convex domains with a convex hole.

1. INTRODUCTION

In the recent years, the problem of estimating the spectral gap related to log-concave probability measures has attracted a lot of attention, culminating in the famous KLS isoperimetric conjecture. More precisely, if μ has some Lebesgue density proportional to e^{-V} on $\mathbb{R}^d$ ($d \geq 2$), where $V : \mathbb{R}^d \to \mathbb{R}$ is some smooth convex potential on $\mathbb{R}^d$, it states that the spectral gap of the log-concave probability measure μ is of order the inverse of the operator norm of the covariance matrix of μ. See for instance the recent lecture notes of Klartag and Lehec [7] for a nice introduction to the topic with historical references and credit, and also [6] for the latest (and sharpest) estimate appearing in the literature, which confirms the conjecture up to some logarithmic prefactor of the dimension. Recall that the spectral gap, or inverse Poincaré constant, if it exists, is defined as the smallest positive eigenvalue of the self-adjoint extension of (minus) the following diffusion operator

$$L = e^V \mathrm{div}(e^{-V}\nabla) = \Delta - \langle \nabla V, \nabla \rangle.$$

Above Δ and ∇ denote the Euclidean Laplacian and gradient, respectively, and $\langle \cdot, \cdot \rangle$ is the scalar product. If the analysis occurs rather on a (connected) compact domain $\Omega \subset \mathbb{R}^d$ with smooth boundary $\partial\Omega$ (say of class $\mathcal{C}^2$), then Neumann boundary conditions enter the game, that is, $\langle \nabla f, \eta \rangle = 0$ on $\partial\Omega$, where η is the outer unit-normal vector. In particular the log-concavity is preserved as soon as Ω is convex, so that most studies appearing in the literature concerns convex domains.

When the domain Ω is not convex, things get complicated since the possible presence of bottlenecks may imply arbitrary small spectral gaps. As far as we know, there are no general result in this context allowing to obtain some information on the spectral gap like for instance some relevant lower bounds. Dealing with this problem on punctured domains, one of the rare recent references beyond the classical work [9] is the paper written

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by Cattiaux and his collaborators [2] in which the authors consider a Gaussian measure. With this study in mind, the present work intends to address this question by considering more general log-concave probability measures on some punctured convex bodies, that is, on some convex domains from which we remove a second convex body. In particular we propose some geometric criteria for which the spectral gap is conveniently controlled from below. As such, this short note completes our previous paper [4] which focuses essentially on convex bodies.

2. A preliminary result

In this part we recall (a simplified version of) Theorem 1.1 in [4] on which our forthcoming main results are based. Let us introduce some notation. Let $\mathcal{C}^\infty(\Omega)$ be the space of infinitely differentiable real-valued functions on Ω. The Jacobian $\mathrm{J}\eta$ of the outer unit-normal vector η, acting at each point $x \in \partial\Omega$ of the boundary as a quadratic form on the tangent space $T_x\Omega = \{m \in \mathbb{R}^d : \langle m, \eta(x)\rangle = 0\}$, is the second fundamental form of the boundary $\partial\Omega$. It is related to a notion of curvature of the boundary, as it will be illustrated in the sequel. When Ω is convex, the second fundamental form is non-negative.
Dealing with matrix inequalities in this paper, we mean that the inequalities have to be understood in the sense of symmetric matrices. The matrix I stands for the identity, $\nabla^2 V$ is the Hessian matrix of the potential V and $\rho(M)$ denotes the smallest eigenvalue of a given symmetric matrix M.
Finally, given a function $g \in L^2(\mu)$, the space of square integrable functions on Ω with respect to μ, denote the variance of g under μ as

$$\mathrm{Var}_\mu(g) = \int_\Omega \left(g - \int_\Omega g\, d\mu\right)^2 d\mu.$$

Theorem 2.1. *On a (connected) compact set $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ with smooth boundary $\partial\Omega$ and outer unit-normal η, we consider a probability measure μ whose Lebesgue density is proportional to e^{-V}, where $V : \Omega \to \mathbb{R}$ is some sufficiently smooth potential on Ω. Let $w : \Omega \to \mathbb{R}^+$ be some smooth function such that the two following assumptions hold:*

- (A_1) *The matrix $\nabla^2 V(x) - \frac{Lw(x)}{w(x)}\, I$ is positive definite for all $x \in \Omega$.*
- (A_2) *The matrix $\mathrm{J}\eta(x) + \frac{\langle \nabla w(x), \eta(x)\rangle}{w(x)}\, I$ is non-negative for all $x \in \partial\Omega$.*

Then the generalized Brascamp-Lieb inequality holds: for all $g \in \mathcal{C}^\infty(\Omega)$,

$$\mathrm{Var}_\mu(g) \leq \int_\Omega \left\langle \nabla g,\, \left(\nabla^2 V - \frac{Lw}{w}\, I\right)^{-1} \nabla g \right\rangle d\mu.$$

In particular the spectral gap $\lambda_1(\Omega, \mu)$ satisfies

$$\lambda_1(\Omega, \mu) \geq \inf_{x \in \Omega} \rho\left(\nabla^2 V(x) - \frac{Lw(x)}{w(x)}\, I\right).$$

Actually, establishing a Brascamp-Lieb type inequality leading to a lower bound on the spectral gap has a long history, starting from the pioneer work of Brascamp and Lieb [5]. See for instance our previous article [4] in which some old and more recent references are given, dealing mainly with log-concave probability measures on the whole Euclidean space or on convex domains.

In view of the two assumptions (A_1) and (A_2), we point out that they do not play the same role: the first one provides the desired estimate on the spectral gap (provided the smallest eigenvalue of the matrix is bounded from below by some positive constant) whereas the second one deals only with the boundary term and does not depend on the measure μ. In particular the assumption (A_2) does not necessarily implies the convexity of the domain Ω, so that there is a room for generalization to some non convex situations as soon as we are able to find some convenient function w such that both assumptions are satisfied. This is the topic of the next section which deals with punctured domains and to which we turn now.

3. Punctured convex bodies

We consider in this part smooth convex bodies punctured by a convex hole. Recall that a smooth convex body is a compact, convex set of $\mathbb{R}^d$ with non-empty interior and smooth boundary. If we add a hole, *i.e.*, a set of positive diameter is removed from the domain, then the convexity is no longer ensured and the classical results on the spectral gap arising in the log-concave setting are no longer available. The connectedness can even be lost so that there may be no spectral gap. Moreover, since there is no monotonicity properties of the spectral gap with respect to the inclusion of domains (except in some very specific situations), one cannot directly compare the spectral gap of the punctured domain (if it exists) to that for the convex body without the hole, making this problem an interesting question. Note that in the seminal paper [9] the authors consider general punctured domains and establish various Poincaré type inequalities. However their forms are quite different from the generalized Brascamp-Lieb inequality appearing in Theorem 2.1 which leads to the desired spectral gap estimate (for instance the variance is defined on the set without the hole and not on the punctured domain itself). Therefore those results will not be discussed in the present work.

The detailed geometrical setting in this section is the following. Given two convex bodies $\Omega_1, \Omega_2 \subset \mathbb{R}^d$ both containing the origin in their interior, we assume that there exists some radius $R_0 > 0$ such that

$$\Omega_1 \subset \mathcal{B}(0, R_0) \subset \Omega_2.$$

The set under consideration is defined as $\Omega = \Omega_2 \backslash \Omega_1$, the domain Ω_2 punctured by Ω_1. As such, the set Ω_1 can be seen as an obstacle in Ω_2, similarly to the terminology used in the paper [2] of Cattiaux and his co-authors. The existence of such a parameter R_0 implies the inequalities $R_1^{\max} \leq R_0 \leq R_2^{\min}$, where for each $i = 1, 2$, the radius $R_i^{\max}$ (resp. $R_i^{\min}$) denotes the smallest (resp. largest) positive number such that $\Omega_i \subset \mathcal{B}(0, R_i^{\max})$ (resp. $\mathcal{B}(0, R_i^{\min}) \subset \Omega_i$). In practice a usual choice for R_0 is any number in the interval $[R_1^{\max}, R_2^{\min}]$ but the choices $R_0 = R_1^{\max}$ or $R_0 = R_2^{\min}$ are the most natural, as we will see in the sequel. The boundary of Ω is $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ and $\partial\Omega_1$ and $\partial\Omega_2$ are called the inner and outer boundaries of Ω, respectively. Dealing with the outer boundary $\partial\Omega_2$, the convexity of Ω_2 and the fact that $0 \in \Omega_2$ imply easily that $\langle x, \eta(x)\rangle \geq 0$ for all $x \in \partial\Omega_2$. However for the inner boundary, the outer unit normal vector η corresponds to the inward unit normal vector for the set Ω_1 so that the sign of the inequality is reversed: $\langle x, \eta(x)\rangle \leq 0$ for all $x \in \partial\Omega_1$.

Now let us introduce some curvature assumptions on the inner and outer boundaries: there exists some numbers $\beta_1, \beta_2 > 0$ such that the following matrix inequalities hold:

$$\mathrm{J}\eta(x) \geq \beta_1 \frac{\langle x, \eta(x)\rangle}{r^2} I, \quad x \in \partial\Omega_1, \tag{3.1}$$

and

$$\mathrm{J}\eta(x) \geq \beta_2 \frac{\langle x, \eta(x)\rangle}{r^2} I, \quad x \in \partial\Omega_2. \tag{3.2}$$

Above and in the remainder of the paper we denote the Euclidean norm $r = |x|$ to simplify the notation. The assumption (3.2), which already appeared in [8] and [4] to derive Brascamp-Lieb type inequalities and spectral gap estimates, controls the convexity of Ω_2. Indeed, it is equivalent to the uniform convexity of Ω_2, *i.e.*, the smallest eigenvalue of the second fundamental form $\mathrm{J}\eta$ of the outer boundary $\partial\Omega_2$, which depends on the space variable, is bounded from below by some positive constant (given in terms of the parameters $\beta_2, R_2^{\min}$ and $R_2^{\max}$). The assumption (3.1) is also related to the convexity of Ω_1 in the sense that the second fundamental form is non-positive on $\partial\Omega_1$, reflecting the fact that we deal with the inner boundary of Ω. As a final comment on those two curvature assumptions, we mention that we always have

$$\beta_1 \geq 1 \geq \beta_2.$$

Although we already gave the argument in [4] ensuring this set of inequalities, let us briefly recall it for completeness. Let $x_0 \in \partial\Omega_2$ be a point intersecting the outer boundary and the sphere of radius $R_2^{\min}$ and centered at the origin. On the one hand the outer unit-normal at x_0 is the same for Ω_2 and $\mathcal{B}(0, R_2^{\min})$ and is given by $\eta(x_0) = x_0/R_2^{\min}$, so that the assumption (3.2) gives

$$\mathrm{J}\eta(x_0) \geq \frac{\beta_2}{R_2^{\min}}\, I.$$

On the other hand at point x_0 the second fundamental form is at most that of the ball $\mathcal{B}(0, R_2^{\min})$, which is $(1/R_2^{\min})I$. Combining those two arguments yields the desired conclusion $\beta_2 \leq 1$. Reversing the signs ($\eta(x_0)$ is changed in $-\eta(x_0)$), the same argument applies for the inner boundary $\partial\Omega_1$ and leads to the inequality $\beta_1 \geq 1$. In particular we have $\beta_2 = 1$ (resp. $\beta_1 = 1$) when Ω_2 (resp. Ω_1) is an Euclidean ball centered at the origin.

We are now in position to state the first main result of this note. In the sequel we assume $R_1^{\max} < R_2^{\min}$, otherwise there might be at least one point which belongs at the same time to the inner and outer boundaries, leading to a kind of trap for the underlying stochastic process. In particular the spectral gap may be arbitrary small. We set $\theta = (R_0 - R_1^{\max})/(R_2^{\min} - R_1^{\max}) \in [0, 1]$,

$$\beta_0^* = (1-\theta)\ \beta_1 + \theta\, \beta_2 > 0, \quad \alpha^* = \frac{\beta_1 - \beta_2}{R_2^{\min} - R_1^{\max}} \geq 0, \tag{3.3}$$

and denote the two functions

$$\varphi_{R_0}(r) = \beta_0^* - \alpha^* \left(r - R_0 + r\, \log\left(\frac{r}{R_0}\right)\right), \quad r \geq 0, \tag{3.4}$$

$$\Phi_{R_0}(r) = \varphi_{R_0}(r)\,(d - 2 - \varphi_{R_0}(r)) + r\,\varphi'_{R_0}(r). \quad r \geq 0. \tag{3.5}$$

Theorem 3.1. *Consider the punctured domain $\Omega = \Omega_2 \backslash \Omega_1 \subset \mathbb{R}^d$, $d \geq 2$, where Ω_1, Ω_2 are two convex bodies both containing the origin in their interior and such that $R_1^{\max} < R_2^{\min}$. Let $R_0 \in [R_1^{\max}, R_2^{\min}]$. Assume that the curvature assumptions* (3.1) *and* (3.2) *hold for some $\beta_1 \geq 1 \geq \beta_2 > 0$. Let μ be the probability measure whose Lebesgue density is proportional to e^{-V}, with $V : \Omega \to \mathbb{R}$ some sufficiently smooth potential on Ω satisfying the following condition: there exists some $\gamma_{R_0} > 0$ such that*

$$\Phi_{R_0}(r) + r^2\, \rho(\nabla^2 V(x)) - \varphi_{R_0}(r)\, \langle \nabla V(x), x\rangle \geq \gamma_{R_0}, \quad x \in \Omega. \tag{3.6}$$

Then the following weighted Poincaré inequality holds: for all $g \in \mathcal{C}^\infty(\Omega)$,

$$\mathrm{Var}_\mu(g) \leq \frac{1}{\gamma_{R_0}} \int_\Omega r^2\, |\nabla g|^2\, d\mu.$$

In particular,

$$\lambda_1(\Omega,\mu) \geq \frac{\gamma_{R_0}}{(R_2^{\max})^2}.$$

Proof. Our goal is to find a convenient function w on Ω for which Theorem 2.1 applies to the present situation. We choose a smooth positive radial function $w(x) = w(r)$ (using an obvious abuse of notation) with r belonging to the admissible interval $[R_1^{\min}, R_2^{\max}]$. Denoting the function

$$\varphi(r) = -\frac{r\, w'(r)}{w(r)}, \quad r \geq 0,$$

we have the preliminary computations:

$$\begin{aligned}\varphi'(r) &= -\frac{w'(r)}{w(r)} - r\,\frac{w''(r)}{w(r)} + r\left(\frac{w'(r)}{w(r)}\right)^2 \\ &= \frac{\varphi(r)}{r} - r\,\frac{w''(r)}{w(r)} + \frac{\varphi(r)^2}{r},\end{aligned}$$

so that for all $x \in \Omega$,

$$\nabla w(x) = \frac{w'(r)\,x}{r} = -\varphi(r)\,w(r)\,\frac{x}{r^2},$$

and

$$\begin{aligned}\Delta w(x) &= w''(r) + (d-1)\,\frac{w'(r)}{r} \\ &= -w(r)\left(\frac{\varphi'(r)}{r} - \frac{\varphi(r)}{r^2} - \frac{\varphi(r)^2}{r^2} + (d-1)\,\frac{\varphi(r)}{r^2}\right) \\ &= -\frac{w(r)}{r^2}\,\Phi(r),\end{aligned}$$

where

$$\Phi(r) = \varphi(r)\,(d-2-\varphi(r)) + r\,\varphi'(r).$$

Now let us consider first the boundary conditions. To ensure the assumption (A_2) of Theorem 2.1, the function w must satisfy for all $x \in \partial\Omega$,

$$\mathrm{J}\eta(x) + \frac{\langle \nabla w(x), \eta(x)\rangle}{w(x)}\, I = \mathrm{J}\eta(x) - \varphi(r)\,\frac{\langle x, \eta(x)\rangle}{r^2}\, I \geq 0.$$

According to the curvature assumptions (3.1) and (3.2), a sufficient condition is thus $\varphi \geq \beta_1$ on $\partial\Omega_1$ and $\varphi \leq \beta_2$ on $\partial\Omega_2$ since $\langle x, \eta(x)\rangle \leq 0$ for all $x \in \partial\Omega_1$ and $\langle x, \eta(x)\rangle \geq 0$ for all $x \in \partial\Omega_2$. Hence this set of inequalities above holds as soon as

$$\begin{cases} \inf_{x\in\partial\Omega_1} \varphi(r) \geq \beta_1, \\ \sup_{x\in\partial\Omega_2} \varphi(r) \leq \beta_2. \end{cases} \tag{3.7}$$

Now let us concentrate on the assumption (A_1) of Theorem 2.1 and its consequence in terms of spectral gap estimate. We have for all $x \in \Omega$:

$$\begin{aligned}\nabla^2 V(x) - \frac{Lw(x)}{w(x)}\, I &= \nabla^2 V(x) + \frac{-\Delta w(x) + \langle \nabla V(x), \nabla w(x)\rangle}{w(x)}\, I \\ &= \frac{-\Delta w(x)}{w(x)}\, I + \nabla^2 V(x) - \varphi(r)\,\frac{\langle \nabla V(x), x\rangle}{r^2}\, I \\ &= \frac{1}{r^2}\left(\Phi(r)\, I + r^2\,\nabla^2 V(x) - \varphi(r)\,\langle \nabla V(x), x\rangle\, I\right).\end{aligned} \tag{3.8}$$

Hence the desired weighted Poincaré inequality and spectral gap estimate hold as soon as the set of inequalities (3.7) is satisfied and the smallest eigenvalue of the expression in (3.8) is bounded from below by some positive constant on Ω. To do so, we have to choose now a convenient non-increasing function w (ensuring that φ is non-negative on $[R_1^{\min}, R_2^{\max}]$) and we consider

$$w(r) = \left(\frac{r}{R_0}\right)^{-\beta(r)} = \exp\left(-\beta(r)\log\left(\frac{r}{R_0}\right)\right),$$

for some relevant function β. Since the curvature assumptions are different depending on whether the inner or outer boundary is considered, the parameter β cannot be constant as in the proof of [4, Corollary 3.1]. An idea is to consider some function $r \mapsto \beta(r)$, which is required to be non-increasing since $\beta_1 \geq \beta_2$, and for instance affine to ensure that the previous computations are tractable: $\beta(r) = \beta_0 - \alpha\,(r - R_0)$ with $\beta_0 > 0$ and $\alpha \geq 0$ to be fixed. We have therefore for all $r \geq 0$,

$$\begin{aligned}\varphi(r) &= \beta(r) + \beta'(r)\, r\, \log\left(\frac{r}{R_0}\right) \\ &= \beta_0 - \alpha\left(r - R_0 + r\,\log\left(\frac{r}{R_0}\right)\right).\end{aligned}$$

Function φ is non-decreasing on $[0, e^{-2}R_0]$ and non-increasing on $[e^{-2}R_0, \infty)$. In particular since $e^{-2}R_0 \leq R_0 \leq R_2^{\min}$, it is non-increasing on $[R_2^{\min}, R_2^{\max}]$. Its supremum and infimum on $[R_1^{\min}, R_2^{\max}]$ satisfy respectively

$$\begin{aligned}\sup_{[R_1^{\min}, R_2^{\max}]} \varphi &= \varphi(e^{-2}R_0)\,\mathbf{1}_{\{R_1^{\min} \leq e^{-2}R_0\}} + \varphi(R_1^{\min})\,\mathbf{1}_{\{R_1^{\min} > e^{-2}R_0\}} \\ &\leq \varphi(e^{-2}R_0) = \beta_0 + \alpha\, R_0\,(1 + e^{-2}),\end{aligned}$$

and

$$\inf_{[R_1^{\min}, R_2^{\max}]} \varphi = \varphi(R_2^{\max}).$$

To see that the infimum above is reached at $R_2^{\max}$, note that it is clear if $R_1^{\min} > e^{-2}\,R_0$ since then φ is non-increasing on $[R_1^{\min}, R_2^{\max}]$, whereas if $R_1^{\min} \leq e^{-2}\,R_0$, then

$$\varphi(R_2^{\max}) \leq \varphi(R_0) = \beta_0 \leq \beta_0 + \alpha\, R_0 = \varphi(0) \leq \varphi(R_1^{\min}),$$

because φ is non-decreasing on $[0, e^{-2}\,R_0]$. Coming back to the set of inequalities (3.7), we have

$$\begin{aligned}\sup_{x \in \partial\Omega_2} \varphi(r) &= \sup_{r \in [R_2^{\min}, R_2^{\max}]} \varphi(r) = \varphi(R_2^{\min}) \\ &\leq \beta_0 - \alpha\,(R_2^{\min} - R_0),\end{aligned}$$

since $\log(R_2^{\min}/R_0) \geq 0$. Moreover,

$$\begin{aligned}\inf_{x \in \partial\Omega_1} \varphi(r) &= \inf_{r \in [R_1^{\min}, R_1^{\max}]} \varphi(r) = \min\{\varphi(R_1^{\min}), \varphi(R_1^{\max})\} \\ &\geq \beta_0 + \alpha\,(R_0 - R_1^{\max}) + \alpha\, R_1^{\min}\,\log\left(\frac{R_0}{R_1^{\max}}\right) \geq \beta_0 + \alpha\,(R_0 - R_1^{\max}),\end{aligned}$$

where to obtain the first inequality above we used the fact that $R_1^{\min} \leq R_1^{\max} \leq R_0$. Choosing then the two non-negative constants β_0 and α as the quantities $\beta_0 = \beta_0^*$ and $\alpha = \alpha^*$ defined in (3.3) entails that $\varphi = \varphi_{R_0}$, the function defined by (3.4). Hence the desired conditions (3.7) hold with φ_{R_0} since

$$\beta_0^* - \alpha^*\,(R_2^{\min} - R_0) = \beta_2 \quad \text{and} \quad \beta_0^* + \alpha^*\,(R_0 - R_1^{\max}) = \beta_1.$$

Moreover in (3.8) we have then $\varphi = \varphi_{R_0}$ and $\Phi = \Phi_{R_0}$ where Φ_{R_0} is defined in (3.5), so that our assumption (3.6) means that the smallest eigenvalue of the matrix appearing in (3.8) is bounded from below by the positive constant γ_{R_0} on Ω. This achieves the proof of the theorem. □

Although the condition (3.6) seems to be difficult to satisfy in general, the choice of some particular potentials V leads to interesting results. A first example deals with the uniform distribution on Ω, meaning that $V \equiv 0$. In this case the function Φ_{R_0} defined in (3.5) is

$$\Phi_{R_0}(r) = \varphi_{R_0}(r)\,(d-1-\varphi_{R_0}(r)) - \varphi_{R_0}(0) - \alpha^*\, r,$$

and a sufficient condition ensuring the assumption (3.6) is

$$\varphi_{R_0}(R_2^{\max})\,(d-1) - \varphi_{R_0}(e^{-2}\,R_0)^2 - \varphi_{R_0}(0) - \alpha^*\,R_2^{\max} \geq \gamma_{R_0}. \tag{3.9}$$

Another interesting example is the standard Gaussian setting, for which $V(x) = |x|^2/2$, $x \in \Omega$. In this case the assumption (3.6) rewrites as

$$\Phi_{R_0}(r) + r^2\,(1-\varphi_{R_0}(r)) \geq \gamma_{R_0}, \quad r \in [R_1^{\min}, R_2^{\max}],$$

and a sufficient condition ensuring this inequality is the following condition:

$$\varphi_{R_0}(R_2^{\max})\,(d-1) - \varphi_{R_0}(e^{-2}\,R_0)^2 - \varphi_{R_0}(0) - \alpha^*\,R_2^{\max} - (R_2^{\max})^2\,(\varphi_{R_0}(e^{-2}\,R_0)-1) \geq \gamma_{R_0}, \tag{3.10}$$

since

$$\sup_{r\in[R_1^{\min},R_2^{\max}]} \varphi_{R_0}(r) \geq \inf_{r\in[R_1^{\min},R_1^{\max}]} \varphi_{R_0}(r) \geq \beta_1 \geq 1.$$

Certainly, the constants γ_{R_0} appearing in the conditions (3.9) and (3.10) strongly depend on $R_1^{\max}, R_2^{\min}, R_2^{\max}$ and on the choice of the key parameter R_0. In any cases a meaningful estimate on the spectral gap holds as soon as the dimension is sufficiently large, depending on those parameters. To observe how those estimates apply, let us consider the particular case when the largest convex body Ω_2 is a centered Euclidean ball, say $\mathcal{B}(0,R_2)$ with radius $R_2 > 0$. An interesting choice of parameter is then $R_0 = R_2^{\min} = R_2$. In this case we have $\beta_2 = 1$ as mentioned earlier, and also $R_2^{\max} = R_2$ so that the parameters of interest in Theorem 3.1 are

$$\varphi_{R_0}(R_2^{\max}) = \beta_0^* = 1.$$

Moreover the constant $\gamma_{R_0}\,(=\gamma_{R_2})$ simplifies but not so much. However if $\Omega_1 = \mathcal{B}(0,R_1)$ with $0 < R_1 < R_2$ then Ω is the crown $\mathcal{C}(0,R_1,R_2)$ centered at the origin and with inner and outer radii R_1 and R_2, so that we get $\beta_1 = 1$, $R_1^{\min} = R_1^{\max} = R_1$, $\varphi_{R_0} \equiv 1$ and the condition (3.6) rewrites as

$$d-3+r^2\,\rho(\nabla^2 V(x)) - \langle \nabla V(x), x\rangle \geq \gamma_{R_0}, \quad x \in \Omega. \tag{3.11}$$

In particular if

$$\rho(\nabla^2 V(x)) \geq \frac{\langle \nabla V(x), x\rangle}{r^2}, \quad x \in \Omega, \tag{3.12}$$

then one can choose in Theorem 3.1 $\gamma_{R_0} = d-3$ which is positive provided $d \geq 4$. Let us summarize the situation of the crown $\mathcal{C}(0,R_1,R_2)$ in the corollary below.

Corollary 3.2. *Consider the crown $\Omega = \mathcal{C}(0,R_1,R_2) \subset \mathbb{R}^d$, $d \geq 4$, with $0 < R_1 < R_2$. Let μ be the probability measure whose Lebesgue density is proportional to e^{-V}, with $V : \Omega \to \mathbb{R}$*

some sufficiently smooth potential satisfying (3.12). *Then the following weighted Poincaré inequality holds: for all* $g \in \mathcal{C}^\infty(\Omega)$,

$$\mathrm{Var}_\mu(g) \leq \frac{1}{d-3} \int_\Omega r^2 \, |\nabla g|^2 \, d\mu.$$

In particular,

$$\lambda_1(\Omega, \mu) \geq \frac{d-3}{R_2^2}.$$

Note that the above weighted Poincaré is somewhat independent from the inner radius R_1 and reads exactly the same as the one obtained in [4] for the ball $\mathcal{B}(0, R_2)$. This is due to the fact that the function $w(r) = R_2/r$ in the proof of Theorem 3.1 is chosen in such a way that it saturates both the boundary assumptions (3.1) and (3.2).

As a by-product of Corollary 3.2, we point out that we are able to obtain some interesting results on the sphere by using the Euclidean approximation on the crown. Let $\mathcal{S}^{d-1}(0, R)$ be the sphere centered at the origin and of radius $R > 0$. Considering only angular functions $f(x) = h(x/r)$ and letting R_1 tends to $R_2 \, (= R)$ yields a spectral gap estimate for the probability measure ν corresponding to the restriction of the measure μ on the sphere $\mathcal{S}^{d-1}(0, R)$. Choosing for instance the potential V to be null or some radial function like in the Gaussian case, one gets the estimate

$$\lambda_1(\mathcal{S}^{d-1}(0, R), \nu) \geq \frac{d-3}{R^2},$$

where ν is the uniform probability measure on $\mathcal{S}^{d-1}(0, R)$. This bound is sharp up to some numerical constant since the exact value of the spectral gap is known and equals $\lambda_1(\mathcal{S}^{d-1}(0, R), \nu) = (d-1)/R^2$. This Euclidean approach is also relevant for other probability measures on $\mathcal{S}^{d-1}(0, R)$, as it can be observed through the following simple example. Let μ be the shifted Gaussian measure $\mathcal{N}_d(a, I)$ on the crown $\mathcal{C}(0, R_1, R)$ for some given $a \in \mathbb{R}^d$. Since $V = |\cdot - a|^2/2$, one obtains in (3.11): for all $x \in \mathcal{C}(0, R_1, R)$,

$$d - 3 + r^2 \, \rho(\nabla^2 V(x)) - \langle \nabla V(x), x \rangle = d - 3 + r^2 - \langle x - a, x \rangle = d - 3 + \langle a, x \rangle,$$

so that the choice $\gamma_R = d - 3 - |a| R$ is relevant as soon as $|a| < \frac{d-3}{R}$. Therefore letting $R_1 \to R$, we get the estimate

$$\lambda_1(\mathcal{S}^{d-1}(0, R), \nu) \geq \frac{d - 3 - |a| R}{R^2},$$

where ν is the probability measure on $\mathcal{S}^{d-1}(0, R)$ obtained by restricting (and normalizing) the Gaussian distribution μ.

Let us come back to the case of a radial probability measure on the crown, for which the potential is of the form $V(x) = V(r)$, $r \in [R_1, R_2]$. We mention that the two-sided estimates of Bobkov presented in [1] are available. Taking the limit in large dimension in (the refined version appearing in [3] of) these estimates entails the exact following asymptotics:

$$\lambda_1(\Omega, \mu) \underset{d \to \infty}{\sim} \frac{d}{\int_\Omega r^2 \, d\mu}. \tag{3.13}$$

After a change in polar coordinates and using Laplace's method, we may obtain for some interesting radial examples on the crown the exact behaviour of the spectral gap in large dimension. In particular this argument concerns the standard Gaussian case for which the assumption (3.12) is satisfied (since $V(r) = r^2/2$ the equality holds in (3.12)). However the Gaussian case has some advantages such as uniform convexity of the potential which allow to go further into the analysis. This is the matter of the next part, extending the previous situation to the non compact situation.

4. The Gaussian setting

In this final part we concentrate our attention on the standard Gaussian distribution μ on the complement of a convex body. In their paper [2], Cattiaux and his coauthors mainly deal with the complement of a finite union of Euclidean balls and consider them as obstacles. When there is only one ball centered at the origin, *i.e.*, $\Omega = \mathbb{R}^d \setminus \mathcal{B}(0, R)$, they obtain the following spectral gap estimate by exploiting the contribution of the one-dimensional radial part:

$$\lambda_1(\Omega, \mu) \geq \frac{d}{2d + R^2}. \tag{4.1}$$

The order of magnitude of their estimate is sharp since (3.13) entails in large dimension the following asymptotics:

$$\lambda_1(\Omega, \mu) \underset{d\to\infty}{\sim} \min\left\{ \frac{d}{R^2}, 1 \right\},$$

which reflects a competition between two regimes, depending on the position of the radius R with respect to the expected distance of the underlying random vector from the origin, which is of order $\sqrt{d}$.

Actually, the non compact situation such as the one emphasized in [2] is not directly covered by Theorem 3.1 since now we have $\Omega_2 = \mathbb{R}^d$ and therefore $R_2^{\max}$ is infinite. Our goal is to show that our methodology can be adapted to the present setting by modifying the function w appearing in the proof of Theorem 3.1. In particular we are able to obtain some bound of the type (4.1) when $\Omega = \mathbb{R}^d \setminus \Omega_1$ with Ω_1 a smooth convex body including the origin in its interior and satisfying the curvature assumption (3.1). To simplify the notation, we denote in the sequel $R_1^{\min} = R_{\min}$, $R_1^{\max} = R_{\max}$ and $\beta_1 = \beta$ in (3.1). We also set $c = R_{\max}/R_{\min}$ (≥ 1).

Theorem 4.1. *Let μ be the standard Gaussian distribution on $\Omega = \mathbb{R}^d \setminus \Omega_1$, the complement of a convex body Ω_1 containing the origin in its interior and satisfying the curvature assumption* (3.1) *for some $\beta \geq 1$.*

- *Assume that $d > 2(\beta + 1)$ and that*

$$\beta < \frac{c^{2\beta} + 1}{2c^{2\beta} - 1}. \tag{4.2}$$

If the parameter $R_{\min}^2$ is in the following range (not empty thanks to (4.2)*):*

$$\frac{2\beta\, c^{2\beta}}{(2\beta - 1)\, c^{2\beta} - 1}(d - 2 - 2\beta) > R_{\min}^2 \geq \frac{\beta + 1 + c^{2\beta}}{\beta}(d - 2 - 2\beta), \tag{4.3}$$

then the spectral gap satisfies:

$$\lambda_1(\Omega, \mu) \geq \frac{2\beta\, c^{2\beta}(d - 2 - 2\beta)}{(1 + c^{2\beta})\, R_{\min}^2} - \left(\frac{(2\beta - 1)\, c^{2\beta} - 1}{1 + c^{2\beta}} \right) > 0. \tag{4.4}$$

- *Assume that $d > 6$. If $\beta \in [1, 2)$ then as soon as*

$$R_{\max}^2 \leq (d - 2 - 2\beta)(1 + 2/\beta), \tag{4.5}$$

we have

$$\lambda_1(\Omega,\mu) \geq \frac{2-\beta}{2+\beta}. \tag{4.6}$$

Proof. Once again we aim at finding some relevant function w on Ω for which Theorem 2.1 applies (this theorem admits a version for non-compact domains Ω even when $\partial\Omega \neq \emptyset$). Since we work on an unbounded domain, we choose this time the following smooth positive radial function $w(x) = w(r) = C + r^{-2\beta}$, $r \geq R_{\min}$, with the positive constant C to be fixed. Dealing with the boundary condition, we have for all $x \in \partial\Omega\,(= \partial\Omega_1)$,

$$\begin{aligned}
\mathrm{J}\eta(x) + \frac{\langle \nabla w(x), \eta(x)\rangle}{w(x)}\, I &= \mathrm{J}\eta(x) + \frac{r\,w'(r)}{w(r)}\,\frac{\langle x, \eta(x)\rangle}{r^2}\, I \\
&\geq \left(\beta + \frac{r\,w'(r)}{w(r)}\right)\frac{\langle x, \eta(x)\rangle}{r^2}\, I \\
&= \left(\beta - \frac{2\beta\, r^{-2\beta}}{C + r^{-2\beta}}\right)\frac{\langle x, \eta(x)\rangle}{r^2}\, I.
\end{aligned}$$

Since $\langle x, \eta(x)\rangle \leq 0$ for all $x \in \partial\Omega$, a sufficient condition ensuring that the right-hand-side in the matrix inequality above is non-negative, and thus the assumption (A_2) of Theorem 2.1, is therefore

$$\beta \leq \inf_{r\in[R_{\min},R_{\max}]} \frac{2\beta}{C\,r^{2\beta}+1},$$

i.e., $C \leq 1/R_{\max}^{2\beta}$. In the sequel of the proof we choose $C = 1/R_{\max}^{2\beta}$.
Now let us focus our attention on the assumption (A_1) of Theorem 2.1 within this choice of constant C, and its consequence for the spectral gap. We have for all $x \in \Omega$:

$$\begin{aligned}
\rho(\nabla^2 V(x)) - \frac{Lw(x)}{w(x)} &= 1 - \frac{w''(r)}{w(r)} - \left(\frac{d-1}{r} - r\right)\frac{w'(r)}{w(r)} \\
&= 1 - \frac{2\beta\,(2\beta+1)\, r^{-2(\beta+1)}}{C + r^{-2\beta}} + \left(\frac{d-1}{r} - r\right)\frac{2\beta\, r^{-(2\beta+1)}}{C + r^{-2\beta}} \\
&= 1 + 2\beta\, R_{\max}^{2\beta}\, \Psi_1(r^2),
\end{aligned}$$

where Ψ_1 stands for the function defined on $(0,\infty)$ by

$$\Psi_1(u) = \frac{d-2-2\beta-u}{u\,(u^\beta + R_{\max}^{2\beta})}.$$

We have

$$\Psi_1'(u) = \frac{\Psi_2(u)}{u^2\,(u^\beta + R_{\max}^{2\beta})^2},$$

where

$$\Psi_2(u) = \beta\, u^{\beta+1} - (d-2-2\beta)\left((\beta+1)\,u^\beta + R_{\max}^{2\beta}\right).$$

This function Ψ_2 satisfies

$$\Psi_2'(u) = \beta\,(\beta+1)\,u^{\beta-1}\,(u - (d-2-2\beta)),$$

hence Ψ_2 is non-increasing on $(0, d-2-2\beta)$ and non-decreasing on $(d-2-2\beta, \infty)$. Since $\Psi_2(0) < 0$, $\Psi_2(d-2-2\beta) < 0$ and $\lim_{u\to\infty}\Psi_2(u) = \infty$, there exists some unique $R_0 > 0$

such that $\Psi_2(R_0^2) = 0$, that is, R_0 satisfies the implicit equation:

$$R_0^2 = (d-2-2\beta)\left(\frac{\beta+1}{\beta} + \frac{1}{\beta}\left(\frac{R_{\max}}{R_0}\right)^{2\beta}\right). \tag{4.7}$$

Moreover, one has $\Psi_2(u) < 0$ if $u < R_0^2$ and $\Psi_2(u) > 0$ if $u > R_0^2$. The minimum of the function Ψ_1 on $(0, +\infty)$ is thus attained at R_0^2 and the function Ψ_1 and is non-increasing on $(0, R_0^2)$ and non-decreasing on (R_0^2, ∞). Note that since Ψ_1 depends on the parameter $R_{\max}$, it is natural that its minimum depends also on this parameter $R_{\max}$.

We now turn to the proof of the first item of the Theorem. We first note that the right-hand-side inequality in (4.3) is equivalent to $\Psi_2(R_{\min}^2) \geq 0$ and thus to $R_{\min} \geq R_0$. In this situation, the function Ψ_1 is non-decreasing on $[R_{\min}^2, \infty)$ and therefore we get

$$\begin{aligned}\lambda_1(\Omega,\mu) &\geq 1 + 2\beta\, R_{\max}^{2\beta}\, \Psi_1(R_{\min}^2)\\ &= 1 + \frac{2\beta\, R_{\max}^{2\beta}\,(d-2-2\beta-R_{\min}^2)}{R_{\min}^2\,(R_{\min}^{2\beta}+R_{\max}^{2\beta})}\\ &= \frac{2\beta\, c^{2\beta}\,(d-2-2\beta)}{(1+c^{2\beta})\,R_{\min}^2} - \frac{(2\beta-1)\,c^{2\beta}-1}{1+c^{2\beta}}.\end{aligned}$$

Finally, this last estimate is positive if and only if the left-hand-side inequality in (4.3) is satisfied and, as mentioned before, the interval for $R_{\min}^2$ in (4.3) is not empty since the inequality (4.2) holds.

Let us deal now with the second item of the Theorem. If we have no guarantee that $R_{\min} \geq R_0$, then we can only use the inequality $\Psi_1(r^2) \geq \Psi_1(R_0^2)$ for all $r \geq R_{\min}$. Hence using the implicit equation (4.7) satisfied by R_0, we have

$$\begin{aligned}\lambda_1(\Omega,\mu) &\geq 1 + 2\beta\, R_{\max}^{2\beta}\, \Psi_1(R_0^2)\\ &= 1 - \frac{2\beta\,(R_{\max}/R_0)^{2\beta}}{1+(R_{\max}/R_0)^{2\beta}}\left(1 - \frac{d-2-2\beta}{R_0^2}\right)\\ &= 1 - \left(1 - \frac{\beta}{\beta+1+(R_{\max}/R_0)^{2\beta}}\right)\frac{2\beta(R_{\max}/R_0)^{2\beta}}{1+(R_{\max}/R_0)^{2\beta}}\\ &= 1 - \frac{2\beta(R_{\max}/R_0)^{2\beta}}{\beta+1+(R_{\max}/R_0)^{2\beta}} \qquad (4.8)\\ &\geq \frac{2-\beta}{2+\beta},\end{aligned}$$

where by monotonicity the last inequality follows if and only if the ratio $R_{\max}/R_0$ is less than 1. Since we have $R_{\max} \leq R_0$ if and only if $\Psi_2(R_{\max}^2) \leq 0$, the latter inequality being also equivalent to (4.5), this concludes the proof. □

Let us comment the potential applications of Theorem 4.1. First, we observe that the constants β and c are invariant by dilation of the convex obstacle Ω_1 with respect to the origin. Moreover we know that in the high-dimensional regime, the Gaussian distribution concentrates near a sphere of radius $\sqrt{d}$. As such, one deduces that the first item of Theorem 4.1 concerns big obstacles whereas the second one deals with small obstacles.

In the case of small obstacles, the spectral gap estimate (4.6) may be replaced by the strongest one (4.8), as observed in the proof. This more general bound is available even if $R_{\max}/R_0 \leq \gamma$ for some constant $\gamma > 1$ (small enough). To be more precise, one must have

$\beta < 2$ and the constant γ must satisfy the inequality

$$\gamma < \left(\frac{\beta+1}{2\beta-1}\right)^{1/2\beta}.$$

Since we always have $R_0^2 \geq (d-2-2\beta)$, an easy sufficient condition to bound the ratio $R_{\max}/R_0$ by γ is therefore $R_{\max} \leq \gamma\sqrt{d-2-2\beta}$, which replaces the inequality (4.5) (recall that the latter is equivalent to the inequality $R_{\max} \leq R_0$). The estimate (4.8) produces also a slightly better bound than (4.6) if $R_{\max}/R_0 \leq \gamma$ with $\gamma < 1$, and it shows that if the domains shrinks to 0 (*i.e.*, $R_{\max} \to 0$) the spectral gaps goes to the value 1, which is expected since it coincides with the spectral gap of the standard Gaussian measure on the whole space $\mathbb{R}^d$ (thus with no obstacle).
For large obstacles, we recall that the interval in the inequality (4.3) is not empty if and only if

$$\beta < \frac{c^{2\beta}+1}{2c^{2\beta}-1}.$$

This inequality depends only on the "shape" of the convex body and is satisfied for convex obstacles that are "close" to a centered ball. Indeed in the case of a centered ball, one has $\beta = 1$, $c = 1$ and thus

$$\beta = 1 < 2 = \frac{c^{2\beta}+1}{2c^{2\beta}-1}.$$

As a final remark of this short note, we provide an explicit example where Theorem 4.1 may apply: the case where $\Omega = \mathbb{R}^d \backslash \Omega_1$ with Ω_1 an Euclidean ball which is not necessarily centered at the origin, for instance $\Omega_1 = \mathcal{B}(a,R)$ with $|a| < R$. The condition $|a| < R$ ensures that the origin is contained in its interior. In this situation, the constants β and c are

$$\beta = \frac{R+|a|}{R} \in [1,2) \quad \text{and} \quad c = \frac{R_{\max}}{R_{\min}} = \frac{R+|a|}{R-|a|}.$$

To compute the (optimal) constant β satisfying the curvature condition (3.1) on the boundary $\partial\Omega$, we proceed as follows. Since $\eta(x) = (a-x)/R$ for all $x \in \partial\Omega$, we have $\mathrm{J}\eta(x) = -(1/R)\, I$ and moreover $x \in \partial\Omega$ rewrites as $x = a - R\,\eta(x)$. Hence

$$\frac{\langle x, \eta(x)\rangle}{r^2} = -\frac{R - \langle a, \eta(x)\rangle}{|a|^2 - 2R\,\langle a, \eta(x)\rangle + R^2}.$$

Since $\eta(x)$ describes the whole unit sphere, the condition (3.1) is satisfied for some $\beta_1 = \beta \geq 1$ if and only if

$$\frac{1}{R} \leq \beta \inf_{u\in[-a,a]} \frac{R-u}{R^2+|a|^2-2Ru} = \beta\,\frac{1}{R+|a|}.$$

To simplify the presentation, we state explicitly the result when the obstacle is the centered ball $\Omega_1 = \mathcal{B}(0,R)$. The bound we obtain is comparable to the sharp one (4.1) given by Cattiaux and its collaborators.

Corollary 4.2. *Let μ be the standard Gaussian distribution on $\mathbb{R}^d \setminus \mathcal{B}(0,R)$ ($d \geq 5$), the complement of the Euclidean ball of radius $R > 0$. Then the spectral gap satisfies*

$$\lambda_1\left(\mathbb{R}^d \setminus \mathcal{B}(0,R), \mu\right) \geq \min\left\{\frac{d-4}{R^2}, \frac{1}{3}\right\}.$$

Proof. In this situation, one has $\beta = 1$, $c = 1$ and the admissible intervals (4.3) and (4.5) become respectively $R^2 \geq 3(d-4)$ and $R^2 \leq 3(d-4)$ which cover the whole half-line $\mathbb{R}^+$. Therefore if $R^2 \geq 3(d-4)$, by (4.4), one has

$$\lambda_1 \left(\mathbb{R}^d \setminus \mathcal{B}(0,R), \mu\right) \geq \frac{d-4}{R^2} = \min\left\{\frac{d-4}{R^2}, \frac{1}{3}\right\},$$

whereas if $R^2 \leq 3(d-4)$, (4.6) provides

$$\lambda_1 \left(\mathbb{R}^d \setminus \mathcal{B}(0,R), \mu\right) \geq \frac{1}{3} = \min\left\{\frac{d-4}{R^2}, \frac{1}{3}\right\},$$

and the results follows. □

References

[1] S.G. Bobkov, Spectral gap and concentration for some spherically symmetric probability measures. *Geometric Aspects of Functional Analysis*, 37-43, Lecture Notes in Math., 1807, Springer, Berlin, 2003.

[2] E. Boissard, P. Cattiaux, A. Guillin and L. Miclo. Ornstein-Uhlenbeck pinball and the Poincaré inequality in a punctured domain. *Séminaire de Probabilités*, XLIX, 1–55, Lecture Notes in Math., 2215, Springer, 2018.

[3] M. Bonnefont, A. Joulin and Y. Ma. Spectral gap for spherically symmetric log-concave probability measures, and beyond. *J. Funct. Anal.*, 270:2456-2482, 2016.

[4] M. Bonnefont and A. Joulin. A note on the spectral gap for log-concave probability measures on convex bodies. *Int. Math. Res. Not.*, 24:14704-14728, 2024.

[5] H.J. Brascamp and E.H. Lieb. On extensions of the Brunn-Minkovski and Prékopa-Leindler theorems, including inequalities for log-concave functions, and with an application to the diffusion equation. *J. Funct. Anal.*, 22:366-389, 1976.

[6] B. Klartag. Logarithmic bounds for isoperimetry and slices of convex sets. *Ars Inveniendi Analytica*, Paper No. 4, 17 pp., 2023.

[7] B. Klartag and J. Lehec. Isoperimetric inequalities in high-dimensional convex sets. To appear in *Bull. Amer. Math. Soc.*, 2025.

[8] A.V. Kolesnikov and E. Milman. Riemannian metrics on convex sets with applications to Poincaré and log-Sobolev inequalities. *Calc. Var. Partial Differ. Equ.*, 55: 1-36, 2016.

[9] E.H. Lieb, R. Seiringer, and J. Yngvason. Poincaré inequalities in punctured domains. *Ann. Math.*, 158:1067-1080, 2003.

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