ESAIM: PROCEEDINGS AND SURVEYS
Editors: Will be set by the publisher

SOME RECENT ADVANCES IN FUNCTIONAL INEQUALITIES USING STOCHASTIC METHODS [*]

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Abstract. This note is a summary of the talks given by the authors during the parallel session "Inégalités fonctionnelles" of the French national conference "Journées MAS 2024", which took place at the University of Poitiers from August 28 to 30, 2024. More precisely, we give a panorama of the current research on functional inequalities involving a probabilistic dynamical approach. Three topics are investigated: 1) in the spirit of the celebrated KLS conjecture for the Poincaré constant, how to estimate the logarithmic Sobolev constant for log-concave random vectors by means of the famous stochastic and classical localization methods; 2) how the Polchinski flow allows to go beyond the classical Bakry-Emery criterion for establishing functional inequalities such as logarithmic Sobolev inequalities or Poincaré inequalities; 3) finally how the use as a key argument of a uniform in time logarithmic Sobolev inequality entails the uniform in time propagation of chaos property for a mean-field type interacting particle system.

Résumé. Cette contribution est un résumé des exposés proposés par les auteurs lors de la session parallèle "Inégalités fonctionnelles" des Journées MAS 2024, qui ont eu lieu à l'Université de Poitiers du 28 au 30 août 2024. Plus précisément, nous proposons un panorama des recherches actuelles sur les inégalités fonctionnelles dont les méthodes utilisées pour les obtenir reposent sur une approche dynamique probabiliste. Trois sujets sont abordés : 1) dans l'esprit de la célèbre conjecture KLS pour la constante de Poincaré, comment estimer la constante de Sobolev logarithmique pour des vecteurs aléatoires log-concaves *via* les fameuses méthodes de localisation, stochastique et classique; 2) comment l'utilisation du flot de Polchinski permet d'établir des inégalités fonctionnelles de type Sobolev logarithmique ou Poincaré dans des situations pour lesquelles le critère classique de Bakry-Emery est inopérant; 3) enfin comment l'utilisation comme argument clé d'une inégalité de Sobolev logarithmique uniforme en temps implique la propriété de propagation du chaos uniforme en temps dans le cas d'un système de particules en interaction de type champ moyen.

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[*] *PB was supported by the ERC grant No. 101001677 "ISOPERIMETRY", AJ is supported by the ANR-23-CE40-0003 Conviviality project of the French National Research Agency, PLB was supported by the Huawei Technologies grant "Huawei Young Talents" and CR is supported by the FP2M federation (CNRS FR 2036)*

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1. Introduction

In this introduction, we present a brief, necessarily incomplete, history of the logarithmic Sobolev inequality, in relation to classical Sobolev inequalities, and discuss its modern developments. This history, begun more than sixty years ago, remains very active and prolific today.

We say that a probability measure μ on the Euclidean space $(\mathbb{R}^n, |\cdot|)$ (respectively a random vector X with distribution μ, denoted $X \sim \mu$) satisfies a logarithmic Sobolev inequality (in short LSI) with constant $\rho > 0$ if for any locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$, one has

$$\mathrm{Ent}_\mu(f^2) \leq 2\rho \int_{\mathbb{R}^n} |\nabla f|^2 d\mu,$$

where ∇ is the Euclidean gradient and the entropy of f^2 with respect to μ is defined by

$$\mathrm{Ent}_\mu(f^2) = \int_{\mathbb{R}^n} f^2 \log(f^2)\, d\mu - \int_{\mathbb{R}^n} f^2\, d\mu \,\log\left(\int_{\mathbb{R}^n} f^2\, d\mu\right).$$

The optimal constant ρ such that the LSI holds is called the logarithmic Sobolev constant and is denoted $\rho_{LS}(\mu)$ (resp. $\rho_{LS}(X)$). For the standard Gaussian measure $d\gamma = e^{-|\cdot|^2/2}/(2\pi)^{n/2}\, d\lambda$ (here λ denotes the Lebesgue measure), the LSI holds with optimal constant $\rho_{LS}(\gamma) = 1$. One of the main features of the above functional inequality is that it tensorizes, meaning that the inequality holds in any dimension with the same constant. As such, it has emerged as a powerful tool for proving infinite-dimensional results in various areas of mathematics (Probability, Analysis, PDE, Convex geometry, Statistics, Optimal transport, Statistical mechanics, Computer Science, Information theory, etc.), see *e.g.* [4,11,56,83,97,104,116]. It was first discovered for the Gaussian measure in Information theory, although stated in an equivalent different form, in Stam's seminal paper [111], as a consequence of the so-called Entropy Power Inequality of Shannon [107]. Its modern developments go back to Gross [53] who first discovered that the LSI for the Gaussian measure is equivalent to the hypercontractivity property of the Ornstein-Uhlenbeck semigroup of Nelson used in quantum field theory [95]. Hypercontractivity is a smoothing property: for $t > 0$ the Ornstein-Uhlenbeck semigroup P_t sends, as a linear operator, $\mathbb{L}^p(\gamma)$ $(p > 1)$ into $\mathbb{L}^q(\gamma)$ with $q \leq 1 + (p-1)e^{2t}$.

In contrast with the LSI, the classical Sobolev inequality (for the Lebesgue measure λ) strongly depends on the dimension $n\,(\geq 2)$. It reads for all $g \in W^{1,p}(\lambda) := \{g\colon \mathbb{R}^n \to \mathbb{R} : g, |\nabla g| \in \mathbb{L}^p(\lambda)\}$ as

$$\left(\int_{\mathbb{R}^n} |g|^{\frac{np}{n-p}} d\lambda\right)^{\frac{n-p}{np}} \leq C_{n,p} \left(\int_{\mathbb{R}^n} |\nabla g|^p d\lambda\right)^{\frac{1}{p}},$$

for $1 \leq p < n$, cf. [51,96,109]. The best constant $C_{n,p}$, that depends on p and n, was computed by Talenti and Aubin [7,115], using rearrangement techniques. For each fixed $1 \leq p < n$ we have asymptotically $C_{n,p} \sim_{n\to\infty} \frac{p}{\sqrt{2\pi e n}}$. The Sobolev inequality is equivalent to the ultracontractivity property of the heat flow (here the heat semigroup sends $\mathbb{L}^1(\lambda)$ into $\mathbb{L}^\infty(\lambda)$) and quantifies the Sobolev embedding $W^{1,p}(\lambda) \subset \mathbb{L}^{\frac{np}{n-p}}(\lambda)$. We observe that $\frac{np}{n-p} \to p$ when $n \to \infty$, showing that the Sobolev embedding becomes more and more trivial as soon as n grows to infinity, while the LSI for the Gaussian distribution, that corresponds to an embedding of the type $W^{1,2}(\gamma) \subset \mathbb{L}^2 \log \mathbb{L}^2(\gamma)$, remains valid in infinite dimension. So, at a heuristic level, the LSI can be seen as what should remain in the infinite dimensional limit of the Sobolev inequalities. Beckner [16] made this heuristic rigorous by precisely proving that the whole family of Sobolev inequalities, properly applied to tensor product functions of the type $g(x) = f(x_1)f(x_2)\ldots f(x_n)$, implies the LSI for the standard Gaussian law in any dimension, with optimal constant 2.

There exist numerous proofs of the LSI and Sobolev inequalities. One important tool goes through interpolation techniques that are very natural and classical in analysis. In the eighties/nineties this tool was widely used in connection with curvature bounds for diffusion operators, as initiated by Bakry and Emery [8, 9] and successfully developed by Ledoux in the past decades to establish the desired functional inequalities under the invariant measure. These interpolation techniques are known nowadays as the semigroup method and can be presented as follows. If the probability measure μ is invariant and reversible for a diffusion process (*i.e.*, a drifted Brownian motion) with semigroup $(P_t)_{t\geq 0}$, the idea is to differentiate in time the map $t \mapsto \int_{\mathbb{R}^n} (\Phi(P_t f))\, d\mu$ that interpolates from time $t = 0$ to the limit $t \to \infty$, between $\int_{\mathbb{R}^n} \Phi(f)\, d\mu$ and, under ergodicity assumption, $\int_{\mathbb{R}^n} \Phi(\int_{\mathbb{R}^n} f\, d\mu)\, d\mu = \Phi(\int_{\mathbb{R}^n} f\, d\mu)$. For example when $\Phi(x) = x^2$, one gets

$$\int_{\mathbb{R}^n} f^2 d\mu - \left(\int_{\mathbb{R}^n} f\, d\mu\right)^2 = -\int_0^\infty \frac{d}{dt} \int_{\mathbb{R}^n} (\Phi(P_t f))\, d\mu\, dt = -\int_0^\infty \int_{\mathbb{R}^n} 2P_t f\, LP_t f\, d\mu\, dt,$$

where L is the infinitesimal generator of the semigroup $(P_t)_{t\geq 0}$, self-adjoint in $\mathbb{L}^2(\mu)$. Integrating by parts one obtains

$$-\int_0^\infty \int_{\mathbb{R}^n} 2P_t f LP_t f\, d\mu\, dt = 2\int_0^\infty \int_{\mathbb{R}^n} |\nabla P_t f|^2 d\mu\, dt.$$

Finally under the famous Bakry-Emery curvature-dimension condition CD(ρ, ∞) with $\rho > 0$, the following sub-commutation relation holds:

$$|\nabla P_t f|^2 \leq e^{-2\rho t} P_t(|\nabla f|^2), \quad t \geq 0, \tag{1.1}$$

leading, after integration in time, to the well-known Poincaré inequality (in short PI)

$$\mathrm{Var}_\mu(f) := \int_{\mathbb{R}^n} f^2 d\mu - \left(\int_{\mathbb{R}^n} f\, d\mu\right)^2 \leq \frac{1}{\rho} \int_{\mathbb{R}^n} |\nabla f|^2 d\mu,$$

i.e., the optimal constant $C_P(\mu)$ (resp. $C_P(X)$ if $X \sim \mu$) appearing in the right-hand-side of the latter inequality, called the Poincaré constant, satisfies

$$C_P(\mu) \leq \frac{1}{\rho}.$$

Moreover one can show that $C_P(\mu)$ is nothing but the inverse of the first positive eigenvalue $\lambda_1(-L)$, called the spectral gap, of the non negative self-adjoint operator $-L$ (note that it may not be a true eigenvalue since extremal functions in the PI may not exist).

Dealing with the LSI, for which we choose $\Phi(x) = x \log x$ on $(0, \infty)$, the same methodology applies and we get under the same strong convexity assumption,

$$\rho_{LS}(\mu) \leq \frac{1}{\rho}.$$

We refer the reader to [4, Chapter 5] or [11] for details. Other choices of function Φ lead to different functional inequalities (Beckner's inequality, Bobkov's functional Gaussian isoperimetry [12, 20], the parabolic Li-Yau inequality [80], etc.). The semigroup method alluded above can be developed in the extended setting of weighted Riemannian manifolds and even more generally of Markov triple [11] for which the square of the gradient above is replaced by an abstract *carré du champ* operator Γ. The recent generalization of Ivanisvili and Volberg [60], further put in more abstract formalism in [44], gives a global approach of the semigroup method under curvature conditions that encompasses, under a single theorem, many known inequalities.

Another motivation for studying these Sobolev type functional inequalities is related to the long-time behavior of the underlying Markovian dynamics, cf. [34]. For instance the PI gives an explicit exponential rate of convergence to equilibrium in $\mathbb{L}^2(\mu)$ of the semigroup while the LSI leads to its exponential decay in relative entropy, as illustrated in Section 4 when dealing with (diffusive) interacting particle systems. See also the connections between functional inequalities and Lyapunov conditions mainly emphasized by Cattiaux and Guillin [10, 35], in the spirit of Meyn-Tweedie theory [86]. In contrast to the diffusive case, the chain rule is not available in discrete spaces so that the situation has to be conveniently adapted. This concerns important models arising in applications such as Markov chains on graphs and discrete spin systems. In this context a typical example is the entropy decay given by a weaker version of the LSI, namely the so-called modified LSI and its variants [27] (all rewrite as the LSI in the continuous setting). On discrete structures all these logarithmic Sobolev type functional inequalities give some relevant information on the mixing time, quantifying the long-time convergence in total variation distance [32, 36, 43, 92]. See also the recent progress [103] in which the hierarchy between LSI and modified LSI can be reversed for reversible Markov chains up to some sparsity parameter of the transition matrix.

In early 2000, deep connections were made between Sobolev type inequalities with mass transportation [24, 33, 42], in particular through the HWI inequality of Otto and Villani [100], relating the (relative) entropy H to the 2-Wasserstein distance W (in the spirit of Talagrand's inequality [114]) and the Fisher information I, that comes as a consequence of the fact that heat flow is the gradient flow of the entropy [99]. Otto's result is at the starting point of the introduction by Lott, Villani [82] and Sturm [112] of a synthetic notion of Ricci curvature lower bounds in general metric measure spaces as convexity of entropy along the geodesics of optimal transport which later revealed to be equivalent to the usual curvature lower bound in the sense of Bakry-Emery, cf. [1–3, 47]. Since the underlying Wasserstein space over discrete spaces does not contain geodesics, this approach has been adapted in [48] to this discrete setting and gave rise to the concept of entropic Ricci curvature, leading to many interesting functional inequalities such as the modified LSI. We also mention the papers [28, 52, 58, 59, 78, 79, 87, 105] based on somewhat similar ideas. In addition to the discrete analogue of the Bakry-Emery curvature-dimension condition and the Ricci-Ollivier curvature [64, 98], we have at our disposal various notions of discrete curvature. However the connection between all these discrete curvatures is still difficult to formulate, even if some recent works on the subject make a step in this direction [49, 63, 66, 70, 81].

Connections between functional inequalities and convex geometry (concentration of measure, isoperimetry, Brunn-Minkowski inequality or its equivalent functional form known as Prékopa-Leindler inequality, Brascamp-Lieb type inequalities, etc.) are also at the heart of many recent developments. We refer to the survey [77] for an account on these connections and mention two papers by Bobkov and Ledoux [25, 26] who proved that Prékopa-Leindler's inequality implies the LSI for the Gaussian measure and the Sobolev inequalities, with optimal constants and recent developments on a functional form of the celebrated Blashke-Santalo inequality on volume product of convex bodies [6], through heat flow [40, 41, 45, 93]. Recent advances also concern the Gaussian correlation inequality of Royen [102], related to inverse Brascamp-Lieb inequalities of Nakamura-Tsuji [94]. We refer the interested reader to [90] and references therein for an account on this topic.

The probability measures under consideration are often log-concave: on $\mathbb{R}^n$ they are of the form $d\mu = e^{-V} d\lambda$ with $V\colon \mathbb{R}^n \to \mathbb{R}$ convex. In fact, convexity plays a crucial role, in particular strong convexity in the form $\nabla^2 V \geq \rho\, \mathrm{Id}$ with $\rho > 0$ (an inequality understood in the sense of quadratic forms ; here ∇^2 denotes the Hessian matrix), an Euclidean version of the Bakry-Emery curvature-dimension condition CD(ρ, ∞), ensures that the LSI holds with optimal constant $\rho_{LS}(\mu) \leq 1/\rho$ as noticed above through the semigroup method. When V is only convex, that is $\rho = 0$, many techniques fail and new ideas had to be discovered. From a spectral point of view related to the self-adjoint operator $L = \Delta - \langle \nabla V, \nabla \rangle$ naturally associated to the measure μ (here Δ is the Laplacian), we mention the intertwining approach [5, 29] which refines the Bakry-Emery theory by allowing to consider a matrix weight in the sub-commutation identity of the form (1.1), in relation to a probabilistic object known as Doob's h-transform. Chosen conveniently, this weight allows to address some interesting examples of (non-necessarily) convex potentials V. Coming back to the convex case, a question rose by Kannan, Lovász and Simonovits [67] asks for proving the PI for log-concave probability measures with an optimal constant $C_P(\mu)$ of order the operator norm of the covariance matrix of μ, *i.e.*, its largest eigenvalue. This is now known as the KLS conjecture and is related to other important questions in high dimensional convex geometry, two of them being the slicing problem of Bourgain [30,31] and the thin-shell (or variance) conjecture, both recently solved by Klartag and Lehec [71, 72]. See also [19] for an alternative proof of the thin-shell conjecture. The KLS conjecture is still open and the most recent development is due to Klartag who proved that $C_P(\mu)$ is of the correct order up to a $\log n$ factor. We refer the reader to his article [69] for references and an historical presentation, see also [73] and Section 2 for more details. These developments rely on a new technique developed by Eldan [46] known as stochastic localization that has connection with the Polchinski flow (renormalization group) [101] put forward by Bauerschmidt and Bodineau [14] in the context of statistical mechanics (see Section 3 for details).

Let us briefly describe the content of this note. First we present in Section 2 a KLS type conjecture related to the LSI (in place of the PI) and discuss this conjecture according to the work emphasized in [18]. In Section 3 we introduce the Polchinski flow and present some known result related to the LSI together with a higher order eigenvalues comparison theorem, as illustrated in [106]. Finally, Section 4 is dedicated to observe how the LSI implies a propagation of chaos property for a mean-field type interacting particle system, a study which has been initiated in [55].

2. Logarithmic Sobolev inequalities for log-concave random vectors

The KLS conjectures states that the Poincaré constant of a log-concave random vector is controlled by its largest variance along any direction. In this part we discuss an analogous conjecture for the logarithmic Sobolev constant.

2.1. A KLS type conjecture for the logarithmic Sobolev constant

Plugging linear functionals in the PI, we find that the Poincaré constant $C_P(\mu)$ is at least the operator norm of the covariance matrix $\mathrm{Cov}(\mu)$ of μ, *i.e.*, its largest eigenvalue. It is easily observed that measures with disconnected support do not satisfy a PI. On the opposite side, the KLS conjectures states that log-concave random vectors have almost optimal Poincaré constant. Below $\mathbb{S}^{n-1}$ stands for the Euclidean unit sphere in $\mathbb{R}^n$ and, given $a, b > 0$, the notation $a \simeq b$ (resp. $a \lesssim b$) means that there exist universal constants $c, C > 0$ such that $c \leq a/b \leq C$ (resp. $a \leq C\, b$).

Conjecture 1 (KLS). *Let μ be a log-concave probability measure on $\mathbb{R}^n$, then*

$$C_P(\mu) \simeq \|\mathrm{Cov}(\mu)\|_{\mathrm{op}} = \sup_{\theta \in \mathbb{S}^{n-1}} \mathrm{Var}_\mu(\langle \cdot, \theta \rangle).$$

The KLS conjecture has attracted a lot of attention since its formulation in 1995 [67]. The best dimensional estimate is due to Klartag [69] who proved that for any log-concave probability measure on $\mathbb{R}^n$,

$$\|\mathrm{Cov}(\mu)\|_{\mathrm{op}} \leq C_P(\mu) \leq C \log(n)\, \|\mathrm{Cov}(\mu)\|_{\mathrm{op}},$$

for some universal constant $C > 0$. Dealing now with the LSI, it is well-known that it is stronger than the PI:

$$C_P(\mu) \leq \rho_{LS}(\mu).$$

Furthermore, while a PI implies exponential concentration, the LSI implies Gaussian concentration. Namely, for any 1-Lipschitz function f, we have

$$\|f\|^2_{\Psi_2(\mu)} \lesssim \rho_{LS}(\mu),$$

where the Orlicz norm related to the Young function $\Psi_2(t) = e^{t^2} - 1$ of the function f is defined as

$$\|f\|_{\Psi_2(\mu)} = \inf\left\{t > 0 : \int_{\mathbb{R}^n} \Psi_2(f/t)\, d\mu \leq 1\right\}.$$

As a consequence, in contrast with the PI, not all log-concave probabilities satisfy a LSI. For instance in the one-dimensional case the Laplace distribution $d\mu = e^{-|\cdot|}/2\, d\lambda$ on $\mathbb{R}$ does not have subgaussian tails, and thus $\rho_{LS}(\mu) = \infty$. Nonetheless, we may formulate a KLS type conjecture in the following form.

Conjecture 2. *Let μ be a log-concave probability measure on $\mathbb{R}^n$, then*

$$\rho_{LS}(\mu) \lesssim \sigma_{SG}^2(\mu) := \sup_{\theta \in \mathbb{S}^{n-1}} \|\langle \cdot - b_\mu, \theta\rangle\|_{\Psi_2(\mu)}^2,$$

where $b_\mu = \int_{\mathbb{R}^n} x\, d\mu(x)$ is the barycenter of μ and $\sigma_{SG}^2(\mu)$ is the worst sub-Gaussian constant of a centered marginal of μ. Up to a universal constant, it is the smallest constant $K > 0$ such that

$$\mu\left(\langle \cdot - b_\mu, \theta\rangle \geq t\right) \leq \exp(-t^2/K^2),$$

for all $t > 0$ and $\theta \in \mathbb{S}^{n-1}$.

By scaling, Conjecture 2 is equivalent to the boundedness of

$$G_n = \sup_\mu \rho_{LS}(\mu),$$

where the supremum runs over all log-concave probabilities μ on $\mathbb{R}^n$ with $\sigma_{SG}(\mu) \leq 1$.

2.2. Dimensional upper-bounds and examples

As a first example, let's examine the case of ν_p, the uniform measure on $\lambda_p B_p$, where B_p is the unit ball of ℓ_p, for $1 \leq p \leq \infty$ and λ_p is chosen so that

$$\mathrm{Vol}(\lambda_p B_p) = 1,$$

with Vol the volume measure in $\mathbb{R}^n$. In that normalization, the covariance of ν_p satisfies the following matrix inequalities:

$$c\,\mathrm{Id} \leq \mathrm{Cov}(\nu_p) \leq C\,\mathrm{Id},$$

where c, C depends neither on p or the dimension. We start by examining the case $p \geq 2$. From the latter matrix estimates we deduce that

$$\sigma_{SG}(\nu_p) \geq c_1,$$

for some constant $c_1 > 0$. On the other hand, as explained in [75], the probability measure ν_p is a Lipschitz image of the standard Gaussian distribution, which satisfies the LSI with optimal constant 1. Thus, in accordance with Conjecture 2, we have

$$\rho_{LS}(\nu_p) \lesssim 1.$$

The case $1 \leq p \leq 2$ is a bit more technical. A closer look at a coordinate marginal of ν_p reveals that

$$\sigma_{SG}^2(\nu_p) \gtrsim n^{2/p-1}.$$

The logarithmic Sobolev constant of ν_p can be extracted from a corresponding isoperimetric inequality proved by Sodin [110], together with an isoperimetric inequality of Barthe [13] for very small sets. We finally find that

$$\rho_{LS}(\nu_p) \lesssim n^{2/p-1}.$$

Our second example is the class of rotationally invariant log-concave probabilities. Recall that an absolutely continuous probability measure μ on $\mathbb{R}^n$ is rotationally invariant if its density with respect to the Lebesgue measure only depends on the Euclidean norm of the space variable, that is, $d\mu = \rho(|\cdot|)\, d\lambda$ with ρ some non negative function on $\mathbb{R}^+$. When $\rho > 0$, the log-concavity property means that $-\log \rho$ is convex and non decreasing on $\mathbb{R}^+$. It was shown for rotationally invariant log-concave probabilities by Bobkov [23] that if $\mathrm{Cov}(\mu) = \mathrm{Id}$ then there exists some universal constant $c > 0$ such that

$$\alpha_\mu(r) \le \exp\left(-cr^2\right), \quad 0 \le r \le \sqrt{n},$$

where α_μ is the concentration function of μ defined by

$$\alpha_\mu(r) = \sup\left\{1 - \mu(A_r)\right\}, \quad r > 0,$$

the supremum running over all measurable sets $A \subset \mathbb{R}^n$ such that $\mu(A) \ge 1/2$. Above A_r is the r-enlargement open set $A_r = \{x \in \mathbb{R}^n : \exists y \in A, |x - y| < r\}$. Now, using a net argument and the latter estimate on the concentration function α_μ, it is not hard to show that

$$\alpha_\mu(r) \le \exp\left(-\frac{cr^2}{\sigma_{SG}^2(\mu)}\right), \quad r > 0.$$

While Gaussian concentration of this form is *a priori* weaker than the LSI, it was proved by Milman [88] that it is actually equivalent under convexity assumptions. Thus Conjecture 2 is valid in the subclass of rotationally invariant measures:

Theorem 3. *Let μ be a rotationally invariant log-concave probability measure on $\mathbb{R}^n$. Then*

$$\rho_{LS}(\mu) \lesssim \sigma_{SG}^2(\mu) = \|\langle\cdot, e_1\rangle\|_{\Psi_2(\mu)}^2,$$

where e_1 is the first element of the canonical basis of $\mathbb{R}^n$.

2.3. Improving Bobkov's results

The first general upper bound was proved by Bobkov [21] : if μ is a centered log-concave probability on $\mathbb{R}^n$, then it satisfies the LSI with optimal constant of order

$$\rho_{LS}(\mu) \lesssim \|\,|X|\,\|_{\Psi_2}^2,$$

where $X \sim \mu$. Here for a given real random variable Y we set

$$\|Y\|_{\Psi_2} = \inf\left\{t > 0 : \mathbb{E}\left[\Psi_2(Y/t)\right] \le 1\right\}.$$

We always have

$$\|\,|X|\,\|_{\Psi_2} \lesssim \sqrt{n}\,\sigma_{SG}(\mu),$$

and this inequality is tight in general since $\mathbb{E}[|X|^2] \le \log(2)\,\|\,|X|\,\|_{\Psi_2}^2$ which is obvious by using Jensen's inequality with the convex function Ψ_2. Hence Bobkov's result can be reformulated as

$$G_n \lesssim n.$$

In [18], two improvements of Bobkov's inequality were proved. The first one is:

Theorem 4. *For all $n \ge 1$,*

$$G_n \lesssim n^{1/2}.$$

In another direction, a result of Bobkov [22] asserts that for a log-concave random vector X,

$$C_P(X) \lesssim \mathrm{Var}(|X|^2)^{1/2}.$$

In the case of the uniform distribution on the Euclidean unit ball, the inequality is tight (both sides are of order $1/n$, up to universal constants). Analogously, the following theorem was shown in [18]. Denote the function $\psi_1(t) = e^{|t|} - 1$ and set for a given random variable Y,

$$\|Y\|_{\Psi_1} = \inf \{t > 0 : \mathbb{E}[\Psi_1(Y/t)] \leq 1\}.$$

Theorem 5. *Let X be a log-concave random vector, then*

$$\rho_{LS}(X) \lesssim \||X|^2 - \mathbb{E}[|X|^2]\|_{\psi_1}.$$

Yet again, the bound is tight, up to constants, when X is uniform over an Euclidean ball (still of order $1/n$). Let us say a word about the proof techniques. The proof of Theorem 4 uses the stochastic localization process. In a nutshell, starting from a probability μ, the stochastic localization process is a measure-valued stochastic process $(\mu_t)_{t\geq 0}$ satisfying, in particular, the following properties:

- $(\mu_t)_{t\geq 0}$ is a probability almost surely with $\mu_0 = \mu$;
- $(\mu_t)_{t\geq 0}$ is a martingale process with respect to the underlying Brownian filtration, *i.e*, for any integrable function f, the process defined by

  $$M_t = \int_{\mathbb{R}^n} f d\mu_t, \quad t > 0,$$

  is a martingale; in particular it has constant expectation:

  $$\mathbb{E}\left[\int_{\mathbb{R}^n} f d\mu_t\right] = \int_{\mathbb{R}^n} f d\mu, \quad t > 0.$$

- μ_t has a Gaussian density with respect to μ proportional to

  $$x \in \mathbb{R}^n \mapsto \exp\left(-\frac{t|x|^2}{2} + \langle c_t, x\rangle\right),$$

  where c_t is a stochastic process (a drifted Brownian motion). Notice that the variance of the Gaussian is deterministic and equal to $(1/t)\,\mathrm{Id}$.

If μ is log-concave, the third point implies that μ_t is t-strongly log-concave, that is, its density p_t with respect to the Lebesgue measure satisfies the matrix inequality

$$\nabla^2(-\log(p_t(x))) \geq t\,\mathrm{Id}, \quad x \in \mathbb{R}^n,$$

leading to the almost sure estimate on its logarithmic Sobolev constant:

$$\rho_{LS}(\mu_t) \leq \frac{1}{t} \quad \text{almost surely.}$$

Let f be a locally Lipschitz function and denote the martingale $M_t = \int_{\mathbb{R}^n} f^2 d\mu_t$, $t \geq 0$. Using the martingale property and Itô calculus, we may write for any $T \geq 0$:

$$\begin{aligned}
\mathrm{Ent}_\mu(f^2) &= \mathbb{E}[\mathrm{Ent}_{\mu_T}(f^2)] + \mathrm{Ent}(M_T) \\
&\leq \frac{2}{T}\,\mathbb{E}\left[\int_{\mathbb{R}^n} |\nabla f|^2 d\mu_t\right] + \mathrm{Ent}(M_T) \\
&= \frac{2}{T}\int_{\mathbb{R}^n} |\nabla f|^2 d\mu + \mathbb{E}\left[\int_0^T \frac{d[M]_t}{2M_t}\right].
\end{aligned}$$

Above the entropy of a positive random variable Y is defined as

$$\mathrm{Ent}(Y) = \mathbb{E}[Y \log(Y)] - \mathbb{E}[Y] \, \log\left(\mathbb{E}[Y]\right),$$

and the stochastic process $([M]_t)_{t\geq 0}$ stands for the quadratic variation of the martingale $(M_t)_{t\geq 0}$. Thus if T is chosen so that

$$\mathrm{Ent}(M_T) = \mathbb{E}\left[\int_0^T \frac{d[M]_t}{2M_t}\right] \leq \frac{\mathrm{Ent}_\mu(f^2)}{2}, \tag{2.1}$$

we would get that

$$\rho_{LS}(\mu) \leq \frac{4}{T}.$$

It can be shown that

$$\frac{d[M]_t}{2M_t} \lesssim \sigma_{SG}^2(\mu_t) \, \mathrm{Ent}_{\mu_t}(f^2). \tag{2.2}$$

Hence, plugging (2.2) into (2.1), and using Jensen's inequality,

$$\mathbb{E}\left[\mathrm{Ent}_{\mu_t}(f^2)\right] \leq \mathrm{Ent}_\mu(f^2),$$

we arrive at the following result.

Proposition 6. *If $\lambda > 0$ is such that*

$$\sigma_{SG}^2(\mu_t) \lesssim \lambda \, \sigma_{SG}^2(\mu) \quad \textit{a.s.}, \quad t > 0,$$

then

$$\rho_{LS}(\mu) \lesssim \lambda \, \sigma_{SG}^2(\mu).$$

This proof strategy is analogous to the one developed for the KLS conjecture, see *e.g.* [73]. However, little is known about the behavior of $\sigma_{SG}^2(\mu_t)$ and in general, one cannot expect an almost sure bound. A challenge is to weaken the requirement of Proposition 6 to a bound with high probability, or to a bound on average. The proof of Theorem 4 is less straightforward. We refer to [18] and [17] for details.

Finally, the proof of Theorem 5 uses the needle decomposition lemma, which reduces the problem to proving a suitable estimate on the logarithmic-Sobolev constant of a one-dimensional log-concave probability measure. Given a log-concave probability measure μ on $\mathbb{R}^n$ and a measurable set A, one can decompose μ into a mixture of one-dimensional log-concave probabilities called needles :

$$\mu = \int_\Omega \mu_I \, d\nu(I) = \mathbb{E}_\nu[\mu_I],$$

for some mixture measure ν defined on a collection Ω of line segments partitioning $\mathbb{R}^n$ and ν-almost surely, μ_I is a one-dimensional log-concave probability measure supported on the segment I with

$$\mu_I(A) = \mu(A).$$

It is then possible to show that a significant fraction, say half, of the μ_I are subgaussian with constant

$$\sigma_{SG}^2(\mu_I) \lesssim \| |X|^2 - \mathbb{E}[|X|^2] \|_{\psi_1}, \tag{2.3}$$

and thus enjoy a LSI with the same constant. We will use the equivalence between the logarithmic isoperimetric constant and the logarithmic Sobolev constant proved by Ledoux [76] in the log-concave setting, that is,

$$\rho_{LS}(\mu) \simeq \left(\inf \frac{\mu^+(B)}{\mu(B)\sqrt{-\log(\mu(B))}}\right)^{-2}, \tag{2.4}$$

where the infimum runs over measurable sets B such that $\mu(B) \leq 1/2$ and

$$\mu^+(B) = \liminf_{r \to 0} \frac{\mu(B^r) - \mu(B)}{r},$$

is the Minkowski content of B. We assume that $\mu(A) \leq 1/2$ and we write

$$\begin{aligned}\mu^+(A) = \mathbb{E}_\nu[\mu_I^+(A)] &\gtrsim \mathbb{E}_\nu \left[\frac{1}{\sqrt{\rho_{LS}(\mu_I)}} \, \mu_I(A) \sqrt{-\log(\mu_I(A))} \right] \\ &= \mathbb{E}_\nu \left[\frac{1}{\sqrt{\rho_{LS}(\mu_I)}} \right] \mu(A) \sqrt{-\log(\mu(A))} \\ &\gtrsim \mathbb{E}_\nu \left[\frac{1}{\sigma_{SG}(\mu_I)} \right] \mu(A) \sqrt{-\log(\mu(A))},\end{aligned}$$

where in the last line we used the one-dimensional inequality

$$\rho_{LS}(\mu_I) \lesssim \sigma_{SG}^2(\mu_I).$$

Using the estimate (2.3), which holds with ν probability $1/2$, we finally obtain :

$$\mu^+(A) \gtrsim \frac{\mu(A)\sqrt{-\log(\mu(A))}}{\sqrt{\||X|^2 - \mathbb{E}[|X|^2]\|_{\psi_1}}},$$

which is equivalent to Theorem 5 according to the equivalence (2.4).

3. Beyond the Bakry-Emery theory: the Polchinski renormalization group

In this part, we briefly present the Polchinski renormalization group in its recent formulation by Bauerschmidt and Bodineau [14]. We mention the main motivation of its introduction, which was the study of the logarithmic Sobolev constant of some continuous infinite dimensional models from Euclidean fields theory. We conclude by showing how it can be seen as a natural strengthening of the famous Bakry-Emery theory [9].

3.1. **The Polchinski renormalization group**

Let us start with a probability distribution ν_0 on $\mathbb{R}^n$ which is given by

$$d\nu_0(\varphi) = e^{-V_0(\varphi)} d\gamma_{C_\infty}(\varphi), \tag{3.1}$$

where γ_{C_∞} denotes the centered Gaussian distribution on $\mathbb{R}^n$ with covariance matrix C_∞ and $V_0 : \mathbb{R}^n \to \mathbb{R}$ is some smooth potential, at least of class $\mathcal{C}^2$. Let us emphasize that we make the unusual choice of variable φ instead of x because in the following section this model will be applied in the setting of the Euclidean fields theory. Moreover the matrix C_∞ is assumed to be positive-definite to simplify the presentation (the usual and relevant framework is semi-definiteness, in particular when some discrete models arising in lattice field theory are considered; see the discussion in the next section).

Our goal is to study the logarithmic Sobolev constant of ν_0. To this aim, let $t \in [0, \infty] \mapsto C_t$ be a smooth family of positive-definite matrices that are increasing in the sense of symmetric matrices, such that $C_0 = 0$ and C_∞ coincides with the covariance matrix of the Gaussian part of ν_0. By using the convolution properties of the Gaussian distribution (*i.e.* the random vector $X + Y$ is Gaussian with covariance matrix $\Sigma_1 + \Sigma_2$ as soon as X and Y are independent Gaussian random vectors with respective covariance matrix Σ_1 and Σ_2), we can then write for all smooth function F,

$$\int_{\mathbb{R}^n} F(\varphi)\, d\nu_0(\varphi) = \int_{\mathbb{R}^n} \left(\frac{1}{Z_t(\varphi)} \int_{\mathbb{R}^n} F(\varphi+\zeta) e^{-V_0(\varphi+\zeta)}\, d\gamma_{C_t}(\zeta) \right) Z_t(\varphi)\, d\gamma_{C_\infty - C_t}(\varphi),$$

where $Z_t(\varphi)$ is the following normalizing constant (with respect to the variable ζ)

$$Z_t(\varphi) = \int_{\mathbb{R}^n} e^{-V_0(\varphi+\zeta)}\, d\gamma_{C_t}(\zeta).$$

In other words, we get the following decomposition for ν_0:

$$\int_{\mathbb{R}^n} F(\varphi)\, d\nu_0(\varphi) = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} F(\zeta)\, d\mu_t^\varphi(\zeta) \right) d\nu_t(\varphi),$$

where the renormalized measure ν_t is given by

$$d\nu_t(\varphi) = Z_t(\varphi)\, d\gamma_{C_\infty - C_t}(\varphi) = e^{-V_t(\varphi)}\, d\gamma_{C_\infty - C_t}(\varphi),$$

depending on the renormalized potential

$$V_t(\varphi) = -\log\left(Z_t(\varphi)\right) = -\log\left(\int_{\mathbb{R}^n} e^{-V_0(\varphi+\zeta)}\, d\gamma_{C_t}(\zeta) \right),$$

and the fluctuation measure μ_t^φ is a random conditional probability distribution related to the so-called Polchinski semigroup. Changing variables from $\varphi + \zeta$ to ζ, the fluctuation measure μ_t^φ has a form somewhat similar to the original one ν_0 (up to the addition of an external field $C_t^{-1}\varphi$), except that the underlying covariance matrix is now C_t. In particular an important observation resides in the fact that we are able to compare their convexity properties: since the associated potentials of ν_0 and μ_t^φ have respective Hessian matrix

$$\nabla^2 V_0 + C_\infty^{-1} \quad \text{and} \quad \nabla^2 V_0 + C_t^{-1},$$

and because $C_t < C_\infty$ (recall that $t \mapsto C_t$ is increasing), one deduces that the potential related to μ_t^φ is more convex than that of ν_0. Note that up to some choice of an adequate random variable φ and time-inversion, it is related to Eldan's stochastic localization emphasized in the previous section, see for instance [15].

While the probability measure μ_t^φ corresponds to the fluctuations of ν_0 at variance scales below C_t, the renormalized measure ν_t corresponds to a ν_0 for which fluctuations at variance scales smaller than C_t have been integrated, and hence the renormalized measures ν_t interpolate between ν_0 and the Dirac delta δ_0. Moreover we mention that one can derive an evolution equation for the renormalized potential V_t, which is the following Hamilton-Jacobi-Bellman equation

$$\partial_t V_t = \frac{1}{2} \sum_{i,j=1}^n (\dot{C}_t)_{ij} \partial^2_{ij} V_t - \frac{1}{2} \sum_{i,j=1}^n (\dot{C}_t)_{ij}\, \partial_i V_t\, \partial_j V_t,$$

and corresponds to an instantiation of the broader Polchinski equation derived by Polchinski in 1984 in his famous paper [101]. As such, the covariance derivatives $\dot{C}_t$ play the role of an inverse metric on $\mathbb{R}^n$, the dot notation standing for the usual time derivative.

Once this formalism has been introduced, Bauerschmidt and Bodineau obtained in [14] the following result, dealing with an estimate on the logarithmic Sobolev constant. Their result is based on a kind of convexity property of the renormalized potential V_t with respect to the inverse metric generated by $\dot{C}_t$. Below we assume to simplify the presentation that the family $(C_t)_{t\geq 0}$ is chosen such that $\dot{C}_0 = \mathrm{Id}$.

Theorem 7. *Assume that for all $t \geq 0$, there exists some $\dot\lambda_t \in \mathbb{R}$ such that*

$$\dot C_t \, \nabla^2 V_t \, \dot C_t - \frac{1}{2}\ddot C_t \geq \dot\lambda_t \, \dot C_t. \tag{3.2}$$

Denoting $\lambda_t = \int_0^t \dot\lambda_s ds$, then ν_0 satisfies the LSI with optimal constant

$$\rho_{LS}(\nu_0) \leq \int_0^\infty e^{-2\lambda_t} \, dt.$$

The Polchinski renormalization group provides a natural and powerful strengthening of the renowned Bakry-Emery theory. For a probability distribution taking the form (3.1) with a convex potential V_0, the LSI is satisfied with optimal constant

$$\rho_{LS}(\nu_0) \leq \frac{1}{\lambda_{\min}(C_\infty^{-1})} = \|\mathrm{Cov}(\nu_0)\|_{\mathrm{op}},$$

where $\lambda_{\min}$ denotes the smallest eigenvalue of the symmetric matrix considered. In this situation, choosing in particular $C_\infty^{-1} = \lambda \, \mathrm{Id}$ for some $\lambda > 0$ and

$$C_t = \int_0^t \exp(-s \, C_\infty^{-1}) \, ds = \frac{1 - e^{-\lambda t}}{\lambda} \, \mathrm{Id}, \quad t \geq 0,$$

and using the fact that the convexity is preserved by the Polchinski equation (*i.e.*, V_t is convex for all $t > 0$ as soon as V_0 is; such a property might be obtained from the Prékopa-Leindler inequality or a maximum principle argument), then (3.2) holds with $\dot\lambda_t = \lambda/2$ and therefore Theorem 7 recovers the classical Bakry-Emery result. However Theorem 7 allows to consider interesting situations beyond this uniformly convex setting. Indeed, the initial potential V_0 is not required to be convex, in contrast to the Bakry–Emery criterion. More precisely, if we are able to prove that the renormalization flow improves the non-convexity of V_0, *i.e.*, V_t becomes more and more convex so that the integral above is finite, then the LSI holds. In general, the analysis of the renormalized potential V_t is model dependent.

3.2. **Logarithmic Sobolev inequality for Euclidean field theory**

This formalism meets the renormalization group approach from Quantum Field Theory, which roughly consists in analyzing the regularity of a scalar *random field*, that is a random function $\varphi : \mathbb{R}^d \to \mathbb{R}$, at different scales. Intuitively, such a random function cannot be a function, because there is no Lebesgue measure in infinite dimension. Instead, it is a random distribution which is therefore not well-defined at any point. Nevertheless if we want to approximate its value at a point, we have to remove the divergent counter-terms, and the closer we get to a point, the more important these counter-terms become. This procedure for changing the scales at which a theory is studied can be formalized as the action of a semigroup, hence the name renormalization group. These procedures were widely used by Wilson in the 1970s to study phase transition phenomena [118].

Since such objects are hard to define rigourously, a possible way to overcome this difficulty is given by the constructive approach, which consists in replacing the continuum $\mathbb{R}^n$ on which such a random field is defined, by a discrete space: the lattice $\varepsilon\mathbb{Z}^n$, and then letting ε go to zero. Defining a random scalar function on this lattice is equivalent to defining a random vector in $\mathbb{R}^\Lambda$, where Λ denotes the number of sites in the lattice, and therefore such a random function is well-defined, and the most studied ones are given by a density of the form (3.1), where C_∞ is the graph Laplacian of the lattice. Since the discrete Laplacian converges towards the continuous one as ε goes to zero, such a procedure as described above will formally allow to define a random field $\varphi : \mathbb{R}^n \to \mathbb{R}$, with density

$$\mathcal{D}\varphi \, \exp\left\{ -\frac{1}{2}|\nabla\varphi|^2 - V_0(\varphi) \right\},$$

where $\mathcal{D}\varphi$ denotes the (ill-defined) Feynman path integral. Let us also mention two alternative approaches to the above constructive one, which are Hairer's regularity structures (see e.g. [57]) and the approach proposed by Gubinelli and his co-authors, based on techniques from paradifferential calculus and on ideas from the theory of controlled rough paths [54].

From this constructive perspective described above, the strength of Bauerschmidt-Bodineau's approach lies in the fact that the logarithmic Sobolev constant $\rho_{LS}(\nu_0)$ in Theorem 7 depends solely on the parameters $\dot{\lambda}_t$ and not on the ambient dimension n. In other words, this constant is independent of the number of lattice sites and hence of the lattice parameter ε. As a result, one can take the limit as $\varepsilon \to 0$ and establish logarithmic Sobolev inequalities for infinite-dimensional continuum models. Notably, without delving into the details, this method has been successfully applied to the two-dimensional sine-Gordon model, associated with the potential $V_0(\varphi) = \cos(\varphi)$, as well as the φ^4 model, corresponding to a quartic potential.

3.3. **Higher order eigenvalues**

As mentioned above, the Bakry-Emery theory has interesting consequences in terms of the spectral gap of the underlying self-adjoint operator. However in the case of a discrete spectrum, the whole spectrum can be addressed too, cf. for instance the recent paper [89]. Since the Polchinski renormalization group approach reinforces this theory, it is therefore expected that the multiscale criterion (3.2) enables also an analysis of the entire spectrum. Let us provide some details. The weighted Laplacian canonically associated with the probability measure ν_0 given by (3.1) is the following second-order differential linear operator:

$$Lf(\varphi) = \Delta f(\varphi) - \left\langle C_\infty^{-1}\varphi + \nabla V_0(\varphi), \nabla f(\varphi) \right\rangle .$$

It is a symmetric and non-positive operator on the space $\mathbb{L}^2(\nu_0)$, hence its unique self-adjoint extension has its spectrum (or rather that of $-L$) denoted $\sigma(-L)$ which is included in $[0,\infty)$. Under mild assumptions on V_0, its spectrum is discrete and zero is an eigenvalue with eigenspace the space of constant functions. Let us denote $\lambda_1(-L)$ the next positive eigenvalue (the spectral gap of $-L$) as well as the higher eigenvalues $\lambda_n(-L)$ counted without multiplicity. When V_0 is convex, the Bakry-Emery criterion ensures that

$$\lambda_1(-L) \geq \lambda_{\min}(C_\infty^{-1}).$$

This inequality can be interpreted as a comparison result between the first eigenvalue of the operator $-L$ related to ν_0 and that of the (unique self-adjoint extension in $\mathbb{L}^2(\gamma_{C_\infty})$ of the) operator related to the Gaussian part γ_{C_∞} of ν_0 given by

$$L_{\mathrm{G}} f(\varphi) = \Delta f(\varphi) - \left\langle C_\infty^{-1}\varphi, \nabla f(\varphi) \right\rangle .$$

In other words the operator L_{G} is nothing but the famous Ornstein-Uhlenbeck semigroup associated to the covariance matrix C_∞ and we have

$$\lambda_1(-L_{\mathrm{G}}) = \lambda_{\min}(C_\infty^{-1}).$$

Using the celebrated Caffarelli contraction principle adapted to the present eigenvalues problem, Milmam proved in [89] that this comparison result also holds for higher eigenvalues when V_0 is convex. A natural question is then to ask how the Polchinski renormalization group approach allows to recover this result. This is the matter of the next result.

Theorem 8. *[74, 89, 106] If the potential V_0 is convex, then for all $i \in \mathbb{N}$,*

$$\lambda_i(-L) \geq \lambda_i(-L_{\mathrm{G}}).$$

Let us give the sketch of the proof. Consider the Ornstein-Uhlenbeck semigroup $(P_t)_{t\geq 0}$ generated by the operator L_{G}, *i.e.*, it is given for all smooth functions F by

$$P_tF(\varphi) = \int_{\mathbb{R}^n} F\left(\varphi \exp(-t\,C_\infty^{-1}) + (\mathrm{Id} - \exp(-2t\,C_\infty^{-1}))^{1/2}\,\zeta\right)\, d\gamma_{C_\infty}(\zeta), \quad t \geq 0.$$

Choosing the covariance decomposition $(C_t)_{t\geq 0}$ as

$$C_t = \int_0^t \exp(-s\,C_\infty^{-1})\, ds, \quad t \geq 0,$$

we can show that

$$P_{t/2}(e^{-V_0}) = e^{-V_t}, \quad t \geq 0.$$

Such an identity appeared first in Shenfeld [108], connecting the renormalized potential to the Ornstein-Uhlenbeck semigroup. Now, by the Courant-Fisher theorem, we have the following variational formula for the eigenvalues

$$\lambda_{i+1}(-L) = \inf_{f\perp E_i} \frac{\int_{\mathbb{R}^n} |\nabla f|^2 d\nu_0}{\int_{\mathbb{R}^n} f^2 d\nu_0},$$

where E_i denotes the space spanned by all the eigenfunctions associated with an eigenvalue $\lambda_k(-L)$ for $k \leq i$. Let us then consider the Rayleigh quotient

$$R_f(t) = \frac{\int_{\mathbb{R}^n} |\nabla Q_t f|^2 d\nu_t}{\int_{\mathbb{R}^n} (Q_t f)^2 d\nu_t}, \quad t \geq 0,$$

where Q_t is the semigroup given by

$$Q_t f = \frac{P_t(fe^{-V_0})}{P_t(e^{-V_0})} = e^{V_{2t}}\, P_t(fe^{-V_0}).$$

Note that we have

$$R_f(0) = \frac{\int_{\mathbb{R}^n} |\nabla f|^2 d\nu_0}{\int_{\mathbb{R}^n} f^2 d\nu_0}.$$

Computing the time derivative of R_f, one can see that

$$\partial_t R_f(t) \leq -\frac{\int_{\mathbb{R}^n} \langle \nabla Q_t f, \dot{C}_t\, \nabla^2 V_t\, \dot{C}_t\, \nabla Q_t f\rangle\, d\nu_t}{\int_{\mathbb{R}^n} (Q_t f)^2 d\nu_t},$$

and since the convexity of V_0 implies that of V_t for all $t > 0$, we get that $t \mapsto R_f(t)$ is non-increasing. By a standard semigroup argument, we deduce that

$$\lambda_{i+1}(-L) \geq \lambda_{i+1}(-L_t), \quad t > 0,$$

where L_t is the operator associated to the probability distribution $P_t(e^{-V_0})\, d\gamma_{C_\infty}$. Letting t go to infinity, the convergence in distribution of the Ornstein-Uhlenbeck process towards the centered Gaussian law with covariance matrix C_∞ yields the desired conclusion.

Theorem 8 is contained in [106] but stated differently. However the idea goes back to the work of Klartag and Putterman [74] who proved the monotonicity of the spectrum of a probability distribution under Gaussian convolution. An alternative approach is to construct a Lipschitz transport map from the Gaussian to our target distribution ν_0, as it is usually done for uniformly convex potentials through the Caffarelli contraction theorem, and more recently by Kim and Milman [68] who used an inverse of the Ornstein-Uhlenbeck flow. Finally, we also mention the work of Shenfeld [108], who explains how to adapt the Kim-Milman construction to infinite-dimensional Euclidean field theory models using the Bauerschmidt-Bodineau multiscale analysis presented in this note.

4. Logarithmic Sobolev inequalities for uniform in time propagation of chaos

The goal of this last section is to discuss an application of the LSI to the study of large interacting particle systems.

4.1. **Motivation**

Consider a system of N interacting particles given by the following system of stochastic differential equations (SDEs):

$$dX_t^{i,N} = \frac{1}{N}\sum_{j\neq i} K(X_t^{i,N} - X_t^{j,N})dt + \sqrt{2\sigma}dB_t^i, \quad i \in \{1, ..., N\}, \tag{4.1}$$

where $X_t^{i,N} \in \mathbb{R}^n$ denotes the position at time t of particle i, $(B^i)_{i=1,...,N}$ are independent standard Brownian motions and $K : \mathbb{R}^n \mapsto \mathbb{R}^n$ is an interaction kernel on which we will give more precise assumptions later. The particles are said to be in mean-field interactions (because of the rescaling $1/N$ ensuring that each one feels a force of order 1) and we assume they are exchangeable, *i.e*, their joint law is invariant by permutation of indices.

The main problem we address is the following: what happens when N goes to infinity? This is linked to a classical question in Statistical Physics aimed at understanding how one can go from a microscopic model, *i.e.*, a description of the system in which we follow the dynamics of each individual particle as in (4.1), to a mesoscopic model, *i.e.*, an evolution equation for the statistical description of the system. The goal is to reduce the study of a (very) large system to the study of a single object. Doing so, we lose some information. In particular, the mesoscopic description gives the probability distribution of one particle in the system, but we do not know *a priori* the joint law of two or more particles, *i.e.*, the correlations in the system in the limit $N \to \infty$. The answer to this problem lies in a phenomenon named Propagation of Chaos (PoC) by Kac [65] (and then generalized by McKean [84]) which states that, as N grows, two given particles become "more and more" statistically independent. Intuitively, this comes from the fact that two particles are only weakly correlated through a force of order $1/N$, and only see each other via their (common) empirical measure. In fact, let us rewrite the system (4.1) in a way that emphasizes this point. Denoting $\mu_t^N = N^{-1}\sum_{i=1}^N \delta_{X_t^{i,N}}$, we have

$$dX_t^{i,N} = K * \mu_t^N(X_t^{i,N})dt + \sqrt{2\sigma}dB_t^i, \quad \text{where} \quad K * \mu(x) := \int_{\mathbb{R}^n} K(x-y)\, d\mu(y).$$

Since they also are identically distributed (by exchangeability), we expect, thanks to some form of law of large numbers, that μ_t^N converges to $\overline{\rho}_t$, the law of one typical particle at the limit $N \to \infty$. We thus identify the candidate for the limit, which is known as a McKean-Vlasov SDE

$$\begin{cases} d\overline{X}_t = K * \overline{\rho}_t(\overline{X}_t)dt + \sqrt{2\sigma}dB_t, \\ \overline{\rho}_t = \text{Law}(\overline{X}_t). \end{cases} \tag{4.2}$$

To the SDEs (4.1) and (4.2) one can link two partial differential equations (PDEs), known in the literature as Fokker-Planck, (forward) Kolmogorov or Liouville equations, which describe the time evolution of the laws of the processes. Denoting ρ_t^N the law of the random vector $(X_t^{1,N}, ..., X_t^{N,N})$ in $\mathbb{R}^{nN}$, the PDE associated to (4.1) writes as

$$\partial_t \rho_t^N = -\sum_{i=1}^N \nabla_{x_i} \cdot \left(\left(\frac{1}{N}\sum_{j\neq i} K(x_i - x_j) \right) \rho_t^N \right) + \sigma \sum_{i=1}^N \Delta_{x_i}\rho_t^N, \tag{4.3}$$

where $\nabla_{x_i}\cdot$ stands for the divergence operator with respect to the variable x_i. Concerning (4.2),

$$\partial_t \overline{\rho}_t = -\nabla \cdot ((K * \overline{\rho}_t)\overline{\rho}_t) + \Delta\overline{\rho}_t. \tag{4.4}$$

The nonlinear PDE (4.4) above would thus be the mesoscopic counterpart to the microscopic system (4.1). The goal is therefore to prove that the particle system (4.1) (or (4.3)) converges in some sense towards the nonlinear object (4.2) (or (4.4)). To be more precise, since *a priori* ρ_t^N is a density on $\mathbb{R}^{nN}$ and $\overline{\rho}_t$ on $\mathbb{R}^n$, we actually prove the convergence of the marginals. Let us denote $\rho_t^{k,N} = \mathrm{Law}(X_t^{1,N}, ..., X_t^{k,N})$ the joint law of the first k particles within the N particle system (with the convention $\rho_t^{N,N} = \rho_t^N$), which by exchangeability is also the joint law of any subset of k particles, and $\overline{\rho}_t^{\otimes k} = \overline{\rho}_t \otimes ... \otimes \overline{\rho}_t$ the non linear limit law $\overline{\rho}_t$ tensorized k times. Our main objective is to show a result of the form

$$\text{PoC:} \qquad \forall k \in \mathbb{N}^*, \qquad \lim_{N\to\infty} \rho_0^{k,N} = \overline{\rho}_0^{\otimes k} \implies \forall t \geq 0, \ \lim_{N\to\infty} \rho_t^{k,N} = \overline{\rho}_t^{\otimes k}. \tag{4.5}$$

Notice that this property yields "independence at the limit", as the joint law converges towards a tensorized law.

We do not discuss the many domains in which one might wish to obtain PoC, nor do we discuss the vast literature around this topic. We instead refer to the classical courses [85, 113], and to the more recent reviews [61] and [37, 38]. Let us only insist on a few "sub-goals". We would like to prove PoC:

- quantitatively: we always see one model (4.1) or (4.4) as an approximation of the other, and we wish to quantify the error made in this approximation.
- uniformly in time: since one observes in (4.5) that PoC is only a limit as N goes to infinity for fixed t, understanding how this convergence behaves with time is then a critical problem, for instance if we wish to compare the (possible) invariant distributions of both models.
- for singular interactions: many physically relevant interaction kernel K are singular in 0 (in the sense $|K(x)|$ tends to infinity as x tends to 0).

In order to quantify PoC, we will need the notion of rescaled relative entropy, defined for all probability measures μ and ν on $\mathbb{R}^{nN}$ by

$$\mathcal{H}_N(\nu, \mu) = \begin{cases} \frac{1}{N} \int_{\mathbb{R}^{nN}} \frac{d\nu}{d\mu} \log \frac{d\nu}{d\mu}\, d\mu & \text{if } \nu \ll \mu, \\ \infty & \text{otherwise.} \end{cases}$$

Note that it corresponds to 1 over N times the entropy $\mathrm{Ent}_\mu(\frac{d\nu}{d\mu})$, the integral being on $\mathbb{R}^{nN}$. Our goal is now to give, in a specific case, an example of a uniform in time LSI as a main crucial tool to prove uniform in time PoC.

4.2. **An example: the 2D vortex model**

Let us give an example concerning the Biot-Savart kernel, which is defined in $\mathbb{R}^2$ by

$$K(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2} = \frac{1}{2\pi}\left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right),$$

and which appears in fluid mechanics when considering the vorticity equation associated to the 2D incompressible Navier-Stokes system. The main result we prove is the following uniform in time PoC.

Theorem 9. *Under some convenient assumptions (satisfied by the Biot-Savart kernel) there are positive constants C_1, C_2 and C_3 such that for all $N \in \mathbb{N}$, all exchangeable probability density ρ_0^N and all $t \geq 0$,*

$$\mathcal{H}_N(\rho_t^N, \overline{\rho}_t^{\otimes N}) \leq C_1\, e^{-C_2 t}\, \mathcal{H}_N(\rho_0^N, \overline{\rho}_0^{\otimes N}) + \frac{C_3}{N}.$$

Note that a convergence of the relative entropy yields a convergence in $\mathbb{L}^1(\lambda)$ and (up to additional technicalities) in Wasserstein distance of order 2. Therefore (with $\rho_0^N = \overline{\rho}_0^{\otimes N}$), we also obtain the existence of some $C > 0$ such that for all $k \in \{1, \dots, N\}$ and all $t \geq 0$,

$$\|\rho_t^{k,N} - \overline{\rho}_t^{\otimes k}\|_{\mathbb{L}^1(\lambda)} + \mathcal{W}_2\left(\rho_t^{k,N}, \overline{\rho}_t^{\otimes k}\right) \leq C\left(\left\lfloor \frac{N}{k} \right\rfloor\right)^{-\frac{1}{2}}.$$

Above the Wasserstein distance of order 2 is defined as

$$\mathcal{W}_2\left(\rho_t^{k,N}, \overline{\rho}_t^{\otimes k}\right) = \left(\inf \int_{\mathbb{R}^{nk}} \int_{\mathbb{R}^{nk}} |x-y|^2\, \Gamma(dx, dy)\right)^{1/2},$$

the infimum running over probability distributions Γ on $\mathbb{R}^{nk} \times \mathbb{R}^{nk}$ such that its marginals are respectively $\rho_t^{k,N}$ and $\overline{\rho}_t^{\otimes k}$.

From now on we work with particles on the two-dimensional torus $\mathbb{T}^2$ to simplify the presentation (and consider a periodized version of the kernel) although the main ideas are also valid in more general situations. Indeed, we refer to the recent series of articles [39, 50, 55, 62, 91, 117] (here in order of preprint appearance) which end up obtaining uniform in time PoC for the 2D vortex model on the entire space $\mathbb{R}^2$, and even with a better speed of convergence in N. The interested reader may find all the technical details in these works. Here we only give an overview of the proof in order to insist on the application of the LSI, and thus consider here the framework of [55].

First let us concentrate on the time derivation of the rescaled relative entropy. The starting point consists in differentiating the rescaled relative entropy along the PDEs. We write

$$\mathcal{H}_N(t) = \mathcal{H}_N(\rho_t^N, \overline{\rho}_t^{\otimes N}), \quad \mathcal{I}_N(t) = \frac{1}{N}\sum_i \int_{\mathbb{T}^{2N}} \rho_t^N \left|\nabla_{x_i} \log \frac{\rho_t^N}{\overline{\rho}_t^{\otimes N}}\right|^2 d\mathbf{x}^N.$$

By direct calculations, we have

$$\begin{aligned}
\frac{d}{dt}\mathcal{H}_N(t) = &-\mathcal{I}_N(t) \\
&- \frac{1}{N^2}\sum_{i,j} \int_{\mathbb{T}^{2N}} \rho_t^N \langle (K(x_i - x_j) - K * \overline{\rho}_t(x_i)), \nabla_{x_i} \log \overline{\rho}_t^{\otimes N}\rangle d\mathbf{x}^N \\
&- \frac{1}{N^2}\sum_{i,j} \int_{\mathbb{T}^{2N}} \rho_t^N \left(\operatorname{div} K(x_i - x_j) - \operatorname{div} K * \overline{\rho}_t(x_i)\right) d\mathbf{x}^N.
\end{aligned}$$

Here, the calculations do not require much assumptions on the kernel, only the fact that the solution $\overline{\rho}_t$ is regular enough to justify them. Since the Biot-Savart kernel is divergence free, we actually get rid of the last line. Furthermore, there exists an explicit matrix field $V \in \mathbb{L}^\infty(\lambda)$ such that $K = \nabla \cdot V$. Therefore, by integration by parts, for all $t \geqslant 0$,

$$\frac{d}{dt}\mathcal{H}_N(t) \leq A_N(t) + \frac{1}{2}B_N(t) - \frac{1}{2}\mathcal{I}_N(t),$$

where

$$\begin{aligned}
A_N(t) :=& \frac{1}{N^2}\sum_{i,j} \int_{\mathbb{T}^{2N}} \rho_t^N \left(V(x_i - x_j) - V * \overline{\rho}_t(x_i)\right) : \frac{\nabla^2_{x_i} \overline{\rho}_t^{\otimes N}}{\overline{\rho}_t^{\otimes N}} d\mathbf{x}^N, \\
B_N(t) :=& \frac{1}{N}\sum_i \int_{\mathbb{T}^{2N}} \rho_t^N \frac{\left|\nabla_{x_i} \overline{\rho}_t^{\otimes N}\right|^2}{|\overline{\rho}_t^{\otimes N}|^2} \left|\frac{1}{N}\sum_j V(x_i - x_j) - V * \overline{\rho}_t(x_i)\right|_f^2 d\mathbf{x}^N.
\end{aligned}$$

Above, given two square matrices A, B, the Hilbert-Schmidt norm $|A|_f$ and the matrix operation $A : B$ are respectively defined by

$$|A|_f = \left(\sum_{i,j} a_{i,j}^2\right)^{1/2}, \quad A : B = \sum_{i,j} a_{i,j}\, b_{i,j}.$$

These two quantities are integrals against ρ_t^N, but we would like them to be against $\overline{\rho}_t^{\otimes N}$ instead, so that we could use some form of law of large numbers. This is possible, using a change of measures, *i.e*, the fact that for μ and ν two probability measures on $\mathbb{T}^{2N}$, we have for all convenient function Φ the following simple inequality: for all $\eta > 0$,

$$\int_{\mathbb{T}^{2N}} \Phi \, d\mu \leq \eta\, \mathcal{H}_N(\mu,\nu) + \frac{\eta}{N} \log \int_{\mathbb{T}^{2N}} e^{N\Phi/\eta}\, d\nu.$$

We then use large deviations estimates from [62] and obtain that for all $t \geqslant 0$,

$$\frac{d}{dt}\mathcal{H}_N(t) \leq C\left(\mathcal{H}_N(t) + \frac{1}{N}\right) - \frac{1}{2}\mathcal{I}_N(t),$$

where

$$C = 2\,\hat{C}_1\,\lambda\,\|V\|_{\mathbb{L}^\infty(\lambda)}\,\|\nabla^2\overline{\rho}_t\,\|_{\mathbb{L}^\infty(\lambda)} + 4\,\hat{C}_2\,\lambda^2\,\|V\|^2_{\mathbb{L}^\infty(\lambda)}\,\|\nabla\overline{\rho}_t\|^2_{\mathbb{L}^\infty(\lambda)},$$

with $\hat{C}_1, \hat{C}_2$ some universal constants, and $\lambda > 0$ such that for all $t \geq 0$ we have $\overline{\rho}_t \in [\lambda^{-1}, \lambda]$ (we can prove this holds provided it is true at time 0, i.e. $\overline{\rho}_0 \in [\lambda^{-1}, \lambda]$, and here for instance we use the fact that we work on the torus). One could stop at this stage, as in [62], and obtain via Gronwall's lemma a control on the relative entropy at time t by the relative entropy at time 0, thus ensuring non uniform in time PoC. We however here wish to use the Fisher information to use the dissipation of entropy and improve on this control.

We therefore have two goals. We wish to have the existence of a constant c such that $\mathcal{H}_N(t) \leq c\mathcal{I}_N(t)$, as well as uniform in time bounds on $\|\nabla\overline{\rho}_t\|_{\mathbb{L}^\infty(\lambda)}$ and $\|\nabla^2\overline{\rho}_t\|_{\mathbb{L}^\infty(\lambda)}$. This last point can be done *via* direct calculations on the PDE, and we in fact obtain that there exists $C > 0$ such that for all $t \geq 0$,

$$\int_0^t \|\nabla\overline{\rho}_t\|^2_{\mathbb{L}^\infty(\lambda)} \leq C \quad \text{and} \quad \int_0^t \|\nabla^2\overline{\rho}_t\|^2_{\mathbb{L}^\infty(\lambda)} \leq C.$$

Then, in order to prove that the Fisher information controls the relative entropy, the use of a LSI is the main tool. Indeed, if $\overline{\rho}_t^{\otimes N}$ satisfies a LSI with constant c, then using the LSI for $f = \sqrt{\frac{\rho_t^N}{\overline{\rho}_t^{\otimes N}}}$ in fact yields

$$\mathcal{H}_N(t) \leq \frac{c}{2}\,\mathcal{I}_N(t). \tag{4.6}$$

Note that we would like c to be independent of N and of t. To obtain such a LSI for $\overline{\rho}_t$, we start by using the fact that the uniform measure u on the torus satisfies a LSI [11, Proposition 5.7.5]. Therefore, since $\lambda^{-1}u \leq \overline{\rho}_t \leq \lambda u$, by Holley-Stroock perturbation lemma [11, Proposition 5.1.6], the measure $\overline{\rho}_t$ also does. Finally, by the tensorization property [11, Proposition 5.2.7], the measure $\overline{\rho}_t^{\otimes N}$ also does (with the same constant c, independent of t and N). We thus obtain (4.6).

What is hidden, both in the proof of the uniform bounds on $\|\nabla\overline{\rho}_t\|_{\mathbb{L}^\infty(\lambda)}$ and $\|\nabla^2\overline{\rho}_t\|_{\mathbb{L}^\infty(\lambda)}$ and in the proof of the LSI, is the fact that the stationary distribution for (4.4) is the uniform distribution on the torus, and that $\overline{\rho}_t$ converges to it. Although we do not explicitly use this property, it morally ensures that $\overline{\rho}_t$ remains a perturbation of the Lebesgue measure provided it began as one, and that its derivatives flatten to 0 sufficiently fast. In a more general setting, say on $\mathbb{R}^2$ for instance, one could add a confining force and prove that the invariant distribution, relative to which the solution should stay bounded, satisfies a LSI.

Finally, there exist constants $C_1, C_2^\infty, C_3 > 0$ and a function $t \mapsto C_2(t) > 0$ (satisfying $\int_0^t C_2(s)ds \le C_2^\infty$ for all $t \ge 0$) such that for all $t \ge 0$,

$$\frac{d}{dt}\mathcal{H}_N(t) \le -(C_1 - C_2(t))\mathcal{H}_N(t) + \frac{C_3}{N}.$$

This implies

$$\mathcal{H}_N(t) \le e^{C_2^\infty - C_1 t}\mathcal{H}_N(0) + \frac{C_3}{C_1 N}e^{C_2^\infty},$$

which allows us to conclude on the uniform in time PoC stated in Theorem 9.

To conclude the short presentation of this part, let us now insist on a few key ideas. First, note that PoC, in the case of singular interaction, highly relies on the regularity of the solution $\overline{\rho}_t$, and the uniformity in time on its long-time behavior. We also note that here we prove a uniform in time LSI for the nonlinear solution because it is somewhat comparable to the Lebesgue measure on the torus, which also satisfies a LSI. As we know, the existence of a LSI is strongly related to the long-time behavior of the system and, in fact, one can (formally) prove uniform in time PoC provided one can prove PoC (non uniformly in time) and a long-time convergence of the processes uniformly in N.

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