A large deviation principle for empirical measures on Polish spaces

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November 23, 2020
Journées doctorants du GDR MEGA

Outline

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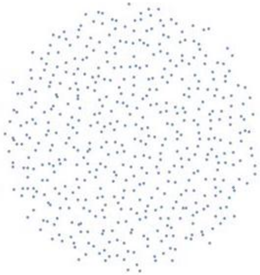
Guiding example

Suppose $(Z_1^{(n)}, \dots, Z_n^{(n)}) \sim \mathbb{P}_n$ where

$$\mathrm{d}\mathbb{P}_n(z_1, \dots, z_n) = \frac{1}{\mathcal{Z}_n} \prod_{i<j}^{n} \|z_i - z_j\|^2 e^{-n \sum_{i=1}^{n} \|z_i\|^2} \mathrm{d}\ell_{\mathbb{C}^n}(z_1, \dots, z_n).$$

Eigenvalues of a Gaussian random matrix, *Ginibre matrix*.



Question

Study the limit behavior of $\hat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{Z_i^{(n)}}$. More precisely,

find an LDP on the set of probability measures $\mathcal{P}(\mathbb{C})$.

[*Hiai* and *Petz* (1998)]

LDP

LDP : Find v_n that goes to ∞ and $I : \mathcal{P}(\mathbb{C}) \to [0, \infty]$ such that

$$\mathbb{P}\left(\hat{\mu}_n \simeq \nu\right) = e^{-v_n(I(\nu)+o(1))}$$

for every $\nu \in \mathcal{P}(\mathbb{C})$. Equivalently,

$$\frac{1}{v_n} \log \mathbb{P}\left(\hat{\mu}_n \simeq \nu\right) = -I(\nu) + o(1).$$

A bit more precisely, for any measurable set $A \subset \mathcal{P}(\mathbb{C})$,

$$-\inf_{\nu \in A^{\circ}} I(\nu) \leq \lim_{n\to\infty} \frac{1}{v_n} \log \mathbb{P}\left(\hat{\mu}_n \in A\right) \leq -\inf_{\nu \in \bar{A}} I(\nu).$$

Laplace principle

LDP regularized version :

$$\lim_{n\to\infty} \frac{1}{v_n} \log \mathbb{E}\left[e^{-v_n f(\hat{\mu}_n)}\right] = - \inf_{\nu\in\mathcal{P}(\mathbb{C})} \{f(\nu) + I(\nu)\}$$

for every $f : \mathcal{P}(\mathbb{C}) \to \mathbb{R}$ continuous and bounded.

Restatement of the problem

Define

- $G(z, w) = -2 \log \|z - w\| + \|z\|^2 + \|w\|^2$;
- $\mathrm{d}\sigma(z) = \dfrac{e^{-\|z\|^2}}{\pi} \mathrm{d}\ell_{\mathbb{C}}(z)$.

We have

$$\prod_{i<j}^{n} \|z_i - z_j\|^2 e^{-n \sum_{i=1}^{n} \|z_i\|^2} \mathrm{d}\ell_{\mathbb{C}^n}(z_1, \ldots, z_n)$$

$$= \pi^n \exp\left(-\sum_{i<j}^{n} G(z_i, z_j) \right) \mathrm{d}\sigma^{\otimes n}(z_1, \ldots, z_n).$$

If we define

$$H_n(z_1, \ldots, z_n) = \frac{1}{n^2} \sum_{i<j}^{n} G(z_i, z_j),$$

interpreted as the total (potential) energy of n particles, then

$$\mathrm{d}\mathbb{P}_n = \frac{1}{\widetilde{\mathcal{Z}}_n} \exp\left(-n^2 H_n\right) \mathrm{d}\sigma^{\otimes n}.$$

How general can H_n be ?

General setting

- M Polish space ;
- σ probability measure on M ;
- $H_n : M^n \to (-\infty, \infty]$ measurable bounded from below ;
- $\{\beta_n\}_n$ sequence of positive numbers.

Let γ_n be the finite measure given by

$$\mathrm{d}\gamma_n = \exp\left(-n\beta_n H_n\right) \mathrm{d}\sigma^{\otimes n}.$$

How the limiting behavior of γ_n
depends on the limiting behavior of H_n ?

Macroscopic energy

Suppose that, for every $\mu \in \mathcal{P}(M)$, the following limit exists :

$$H(\mu) = \lim_{n\to\infty} \int_{M^n} H_n \mathrm{d}\mu^{\otimes n},$$

and define the relative entropy

$$\mathrm{Ent}_\sigma(\mu) = \int_M \left(\frac{\mathrm{d}\mu}{\mathrm{d}\sigma}\right) \log \left(\frac{\mathrm{d}\mu}{\mathrm{d}\sigma}\right) \mathrm{d}\sigma.$$

In the Ginibre case,

$$\begin{aligned} H(\mu) &= \frac{1}{2} \int_{\mathbb{C}\times\mathbb{C}} G(z,w)\mathrm{d}\mu(z)\mathrm{d}\mu(w) \\ &= -\int_{\mathbb{C}\times\mathbb{C}} \log\|z-w\|\mathrm{d}\mu(z)\mathrm{d}\mu(w) + \int_{\mathbb{C}} \|z\|^2 \mathrm{d}\mu(z). \end{aligned}$$

Laplace principle goal

For $\mathbf{x} = (x_1, \ldots, x_n)$ define

$$\hat{\mu}_{\mathbf{x}} = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i}.$$

Goal : Laplace principle

If $\beta_n \to \beta \in (0, \infty]$, for $f : \mathcal{P}(M) \to \mathbb{R}$ bounded continuous,

$$\lim_{n\to\infty} \frac{1}{n\beta_n} \log \int_{M^n} e^{-n\beta_n f(\hat{\mu}_{\mathbf{x}})} \mathrm{d}\gamma_n(\mathbf{x}) = - \inf_{\mu \in \mathcal{P}(M)} \left\{ f(\mu) + H(\mu) + \frac{1}{\beta} \mathrm{Ent}_\sigma(\mu) \right\}.$$

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Main step : *Dupuis* and *Ellis* approach to LDP

Lemma (Legendre transform of the entropy)

- *E measurable space,*
- *$\nu \in \mathcal{P}(E)$ and*
- *$g : E \to (-\infty, \infty]$ measurable bounded from below.*

Then

$$\log \int_E e^{-g} \mathrm{d}\nu = - \inf_{\tau \in \mathcal{P}(E)} \left\{ \int_E g \mathrm{d}\tau + \mathrm{Ent}_\nu(\tau) \right\}.$$

$$\frac{1}{n\beta_n} \log \int_{M^n} e^{-n\beta_n \left(f(\hat{\mu}_{\mathbf{x}}) + H_n(\mathbf{x}) \right)} \mathrm{d}\sigma^{\otimes n}(\mathbf{x})$$

$$= - \inf_{\tau \in \mathcal{P}(M^n)} \left\{ \int_{M^n} \left(f(\hat{\mu}_{\mathbf{x}}) + H_n(\mathbf{x}) \right) \mathrm{d}\tau(\mathbf{x}) + \frac{1}{n\beta_n} \mathrm{Ent}_{\sigma^{\otimes n}}(\tau) \right\}.$$

Laplace principle goal II

New goal : Convergence of the infima

If $\beta_n \to \beta \in (0, \infty]$, for $f : \mathcal{P}(M) \to \mathbb{R}$ bounded continuous,

$$\inf_{\tau \in \mathcal{P}(M^n)} \left\{ \int_{M^n} \Big(f(\hat{\mu}_{\mathbf{x}}) + H_n(\mathbf{x}) \Big) \mathrm{d}\tau(\mathbf{x}) + \frac{1}{n\beta_n} \mathrm{Ent}_{\sigma^{\otimes n}}(\tau) \right\}$$

$$\xrightarrow[n \to \infty]{} \inf_{\mu \in \mathcal{P}(M)} \left\{ f(\mu) + H(\mu) + \frac{1}{\beta} \mathrm{Ent}_{\sigma}(\mu) \right\}.$$

Notion of limit

- Sequence $\{H_n\}_n$ **uniformly bounded from below**,
- $H : \mathcal{P}(M) \to (-\infty, \infty]$.

Definition (Macroscopic limit)

H is the macroscopic limit of H_n if

- $\forall \mu \in \mathcal{P}(M)$

$$\lim_{n\to\infty} \int_{M^n} H_n \mathrm{d}\mu^{\otimes n} = H(\mu) \quad \text{and}$$

- whenever $\frac{1}{n}\sum_{i=1}^{n} \delta_{x_i} \to \mu$

$$\liminf_{n\to\infty} H_n(x_1, \ldots, x_n) \geq H(\mu).$$

This notion of convergence suffices !

A Laplace principle

Theorem (Laplace principle for positive temperature)

Suppose

- H *is the macroscopic limit of* H_n *and*
- β_n *converges to some* $\beta \in (0, \infty)$.

Then, for every bounded continuous $f : \mathcal{P}(M) \to \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{n\beta_n} \log \int_{M^n} e^{-n\beta_n f(\hat{\mu}_{\mathbf{x}})} \mathrm{d}\gamma_n(\mathbf{x}) = -\inf_{\mu \in \mathcal{P}(M)} \left\{ f(\mu) + H(\mu) + \frac{1}{\beta} \mathrm{Ent}_\sigma(\mu) \right\}.$$

If $\iota = \inf \left(H + \frac{1}{\beta}\mathrm{Ent}_\sigma \right) < \infty$, it implies an LDP with

$$\text{rate function } H + \frac{1}{\beta}\mathrm{Ent}_\sigma - \iota \text{ and speed } n\beta_n$$

Positive temperature case limit

Suppose that

$$(X_1^{(n)}, \dots, X_n^{(n)}) \sim \frac{\gamma_n}{\gamma_n(M^n)}.$$

Under the conditions of the preceding theorem :

Theorem (Limit of empirical measures)

If $H + \frac{1}{\beta}\mathrm{Ent}_\sigma$ has a unique minimizer μ_{eq},

$$\frac{1}{n}\sum_{i=1}^{n} \delta_{X_i^{(n)}} \xrightarrow[n\to\infty]{\text{a.s.}} \mu_{\mathrm{eq}}.$$

Infinite β

What happens when $\beta_n \to \infty$? Two more conditions.

- $\{H_n\}_n$ **confining** : Let $\mathbf{x}_n = (x_1, \dots, x_n)$.

$$\liminf_{n\to\infty} H_n(\mathbf{x}_n) < \infty \quad \Longrightarrow \quad \{\hat{\mu}_{\mathbf{x}_n}\}_n \text{ is precompact in } \mathcal{P}(M).$$

- H **regular** : If $H(\mu) < \infty$, there exists $\mu_n \to \mu$ such that

$$\forall n, \, \mathrm{Ent}_\sigma(\mu_n) < \infty \quad \text{and} \quad \lim_{n\to\infty} H(\mu_n) = H(\mu).$$

Another Laplace principle

Theorem (Laplace principle for zero temperature)

Suppose

- H *is the macroscopic limit of* H_n,
- β_n *tends to infinity,*
- $\{H_n\}_n$ *is confining and*
- H *is regular.*

Then, for every bounded continuous $f : \mathcal{P}(M) \to \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{n\beta_n} \log \int_{M^n} e^{-n\beta_n f(\hat{\mu}_{\mathbf{x}})} \mathrm{d}\gamma_n(\mathbf{x}) = - \inf_{\mu\in\mathcal{P}(M)} \{f(\mu) + H(\mu)\} .$$

Zero temperature case limit

Suppose that

$$(X_1^{(n)}, \ldots, X_n^{(n)}) \sim \frac{\gamma_n}{\gamma_n(M^n)}.$$

Under the conditions of the preceding theorem :

Theorem (Limit of empirical measures)

If H has a unique minimizer μ_{eq},

$$\frac{1}{n}\sum_{i=1}^{n} \delta_{X_i^{(n)}} \xrightarrow[n\to\infty]{\mathrm{a.s.}} \mu_{\mathrm{eq}}.$$

Γ-convergence

Theorem (Γ-convergence)

Suppose

- *H is the macroscopic limit of H_n and*
- *$\{H_n\}_n$ is confining.*

Then, for every bounded continuous function $f : \mathcal{P}(M) \to \mathbb{R}$

$$\lim_{n\to\infty} \inf_{\mathbf{x}_n \in M^n} \{f(\hat{\mu}_{\mathbf{x}_n}) + H_n(\mathbf{x}_n)\}$$

$$= \inf_{\mu \in \mathcal{P}(M)} \{f(\mu) + H(\mu)\}.$$

Deterministic case limit

Suppose that

$$\lim_{n\to\infty}\left[H_n(X_1^{(n)},\dots,X_n^{(n)}) - \inf H_n\right] = 0.$$

Under the conditions of the preceding theorem :

Theorem (Limit of empirical measures)

If H has a unique minimizer μ_{eq},

$$\frac{1}{n}\sum_{i=1}^{n}\delta_{X_i^{(n)}} \xrightarrow[n\to\infty]{} \mu_{\mathrm{eq}}.$$

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Two-body interaction

Suppose H_n is given by

$$H_n(x_1, \dots, x_n) = \frac{1}{n^2} \sum_{i<j} G(x_i, x_j)$$

for some $G : M \times M \to (-\infty, \infty]$.

- $H(\mu) = \frac{1}{2} \int_{M\times M} G(x, y)\mathrm{d}\mu(x)\mathrm{d}\mu(y)$.
- G bounded from below $\Longrightarrow \{H_n\}_n$ unif. bounded from below.
- G lower semicont. $\Longrightarrow H$ is the macroscopic limit of H_n.
- $G(x, y) \to \infty$ when $x, y \to \infty \Longrightarrow \{H_n\}_n$ confining.
- H regular : enough to ask $\mu_n \ll \sigma$ instead of $\mathrm{Ent}_\sigma(\mu_n) < \infty$.

k-body interaction

$G : M^k \to (-\infty, \infty]$ lower semicont. and bounded from below.

$$H_n(x_1, \dots, x_n) = \frac{1}{n^k} \sum_{\{i_1,\dots,i_k\} \subset \{1,\dots,n\}} G(x_{i_1}, \dots, x_{i_k}).$$

Macroscopic limit

$$H(\mu) = \frac{1}{k!} \int_{M^k} G \mathrm{d}\mu^{\otimes k}.$$

Random polynomial energy term

$G : M \times M \to (-\infty, \infty]$ and $\nu \in \mathcal{P}(M)$

$$H_n(x_1, \dots, x_n) = \frac{n+1}{n^2} \log \left(\int_M e^{-\sum_{i=1}^n G(x_i, x)} \mathrm{d}\nu(x) \right).$$

This term appears for Gaussian random polynomials !

Under some conditions, the macroscopic limit is

$$H(\mu) = - \inf_{x \in \operatorname{supp} \nu} \left\{ \int_M G(x, y) \mathrm{d}\mu(y) \right\}.$$

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Wiley Series in Probability and Statistics : Probability and Statistics. John Wiley & Sons, Inc., New York, 1997.

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Thank you for your attention !

Concrete examples

Ginibre ensemble. $\{X_{i,j}\}_{i,j\geq 1}$ i.i.d. complex standard Gaussians. Define

$$\mathbf{X}_n = \left(\frac{X_{i,j}}{\sqrt{n}}\right)_{1\leq i,j\leq n}.$$

Law of eigenvalues of $\mathbf{X}_n$: $M = \mathbb{C}$ with $\beta_n = n$,

$$G(z,w) = -2\log\|z-w\| + \|z\|^2 + \|w\|^2$$

$$\text{and} \qquad \mathrm{d}\sigma(z) = \frac{e^{-\|z\|^2}}{\pi}\mathrm{d}\ell_{\mathbb{C}}(z).$$

LDP : [*Hiai* and *Petz* (1998)].

Spherical ensemble. $\tilde{\mathbf{X}}_n \sim \mathbf{X}_n$ independent. Define

$$\mathbf{Y}_n = \mathbf{X}_n \tilde{\mathbf{X}}_n^{-1}.$$

Law of eigenvalues of $\mathbf{Y}_n$: $M = \mathbb{C}$ with $\beta_n = n$,

$$G(z, w) = -2 \log |z - w| + \log(1 + |z|^2) + \log(1 + |w|^2)$$

$$\text{and} \qquad \mathrm{d}\sigma(z) = \frac{1}{\pi(1 + |z|^2)^2} \mathrm{d}\ell_{\mathbb{C}}(z).$$

LDP : [*Hardy* (2012)].

Gaussian Kac polynomials. $\{a_i\}_{i\geq 0}$ i.i.d. complex standard Gaussians. Define

$$\mathbf{p}_n(z) = \sum_{i=0}^{n} a_i z^i.$$

Law of zeros of $\mathbf{p}_n$: $M = \mathbb{C}$ with $\beta_n = n$,

$$G(z,w) = -2\log|z-w| + 2\log_+|z| + 2\log_+|w|,$$
$$\mathrm{d}\sigma(z) = \frac{1}{2\pi}\min(1,|z|^{-4})\mathrm{d}\ell_{\mathbb{C}}(z),$$

with the extra term for ν the uniform measure on the unit circle.

LDP : [*Zeitouni* and *Zelditch* (2010)].