Longest Common Subsequence of Random Permutations

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Longest common subsequence

- $\mathfrak{S}_n$: the group of permutations of $\{1, \ldots, n\}$.
- $(\sigma(i_1), \ldots, \sigma(i_k))$ a subsequence of σ of length k if $i_1 < i_2 < \cdots < i_k$.
- $LCS(\sigma, \rho)$: the length of the longest common subsequence of σ and ρ.
- For example:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 4 & 5 \end{pmatrix}, \quad \rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 5 & 2 & 3 \end{pmatrix},$$

$LCS(\sigma, \rho) = 3.$

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Conjecture (Bukh and Zhou (2016))

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Houdré and Xu (2018) found a counter example ! But conjecture that it is asymptotically true i.e.

$$\liminf_{n\to\infty} \frac{\mathbb{E}(LCS(\sigma_n, \rho_n))}{\sqrt{n}} \geq \lim_{n\to\infty} \frac{\mathbb{E}(LCS(\sigma_{unif,n}, \rho_{unif,n}))}{\sqrt{n}} = 2.$$

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- $(\sigma(i_1),\dots,\sigma(i_k))$ increasing subsequence of σ of length k if $i_1 < i_2 < \cdots < i_k$ and $\sigma(i_1) < \cdots < \sigma(i_k)$.
- $LIS(\sigma)$: the length of the longest increasing subsequence of σ.
- Example:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 1 & 2 & 4 & 7 & 5 & 6 \end{pmatrix}$$

  $LIS(\sigma) = 5$.
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Conjecture (Ulam (1961))

If $\sigma_n \sim U_{\mathfrak{S}_n}$ *then*

$$\lim_{n\to\infty} \frac{\mathbb{E}(LIS(\sigma_n))}{\sqrt{n}} = c.$$

Theorem (Vershik and Kerov (1977); Logan and Shepp (1977))

If $\sigma_n \sim U_{\mathfrak{S}_n}$ *then*

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Theorem (Baik, Deift, and Johansson (1999))

If $\sigma_n \sim U_{\mathfrak{S}_n}$ *then*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{LIS(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s\right) = F_2(s).$$

F_2: CDF of the GUE Tracy-Widom distribution.

Main result

Theorem (K. (2020))

Assume that for any $n \geq 1$, σ_n and ρ_n are two i.i.d. conjugation invariant random permutations of size n. Then

$$\liminf_{n\to\infty} \frac{\mathbb{E}(LCS(\sigma_n, \rho_n))}{\sqrt{n}} \geq 2.$$

Corollary

There exists n_0 such that for any $n > n_0$, for any σ_n and ρ_n two i.i.d. conjugation invariant random permutations of size n

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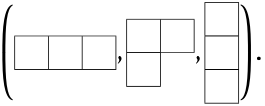
Young diagram

Definition (Young diagram)

$\lambda = (\lambda_i)_{i \geq 1} \in \mathbb{N}^{\mathbb{N}^*}$ is a Young diagram of size n if

- $\forall i \geq 1,\ \lambda_{i+1} \leq \lambda_i$,
- $\sum_{i=1}^{\infty} \lambda_i = n$.

Example: Young diagrams of size 3 are
$\mathbb{Y}_3 = (3, \underline{0}), (2, 1, \underline{0}), (1, 1, 1, \underline{0})$
or


Young tableau

Definition (Young tableau)

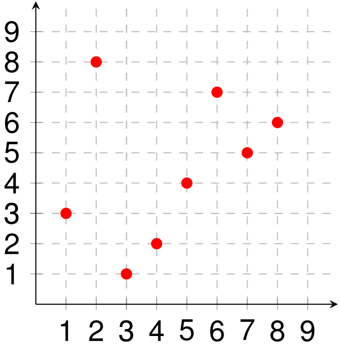
A Young tableau of shape λ is a filling of the boxes of λ using the entries $\{1, 2, \ldots, n\}$ and the entries in each row and each column are increasing.

- Example: Young tableaux of shape $\begin{array}{|c|c|c|}\hline & & \\ \hline \\ \cline{1-1}\end{array}$ are

  $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$.
- $dim(\lambda)$ = The number of Young tableaux of shape λ.
- Example: $dim\left(\begin{array}{|c|c|c|}\hline & & \\ \hline \\ \cline{1-1}\end{array}\right) = 3$.
- $dim(\lambda)$ = dimension of the irreducible representation of $\mathfrak{S}_n$ indexed by λ.
- $\sum_{\lambda \in \mathbb{Y}_n} dim(\lambda)^2 = \mathrm{card}(\mathfrak{S}_n) = n!$.

Viennot's geometric construction

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 1 & 2 & 4 & 7 & 5 & 6 \end{pmatrix}.$$



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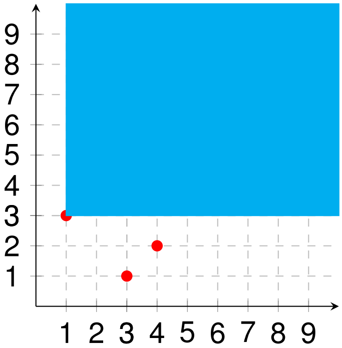

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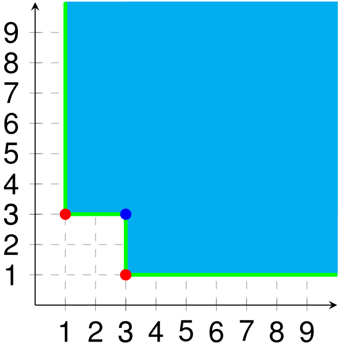

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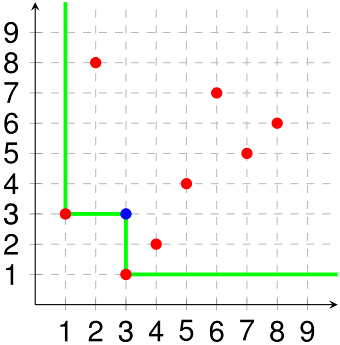

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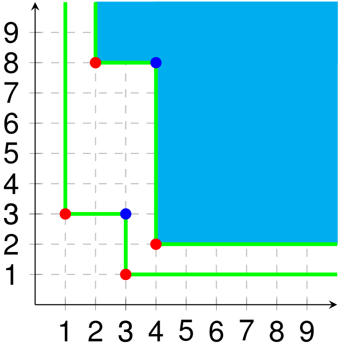

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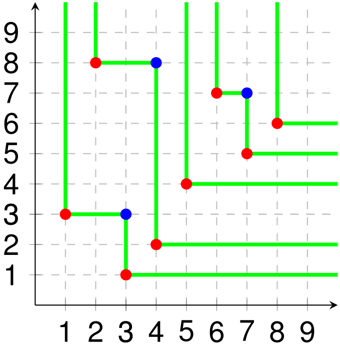

1	2	4	5	6

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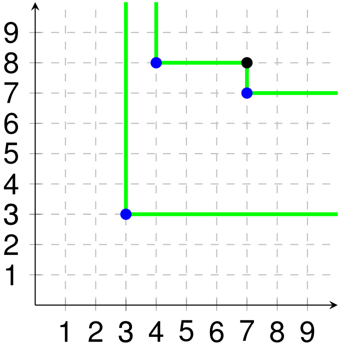

1	2	4	5	6
3	7			

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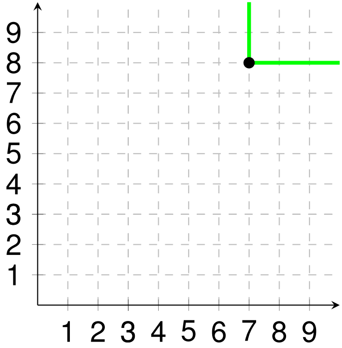


1	2	4	5	6
3	7			
8				

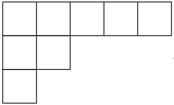
,

1	2	5	6	8
3	4			
7				

Robinson-Schensted correspondence

- One-to-one correspondence between permutations and pairs of standard Young tableaux of the same shape.
- We denote by $\lambda(\sigma) := (\lambda_i(\sigma))_{i\geq 1}$ the shape of the image of σ by this correspondence. For example, if

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 1 & 2 & 4 & 7 & 5 & 6 \end{pmatrix} \quad \text{then} \quad \lambda(\sigma) =$$



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- $LIS(\sigma) = \lambda_1(\sigma)$.
- $\{\lambda_i - i\}_{i\geq 1}$ is determinantal (when the size is a random Poisson variable).

Greene

Let
$\sigma \in \mathfrak{S}_n$. We denote by

$$\mathfrak{I}_1(\sigma) := \{s \subset \{1,2,\ldots,n\};\ \forall i,j \in s,\ (i-j)(\sigma(i)-\sigma(j)) \geq 0\},$$
$$\mathfrak{I}_{k+1}(\sigma) := \{s \cup s',\ s \in \mathfrak{I}_k,\ s' \in \mathfrak{I}_1\},$$

For example, for

$$\sigma_{ex,3} := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \mathfrak{I}_1(\sigma_{ex,3}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}\}$$

and

$$\mathfrak{I}_2(\sigma_{ex,3}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

Greene

We have the following.

Proposition (Greene (1974))

For any permutation $\sigma \in \mathfrak{S}_n$,

$$\max_{s \in \mathfrak{I}_i(\sigma)} \mathrm{card}(s) = \sum_{k=1}^{i} \lambda_k(\sigma).$$

In particular,

$$\max_{s \in \mathfrak{I}_1(\sigma)} \mathrm{card}(s) = \lambda_1(\sigma) = \mathrm{LIS}(\sigma).$$

Russian notations

- Rotate the diagram by $\frac{5\pi}{4}$.
- Complete the high function by $x \to |x|$.
- We denote by L_λ the resulting function.

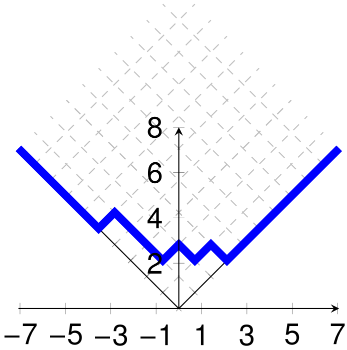

Figure: $L_{(5,2,1,\underline{0})}$

Vershik-Kerov-Logan-Shepp shape

Theorem (Vershik and Kerov (1977); Logan and Shepp (1977))

If $\sigma_n \sim U_{\mathfrak{S}_n}$, *then for any* $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{s\in\mathbb{R}} \left| \frac{1}{\sqrt{2n}} L_{\lambda(\sigma_n)}\left(s\sqrt{2n}\right) - \Omega(s) \right| < \varepsilon \right) = 1,$$

where

$$\Omega(s) := \begin{cases} \frac{2}{\pi}\left(s \arcsin(s) + \sqrt{1-s^2}\right) & \text{if } |s| < 1 \\ |s| & \text{if } |s| \geq 1 \end{cases}.$$

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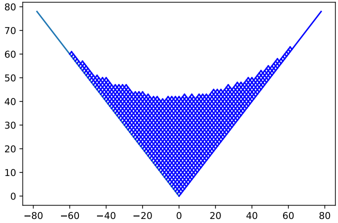

Figure: Typical Young diagram under the Plancherel distribution

Idea of the proof

Let $\#_1(\sigma)$ be the number of fixed points of σ and $\#(\sigma)$ the number of its cycles. If $\mathbb{E}(\#_1(\sigma_n)) > \sqrt{2}n^{\frac{3}{4}}$ then

$$\mathbb{E}(LCS(\sigma_n, \rho_n) > card(\{i : \sigma_n(i) = \rho_n(i) = i\}) > 2\sqrt{n}.$$

If $\mathbb{E}(\#_1(\sigma_n)) \leq \sqrt{2}n^{\frac{3}{4}}$ then $\frac{\#(\sigma_n \rho_n^{-1})}{n} \xrightarrow[n\to\infty]{\mathbb{P}} 0$.

Theorem (K. (2018))

Assume that the sequence of random permutations $(\sigma_n)_{n\geq 1}$ satisfies:

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Let $\#_1(\sigma)$ be the number of fixed points of σ and $\#(\sigma)$ the number of its cycles. If $\mathbb{E}(\#_1(\sigma_n)) > \sqrt{2}n^{\frac{3}{4}}$ then

$$\mathbb{E}(LCS(\sigma_n, \rho_n) > card(\{i : \sigma_n(i) = \rho_n(i) = i\}) > 2\sqrt{n}.$$

If $\mathbb{E}(\#_1(\sigma_n)) \leq \sqrt{2}n^{\frac{3}{4}}$ then $\frac{\#(\sigma_n \rho_n^{-1})}{n} \xrightarrow[n\to\infty]{\mathbb{P}} 0$.

Theorem (K. (2018))

Assume that the sequence of random permutations $(\sigma_n)_{n\geq 1}$ satisfies:

- *For all positive integer n, σ_n is conjugation invariant.*
- *The number of cycles is such that: for all $\varepsilon > 0$, $\frac{\#(\sigma_n)}{n} \xrightarrow[n\to\infty]{\mathbb{P}} 0$.*

Then for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{s\in\mathbb{R}} \left| \frac{1}{\sqrt{2n}} L_{\lambda(\sigma_n)}\left(s\sqrt{2n}\right) - \Omega(s) \right| < \varepsilon\right) = 1.$$

General case

Theorem (K. 20+)

Assume that for any $n \geq 1$, σ_n and ρ_n are independent and their distributions are conjugation invariant. Then

$$\liminf_{n\to\infty} \frac{\mathbb{E}(\{LCS(\sigma_n, \rho_n))}{\sqrt{n}} \geq 2\sqrt{13} - 6 \simeq 1.21\ldots.$$

1.21 improves 0.564 in (K., 2020) but we conjecture is 2 is the best possible bound.

Corollary

There exists n_1 such that for any $n > n_1$, for any σ_n and ρ_n independent conjugation invariant random permutations of size n

$$\mathbb{E}(LCS(\sigma_n, \rho_n)) \geq \sqrt{n}.$$

General case

Proposition

Assume that for any $n \geq 1$, σ_n *and* ρ_n *are independent and* σ_n *is conjugation invariant. Then*

$$\liminf_{n\to\infty} \frac{\mathbb{E}(LCS(\sigma_n, \rho_n))}{\sqrt{n}} \geq G^{-1}\left(\liminf_{n\to\infty} \frac{\mathbb{E}(\#(\sigma_n))}{2n}\right),$$

where , $G(x) = \int_{-1}^{1} \left(\Omega(s) - \left|s + \frac{x}{2}\right| - \frac{x}{2}\right)_+ \mathrm{d}s,$

$$\Omega(s) := \begin{cases} \frac{2}{\pi}\left(s \arcsin(s) + \sqrt{1-s^2}\right) & \textit{if } |s| < 1 \\ |s| & \textit{if } |s| \geq 1 \end{cases}. \tag{1}$$

In particular, if $\lim_{n\to\infty} \mathbb{E}\left(\frac{\#(\sigma_n)}{n}\right) = 0$, *we have*

$$\liminf_{n\to\infty} \frac{\mathbb{E}(LCS(\sigma_n, \rho_n))}{\sqrt{n}} \geq 2.$$

General case

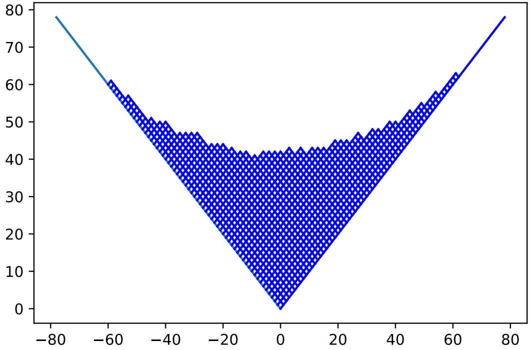
Proposition

Assume that for any $n \geq 1$, σ_n and ρ_n are independent and the law of σ_n is conjugation invariant.

- *If* $\frac{\#(\sigma_n)}{n^{\frac{1}{6}}} \xrightarrow{\mathbb{P}} 0$, *then for any* $s \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{LCS(\sigma_n, \rho_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s\right) = F_2(s),$$

  where F_2 *is the cumulative distribution function of the Tracy-Widom distribution.*
- *If* $\frac{\#(\sigma_n)}{\sqrt{n}} \xrightarrow{\mathbb{P}} 0$, *then* $\frac{LCS(\sigma_n,\rho_n)}{\sqrt{n}} \xrightarrow{\mathbb{P}} 2$.
- *If* $\lim_{n\to\infty} \mathbb{E}\left(\frac{\#(\sigma_n)}{\sqrt{n}}\right) = 0$, *then* $\lim_{n\to\infty} \frac{\mathbb{E}(LCS(\sigma_n,\rho_n))}{\sqrt{n}} = 2$.



Thank you for your attention