Asymptotic expansion of smooth functions in polynomials in deterministic matrices and iid GUE matrices.

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Based on the following work,
Asymptotic expansion of smooth functions in polynomials in deterministic matrices and iid GUE matrices.

Definition

Let W_N be a random matrix of size N such that:

- *Diagonal coefficents are independent centered gaussian random variable of variance 1.*
- *The real part and the imaginary part of the upper diagonal coefficents are independent centered gaussian random variable of variance 1/2.*
- *The lower diagonal coefficents are chosen such that W_N is self-adjoint.*

Then $X_N = N^{-1/2} W_N$ is a GUE random matrix of size N.

If $(a_{i,j})_{1\leq i,j\leq N}$ and $(b_{i,j})_{1\leq i,j\leq N}$ are independent centered gaussian random variable of variance 1, then

$$X_N = \frac{1}{\sqrt{N}} \begin{pmatrix} a_{1,1} & \frac{a_{1,2}+\mathbf{i}b_{1,2}}{\sqrt{2}} & \cdots & \frac{a_{1,N}+\mathbf{i}b_{1,N}}{\sqrt{2}} \\ \frac{a_{2,1}-\mathbf{i}b_{2,1}}{\sqrt{2}} & a_{2,2} & \cdots & \frac{a_{2,n}+\mathbf{i}b_{2,N}}{\sqrt{2}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{N,1}-\mathbf{i}b_{N,1}}{\sqrt{2}} & \frac{a_{N,2}-\mathbf{i}b_{N,2}}{\sqrt{2}} & \cdots & a_{N,N} \end{pmatrix}.$$

Theorem

Let X_N be a GUE random matrix of size N, let $\lambda_1 \leq \cdots \leq \lambda_N$ be its eigenvalues, then for $A \in \mathbb{R}^N$,

$$\mathbb{P}\Big(\lambda_1, \ldots, \lambda_N \in A\Big) = \frac{1}{Z_N} \int_A \mathbf{1}_{x_1 \leq \cdots \leq x_N} \prod_{1 \leq i < j \leq N} |x_j - x_i|^2 \prod_{i=1}^{N} e^{-\frac{x_i^2}{2}} \, dx$$

where Z_N is a renormalization constant.

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where Z_N is a renormalization constant.

- However for most models we do not have an exact formula for the distribution of eigenvalues. Hence we try to answer easier questions. To begin with, given a self-adjoint random matrix M, how many eigenvalues is there roughly in a given interval of $\mathbb{R}$?

The empirical measure of a matrix

Definition

Let M be a self-adjoint matrix of size N, let $\lambda_1, \dots, \lambda_N$ be its eigenvalues, we define the empirical measure μ_M by

$$\mu_M = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i},$$

where δ_λ is the dirac measure in λ.

Let f be a continuous function on the real line, then

$$\mu_M(f) = \frac{1}{N} \sum_{i=1}^{N} f(\lambda_i) = \frac{1}{N} \operatorname{Tr}\left(f(M)\right).$$

Consequently, we have the equivalent definition,

$$\mu_M : f \in \mathcal{C}^0(\mathbb{R}) \mapsto \frac{1}{N} \operatorname{Tr}\left(f(M)\right).$$

The semicircular variable

We consider

- $H = l^2(\mathbb{N}) = \{(u_n)_{n\in\mathbb{N}} \mid \sum_n |u_n|^2 < \infty\}$,
- $\mathcal{A} = B(H)$ the space of continuous linear operator on H,
- $\tau : a \in \mathcal{A} \mapsto \langle a(e_1), e_1\rangle \in \mathbb{C}$ where $e_1 = (1, 0, 0, \dots)$.

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For example, we have $l : (u_n)_{n\in\mathbb{N}} \mapsto (u_{n+1})_{n\in\mathbb{N}}$ and $l^* : (u_n)_{n\in\mathbb{N}} \mapsto (u_{n-1})_{n\in\mathbb{N}}$ with $u_{-1} = 0$. Euristically,

$$l = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \ddots \\ 0 & 0 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \quad l^* = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \ddots \\ 0 & 1 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

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Definition

We say that $x \in \mathcal{A}$ is a semicircular variable if it is self-adjoint and one has for any $k \in \mathbb{N}$,

$$\tau(x^k) = \int t^k d\sigma(t),$$

where $d\sigma(t) = \frac{1}{2\pi}\sqrt{4 - t^2}\mathbf{1}_{[-2,2]}dt$ is the semicircle distribution. In particular $l + l^*$ is a semicircular variable.

The spectral measure

Let $a \in \mathcal{A}$ be self-adjoint:

- For any polynomial P, $\|P(a)\| \leq \|P\|_{\mathcal{C}^0([-\|a\|,\|a\|])}$.
- Consequently if P_n converges uniformly on $[-\|a\|,\|a\|]$ towards a function f, $P_n(a)$ is a Cauchy sequence and hence converges towards an operator $f(a)$.
- The map $f \in \mathcal{C}^0([-\|a\|,\|a\|]) \mapsto \tau(f(a)) \in \mathbb{C}$ is a positive linear functional on $\mathcal{C}^0([-\|a\|,\|a\|])$, hence by Riesz-Markov-Kakutani representation theorem there exists a unique measure ν_a such that for ay continuous functions f,

$$\tau(f(a)) = \int_{\mathbb{R}} f \, d\nu_a.$$

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For example the spectral measure of a semicircular variable is the semicircle distribution, that is for any continuous function f,

$$\tau(f(x)) = \frac{1}{2\pi} \int_{-2}^{2} f(t)\sqrt{4-t^2}dt,$$

It has been known for a long time that the empirical measure of a GUE random matrices X_N converges almost surely towards the spectral measure of a semicircular variable x, that is almost surely $\frac{1}{N}\operatorname{Tr}(f(X_N)) \underset{N\to+\infty}{\longrightarrow} \tau(f(x))$.

Classical and free independance

Definition

Let $\mathcal{A}$ be a σ-algebra endowed with an expectation $\mathbb{E}$, the sub-σ-algebras $\mathcal{A}_1 \dots, \mathcal{A}_n$ are said to be independent if for any $I \subset [1, n]$ and $A_i \in \mathcal{A}_i$,

$$\mathbb{E}\left[\prod_{i \in I} (\mathbf{1}_{A_i} - \mathbb{E}[\mathbf{1}_{A_i}])\right] = 0$$

Definition

Let $\mathcal{A}$ and τ be as previously defined, the sub-algebras $\mathcal{A}_1 \dots, \mathcal{A}_n$ are said to be freely independent if for any $k \geq 1$, for any $i_1, \dots, i_k$ such that for any j, $i_j \neq i_{j+1}$ and $a_j \in \mathcal{A}_{i_j}$,

$$\tau\left(\prod_{1 \leq j \leq k} (a_j - \tau(a_j))\right) = 0$$

Seminal works

Theorem (D. Voiculescu, 1991)

Let $X^N = (X_1^N, \ldots, X_d^N)$ be independent GUE matrices, $x = (x_1, \ldots, x_d)$ be a system of free semicircular variables, let P be a self-adjoint polynomial. Then almost surely the empirical measure of $P(X^N)$ converges in distribution towards the spectral measure of $P(x)$. That is almost surely for any continuous function f,

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{Tr}\Big(f(P(X^N)) \Big) = \tau\Big(f(P(x)) \Big) .$$

Consequences on the eigenvalues

We fix P a self-adjoint polynomial, then $\mu_{P(X^N)}$ converges towards $\nu_{P(x)}$. Thus with $\sigma(P(X^N))$ the spectrum of $P(X^N)$, almost surely,

$$\frac{\#\left\{\lambda \in \sigma(P(X^N)) \;\middle|\; a \leq \lambda \leq b\right\}}{N} = \mu_{P(X^N)}([a,b]) \quad \xrightarrow[N\to\infty]{} \quad \nu_{P(x)}([a,b]).$$

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However we are usually interested in the following questions, which are not answered by Voiculescu's Theorem.

1. If the size of the interval $[a,b]$ converges towards 0 when N goes to infinity, what is the average number of eigenvalues in this interval?
2. Can we show that there is no outlier? I.e. that almost surely, for any $\varepsilon > 0$, for N large enough, $\sigma(P(X^N)) \subset \sigma(P(x)) + \varepsilon$.

Partial answers

- The first questions was studied for numerous models of random matrices, but only in the single matrix case, i.e. $P(X) = X$. We do not tackle this question in this talk, albeit our main result can give a result for general polynomials for interval of the size $N^{-\alpha}$ for α well-chosen.

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- As for the second question, the starting idea is the following, let f be a non-negative function such that f is equal to 1 on the interval $[a, b]$, then

$$\mathbb{P}\Big(\sigma(P(X^N)) \cap [a,b]\Big) \leq \mathbb{P}\Big(\operatorname{Tr}\Big(f(P(X^N))\Big) \geq 1\Big) \leq \mathbb{E}\Big[\operatorname{Tr}\Big(f(P(X^N))\Big)\Big].$$

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- The crux to prove Voiculescu's result is to show that

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\operatorname{Tr}\Big(f(P(X^N))\Big)\right] = \tau\Big(f(P(x))\Big).$$

- The crux to study where the spectrum is located is to show that

$$\mathbb{E}\left[\frac{1}{N}\operatorname{Tr}\Big(f(P(X^N))\Big)\right] = \tau\Big(f(P(x))\Big) + \mathcal{O}(N^{-2}).$$

A finite order Taylor expansion

- Naturally one can wonder what happens at the next order. More precisely, could we write this expectation as a finite order Taylor expansion. That is, can we prove that for any k, if f is smooth enough, there exist deterministic constants $\alpha_i^P(f)$ such that

$$\mathbb{E}\left[\frac{1}{N}\mathrm{Tr}\left(f(P(X_1^N,\dots,X_d^N))\right)\right] = \sum_{i=0}^{k} \frac{\alpha_i^P(f)}{N^{2i}} + \mathcal{O}(N^{-2k-2}).$$

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- In the case where f is a polynomial and $d = 1$, Harer and Zagier gave a positive answer in 1986. They proved that given $\mathcal{M}_g^k$ the number of maps of genus g, one vertex and $k/2$ edges.

$$\mathbb{E}\left[\frac{1}{N}\mathrm{Tr}\left((X_1^N)^k\right)\right] = \sum_{g\in\mathbb{N}}\frac{1}{N^{2g}}\mathcal{M}_g^k.$$

- Haagerup and Thorbjørnsen gave a positive answer in 2010 for the specific case of a single GUE matrix.

The main result: the case of independent GUE matrices

Theorem (P., 2020)

Let the following objects be given,

- $X^N = (X_1^N, \ldots, X_d^N)$ *independent GUE matrices in* $\mathbb{M}_N(\mathbb{C})$,
- $Z^N = (Z_1^N, \ldots, Z_q^N)$ *deterministic matrices in* $\mathbb{M}_N(\mathbb{C})$ *whose norm is uniformly bounded over* $\mathbb{N}$,
- P *a self-adjoint polynomial,*
- $f \in \mathcal{C}^{4k+6}(\mathbb{R})$.

Then there exist determinitic constants $(\alpha_i^P(f, Z^N))_{i\in\mathbb{N}}$ *such that,*

$$\mathbb{E}\left[\frac{1}{N}\mathrm{Tr}_N\left(f(P(X^N, Z^N))\right)\right] = \sum_{0\le i\le k} \frac{1}{N^{2i}}\alpha_i^P(f, Z^N) + \mathcal{O}(N^{-2(k+1)}).$$

Besides, if the support of f and the spectrum of $P(x, Z^N)$ *are disjoint, then for any i,* $\alpha_i^P(f, Z^N) = 0$.

Consequences

Corollary (P., 2020)

Let X^N be independent GUE matrices of size N, x be a free semicircular system and P a self-adjoint polynomial. Given $\alpha < 1/2$, almost surely for N large enough,

$$\sigma\left(P(X^N)\right) \subset \sigma\left(P(x)\right) + N^{-\alpha},$$

where $\sigma(X)$ is the spectrum of X.

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Corollary (P., 2020)

Let X^N be a vector of independent GUE matrices of size N, x be a free semicircular system and P a polynomial. Then there exist a constant C such that for N large enough,

$$\mathbb{P}\left(\frac{\sqrt{N}}{\ln^4 N}\left(\left\|P(X^N)\right\| - \|P(x)\|\right) \geq C\left(\delta+1\right)\right) \leq e^{-N} + e^{-\delta^2 \ln^8 N}.$$

Idea of the proof

We want to show the following formula:

$$\mathbb{E}\left[\frac{1}{N}\operatorname{Tr}_N\left(f(P(X^N))\right)\right] = \sum_{0\leq i\leq k}\frac{1}{N^{2i}}\alpha_i^P(f) + \mathcal{O}(N^{-2(k+1)}).$$

- First, thanks to Fourrier transform we can assume that f is of the form $f_z : x \in \mathbb{R} \to e^{\mathbf{i}xz}$.

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- Secondly we set Q a polynomial in $X_1, \dots, X_d, Y_1, \dots, Y_p$ and $X_t^N = e^{-t/2}X^N + (1-e^{-t})^{1/2}x$, then given a system of p free semicircular y, for any polynomial Q,

$$\mathbb{E}\left[\tau_N\Big(Q(X^N, y)\Big)\right] = \tau\Big(Q(x,y)\Big) - \int_0^\infty \mathbb{E}\left[\frac{d}{dt}\tau_N\Big(Q(X_t^N, y)\Big)\right] dt.$$

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- Then we show that there is a deterministic operator $T_{t,p}$ on the space of polynomials such that if $\widetilde{y}$ is a system of $2p+2d$ free semicircular

$$\mathbb{E}\left[\frac{d}{dt}\tau_N\Big(Q(X_t^N,y)\Big)\right] = \frac{e^{-t}}{N^2}\mathbb{E}\left[\tau_N\Big(T_{t,p}(Q)(X^N,\widetilde{y})\Big)\right].$$

- We then proceed by induction.