# Practical

**Exercise 1.** *Let $\mathcal{X} = (X_1, X_2, X_3)$ be 3 iid random variables $\mathcal{N}(0,1)$ distributedy and $a_1, a_2, a_3, a_4$ four real numbers. We consider the funcion $G$ defined by*

$$G(X_1, X_2, X_3) = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_1X_2.$$

1. *Theoretical part*

   (a) *Assume that $a_3 = a_4 = 0$.*
   *and show that*

$$\begin{aligned}
\mathbb{E}(G(\mathcal{X})|X_1) &= a_1X_1,\\
\mathbb{E}(G(\mathcal{X})|X_2) &= a_2X_2\\
\mathbb{E}(G(\mathcal{X})|X_3) &= 0,\\
\mathbb{E}(G(\mathcal{X})|X_1, X_2) &= a_1X_1 + a_2X_2,\\
\mathbb{E}(G(\mathcal{X})|X_1, X_3) &= a_1X_1\\
\mathbb{E}(G(\mathcal{X})|X_2, X_3) &= a_2X_2
\end{aligned}$$

   *Compute $S^1,\ S^2,\ S^3,\ S^{1,2},\ S^{1,3},\ S^{2,3},\ S^{1,2,3}$*

   (b) *Show that in the general case*

$$\begin{aligned}
S^1 &= \frac{a_1^2}{a_1^2 + a_2^2 + a_3^2 + a_4^2},\\
S^2 &= \frac{a_2^2}{a_1^2 + a_2^2 + a_3^2 + a_4^2},\\
S^3 &= \frac{a_3^2}{a_1^2 + a_2^2 + a_3^2 + a_4^2},\\
S^{1,2} &= \frac{a_1^2 + a^2 + a_4^2}{a_1^2 + a_2^2 + a_3^2 + a_4^2}.
\end{aligned}$$

2. *Simulation part*

   (a) *Create. program using the Pick-freeze's method in order to estimate the different indices*

   (b) *Illustrate le LLN and the CLT and give an estimate of the limiting variance in the CLT theorem.*

**Exercise 2** (Ishigami Function). *The Ishigami function is defined bye*

$$Y = G(X_1, X_2, X_3) = \sin X_1 + 7\sin^2 X_2 + 0.1X_3^4 \sin X_1 \tag{1}$$

*where* $(X_j)_{j=1,2,3}$ *are iid and uniforml distributed on* $[-\pi; \pi]$.

1. *ShÃ w that*
$$S^1 = 0.3139, \quad S^2 = 0.4424, \quad S^3 = 0.$$
2. *We want to know* $S^{1,2} > S^3$ *(we assume here that we don't know how to compute the theoretical values)*
   (a) *Use the Pick-freeze method's to estimate jointly* $\left(S^{1,2}, S^3\right)$.
   (b) *Thanks to the Delta method we know that*
   $$\sqrt{n}\left(\begin{pmatrix} S_n^{1,2} \\ S_n^3 \end{pmatrix} - \begin{pmatrix} S^{1,2} \\ S^3 \end{pmatrix}\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Gamma).$$
   *Who is* $\Gamma$*?*
   (c) *Give an estimator of* $\Gamma$
   (d) *Perform a statistical test of level* $\alpha = 5/100$ *to test* $H_0 : S^{1,2} \leqslant S^3$ *against* $H_1 : S^{1,2>} S^3$.
   (e) *Same questions to test* $H_0 : S^3 = 0$ *against* $H_1 : S^3 > 0$.

**Exercise 3** (Sobol G-funcion (**)). *Let* $X_1, \ldots, X_d$ *be iid RV uniformly disteibuted on* $[0, 1]$ *and let* $a_1, \ldots, a_d$ *be real numbers the Sobol G function is defined by*

$$Y = g_{sobol}(X_1, \ldots, X_d) = \prod_{k=1}^{d} g_k(X_k) \tag{2}$$

*with* $g_k(X_k) = \frac{|4X_k - 2| + a_k}{1 + a_k}$.
*Compute and estimate* $S^i$*for* $i \in \{1, \ldots, d\}$. *Give an illustration of the convergence in the CLT and give an estimation of the limiting variance.*

**Exercise 4.** *Bréguet's formula gives the cunsumption of an airplane as a function of several inputs :*

$$M_{fuel} = M\left(e^{\frac{SFC\cdot g\cdot Ra}{V\cdot F}10^{-3}} - 1\right). \tag{3}$$

*Variables fijas*

- $M$ *: = mass of the plane*
- $g$ *: universal gravitational constant*
- $Ra$ *: Range = distance of the flight*

*the Random inputs are*

- $V$ *: Cruise speed = speed of the plane*
- $F$ *: Lift-to-drag ratio = aerodynamic coefficient*
- $SFC$ *: Specific Fuel Consumption = engine's performance.*

*The distribution of* $V$, $F$ *yand*$SFC$*are given by the expert as*

| variable | distribution | parameters |
|---|---|---|
| $V$ | *Uniforme* | $(V_{min}, V_{max})$ |
| $F$ | *Beta* | $(7, 2, F_{min}, F_{max})$ |
| $SFC$ | $\theta_2\, e^{-\theta_2(u-\theta_1)}\, \mathbb{1}_{[\theta_1,+\infty[}$ | $\theta_1 = 17.23, \theta_2 = 3.45$ |

| variable | mean valuel | min | max |
|---|---|---|---|
| $V$ | **231** | 226 | 234 |
| $F$ | **19** | 18.7 | 19.05 |

*The airplane manufacturer wants to know if he has to improve the engine or the geometry of its plane. So he wants to know if* $S^{SFC} \geqslant S^F$*. Built the statistical test to test* $H_0$ : $S^{SFC} \geqslant S^F$ *against* $H_1$ : $S^{SFC} \leq S^F$.
*Conclude.*

**Exercise 5.** *(**)*
*Let*

$$Y = \exp\{X_1 + 2X_2\}, \tag{4}$$

*with* $X_1$ *y* $X_2$ *are iid* $\mathcal{N}(0,1)$. *distributed RV.*

1. *Show that*

   $$f_Y(y) = \frac{1}{\sqrt{10\pi}y} e^{-(\ln y)^2/10} \mathbb{1}_{\mathbb{R}^+}(y) \quad \text{and} \quad F_Y(y) = \Phi\left(\frac{\ln y}{\sqrt{5}}\right),$$

   *where*

   $$\Phi(x) := \int_{-\infty}^{x} e^{-t^2/2} dt/\sqrt{2\pi}.$$

   *Show that the Cramér-von Mises indices* $S^1_{2,CVM}$ *and* $S^2_{2,CVM}$ *are*

   $$S^1_{2,CVM} = \frac{6}{\pi} \arctan 2 - 2 \approx 0.1145$$
   $$S^2_{2,CVM} = \frac{6}{\pi} \arctan \sqrt{19} - 2 \approx 0.5693.$$

2. *Give a program that estimate these values.*