# Superconcentration inequalities for centered Gaussian stationnary processes

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- ▶ Superconcentration inequality for stationary Gaussian sequences.
- Main result (abstract theorem)
- ▶ Tools and sketch of the proof.

▶ What is superconcentration ?

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sharp inequality, does not depend on  $\Gamma$ .

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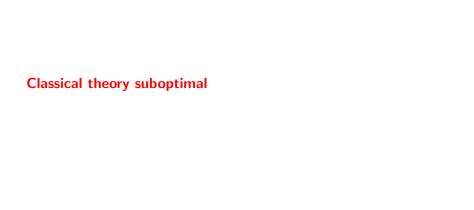
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In fact,

$$\operatorname{Var}(M_n) \leq \frac{C}{\log n}, \quad C > 0$$



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Lot of different models

Largest eigenvalue in random matrix theory.

▶ *M* random matrix from the GUE, namely

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- ▶ In fact,  $Var(\lambda_{max}) \leq \frac{C}{N^{4/3}}$  [Ledoux-Rider '10].

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- First passage in percolation theory.

#### Classical theory suboptimal

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Lot of different models

- Largest eigenvalue in random matrix theory.
- Branching random walk.
- ▶ Discrete Gaussian free field on  $\mathbb{Z}^d$ .
- Free energy in spin glasses theory (SK model).
- First passage in percolation theory.
- Stationary Gaussian sequences.

- ▶ What is superconcentration ?
- ► Convergence of extremes (Gaussian case).

### Theorem [Berman '64]

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## Proposition [Chatterjee '14]

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## Gaussian concentration inequality [Borel-Sudakov-Tsirelson '76]

Take  $X \sim \mathcal{N}(0, Id)$  and  $F : \mathbb{R}^n \to \mathbb{R}$  Lipschitz.

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Reflects asymptotics Gumbel

#### Recall

## Theorem [Berman 64']

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- Reflects Gumbel asymptotics.
- ▶ Implies  $Var(\max_i X_i) \leq \frac{C}{\log n}$  (optimal).

## Tools

Proof?

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Chatterjee's scheme of proof for the variance at the exponential level.

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Chatterjee's scheme of proof for the variance at the exponential level.

General theorem implies superconcentration inequality for Gaussian stationary sequences.

- What is superconcentration ?
- Convergence of extremes (Gaussian case).
- ► Superconcentration inequality for stationary Gaussian sequences.
- Main result (abstract theorem)

## General theorem [T. '15]

 $X=(X_1,\ldots,X_n)\sim \mathcal{N}(0,\Gamma)$  Assume that for some  $r_0\geq 0$ , there exists a covering  $\mathcal{C}(r_0)$  of  $\{1,\ldots,n\}$  verifying :

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Explanation : if  $\Gamma = Id$ , choose  $r_0 > 0$  then  $\mathcal{C}(r_0) = \{\{1\}, \dots \{n\}\}$ . Indeed, if  $\Gamma_{ii} > 0$  then i = j.

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Explanation :  $\mathcal{C}(r_0) = \left\{\{1\}, \ldots \{n\}\right\}$  partition of  $\{1, \ldots, n\}$ , so  $\sum_i 1_{\{l=i\}} = 1$ . In general,  $\mathcal{C}(r_0)$  "slightly bigger" than a partition and  $C \geq 1$ .

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Let  $\rho(r_0) = \max_{D \in \mathcal{C}(r_0)} \mathbb{P}(I \in D)$ .

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- ▶ there exists  $C \ge 1$  such that, a.s.,  $\sum_{D \in \mathcal{C}(r_0)} 1_{\{I \in D\}} \le C$ , where  $I = \operatorname{argmax}_i X_i$ .

Let  $\rho(r_0) = \max_{D \in \mathcal{C}(r_0)} \mathbb{P}(I \in D)$ . Then, for every  $\theta \in \mathbb{R}$ ,

$$\operatorname{Var}\left(e^{\theta M_n/2}\right) \leq C \frac{\theta^2}{4} \left(r_0 + \frac{1}{\log\left(1/\rho(r_0)\right)}\right) \mathbb{E}\left[e^{\theta M_n}\right].$$

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(Stationary case : 
$$K_{r_0} = 1/\log n$$
)

#### Recall

### Theorem [T. '15]

 $(X_i)_{i\geq 0}$  centered stationary Gaussian sequence, covariance function  $\phi$ . Assume  $\phi(n)=o(\log n)$   $(n\to\infty)$  and technicals hypothesis, then

$$\mathbb{P}(|M_n - \mathbb{E}[M_n]| \ge t) \le 6e^{-ct\sqrt{\log n}}, \quad t \ge 0.$$

- What is superconcentration ?
- Convergence of extremes (Gaussian case).
- Superconcentration inequality for stationary Gaussian sequences.
- Main result (abstract theorem)
- Tools and sketch of the proof.

# Key steps of the proof

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- ▶ Remember  $C(r_0)$  is "slightly bigger" than a partition of  $\mathbb{R}^n$ .

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Hypercontractivity alone doesn't work .

