About superconcentration and related topics : a short survey

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Outline of the Talk

- Introduction
- Superconcentration for stationary Gaussian sequences
- Around Talagrand's inequalities
- Application in Boolean analysis
- Optimal transport approach

Introduction

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Concentration theory : effective tool in various mathematical areas

- Probability in high dimension
- Probability in Banach spaces
- Empirical process
- Mechanical statistics
- $\dots$

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Lack of precision for particular example ?

Standard Gaussian measure

γ_n standard Gaussian measure on $\mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$ smooth enough

Poincaré's inequality

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If $X \sim \mathcal{N}(0, \Gamma)$ then

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At this level of generality, this inequality is sharp but does not depend on Γ. problem ?

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Poincaré's inequality sub-optimal for some functionals = Superconcentration (Chatterjee)

Branching Random Walk

- $\mathcal{T}$ binary tree with depth n.
- X_e *i.i.d.* $\mathcal{N}(0,1)$ on each edge e.
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Tools : modified second moment method combined with comparison arguments (very technicals proof).

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Largest eigenvalue

$$\lambda_{\max} = \sup_{|u|=1} \sum_{i,j=1}^{n} X_{ij} u_i \overline{u_j}$$

Relevant regime : $\sigma^2 \sim 1/n$.

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Theorem [Tracy-Widom]

$$n^{2/3}(\lambda_{\mathsf{max}} - 1) \xrightarrow{\mathcal{L}} TW$$

Other examples

- First time passage in percolation theory.
- Free energy in spin glass theory (REM, GREM, SK, . . .).
- Discrete Gaussian Free Field $\mathbb{Z}^2$.
- Order statistics from an i.i.d. sample (maximum, median,. . .).
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- Common properties ? Is it possible, in general, to improve (even slightly) upon classical concentration ?

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Approach of my thesis : semi-groups interpolation and hypercontractive arguments.

Attention : Hypercontractivity = logarithmic gain (sub-linearity)

Superconcentration for stationary Gaussian sequences

Stationary Gaussian sequences

$(X_n)_{n\geq 0}$ centered stationary Gaussian sequence, with covariance function $\mathbb{E}[X_i X_j] = \phi(|i-j|)$ où $\phi : \mathbb{N} \to \mathbb{R}_+$.

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Extreme theory [Berman]

If $\phi(n) \log n \underset{n\to\infty}{\longrightarrow} 0$ then

$$\sqrt{2\log n}\big(M_n - b_n\big) \xrightarrow{\mathcal{L}} \Lambda_0$$

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Gumbel's distribution : $\mathbb{P}(\Lambda_0 \geq t) = 1 - e^{-e^{-t}}$ $\quad$ ($\sim e^{-t}$ for t large enough)

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Variance

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Tools : variance representation by semi-groups and hypercontractivity.

Talagrand's inequality : bounding the variance

γ_n standard Gaussian measure on $\mathbb{R}^n$.

Theorem [Talagrand]

$f\ :\ \mathbb{R}^n \to \mathbb{R}$ smooth enough

$$\mathrm{Var}_{\gamma_n}(f) \leq C \sum_{i=1}^{n} \frac{\|\partial_i f\|_2^2}{1 + \log \frac{\|\partial_i f\|_2}{\|\partial_i f\|_1}}$$

Improve upon Poincaré's inequality.

Proof ?

Semi-group

Ornstein-Uhlenbeck's semi-group

$$P_t(f)(x) = \int_{\mathbb{R}^n} f(e^{-t}x + \sqrt{1 - e^{-2t}}y)d\gamma_n(y) \quad t \geq 0, x \in \mathbb{R}^n$$

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Hypercontractivity

$$\|P_t f\|_q \leq \|f\|_{p(t)}, \quad p(t) = (q-1)e^{-2t} + 1, t > 0$$

Note : $p(t) < q$ (improve upon Jensen's inequality).

Representation formula

Interpolation by semi-group

$$\mathrm{Var}_{\gamma_n}(f) \;=\; 2\int_0^{\infty} e^{-2t}\int_{\mathbb{R}^n} |P_t \nabla f|^2 d\gamma_n dt$$

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Hypercontractivity

Pour $i = 1, \ldots, n$

$$\|P_t(\partial_i f)\|_2 \leq \|\partial_i f\|_{p(t)} \quad p(t) = 1 + e^{-2t},\ t > 0.$$

It implies Talagrand's inégality (after some interpolation arguments based on Hölder's inequality)

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Then

$$\mathrm{Var}(M_n) \leq C \left(r_0 + \frac{1}{\log 1/\rho(r_0)} \right)$$

Chatterjee's Theorem : a sketch of proof

$X \sim \mathcal{N}(0,\Gamma)$, $f : \mathbb{R}^n \to \mathbb{R}$ smooth enough

Variance representation

$$\mathrm{Var}\big(f(X)\big) = 2\int_0^\infty e^{-2t} \sum_{i,j=1}^{n} \Gamma_{ij}\mathbb{E}[\partial_j f(X) P_t(\partial_i f)(X)]dt.$$

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Sketch of proof

- Γ satisfies a « covering »property (which allows one to gather the Γ_{ij} in pack of same « size »).
- $(P_t)_{t\geq 0}$ is hypercontractive, it can be used to control the size (in L^p-norm) of each of these packs.

Stationnary Gaussian sequences

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Recall

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Non-asymptotic concentration inequality ?

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Non-asymptotic concentration inequality ?

Goal

- $\mathbb{P}(\sqrt{2\log n}(M_n - b_n) \geq t) \leq \psi_1(t), \quad t \geq 0$
- $\mathbb{P}(\sqrt{2\log n}(M_n - b_n) \leq -t) \leq \psi_2(t), \quad t \geq 0$

with ψ_i, $i = 1, 2$ reflecting Gumbel's asymptotics.

Gaussian concentration

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a L-Lipschitz function and $X \sim \mathcal{N}(0, I_d)$ then

Theorem [Borell, Sudakov-Tsirel'son]

$$\mathbb{P}\Big(|f(X) - \mathbb{E}[f(X)]| \geq t\Big) \leq 2e^{-t^2/2L}$$

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$\mathbb{P}(\sqrt{2\log n}|M_n - \mathbb{E}[M_n]| \geq t) \leq 2e^{-t^2/4\log n}$ (classical theory)

- The Gaussian decay is not reflecting the behavior of the limiting distribution.

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Superconcentration inequality ?

Stationary Gaussian sequences

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- Corresponds to Gumbel's asymptotics (t large),

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- Up to numerical constant, same result holds with b_n instead of $\mathbb{E}[M_n]$.
- Corresponds to Gumbel's asymptotics (t large),
- Implies optimal bounds on the variance,

Stationary Gaussian sequences

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- Corresponds to Gumbel's asymptotics (t large),
- Implies optimal bounds on the variance,
- Consequence of a more general Theorem which holds for a large class of stationary Gaussian fields.

Sketch of proof

Goal : $\mathbb{P}\Big(\sqrt{2\log n}|M_n - \mathbb{E}[M_n]| \geq t\Big) \leq 3e^{-ct}$

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We would like to obtain (1) for $Z = M_n = \max_{i=1,\dots,n} X_i$ with $K \sim \mathrm{Var}(M_n) \sim C/\log n$.

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Proof : Extension of Chatterjee's Theorem at an exponential level.

Talagrand's inequalities at higher order

Talagrand's inequality

Recall : γ_n standard Gaussian measure on $\mathbb{R}^n$.

Theorem [Talagrand]

$f : \mathbb{R}^n \to \mathbb{R}$ smooth enough

$$\mathrm{Var}_{\gamma_n}(f) \leq C \sum_{i=1}^{n} \frac{\|\partial_i f\|_2^2}{1 + \log \frac{\|\partial_i f\|_2}{\|\partial_i f\|_1}}$$

can be obtained from a variance representation formula together with a d'hypercontractive property.

Talagrand's inequality at higher order

Question :

Alternative representation formula

$$\Downarrow$$

Talagrand's inequality at order 2 ?

Representation formula

$f : \mathbb{R}^n \to \mathbb{R}$ smooth enough, $|\cdot|$ Euclidean norm.

Variance representation

$$\begin{aligned}\mathrm{Var}_{\gamma_n}(f) &= \left|\int_{\mathbb{R}^n} \nabla f \, d\gamma_n\right|^2 \\ &+ 2\int_0^\infty e^{-2u}(1-e^{-2u}) \int_{\mathbb{R}^n} \left|P_u(\nabla^2 f)\right|^2 d\gamma_n du\end{aligned}$$

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- Decomposition in L^2 (Hermite's polynomials) + integral remainder term
- Similar to previous works of various authors : Houdré, Kagan, Perez-Abreu, Ledoux, ...
- Inverse Poincaré's inequality straighforward.

Proof

Representation formula

$$\mathrm{Var}_{\gamma_n}(f) = 2\int_0^{\infty} e^{-2t} \int_{\mathbb{R}^n} |P_t \nabla f|^2 d\gamma_n dt$$

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$s \to \infty$ by ergodicity $K(\infty) = \left| \int_{\mathbb{R}^n} \nabla f \, d\gamma_n \right|^2.$

By integration by parts

$$\left(\int_{\mathbb{R}^n} f(-Lf) d\gamma_n = \int_{\mathbb{R}^n} |\nabla f|^2 d\gamma_n, \quad L = \Delta - x \cdot \nabla \right)$$

and commutation property ($\nabla P_t = e^{-t} P_t \nabla$, $t \geq 0$)

$$K'(u) = \frac{d}{du} \int_{\mathbb{R}^n} |P_u \nabla f|^2 d\gamma_n =$$

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Finally

$$K(t) = \left| \int_{\mathbb{R}^n} \nabla f \, d\gamma_n \right|^2 + 2 \int_t^{\infty} e^{-2u} \int_{\mathbb{R}^n} e^{-2u} |P_u \nabla^2 f|^2 d\gamma_n du$$

Substitute the expression of $K(t)$ in the representation formula to conclude.

First iteration

$$\begin{aligned}\mathrm{Var}_{\gamma_n}(f) &= \left| \int_{\mathbb{R}^n} \nabla f \, d\gamma_n \right|^2 \\ &+ 2\int_0^\infty e^{-2u}(1-e^{-2u}) \int_{\mathbb{R}^n} \left| P_u(\nabla^2 f) \right|^2 d\gamma_n du\end{aligned}$$

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Iterate the procedure (set $K_2(t) = \int_{\mathbb{R}^n} \left|P_u(\nabla^2 f)\right|^2 d\gamma_n \ldots$)

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Iterate the procedure (set $K_2(t) = \int_{\mathbb{R}^n} \left| P_u(\nabla^2 f) \right|^2 d\gamma_n \dots$)

Iteration at order p

$p \geq 1$

$$\mathrm{Var}_{\gamma_n}(f) = \sum_{k=1}^{p} \frac{1}{k!} \left| \int_{\mathbb{R}^n} \nabla^k f \, d\gamma_n \right|^2 + \frac{2}{p!} \int_0^\infty e^{-2t}(1-e^{-2t})^p \int_{\mathbb{R}^n} \left| P_t(\nabla^{p+1} f) \right|^2 d\gamma_n dt$$

Talagrand's inequality at order 2

Control the remainder term with $(P_t)_{t\geq 0}$'s hypercontractivity .

$$
\begin{aligned}
R &= 2\sum_{i,j=1}^{n}\int_0^{\infty} e^{-2u}(1-e^{-2u})\int_{\mathbb{R}^n}\Big[P_u\big(\partial_{ij}f\big)\Big]^2 d\gamma_n du \\
&= 2\sum_{i,j=1}^{n}\int_0^{\infty} e^{-2u}(1-e^{-2u})\|P_u\big(\partial_{ij}f\big)\|_2^2 du
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Follow the proof of Talagrand's inequality and get an improvement thanks to the factor $1-e^{-2u}$

Inégalités Talagrand d'ordre 2

Théorème [T.]

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Remark : this inequality can be obtained at any order $p \geq 1$.

Boolean Analysis

Analyse booléenne

Historically, Talagrand's inequality has been obtained on $C_n = \{-1, 1\}^n$ with $\mu^n = (\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1)^{\otimes n}$.

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Theorem [Talagrand]

$f \, : \, C_n \to \{0,1\}$

$$\mathrm{Var}_{\mu^n}(f) \leq C \sum_{i=1}^{n} \frac{\|D_i f\|_2^2}{1 + \log \frac{\|D_i f\|_2}{\|D_i f\|_1}}$$

with $D_i f(x) = \frac{f(x) - f\left(\tau_i(x)\right)}{2}$ $\quad \tau_i(x) = (x_1, \dots, -x_i, \dots, x_n)$, $x \in C_n$.

Influence and KKL's Theorem

$f : C_n \to \{0,1\}, \quad \mu^n = (\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1)^{\otimes n}$

Influence

$$I_i(f) = \mathbb{P}\big(f(X) \neq f\big(\tau_i(X)\big)\big), \quad \mathcal{L}(X) = \mu^n$$

Probability that the i-th coordonnate is pivotal for input X

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Theorem [Kalai-Kahn-Linial]

$$\forall f\ :\ C_n \to \{0,1\}, \exists i \in \{1,\dots,n\}\quad I_i(f) \geq c\frac{\log n}{n}$$

(f is implicitly assumed to be centered. This inequality is optimal on Tribes functions)

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KKL's Theorem can be proved by Talagrand's inequality

Talagrand's inequality

$f \ : \ C_n \to \{0, 1\}$

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(Up to numerical constant)

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Talagrand's inequality in terms of influences

$$\mathrm{Var}_{\mu^n}(f) \leq C \sum_{i=1}^{n} \frac{I_i(f)}{1 + \log \frac{1}{1/\sqrt{I_i(f)}}}.$$

Application : Kahn-Kalai-Linial's Theorem

If it exists $i \in \{1, \ldots, n\}$ such that $I_i(f) \geq \frac{C}{\sqrt{n}}$ then $I_i(f) \geq C\frac{\log n}{n}$.

Application : Kahn-Kalai-Linial's Theorem

If it exists $i \in \{1, \dots, n\}$ such that $I_i(f) \geq \frac{C}{\sqrt{n}}$ then $I_i(f) \geq C\frac{\log n}{n}$.

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Talagrand's inequality yields that

$$\exists i \in \{1, \ldots, n\} \quad \text{s.t.} \quad \frac{C}{n} \leq \frac{I_i(f)}{1+\log \frac{1}{1/\sqrt{I_i(f)}}} \qquad (2)$$

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IIt is enough to use (1) to deduce $\frac{C}{n} \leq \frac{I_i(f)}{\log n}$ from (2).

Influence of order 2

$f : \{-1,1\}^n \to \{0,1\}$ define

Influence of order 2

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Talagrand's inequality of ordre 2 on the cube ?

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Talagrand's inequality of ordre 2 on the cube ? Yes ! (same proof as the Gaussian case with two additionally technical issues)

On the Cube

Talagrand's inequality at order 2 [T.]

$$\mathrm{Var}_{\mu^n}(f) \quad \leq \quad C \sum_{i=1}^{n} \|D_i f\|_p^2 + C \sum_{i \neq j} \frac{\|D_{ij} f\|_2^2}{\left[1 + \log \frac{\|D_{ij} f\|_2}{\|D_{ij} f\|_1}\right]^2}$$

with $1 < p < 2$.

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Application : proof of KKL's Theorem type at order 2

KKL's Theorem at order 2

$f : C_n \to \{0,1\}$

KKL's Theorem at order 2 [T.]

Either $\exists i \in \{1, \ldots, n\}$

$$I_i(f) \geq c \left(\frac{1}{n}\right)^{1/1+\eta(p)} \quad 0 < \eta(p) < 1$$

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or $\exists i \neq j \in \{1,\ldots,n\}$

$$I_{(i,j)}(f) \geq c\left(\frac{\log n}{n}\right)^2$$

with $c > 0$ a numerical constant.

Proof : same method as KKL's Theorem
(Tribes functions also optimal for the second alternative)

Thanks for your attention

Superconcentration for product measures and Optimal Transport

Basics Facts

μ_n symmetric exponential measure on $\mathbb{R}^n$, γ_n standard Gaussian measure on $\mathbb{R}^n$.

Monotone rearrangement

$$\mu_n \xrightarrow{T} \gamma_n$$

where $T(x_1, \dots, x_n) = \big(t(x_1), \dots, t(x_n)\big)$ with $t : \mathbb{R} \to \mathbb{R}$ s.t.

$$\int_{-\infty}^{x} d\mu_1 = \int_{-\infty}^{t(x)} d\gamma_1$$

Notice : $\mathrm{Var}_{\gamma_n}(f) = \mathrm{Var}_{\mu_n}(f \circ T)$.

Weighted Poincaré's inequality

Poincaré's inequality for the Exponential measure

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Recall that $\mu_n \xrightarrow{T} \gamma_n$ with $T(x_1, \ldots, x_n) = \big(t(x_1), \ldots, t(x_n)\big)$

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Recall that $\mu_n \xrightarrow{T} \gamma_n$ with $T(x_1, \ldots, x_n) = \big(t(x_1), \ldots, t(x_n)\big)$ Estimate the behavior of $t' \circ t^{-1}$ to bound the variance of f under γ_n

Sample result

Standard Gaussian measure example [T.]

$$\mathrm{Var}_{\gamma_n}(f) \leq C \sum_{i=1}^{n} \int_{\mathbb{R}^n} (\partial_i f(x))^2 \left(\frac{1}{1+|x_i|} \right)^2 d\gamma_n(x)$$

- It can provide application in Superconcentration) of Gozlan's theoretical study of weighted Poincaré's inequalities

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- Great flexibility of the method : large choice of measure (log-concave, uniform,. . .), various choice of functionals (médiane, maximum, l^p-norms, largest eigenvalue (in moduli) of Ginibre ensemble . . .).

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- Extension to an exponentiel level in order to obtain deviation inequalities (similar to Boucheron/Thomas's work on order statistics).
- Transport of isoperimetrics inequalities in order to obtain more precise left deviations inequalities.