Superconcentration inequalities for centered Gaussian stationnary processes

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Outline

- What is superconcentration ?
- Convergence of extremes (Gaussian case).
- Superconcentration inequality for stationary Gaussian sequences.
- Main result (abstract theorem)
- Tools and sketch of the proof.

- What is superconcentration ?

$$X = (X_1, \ldots, X_n) \sim \mathcal{N}(0, \Gamma)$$

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In fact,

$$\mathrm{Var}(M_n) \leq \frac{C}{\log n}, \quad C > 0$$

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Lot of different models

- Largest eigenvalue in random matrix theory.

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- M random matrix from the GUE, namely

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- In fact, $\mathrm{Var}(\lambda_{\max}) \leq \frac{C}{N^{4/3}}$ [Ledoux-Rider '10].

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- Stationary Gaussian sequences.

- What is superconcentration ?
- Convergence of extremes (Gaussian case).

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Proposition [Chatterjee '14]

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- Superconcentration inequality for stationary Gaussian sequences.

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- Reflects Gumbel asymptotics.
- Implies $\mathrm{Var}(\max_i X_i) \leq \frac{C}{\log n}$ (optimal).

Tools

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Chatterjee's scheme of proof for the variance at the exponential level.

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Chatterjee's scheme of proof for the variance at the exponential level.

General theorem implies superconcentration inequality for Gaussian stationary sequences.

- What is superconcentration ?
- Convergence of extremes (Gaussian case).
- Superconcentration inequality for stationary Gaussian sequences.
- Main result (abstract theorem)

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Explanation : if $\Gamma = Id$, choose $r_0 > 0$ then $\mathcal{C}(r_0) = \Big\{\{1\}, \ldots \{n\}\Big\}$. Indeed, if $\Gamma_{ij} > 0$ then $i = j$.

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Let $\rho(r_0) = \max_{D \in \mathcal{C}(r_0)} \mathbb{P}(I \in D)$. Then, for every $\theta \in \mathbb{R}$,

$$\mathrm{Var}\Big(e^{\theta M_n/2}\Big) \leq C\, \frac{\theta^2}{4} \Big(r_0 + \frac{1}{\log\left(1/\rho(r_0)\right)}\Big) \mathbb{E}\left[e^{\theta M_n}\right].$$

Exponential Poincaré inequality

Conclusion ?

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implies

$$\mathbb{P}\left(|Z - \mathbb{E}[Z]| \geq t\right) \leq 6e^{-ct/\sqrt{K}}, \quad t > 0$$

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(Stationary case : $K_{r_0} = 1/\log n$)

Recall

Theorem [T. '15]

$(X_i)_{i\geq 0}$ centered stationary Gaussian sequence, covariance function ϕ. Assume $\phi(n) = o(\log n)$ $(n \to \infty)$ and technicals hypothesis, then

$$\mathbb{P}\left(|M_n - \mathbb{E}[M_n]| \geq t\right) \leq 6e^{-ct\sqrt{\log n}}, \quad t \geq 0.$$

- What is superconcentration ?
- Convergence of extremes (Gaussian case).
- Superconcentration inequality for stationary Gaussian sequences.
- Main result (abstract theorem)
- Tools and sketch of the proof.

Key steps of the proof

Main steps

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- Proper use of the covering $\mathcal{C}(r_0)$.

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Key steps for the proof

Denote $I = \mathrm{argmax_i X_i}$ and $I^t = \mathrm{argmax_i X_i^t}$.

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Use of Hölder's inequality

$$\begin{aligned}\mathbb{E}\left[e^{\theta F^t/2}1_{\{I,I^t\in D\}}\right] &= \mathbb{E}\left[e^{\theta M_n/2}1_{\{I\in D\}}\, e^{\theta M_n^t/2}1_{\{I^t\in D\}}\right] \\ &\leq \mathbb{E}\left[e^{\theta p M_n/2}1_{\{I\in D\}}\right]^{1/p}\,\mathbb{E}\left[e^{\theta q M_n^t/2}1_{\{I^t\in D\}}\right]^{1/q}.\end{aligned}$$

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By hypercontractivity,

$$\mathbb{E}\left[e^{\theta q M_n^t/2}1_{\{I^t\in D\}}\right]^{1/q} = \mathbb{E}[Q_t^q(e^{\theta M_n/2}1_{\{I\in D\}})]^{1/q}$$

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by Hölder's inequality

$$\mathbb{E}\left[e^{\theta F^t/2} 1_{\{I, I^t \in D\}}\right] \leq \mathbb{P}(I \in D)^{\frac{2-p}{p}}\, \mathbb{E}\left[e^{\theta M_n} 1_{\{I\in D\}}\right].$$

Key steps for the proof

Finally with the property of the covering $\mathcal{C}(r_0)$.

$$\mathcal{I} \leq \left(r_0 + C\rho(r_0)^{\frac{2-p}{p}}\right)\mathbb{E}[e^{\theta M_n}].$$

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- Gaussian stationary processes on $\mathbb{R}^d$.

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- Log-concave measures on $\mathbb{R}^n$ with convexity assumptions.

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Hypercontractivity relevant Gaussian processes $\simeq$ independent case (variance $\sim \frac{1}{\log n}$).

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Hypercontractivity alone doesn't work .

Thank you for your attention.