Superconcentration and Optimal Transport

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Outline of the Talk

- Introduction
- Basic facts on monotone rearrangement
- Transporting Poincaré inequalities
- Application in Superconcentration
- Extreme Theory and non-asymptotic deviation inequalities
- Transporting isoperimetric inequalities.

Introduction

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Concentration theory : effective tool in various mathematical areas

- Probability in high dimension
- Probability in Banach spaces
- Empirical process
- Mechanical statistics
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Lack of precision for particular example ?

Standard Gaussian measure

γ_n standard Gaussian measure on $\mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$ smooth enough

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Consequence

If $X \sim \mathcal{N}(0, \Gamma)$ then

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At this level of generality, this inequality is sharp but does not depend on Γ. problem ?

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Poincaré's inequality sub-optimal for some functionals = Superconcentration (Chatterjee)

Branching Random Walk

- $\mathcal{T}$ binary tree with depth n.
- X_e $i.i.d.$ $\mathcal{N}(0,1)$ on each edge e.
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Tools : modified second moment method combined with comparison arguments (very technicals proof).

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Largest eigenvalue

$$\lambda_{\max} = \sup_{|u|=1} \sum_{i,j=1}^{n} X_{ij} u_i \overline{u_j}$$

Relevant regime : $\sigma^2 \sim 1/n$.

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Theorem [Tracy-Widom]

$$n^{2/3}(\lambda_{\max} - 1) \xrightarrow{\mathcal{L}} TW$$

Other examples

- First time passage in percolation theory.
- Free energy in spin glass theory (REM, GREM, SK, ...).
- Discrete Gaussian Free Field $\mathbb{Z}^2$.
- Order statistics from an i.i.d. sample (maximum, median,...).
- l^p, $p > 2$ norm of standard Gaussian vector.
- Largest particule (in moduli) of Coulomb gazes.
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- Each models, ad-hoc methods, sometimes very technicals
- Common properties ? Is it possible, in general, to improve (even slightly) upon classical concentration ?

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We will consider product measures : $\mu_1 \otimes \ldots \otimes \mu_n$ and $\nu_1 \otimes \ldots \otimes \nu_n$.

Basic facts on monotone rearrangement

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Let ν be a probability measure on $\mathbb{R}$ with density h w.r.t. the Lebesgue measure and cumulative distribution function H.

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Monotone rearrangement

Let $t : \mathbb{R} \to \mathbb{R}$ be the function pushing μ onto ν
i.e.

$$\int_{-\infty}^{x} d\mu = \int_{-\infty}^{t(x)} d\nu$$

Monotone rearrangement on the real line

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Notation : hazard function associated to μ : $\kappa_\mu(x) = \frac{g(x)}{1-G(x)}$

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$$\text{thus} \quad t'(x) = \frac{\kappa_\mu(x)}{\kappa_\nu\big(t(x)\big)}$$

Transport of product measure in $\mathbb{R}^n$

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Step 3 : Set $T : \mathbb{R}^n \to \mathbb{R}^n$ as

$$T(x_1, \ldots, x_n) = \big(t(x_1), \ldots, t(x_n)\big) \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n$$

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Notice : T transports μ^n onto γ_n and $\mathrm{Var}_{\gamma_n}(f) = \mathrm{Var}_{\mu^n}(f \circ T)$ for $f : \mathbb{R}^n \to \mathbb{R}$ smooth enough

Transporting Poincaré inequality

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Estimate the behavior of $t' \circ t^{-1}$ (which can be expressed in terms of κ_μ and κ_{γ_1}) to bound the variance of f under γ_n

Sample result

Lemma

With the preceding setting, we have the following estimates

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Notice : Houdré-Bobkov et Bobkov-Ledoux already obtained the preceding inequality (in dimension 1) by other means. It is also an explicit version of Gozlan's theoretical work on weighted Poincaré's inequalities.

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&\leq \frac{C''}{1+\log n}
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Product measure with non-identical factors

Coulomb Gazes : density of $\{z_1, \dots, z_n\}$

$$(z_1, \dots, z_n) \in \mathbb{C}^n \mapsto \prod_{j=1}^{n} e^{-n|z_j|^\alpha} \prod_{1 \le j < l \le n} |z_j - z_k|^2, \quad \alpha \ge 1$$

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Lemma [Rider] : representation of the largest (in moduli) particule

$$|z|_{(1)} \overset{\mathcal{L}}{=} \max_{i=1,\ldots,n} R_i,$$

R_i all independent with density proportional to $t \mapsto t^{2i-1} e^{-nt^\alpha} 1_{t \geq 0}$.

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Non-asymptotic variance bounds

$$\mathrm{Var}\big(|z|_{(1)}\big) \leq \frac{C_\alpha}{n \log n}$$

Sharp with respect to some asymptotic results from Rider and Chafaï-Péché.

Extreme Theory and non-asymptotic deviation inequalities

Convergence of Extremes

Recall the following fact, in the Gaussian case,

$$\sqrt{2\log n}(M_n - b_n) \xrightarrow[\mathcal{L}]{} \Lambda_0, \quad n \to \infty$$

with $\mathbb{P}(\Lambda_0 \geq x) = 1 - e^{-e^{-t}}$, $t \in \mathbb{R}$ (Gumbel distribution).

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What about deviation inequalities ?

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Recall the following fact, in the Gaussian case,

$$\sqrt{2\log n}(M_n - b_n) \underset{\mathcal{L}}{\longrightarrow} \Lambda_0, \quad n \to \infty$$

with $\mathbb{P}(\Lambda_0 \geq x) = 1 - e^{-e^{-t}}$, $t \in \mathbb{R}$ (Gumbel distribution).

What about deviation inequalities ?

i.e. $\mathbb{P}\left(\sqrt{\log n}(M_n - \mathbb{E}[M_n]) \geq t \right) \leq Ce^{-ct}$

It should reflect the size of the variance of M_n and the asymptotics of Λ_0 (here on the right tail).

Extension to an exponential level : two further arguments

Lemma

$$\text{If } \mathrm{Var}(e^{\theta Z/2}) \leq \frac{\theta^2}{4} K \mathbb{E}[e^{\theta Z}] \quad \theta > 0$$

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Lemma

Let $f : \mathbb{R}^n \to \mathbb{R}$ non-increasing and $g : \mathbb{R}^n \to \mathbb{R}$ non-decreasing, then

$$\mathbb{E}\big[f(X)g(X)\big] \leq \mathbb{E}\big[f(X)\big]\mathbb{E}\big[g(X)\big], \quad X = (X_1, \ldots, X_n)$$

with X_i independent random variables.

Application in the Gaussian case

Standard Gaussian measure example [T.]

Combine all of this

$$\mathrm{Var}_{\gamma_n}(f) \leq C \sum_{i=1}^{n} \int_{\mathbb{R}^n} (\partial_i f(x))^2 \left(\frac{1}{1+|x_i|} \right)^2 d\gamma_n(x)$$

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Step 1 : apply to $f(x) = e^{\frac{\theta}{2} \max_{i=1,\dots,n} x_i}, \theta > 0$ to get

$$\mathrm{Var}(e^{\theta M_n/2}) \leq C \frac{\theta^2}{4} \mathbb{E}\left[e^{\theta M_n} \frac{1}{1+(M_n)^2} \right]$$

(we used again the fact $(A_i)_{i=1,\dots,n}$ is a partition).

Application in the Gaussian case

Step 2 : $(x_1, \ldots, x_n) \mapsto \frac{1}{1+\max_{i=1,\ldots,n} x_i}$ is a non-increasing function, so apply Harris's Lemma :

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Notice : all we needed was a bound on the variance of M_n and the fact that the map $t' \circ t^{-1}(x)$ was dominated by a non-increasing function.

Transporting Isoperimetric inequalities

Left tail of Gumbel's distribution

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Is it possible to transport stronger functional inequalities to obtain something relevant in the domain of attraction of Gumbel's distribution ?

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Classical Euclidean isoperimetric problem : at volume being fixed what are the borel sets that minimize the surface area ?

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with $A_r = A + rB_2$ and B_2 stands for the unitary Euclidean ball.

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Notice : if $\gamma_n(A) \geq 1/2 = \Phi(0)$ then $\gamma_n(A_r) \geq \Phi(0 + r) \geq 1 - e^{-r^2/2}$.

From isoperimetry to concentration

It can be equivalently stated in terms of function.

Gaussian concentration for Lipschitz function

$$\gamma_n\Big(\big|f - \mathrm{Med}(f)\big| \geq t\Big) \leq 2e^{-t^2/2L^2}, \quad t \geq 0$$

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$$\mathbb{P}\Big(\big|M_n - \mathbb{E}[M_n]\big| \geq t\Big) \leq Ce^{-t^2/2}, \quad t \geq 0$$

It does not reflect the size of $\mathrm{Var}(M_n)$ neither the asymptotics of the Gumbel !

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Remark : reflects the size of $\mathrm{Var}(M_n)$ and the right tail of Gumbel's distribution (but not the left tail !). Similar results for **correlated** Gaussian random variables have been obtained by Tanguy.

Reaching the left tail in Gumbel's domain of attraction

One way to reach the asymptotics of the left tail of the Gumbel's distribution is to use another isoperimetric inequality. Bobkov obtained an isoperimetric inequality for the Exponential measure on $\mathbb{R}_+^n$.

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Sharp with respect to Extreme theory (left tail of Gumbel's distribution). Still work for log-concave measure on $\mathbb{R}_+^n$.

Thank you for your attention