Superconcentration inequalities for centered Gaussian stationnary processes

Kevin Tanguy

Toulouse University

June 21, 2016

Outline

- What is superconcentration ?
- Convergence of extremes (Gaussian case).
- Superconcentration inequality for stationary Gaussian sequences.
- Tools and sketch of the proof.

- What is superconcentration ?

$X = (X_1, \ldots, X_n) \sim \mathcal{N}(0, \Gamma)$

$X = (X_1, \ldots, X_n) \sim \mathcal{N}(0, \Gamma)$ with $\mathbb{E}\left[X_i^2\right] = 1$.

$X = (X_1, \ldots, X_n) \sim \mathcal{N}(0, \Gamma)$ with $\mathbb{E}\left[X_i^2\right] = 1$.

Set $M_n = \max_i X_i$.

$X = (X_1, \ldots, X_n) \sim \mathcal{N}(0, \Gamma)$ with $\mathbb{E}\left[X_i^2\right] = 1$.

Set $M_n = \max_i X_i$.

Variance upper bound

$$\mathrm{Var}(M_n) \leq ?$$

$X = (X_1, \ldots, X_n) \sim \mathcal{N}(0, \Gamma)$ with $\mathbb{E}\left[X_i^2\right] = 1$.

Set $M_n = \max_i X_i$.

Variance upper bound

$$\mathrm{Var}(M_n) \leq ?$$

Classical concentration theory

$$\mathrm{Var}(M_n) \leq \max_i \mathrm{Var}(X_i).$$

$X = (X_1, \dots, X_n) \sim \mathcal{N}(0, \Gamma)$ with $\mathbb{E}\left[X_i^2\right] = 1$.

Set $M_n = \max_i X_i$.

Variance upper bound

$$\mathrm{Var}(M_n) \leq ?$$

Classical concentration theory

$$\mathrm{Var}(M_n) \leq \max_i \mathrm{Var}(X_i).$$

sharp inequality,

$X = (X_1, \ldots, X_n) \sim \mathcal{N}(0, \Gamma)$ with $\mathbb{E}\left[X_i^2\right] = 1$.

Set $M_n = \max_i X_i$.

Variance upper bound

$$\mathrm{Var}(M_n) \leq ?$$

Classical concentration theory

$$\mathrm{Var}(M_n) \leq \max_i \mathrm{Var}(X_i).$$

sharp inequality, does not depend on Γ.

Take $\Gamma = Id$ (X_i's independent).

Take $\Gamma = Id$ (X_i's independent).

$$\mathrm{Var}(M_n) \leq 1$$

Take $\Gamma = Id$ (X_i's independent).

$$\mathrm{Var}(M_n) \leq 1$$

In fact,

$$\mathrm{Var}(M_n) \leq \frac{C}{\log n}, \quad C > 0$$

Classical theory suboptimal

Classical theory suboptimal

Chatterjee's terminology : superconcentration phenomenon.

Classical theory suboptimal

Chatterjee's terminology : superconcentration phenomenon.

Lot of different models

- Largest eigenvalue in random matrix theory.

Classical theory suboptimal

Chatterjee's terminology : superconcentration phenomenon.

Lot of different models

- Largest eigenvalue in random matrix theory.

Classical theory suboptimal

Chatterjee's terminology : superconcentration phenomenon.

Lot of different models

- Largest eigenvalue in random matrix theory.
- Branching random walk.

Branching random walk.

- Take a binary tree of depth N.
- Put X_e $i.i.d.$ $\mathcal{N}(0,1)$ on each edge e.
- Take a path π from the top to the bottom of the tree, $X_\pi = \sum_{e \in \pi} X_e$.

Branching random walk.

- Take a binary tree of depth N.
- Put X_e $i.i.d.$ $\mathcal{N}(0,1)$ on each edge e.
- Take a path π from the top to the bottom of the tree, $X_\pi = \sum_{e\in\pi} X_e$.

- Classical theory : $\mathrm{Var}(\max_\pi X_\pi) \leq N \quad (X_\pi \sim \mathcal{N}(0,N))$.

Branching random walk.

- Take a binary tree of depth N.
- Put X_e $i.i.d.$ $\mathcal{N}(0,1)$ on each edge e.
- Take a path π from the top to the bottom of the tree, $X_\pi = \sum_{e\in\pi} X_e$.

- Classical theory : $\mathrm{Var}(\max_\pi X_\pi) \leq N \quad (X_\pi \sim \mathcal{N}(0,N))$.
- In fact, $\mathrm{Var}(\max_\pi X_\pi) \leq C$ [Bramson-Ding-Zeitouni].

Classical theory suboptimal

Chatterjee's terminology : superconcentration phenomenon.

Lot of different models

- Largest eigenvalue in random matrix theory.
- Branching random walk.

Classical theory suboptimal

Chatterjee's terminology : superconcentration phenomenon.

Lot of different models

- Largest eigenvalue in random matrix theory.
- Branching random walk.
- Discrete Gaussian free field on $\mathbb{Z}^d$.

Classical theory suboptimal

Chatterjee's terminology : superconcentration phenomenon.

Lot of different models

- Largest eigenvalue in random matrix theory.
- Branching random walk.
- Discrete Gaussian free field on $\mathbb{Z}^d$.
- Free energy in spin glasses theory (SK model).

Classical theory suboptimal

Chatterjee's terminology : superconcentration phenomenon.

Lot of different models

- Largest eigenvalue in random matrix theory.
- Branching random walk.
- Discrete Gaussian free field on $\mathbb{Z}^d$.
- Free energy in spin glasses theory (SK model).
- First passage in percolation theory.

Classical theory suboptimal

Chatterjee's terminology : superconcentration phenomenon.

Lot of different models

- Largest eigenvalue in random matrix theory.
- Branching random walk.
- Discrete Gaussian free field on $\mathbb{Z}^d$.
- Free energy in spin glasses theory (SK model).
- First passage in percolation theory.
- Stationary Gaussian sequences.

- What is superconcentration ?
- Convergence of extremes (Gaussian case).

Stationary Gaussian sequences

Theorem [Berman '64]

$(X_i)_{i\geq 0}$ centered normalized stationary Gaussian sequence with covariance function ϕ.

Stationary Gaussian sequences

Theorem [Berman '64]

$(X_i)_{i\geq 0}$ centered normalized stationary Gaussian sequence with covariance function ϕ. Assume $\phi(n) = o(1/\log n)$ $(n \to \infty)$,

Stationary Gaussian sequences

Theorem [Berman '64]

$(X_i)_{i\geq 0}$ centered normalized stationary Gaussian sequence with covariance function ϕ. Assume $\phi(n) = o(1/\log n)$ $(n \to \infty)$, then

$$a_n(M_n - b_n) \xrightarrow{d} G, \quad (n \to \infty)$$

Stationary Gaussian sequences

Theorem [Berman '64]

$(X_i)_{i\geq 0}$ centered normalized stationary Gaussian sequence with covariance function ϕ. Assume $\phi(n) = o(1/\log n)$ $(n \to \infty)$, then

$$a_n(M_n - b_n) \xrightarrow{d} G, \quad (n \to \infty)$$

where $a_n = \sqrt{2 \log n}$

Stationary Gaussian sequences

Theorem [Berman '64]

$(X_i)_{i\geq 0}$ centered normalized stationary Gaussian sequence with covariance function ϕ. Assume $\phi(n) = o(1/\log n)$ $(n \to \infty)$, then

$$a_n(M_n - b_n) \xrightarrow{d} G, \quad (n \to \infty)$$

where $a_n = \sqrt{2\log n}$ and $\mathbb{P}(G \geq t) = 1 - e^{-e^{-t}}$, $t \in \mathbb{R}$ (Gumbel).

Stationary Gaussian sequences

Theorem [Berman '64]

$(X_i)_{i\geq 0}$ centered normalized stationary Gaussian sequence with covariance function ϕ. Assume $\phi(n) = o(1/\log n)$ $(n \to \infty)$, then

$$a_n(M_n - b_n) \stackrel{d}{\to} G, \quad (n \to \infty)$$

where $a_n = \sqrt{2 \log n}$ and $\mathbb{P}(G \geq t) = 1 - e^{-e^{-t}}$, $t \in \mathbb{R}$ (Gumbel).

Note : $\mathbb{P}(G \geq t) \sim e^{-t}$

Results from classical theory

$$\mathrm{Var}(M_n) \leq 1.$$

Results from classical theory

$$\mathrm{Var}(M_n) \leq 1.$$

Question :

- Correct bound ?

Results from classical theory

$$\mathrm{Var}(M_n) \leq 1.$$

Question :

- Correct bound ?

Proposition [Chatterjee '14]

$$\mathrm{Var}(M_n) \leq \frac{C}{\log n}.$$

- What is superconcentration ?
- Convergence of extremes (Gaussian case).
- Superconcentration inequality for stationary Gaussian sequences.

Question : **(super)concentration inequality reflecting convergence of extremes ?**

Question : **(super)concentration inequality reflecting convergence of extremes ?**

Similar as

- $Y_1, \ldots, Y_n$ i.i.d Rademacher $\big(\mathbb{P}(Y_1 = 1) = \mathbb{P}(Y_1 = -1) = 1/2\big)$

Question : **(super)concentration inequality reflecting convergence of extremes ?**

Similar as

- $Y_1, \ldots, Y_n$ i.i.d Rademacher $\quad \big(\mathbb{P}(Y_1 = 1) = \mathbb{P}(Y_1 = -1) = 1/2\big)$
- $n^{-1/2} \sum_{i=1}^{n} X_i \xrightarrow{d} \mathcal{N}(0,1) \quad$ (CLT)

Question : **(super)concentration inequality reflecting convergence of extremes ?**

Similar as

- $Y_1, \ldots, Y_n$ i.i.d Rademacher $\quad \big(\mathbb{P}(Y_1 = 1) = \mathbb{P}(Y_1 = -1) = 1/2\big)$
- $n^{-1/2} \sum_{i=1}^n X_i \xrightarrow{d} \mathcal{N}(0,1) \quad$ (CLT)
- $\mathbb{P}(n^{-1/2} \sum_{i=1}^n X_i \geq t) \leq e^{-t^2/2}$, $t \geq 0$.

(Super)concentration inequality ?

Gaussian concentration inequality [Borel-Sudakov-Tsirelson '76]

Take $X \sim \mathcal{N}(0, Id)$ and $F : \mathbb{R}^n \to \mathbb{R}$ Lipschitz.

(Super)concentration inequality ?

Gaussian concentration inequality [Borel-Sudakov-Tsirelson '76]

Take $X \sim \mathcal{N}(0, Id)$ and $F : \mathbb{R}^n \to \mathbb{R}$ Lipschitz.

$$\mathbb{P}\Big(\big|F(X) - \mathbb{E}[F(X)]\big| \geq t\Big) \leq 2e^{-t^2/2\|F\|_{Lip}^2},\ t \geq 0$$

(Super)concentration inequality ?

Gaussian concentration inequality [Borel-Sudakov-Tsirelson '76]

Take $X \sim \mathcal{N}(0, Id)$ and $F : \mathbb{R}^n \to \mathbb{R}$ Lipschitz.

$$\mathbb{P}\Big(\big|F(X) - \mathbb{E}[F(X)]\big| \geq t\Big) \leq 2e^{-t^2/2\|F\|^2_{Lip}},\ t \geq 0$$

If $F(x) = \max_i x_i$, $\|F\|^2_{Lip} \leq 1$

$$\mathbb{P}\Big(a_n\big|M_n - \mathbb{E}[M_n]\big| \geq t\Big) \leq 2e^{-\frac{t^2}{2a_n^2}},\ t \geq 0,\ a_n = \sqrt{2\log n}$$

(Super)concentration inequality ?

Gaussian concentration inequality [Borel-Sudakov-Tsirelson '76]

Take $X \sim \mathcal{N}(0, Id)$ and $F : \mathbb{R}^n \to \mathbb{R}$ Lipschitz.

$$\mathbb{P}\Big(\big|F(X) - \mathbb{E}[F(X)]\big| \geq t\Big) \leq 2e^{-t^2/2\|F\|_{Lip}^2}, \ t \geq 0$$

If $F(x) = \max_i x_i$, $\|F\|_{Lip}^2 \leq 1$

$$\mathbb{P}\Big(a_n\big|M_n - \mathbb{E}[M_n]\big| \geq t\Big) \leq 2e^{-\frac{t^2}{2a_n^2}}, \ t \geq 0, \ a_n = \sqrt{2\log n}$$

does not reflect Gumbel asymptotics.

Question : **(super)concentration inequality reflecting convergence of extremes ?**

Question : **(super)concentration inequality reflecting convergence of extremes ?**

Theorem [T. '15]

$(X_i)_{i\geq 0}$ centered stationary Gaussian sequence, covariance function ϕ.
Assume $\phi(n) = o(1/\log n)$ $(n \to \infty)$ and technicals hypothesis, then

$$\mathbb{P}\left(|M_n - \mathbb{E}[M_n]| \geq t\right) \leq 6e^{-ct\sqrt{\log n}}, \quad t \geq 0.$$

Question : **(super)concentration inequality reflecting convergence of extremes ?**

Theorem [T. '15]

$(X_i)_{i\geq 0}$ centered stationary Gaussian sequence, covariance function ϕ. Assume $\phi(n) = o(1/\log n)$ $(n \to \infty)$ and technicals hypothesis, then

$$\mathbb{P}\left(|M_n - \mathbb{E}[M_n]| \geq t\right) \leq 6e^{-ct\sqrt{\log n}}, \quad t \geq 0.$$

Remark : same inequality holds with b_n instead of $\mathbb{E}\left[M_n\right]$.

Question : **(super)concentration inequality reflecting convergence of extremes ?**

Theorem [T. '15]

$(X_i)_{i \geq 0}$ centered stationary Gaussian sequence, covariance function ϕ. Assume $\phi(n) = o(1/\log n)$ $(n \to \infty)$ and technicals hypothesis, then

$$\mathbb{P}\left(|M_n - \mathbb{E}[M_n]| \geq t\right) \leq 6e^{-ct\sqrt{\log n}}, \quad t \geq 0.$$

Remark : same inequality holds with b_n instead of $\mathbb{E}\left[M_n\right]$.

$$\mathbb{P}\left(\sqrt{\log n}|M_n - b_n| \geq t\right) \leq Ce^{-ct}, \quad t \geq 0.$$

Question : **(super)concentration inequality reflecting convergence of extremes ?**

Theorem [T. '15]

$(X_i)_{i \geq 0}$ centered stationary Gaussian sequence, covariance function ϕ. Assume $\phi(n) = o(1/\log n)$ $(n \to \infty)$ and technicals hypothesis, then

$$\mathbb{P}\left(|M_n - \mathbb{E}[M_n]| \geq t\right) \leq 6e^{-ct\sqrt{\log n}}, \quad t \geq 0.$$

Remark : same inequality holds with b_n instead of $\mathbb{E}\left[M_n\right]$.

$$\mathbb{P}\left(\sqrt{\log n}|M_n - b_n| \geq t\right) \leq Ce^{-ct}, \quad t \geq 0.$$

- Reflects Gumbel asymptotics

Recall

Theorem [Berman 64']

$(X_i)_{i\geq 0}$ centered normalized stationary Gaussian sequence with covariance function ϕ. Assume $\phi(n) = o(\log n)$ $(n \to \infty)$, then

$$a_n(M_n - b_n) \xrightarrow{d} G, \quad (n \to \infty)$$

where $a_n = \sqrt{2\log n}$ and $\mathbb{P}(G \geq t) = 1 - e^{-e^{-t}}$, $t \in \mathbb{R}$ (Gumbel).

Note : $\mathbb{P}(G \geq t) \sim e^{-t}$

Theorem [T. 15']

$(X_i)_{i\geq 0}$ centered stationary Gaussian, covariance function ϕ. Assume $\phi(n) = o(\log n)$ $(n \to \infty)$ and technicals hypothesis, then

$$\mathbb{P}\left(|M_n - \mathbb{E}[M_n]| \geq t\right) \leq 6e^{-ct\sqrt{\log n}}, \quad t \geq 0.$$

Remark : same inequality holds with b_n instead of $\mathbb{E}\left[M_n\right]$.

- Reflects Gumbel asymptotics.

Theorem [T. 15']

$(X_i)_{i\geq 0}$ centered stationary Gaussian, covariance function ϕ. Assume $\phi(n) = o(\log n)$ $(n \to \infty)$ and technicals hypothesis, then

$$\mathbb{P}\left(|M_n - \mathbb{E}[M_n]| \geq t\right) \leq 6e^{-ct\sqrt{\log n}}, \quad t \geq 0.$$

Remark : same inequality holds with b_n instead of $\mathbb{E}\left[M_n\right]$.

- Reflects Gumbel asymptotics.
- Implies $\mathrm{Var}(\max_i X_i) \leq \frac{C}{\log n}$ (optimal).

Tools

Proof ?

Proof ?

Chatterjee's scheme of proof for the variance at the exponential level.

Tools

Proof ?

Chatterjee's scheme of proof for the variance at the exponential level.

General theorem implies superconcentration inequality for Gaussian stationary sequences.

Key steps of the proof

Main steps

- Semigroup representation of the variance

Key steps of the proof

Main steps

- Semigroup representation of the variance
- Hypercontractivity

Key steps of the proof

Main steps

- Semigroup representation of the variance
- Hypercontractivity
- Proper use of a covering $\mathcal{C}$ of $\{1, \ldots n\}$.

Interpolation, independant case

Ornstein-Uhlenbeck semigroup

$$P_t f(x) = \int_{\mathbb{R}^n} f(xe^{-t} + \sqrt{1 - e^{-2t}}y) d\gamma(y), \; t \geq 0, \; x \in \mathbb{R}^n.$$

Interpolation, independant case

Ornstein-Uhlenbeck semigroup

$$P_t f(x) = \int_{\mathbb{R}^n} f(xe^{-t} + \sqrt{1-e^{-2t}}y)d\gamma(y),\ t \geq 0,\ x \in \mathbb{R}^n.$$

Heat equation : $\partial_t P_t f = L P_t f$ where $L = \Delta - x \cdot \nabla$.

Interpolation, independant case

Ornstein-Uhlenbeck semigroup

$$P_t f(x) = \int_{\mathbb{R}^n} f(xe^{-t} + \sqrt{1 - e^{-2t}}y)d\gamma(y), \; t \geq 0, \; x \in \mathbb{R}^n.$$

Heat equation : $\partial_t P_t f = L P_t f$ where $L = \Delta - x \cdot \nabla$.

$$\mathrm{Var}(f) = \mathbb{E}[f^2] - \mathbb{E}[f]^2 \quad =$$

Interpolation, independant case

Ornstein-Uhlenbeck semigroup

$$P_t f(x) = \int_{\mathbb{R}^n} f(xe^{-t} + \sqrt{1-e^{-2t}}y) d\gamma(y),\ t \geq 0,\ x \in \mathbb{R}^n.$$

Heat equation : $\partial_t P_t f = L P_t f$ where $L = \Delta - x \cdot \nabla$.

$$\mathrm{Var}(f) = \mathbb{E}[f^2] - \mathbb{E}[f]^2 \quad = \quad \mathbb{E}\big[(P_0 f)^2\big] - \big(\mathbb{E}[P_\infty f]\big)^2$$

Interpolation, independant case

Ornstein-Uhlenbeck semigroup

$$P_t f(x) = \int_{\mathbb{R}^n} f(xe^{-t} + \sqrt{1 - e^{-2t}} y) d\gamma(y),\ t \geq 0,\ x \in \mathbb{R}^n.$$

Heat equation : $\partial_t P_t f = L P_t f$ where $L = \Delta - x \cdot \nabla$.

$$\begin{aligned} \mathrm{Var}(f) = \mathbb{E}[f^2] - \mathbb{E}[f]^2 &= \mathbb{E}\left[(P_0 f)^2\right] - \left(\mathbb{E}[P_\infty f]\right)^2 \\ &= -\int_0^\infty \frac{d}{dt} \mathbb{E}\left[(P_t f)^2\right] dt \end{aligned}$$

Interpolation, independant case

Ornstein-Uhlenbeck semigroup

$$P_t f(x) = \int_{\mathbb{R}^n} f(xe^{-t} + \sqrt{1-e^{-2t}}y)d\gamma(y),\ t \geq 0,\ x \in \mathbb{R}^n.$$

Heat equation : $\partial_t P_t f = LP_t f$ where $L = \Delta - x \cdot \nabla$.

$$\begin{aligned}
\mathrm{Var}(f) = \mathbb{E}[f^2] - \mathbb{E}[f]^2 &= \mathbb{E}\big[(P_0 f)^2\big] - \big(\mathbb{E}[P_\infty f]\big)^2 \\
&= -\int_0^\infty \frac{d}{dt}\mathbb{E}\big[(P_t f)^2\big]dt \\
&= -2\int_0^\infty \mathbb{E}\big[(P_t f)(\partial_t P_t f)\big]dt = \dots
\end{aligned}$$

Interpolation, independant case

Ornstein-Uhlenbeck semigroup

$$P_t f(x) = \int_{\mathbb{R}^n} f(xe^{-t} + \sqrt{1 - e^{-2t}} y) d\gamma(y),\ t \geq 0,\ x \in \mathbb{R}^n.$$

Heat equation : $\partial_t P_t f = L P_t f$ where $L = \Delta - x \cdot \nabla$.

$$\begin{aligned}
\mathrm{Var}(f) = \mathbb{E}[f^2] - \mathbb{E}[f]^2 &= \mathbb{E}\big[(P_0 f)^2\big] - \big(\mathbb{E}[P_\infty f]\big)^2 \\
&= -\int_0^\infty \frac{d}{dt} \mathbb{E}\big[(P_t f)^2\big] dt \\
&= -2 \int_0^\infty \mathbb{E}\big[(P_t f)(\partial_t P_t f)\big] dt = \dots \\
&= 2 \int_0^\infty e^{-2t} \sum_{i=1}^n \mathbb{E}\big[(P_t \partial_i f)^2\big] dt
\end{aligned}$$

Hypercontractivity

P_t hypercontractive,

$$\mathbb{E}\left[(P_t h)^2\right]^{1/2} \leq \mathbb{E}[h^{p_t}]^{1/p_t},$$

with $p_t = 1 + e^{-2t} < 2$.

Hypercontractivity

P_t hypercontractive,

$$\mathbb{E}\left[(P_t h)^2\right]^{1/2} \leq \mathbb{E}[h^{p_t}]^{1/p_t},$$

with $p_t = 1 + e^{-2t} < 2$.

Apply to $h = \partial_i(e^{\theta M_n})$, $\theta \in \mathbb{R}$ and combined with concentration Lemma.

Hypercontractivity

P_t hypercontractive,

$$\mathbb{E}\left[(P_t h)^2\right]^{1/2} \leq \mathbb{E}[h^{p_t}]^{1/p_t},$$

with $p_t = 1 + e^{-2t} < 2$.

Apply to $h = \partial_i(e^{\theta M_n})$, $\theta \in \mathbb{R}$ and combined with concentration Lemma.

General case similar, with further step to treat correlation.

Conclusion

Similar results

- Gaussian stationary sequences with other covariance behavior.
- Gaussian stationary processes on $\mathbb{R}^d$.
- discrete Gaussian free field on $\mathbb{Z}^d$, $d \geq 3$

Conclusion

Similar results

- Gaussian stationary sequences with other covariance behavior.
- Gaussian stationary processes on $\mathbb{R}^d$.
- discrete Gaussian free field on $\mathbb{Z}^d$, $d \geq 3$
  (infinite group of finite type+volume growth condition ok).

Conclusion

Similar results

- Gaussian stationary sequences with other covariance behavior.
- Gaussian stationary processes on $\mathbb{R}^d$.
- discrete Gaussian free field on $\mathbb{Z}^d$, $d \geq 3$
  (infinite group of finite type+volume growth condition ok).
- Uniform measure on the sphere $\mathcal{S}^{n-1} \subset \mathbb{R}^n$.
- Log-concave measures on $\mathbb{R}^n$ with convexity assumptions.

Conclusion

Hypercontractivity relevant Gaussian processes $\simeq$
independent case (variance $\sim \frac{1}{\log n}$).

Conclusion

Hypercontractivity relevant Gaussian processes $\simeq$ independent case (variance $\sim \frac{1}{\log n}$).

discrete Gaussian free field on $\mathbb{Z}^2$ completely different behavior.

- $\mathrm{Var}(M_n) = O(1)$ [Bramson-Ding-Zeitouni]
- convergence in distribution Gumbel randomly shifted [Bramson-Ding-Zeitouni '15].

Hypercontractivity alone doesn't work .

Thank you for your attention.