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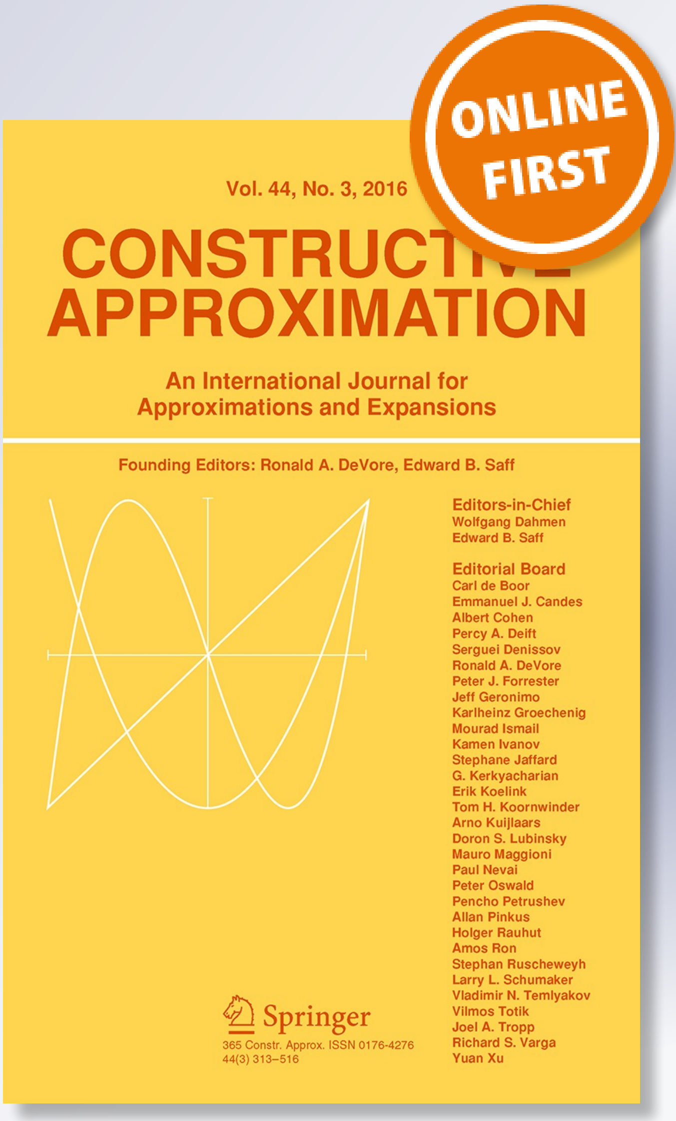
ISSN 0176-4276

Constr Approx
DOI 10.1007/s00365-016-9359-x



Springer

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Estimates for the Bohr Radius of a Faber–Green Condenser in the Complex Plane

P. Lassère[1] · **E. Mazzilli**[2]

Received: 1 October 2015 / Revised: 16 June 2016 / Accepted: 29 August 2016

Abstract We give some upper and lower estimates for the Bohr radius of a Faber–Green condenser in the complex plane.

Keywords Functions of a complex variable · Inequalities · Schauder basis

Mathematics Subject Classification 30B10 · 30A10

1 Introduction

The aim of this paper is to give some estimates for the Bohr radius of a Faber–Green condenser. Let us recall the classical Bohr theorem for the unit disk:

Classical Bohr's Theorem [5] *Let* $f(z) = \sum_n a_n z^n$ *be holomorphic on the unit disc* $\mathbb{D} := \{z \in \mathbb{C}\colon |z| < 1\}$. *If* $|f(z)| < 1$ *for all* $z \in \mathbb{D}$, *then* $\sum_n |a_n| \cdot |z^n| < 1$ *for all* $|z| < 1/3$. *Moreover, for all* $\varepsilon > 0$, *there exists a holomorphic function*

"In memory of Professor Nguyen Thanh Van".

Communicated by Edward B. Saff.

✉ P. Lassère
lassere@math.ups-tlse.fr

E. Mazzilli
Emmanuel.Mazzilli@math.univ-lille1.fr

1 Institut de Mathématiques, UMR CNRS 5580, Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse, France

2 Université Lille 1, 59655 Villeneuve d'Ascq Cedex, France

Published online: 27 December 2016

$f_\varepsilon(z) = \sum_n a_n^\varepsilon z^n$ *on* $\mathbb{D}$ *satisfying* $|f_\varepsilon(z)| < 1$ *for all* $z \in \mathbb{D}$*, but* $\sum_n |a_n^\varepsilon| \cdot |z^n| > 1$ *on* $|z| = \varepsilon + 1/3$.

For the last 20 years, this result has been generalized in many ways: to polynomials in one complex variable by Guadarrama [12], Fournier [11], and Chu [6]; to several complex variables by Boas and Khavinson [4]; to the polydisk by Defant et al. [8]; to complex manifolds by Aytuna and Djakov [2]; by Aizenberg et al. in functional analysis [1]; and to operator algebras by Dixon [9] and Paulsen et al. [17]. For a survey of literature on Bohr's phenomenon, see Bénéteau et al. [3].

In this paper, we focus on the Bohr radius of a condenser in the complex plane. For the convenience of the reader, let us recall the definition introduced in [15] (see Kaptanoglu and Sadik [14] for a partial approach in their seminal work).

Set $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$ and let $\mathscr{O}(\mathbb{D}_r)$ be the space of holomorphic functions on $\mathbb{D}_r$. We can reformulate the classical Bohr theorem in the following way.

The real 3 *is the smallest* $r > 1$ *such that: if* $f(z) = \sum_n a_n z^n \in \mathscr{O}(\mathbb{D}_r)$, $|f(z)| < 1$ *on* $\mathbb{D}_r$*, then* $\sum_n |a_n| \cdot |z^n| < 1$ *for all* $z \in \mathbb{D}$.

This approach can be easily generalized for an arbitrary continuum (we recall that a continuum $K \subset \mathbb{C}$ is a compact set in $\mathbb{C}$ that contains at least two points and such that $\overline{\mathbb{C}} \setminus K$ is simply connected) if we notice that the discs $\mathbb{D}_r$ are, for $r > 1$, the levels sets of the Green function with pole at ∞ of $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$.

Given a continuum $K \subset \mathbb{C}$, by the Riemann mapping theorem, $\overline{\mathbb{C}} \setminus K$ has a Green function Φ_K with pole at ∞ and level sets $(\Omega_r^K)_{r>1}$. The sets $(K, (\Omega_r^K)_{r>1})$ will be called a Green-condenser. To achieve the construction, we have to ensure two things. First, we need to replace the Taylor basis $(z^n)_{n \geq 0}$ by a common basis $(\varphi_n)_{n \geq 0}$ for the spaces $\mathscr{O}(\Omega_r^K)$ (thanks to the general theory of common bases, there are many [21]) equipped with the usual compact convergence topology. We then consider a Green-condenser $(K, (\Omega_r^K)_{r>1}(\varphi_n)_{n \geq 0})$, where $(\varphi_n)_{n \geq 0}$ is a common basis for the spaces $\mathscr{O}(\Omega_r^K)$. Second, we will use the following result:

Theorem [2,15] *For a Green-condenser* $(K, (\Omega_r^K)_{r>1}, (\varphi_n)_{n \geq 0})$*, there always exists* $r > 1$ *such that if* $f = \sum_n a_n \varphi_n \in \mathscr{O}(\Omega_r^K)$ *satisfies* $|f| < 1$ *on* Ω_r^K*, then* $\sum_n |a_n| \cdot \|\varphi_n\|_K < 1$.

Note that in fact we obtained the result with the additional hypothesis that there exists $a \in K$ such that $\varphi_n(a) = 0$ for all $n \geq 1$, and in [2], Aytuna and Djakov relax this assumption even in a more general context. We can now define the Bohr radius for any condenser.

The ***Bohr radius*** $B(K)$ *of* $(K, (\Omega_r^K)_{r>1}, (\varphi_n)_{n \geq 0})$ *is the infimum of all* $r > 1$ *such that* Ω_r^K *satisfies the previous theorem.*

In the rest of the paper, we always work with $(F_{K,n})_{n \geq 0}$ the Faber basis for K (see the definition in the next section) and hence with the Faber–Green condenser $(K, (\Omega_r^K)_{r>1}, (F_{K,n})_{n \geq 0})$. In general, it is not possible to calculate the exact value of $B(K)$ for an arbitrary continuum K. We know only the exact value of $B(K)$ in two cases: $K = \mathbb{D}$, of course, and for the elliptic condenser $K = [-1, 1]$. Even in the elliptic case, the proof is difficult (see [16]). The level sets $\Omega_r^{[-1,1]}$ of the Green function of $\overline{\mathbb{C}} \setminus [-1, 1]$ are ellipses of loci $-1, 1$ and eccentricity $\varepsilon = \frac{2r}{1+r^2}$ (the "big level sets" tend to "big discs" as $r \to \infty$). For this particular condenser, it is easy to deduce from [16] the exact value of $B(\Omega_r^{[-1,1]})$ for all $r > 1$. Furthermore, we then can observe that $r \mapsto B(\Omega_r^{[-1,1]})$ is a decreasing function and tends to 3 as r tends to ∞. In fact, this last property is true for all condensers $(K, (\Omega_r^K)_{r>1}, (F_{K,n})_{n\geq 0})$ (see Theorem 2). Let us point out that the classical fact that "big level sets" tend to "big discs" as $r \to \infty$ is not enough to deduce this property. We have to analyze carefully the behavior of Faber polynomials and of the Bohr radius $B(\overline{\Omega_r^K})$ for r large (see the proof of Theorem 2).

In this paper, we give some estimates for $B(K)$. The main results of the paper are:

Theorem 1 (uniform upper bound for $B(K)$. See Sect. 3 for exact estimates.)

(1) *For every continuum $K \subset \mathbb{C}$, we have $B(K) \lesssim 13.8$.*
(2) *Moreover, if K is convex, then $B(K) \lesssim 5.26$.*

Remark If K is the unit disk, then $B(K) = 3$, and if $K = [-1, 1]$, $B(K) \simeq 5.1284$ (see [16]).

Theorem 2 *For every continuum $K \subset \mathbb{C}$, we have*

$$\lim_{r\to\infty} B(\overline{\Omega_r^K}) = 3.$$

In a particular class of Faber–Green condenser (the positive class), we show the following result:

Theorem 3 *For any positive Faber–Green condenser, we have*

$$B(K) \geq 3.$$

Moreover, if K is a positive Faber–Green condenser, then $B(K) = 3$ if and only if K is a closed disk.

The paper is organized as follows. The next section provides the background on Faber's polynomials. In Sect. 3, we prove Theorem 1 and some other estimates of $B(K)$ when K is the interior of a Jordan's curve or a m-cusped hypocycloid. Section 4 is devoted to the proof of the Theorem 2, and in the last section, we define the positive class for Faber–Green condenser and prove Theorem 3.

2 Faber Polynomials

This section is devoted to Faber polynomials and their properties. The classic reference on this topic is the book of Suetin [20].

First let us recall the construction of the Faber polynomials for a continuum $K \subset \mathbb{C}$. Given a continuum $K \subset \mathbb{C}$, there exists a unique Riemann mapping

$$\Phi_K : \overline{\mathbb{C}} \backslash K \to \overline{\mathbb{C}} \backslash \mathbb{D}$$

normalized by

$$\Phi_K(\infty) = \infty \quad \text{and} \quad \Phi_K'(\infty) := \lim_{z\to\infty} \frac{\phi_K(z)}{z} = \gamma > 0,$$

where γ is the logarithmic capacity or the transfinite diameter of K. In a neighborhood of the point $z = \infty$, we have the Laurent expansion

$$\Phi_K(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} +, \cdots .$$

Thus

$$\begin{aligned}\Phi_K^n(z) &= \left(\gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + \cdots\right)^n \\ &= \gamma^n z^n + a_{n-1}^{(n)} z^{n-1} + a_{n-2}^{(n)} z^{n-2} + \cdots + a_1^{(n)} z + a_0^{(n)} + \sum_{j\geq 1} \frac{b_j^{(n)}}{z^j}.\end{aligned}$$

The nth Faber polynomial $F_{K,n}$ is now defined by taking the polynomial part of the Laurent expansion of Φ_K^n. For the sum of negative powers of z, we write

$$E_{K,n}(z) := \sum_{j\geq 1} \frac{b_j^{(n)}}{z^j} = \Phi_K^n(z) - F_{K,n}(z).$$

As illustrated in Fig. 1, if $R > 1$, the circle $C(0, R)$ is mapped by Φ_K^{-1} onto a closed regular analytic curve Γ_R. This is the boundary of the bounded domain $\Omega_R^K = \{z \in \mathbb{C}\backslash K \colon |\Phi_K(z)| < R\} \cup K$, which is usually called the R-Green level set of K.

Finally, the Faber–Green condenser is $(K, (\Omega_R^K)_{R>1}, (F_{k,n})_{n\geq 0})$. For a Faber–Green condenser, the situation is fairly like the Taylor one for the disk $(\overline{\mathbb{D}(0,1)}, (D(0, R))_{R>1}, (z^n)_{n\geq 0})$ in the following way ([20], chapter 1): for all $f \in \mathscr{O}(\Omega_R^K)$,

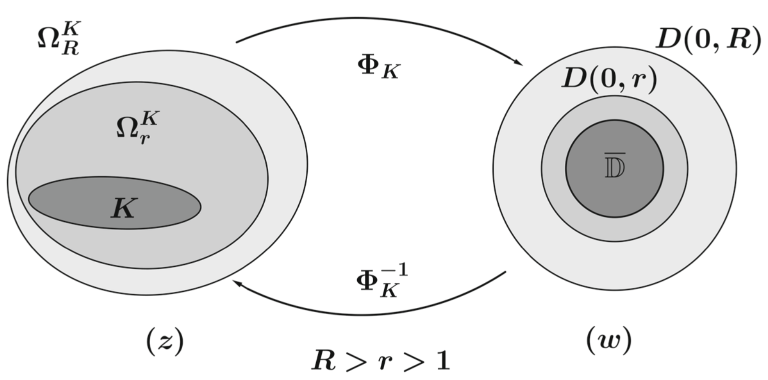

Fig. 1 The biholomorphism Φ_K

there exists a unique sequence $(a_n)_n$ of complex numbers such that $f = \sum_n a_n F_{K,n}$ in $\mathscr{O}(\Omega_R^K)$ equipped with its natural compact convergence topology. Moreover, for $f \in \mathscr{O}(K)$, $\limsup_n |a_n|^{1/n} = R^{-1}$ if and only if R is the largest Green-level set such that $f \in \mathscr{O}(\Omega_R^K)$.
Some examples:

- If K is the unit disk $\mathbb{D}$, then $\Phi_K(z) = z$. Hence, in this case, the Faber polynomials coincide with the Taylor polynomials $F_{K,n}(z) = z^n$, and the Faber–Green level sets are concentric disks $\Omega_R^K = D(0, R)$.
- For $K = [-1, 1]$, we have $\Phi_K(z) = z + \sqrt{z^2 - 1}$, $z \in \mathbb{C}\backslash K$ (where the branch of the square root is taken so that $\Phi'_K(\infty) = 2$). In this example, the Faber polynomials are the Chebyshev polynomials of the first kind $F_{K,n} = T_n$, and the level sets are ellipses.

We can also replace K by one of its level sets $\overline{\Omega_R^K}$. It is not difficult to observe that

$$F_{K,n}(z) = R^n F_{\overline{\Omega_R^K},n}(z),$$

and we will often use this formula.

As usual when dealing with Faber polynomials, it is better to work with the variable $w = \Phi_K(z)$ which lives in the annulus $D(0, R)\backslash\overline{\mathbb{D}}$ when $z \in \Omega_R^K\backslash K$. In this new coordinate, one has:

$$f(\Phi_K^{-1}(w)) = f(z), \quad \forall\, z = \Phi_K^{-1}(w) \in \Omega_R^K\backslash K \ \text{ and } \ w \in D(0, R)\backslash\overline{\mathbb{D}},$$

$$F_{K,n}(z) = F_{K,n}(\Phi_K^{-1}(w)) = w^n + \sum_{j\geq 1}^{\infty} \frac{\alpha_j^{(n)}}{w^j},$$

$$\Phi_K^{-1}(w) := \frac{w}{\gamma} + \beta_0 + \sum_{j\geq 1}^{\infty} \frac{\beta_j}{w^j}, \quad \forall\, |w| > 1.$$

3 Caratheodory-Type Inequalities and Uniform Bounds

3.1 Caratheodory-Type Inequalities

Proposition 1 *For all $R > 1$ and $f = \sum_{n\geq 0} a_n F_{K,n} \in \mathscr{O}(\Omega_R^K)$ such that $\mathtt{re}(f(z)) \geq 0$, $z \in \Omega_R^K$, we have the Caratheodory-type inequalities:*

$$|a_n| \leq \frac{2\mathtt{re}(a_0)}{R^n - 1}, \quad \forall\, n \geq 1. \tag{1}$$

Proof • First, suppose that $f = \sum_{n\geq 0} a_n F_{K,n} \in \mathscr{O}(\overline{\Omega_R^K})$. We have, for all $1 < r < R$,

$$\int_{C_{(0,R)}} f(\Phi_K^{-1}(w))\, w^{n-1}\mathrm{d}w - \int_{C_{(0,r)}} f(\Phi_K^{-1}(w))w^{n-1}\mathrm{d}w = 0. \tag{2}$$

On the other hand, because of the uniform convergence on compacts sets in $\overline{\Omega_R^K}\backslash K$,

$$\int_{C(0,R)} \overline{f(\Phi_K^{-1}(w))}\, w^{n-1}\mathrm{d}w - \int_{C(0,r)} \overline{f(\Phi_K^{-1}(w))}\, w^{n-1}\mathrm{d}w$$
$$= \sum_{j\geq 0}\left(\int_{C(0,R)} \overline{a_j}\cdot \overline{F_{K,j}(\Phi_K^{-1}(w))}\, w^{n-1}\mathrm{d}w - \int_{C(0,r)} \overline{a_j}\cdot \overline{F_{K,j}(\Phi_K^{-1}(w))}\, w^{n-1}\mathrm{d}w\right).$$

But, $F_{K,j}(\Phi_K^{-1}(w)) = w^j + \sum_{k\geq 1}\frac{b_j^k}{w^k}$; thus

$$\int_{C(0,r)} \overline{F_{K,j}(\Phi_K^{-1}(w))}\, w^{n-1}\mathrm{d}w = \begin{cases} 0 & \text{if } j\neq n,\\ 2i\pi r^{2n} & \text{if } j = n,\end{cases}$$

and

$$\int_{C(0,R)} \overline{f(\Phi_K^{-1}(w))}\, w^{n-1}\mathrm{d}w - \int_{C(0,r)} \overline{f(\Phi_K^{-1}(w))} w^{n-1}\mathrm{d}w = 2i\pi\overline{a_n}(R^{2n}-r^{2n}). \quad (3)$$

Then, (2)+(3) gives:

$$\int_{C(0,R)} 2\mathtt{re}\left(f(\Phi_K^{-1}(w))\right) w^{n-1}\mathrm{d}w - \int_{C(0,r)} 2\mathtt{re}(f(\Phi_K^{-1}(w)))w^{n-1}\mathrm{d}w$$
$$= 2i\pi\overline{a_n}(R^{2n}-r^{2n}).$$

The real part of f is positive for all $z\in\overline{\Omega_R^K}$; we get

$$\begin{aligned}\left|\overline{a_n}\cdot(R^{2n}-r^{2n})\right| &\leq \frac{1}{\pi}\int_{C(0,R)} \mathtt{re}(f(\Phi_K^{-1}(w)))|w^{n-1}|\cdot|\mathrm{d}w| \\ &\quad + \frac{1}{\pi}\int_{C(0,r)} \mathtt{re}(f(\Phi_K^{-1}(w)))|w^{n-1}|\cdot|\mathrm{d}w| \\ &= 2\mathtt{re}(a_0)(R^n+r^n)\end{aligned}$$

and

$$|a_n| \leq 2\mathtt{re}(a_0)\frac{R^n+r^n}{R^{2n}-r^{2n}}, \quad \forall n\geq 1,\ 1<r<R.$$

Let $r\to 1$:

$$|a_n| \leq 2\mathtt{re}(a_0)\frac{R^n+1}{R^{2n}-1} = \frac{2\mathtt{re}(a_0)}{R^n-1}, \quad \forall n\geq 1.$$

- If $f\in\mathcal{O}(\Omega_R^K)$, then we get the previous inequality for every $R'<R$. It suffices then to take the limits when R' goes to R. □

As a corollary, we deduce the estimates:

Theorem 1 *For every continuum K, we have*

$$B(K) \le \inf\left\{R > 1\colon \sum_{n\ge 1} \frac{4\sqrt{n\ln(n)+2n}}{R^n - 1} \le 1\right\} \lesssim 13,8. \tag{4}$$

For every convex continuum K, we have

$$B(K) \le \inf\left\{R > 1\colon \sum_{n\ge 1} \frac{4}{R^n - 1} \le 1\right\} \lesssim 5,26. \tag{5}$$

Proof Let $f = \sum_{n\ge 0} a_n F_{K,n} \in \mathscr{O}(\Omega_R^K)$ with $f(\Omega_R^K) \subset \mathbb{D}$. Up to a rotation, we can always suppose $a_0 \in \mathbb{R}^+$. Then the real part of $g := 1 - f$ is positive on Ω_R^K, and we can apply Proposition 1 to

$$g(z) = 1 - a_0 + \sum_{n\ge 1} a_n F_{K,n}(z).$$

This gives

$$\sum_{n\ge 0} |a_n| \cdot \|F_{K,n}\|_K \le a_0 + 2(1 - a_0) \sum_{n\ge 1} \frac{\|F_{K,n}\|_K}{R^n - 1}.$$

So

$$\sum_{n\ge 1} \frac{2\|F_{K,n}\|_K}{R^n - 1} \le 1 \quad \Longrightarrow \quad \sum_{n\ge 0} |a_n| \cdot \|F_{K,n}\|_K \le 1$$

and $R \ge B(K)$. This gives immediately

$$B(K) \le \inf\left\{R > 1\colon \sum_{n\ge 1} \frac{\|F_{K,n}\|_K}{R^n - 1} \le 1/2\right\}.$$

For K convex, we have $1 \le \|F_{K,n}\|_K \le 2$ [18]. An easy computation gives

$$B(K) \le \inf\left\{R > 1\colon \sum_{n\ge 1} \frac{4}{R^n - 1} \le 1\right\} \lesssim 5.26.$$

If K is no more convex, then the sequence $(\|F_{K,n}\|_K)_n$ is no more bounded but cannot grow too fast. Crudely we have [18]

$$1 \le \|F_{K,n}\|_K \le 2\sqrt{n\ln(n) + 2n},$$

and therefore

$$B(K) \leq \inf \left\{ R > 1: \sum_{n \geq 1} \frac{4\sqrt{n \ln(n) + 2n}}{R^n - 1} \leq 1 \right\}.$$

Using Maple, $B(K) \lesssim 13.8$. □

3.2 Example of the *m*-Cusped Hypocycloid

The results of Sect. 3 are particulary useful when we can calculate exactly the norm of $F_{K,n}$. It is often the case when

$$\Phi_K^{-1}(w) := \frac{w}{\gamma} + \beta_0 + \sum_{j \geq 1}^{\infty} \frac{\beta_j}{w^j},$$

where β_j, $j \geq 1$, are real and nonnegative (in fact, it is the definition of the positive class of condenser, see Sect. 5). In this case, we have for the continuum K ([7], Theorem 3.1):

$$F_{K,n}(\Phi_K^{-1}(w)) = w^n + \sum_{j \geq 1}^{\infty} \frac{\alpha_j^{(n)}}{w^j} \quad \text{where} \quad \alpha_j^{(n)} \geq 0.$$

The m-cusped hypocycloids H_m are in the positive class and satisfy the last property. Hypocycloids are starlike domains but not convex. Let us briefly recall the basic definitions and simple properties of the m-cusped hypocycloids. H_m is the bounded region delimited by the closed curve C_m defined by the equation

$$z = \exp(i\theta) + \frac{1}{m-1} \exp(-(m-1)i\theta), \quad m = 2, 3, \ldots.$$

The curve C_m is the trajectory of a point on the unit disk rolling without sliding in a larger disc of radius m. For $m = 2$, $H_2 = [-2, 2]$, and we can calculate the exact value of $B([-2, 2])$ [16]. If $m \geq 3$, it is straightforward to verify that

$$\Phi_{H_m}^{-1}(w) = w + \frac{1}{(m-1)w^{m-1}},$$

and $\Phi_{H_m}^{-1}$ admits a continuous extension on the unit circle which gives a topological mapping of the unit circle onto C_m. The coefficients $\alpha_j^{(n)}$ of $F_{H_m,n}$ are all positive, and the series $\sum_j \alpha_j^{(n)}$ converge absolutely (see [13]). This implies

$$\|F_{H_m,n}\|_{H_m} = |F_{H_m,n}(\Phi_{H_m}^{-1}(1))|.$$

In [13], it is implicitly proved that

$$\|F_{H_3,n}\|_{H_3} = 2 + \left(\frac{-1}{2}\right)^n := M_{3,n} \quad \text{and} \quad \|F_{H_4,n}\|_{H_4} = 2 + \frac{\lambda^n + \bar{\lambda}^n}{3^{n/2}} := M_{4,n},$$

where $\lambda = \frac{1}{\sqrt{3}}(-1 + \sqrt{2}i)$.

We can now give the upper bound for $B(H_3)$ using the same methods as in Theorem 1.

Corollary 1 *Let $i = 3, 4$. Then for H_i, we have the estimates*

$$B(H_i) \leq \inf \left\{ R > 1 \colon \sum_{n \geq 1} \frac{2M_{i,n}}{R^n - 1} \leq 1 \right\}.$$

In particular,

$$B(H_3) \leq 4.919167\ldots$$

Remark For $m > 4$, we can prove the following:

$$\|F_{H_m,n}\|_{H_m} \leq \left(\frac{m}{m-1}\right)^{m-1},$$

which is not optimal. The upper bound obtained for $B(H_m)$ with this estimate is not precise.

3.3 Angular Measure

In this subsection, we use classical notions (see [19]). For the convenience of the reader we recall these concepts here.

Suppose that Γ is a rectifiable Jordan curve, and let Ω be the interior of the bounded domain delimited by Γ. To almost every point of such curve Γ, we associate two angles as follows (illustrated in Fig. 2):

- Let s be the curvilinear coordinate of Γ. Then, for almost every s, we can define the tangent vector at s to Γ. The first angle $\sigma(s)$ will be the angle between the real axis and this tangent vector.
- For the second one, fix arbitrarily a point $z_0 = \Phi(e^{i\varphi}) \in \Gamma$. Define $v(\theta, \varphi)$ for $z = \Phi(e^{i\theta}) \in \Gamma$ as $v(\theta, \varphi) := \texttt{arg}(z - z_0)$.

If we suppose that $s \mapsto \sigma(s)$ is a function of bounded variation on $[0, l]$ (l is the length of Γ), we can associate with this function a unique finite total variation measure denoted also by σ. Then we define

$$V(\Gamma) := \int_0^l d|\sigma|(s).$$

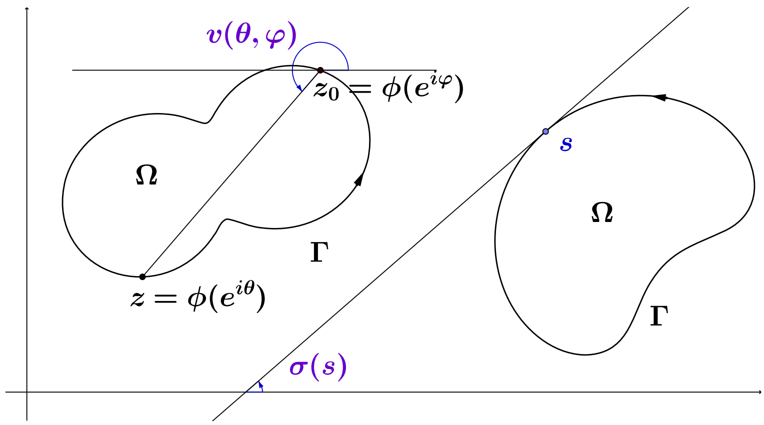

Fig. 2 The angular measure

If σ is of bounded variation, then $\theta \mapsto v(\theta, \varphi)$ is also of bounded variation, and it is not hard to see that

$$\int_0^{2\pi} d|v|(\theta) \leq V(\Gamma).$$

We can then state the main result on the norm of Faber's polynomials using angle functions:

Proposition 2 [19] *Suppose that* Γ *is a rectifiable Jordan curve and* σ *of bounded variation. Then*

$$F_{\overline{\Omega},n}\left(\Phi(e^{i\varphi})\right) = \frac{1}{\pi}\int_0^{2\pi} e^{in\theta}\mathrm{d}v(\theta), \text{ and}$$

thus

$$\|F_{\overline{\Omega},n}\|_{\overline{\Omega}} \leq \frac{V(\Gamma)}{\pi}.$$

We can give another corollary of Theorem 1:

Corollary 2 *Suppose* Γ *and* Ω *are as before. Then, we have the estimate for the Bohr radius*

$$B(\overline{\Omega}) \leq \inf\left\{ R > 1\colon \sum_{n\geq 1} \frac{2V(\Gamma)}{\pi(R^n - 1)} \leq 1 \right\}.$$

Remark The quantity $V(\Gamma)$ is often easy to calculate or at least to estimate. Let us mention two examples:

- If Γ is convex, then obviously $V(\Gamma) = 2\pi$. Note that, in this case, we get again the second part of Theorem 1.
- If Γ is a finite union of polygonal arcs, then the calculation of $V(\Gamma)$ is particulary simple. In the case of nonconvex such Γ, the angular approach gives a better estimate than the general estimate of Theorem 1.

4 Behavior of $B(\overline{\Omega_r^K})$ when $r \to \infty$

To simplify the notation when we consider $\overline{\Omega_r^K}$ as a continuum, we will write Ω_r^K. Thus, we write $F_{\Omega_r^K,n}$ instead of $F_{\overline{\Omega_r^K},n}$, the level sets $\Omega_R^{\Omega_r^K}$ instead of $\Omega_R^{\overline{\Omega_r^K}}$, $\Phi_{\Omega_r^K}$ instead of $\Phi_{\overline{\Omega_r^K}}$, and $B(\Omega_r^K)$ instead of $B(\overline{\Omega_r^K})$.

Let $K \subset \mathbb{C}$ be a regular compact set, $r > 1$. So $\Phi_{\Omega_r^K} = r^{-1}\Phi_K$, and therefore

$$F_{\Omega_r^K,n}(z) = \left(\frac{\Phi_K(z)}{r^n}\right)^n + \frac{E_n(z)}{r^n}, \quad \forall z \in \mathbb{C}\backslash K,$$

or, in the $w = r^{-1}\Phi_K(z)$ coordinate,

$$F_{\Omega_r^K,n}(\Phi_K^{-1}(rw)) = w^n + \frac{E_n(\Phi_K^{-1}(rw))}{r^n}, \quad \forall |w| > 1. \tag{6}$$

Remember that ([20], p. 43), for all $1 < r_0 < r$, we have the uniform estimate

$$|E_n(z)| \le \frac{r_0^n \,\mathtt{length}(\partial\Omega_{r_0}^K)}{2\pi \,\mathtt{dist}\left(\partial\Omega_{r_0}^K, \partial\Omega_r^K\right)}, \quad \forall z \in \mathbb{C}\backslash\Omega_r^K,\ n \in \mathbb{N}, \tag{7}$$

where $\mathtt{length}(\partial\Omega_{r_0}^K)$ denotes the arclength of the level line $\{|\Phi_K| = r_0\}$.

Now let $0 < r_1 < 1 < r$ and $R > 1$. Consider

$$f_{r_1}(z) = -r_1 + \left(\frac{1}{r_1} - r_1\right) \sum_{n\ge 1} \frac{r_1^n}{R^n} F_{\Omega_r^K,n}(z).$$

Then $f_{r_1} \in \mathscr{O}(\Omega_{R/r_1}^{\Omega_r^K}) \subset \mathscr{O}(\overline{\Omega_R^{\Omega_r^K}})$, and we have the following:

Lemma 1 *Let $r_0' > r_0 > 1$. There exists $M > 0$ such that*

$$\sup_{z\in\partial\Omega_R^{\Omega_r^K}} |f_{r_1}(z)| \le 1 + M\left(\frac{1}{r_1} - r_1\right)\cdot\frac{1}{r}, \quad \forall r > r_0',\ 0 < r_1 < 1.$$

Proof The point $z \in \partial\Omega_R^{\Omega_r^K}$ if and only if $w = r^{-1}\Phi_K(z) = Re^{i\theta}$. So, using (6),

$$\begin{aligned}
f_{r_1}(\Phi_K^{-1}(rw)) &= -r_1 + \left(\frac{1}{r_1} - r_1\right)\sum_{n\geq 1}\frac{r_1^n}{R^n}\left(w^n + \frac{E_n(\Phi_K^{-1}(rw))}{r^n}\right)\\
&= -r_1 + \left(\frac{1}{r_1} - r_1\right)\sum_{n\geq 1} r_1^n e^{in\theta}\\
&\quad + \left(\frac{1}{r_1} - r_1\right)\sum_{n\geq 1}\frac{r_1^n}{R^n r^n}\cdot E_n(\Phi_K^{-1}(rw))\\
&= \frac{e^{i\theta} - r_1}{1 - r_1 e^{i\theta}} + \left(\frac{1}{r_1} - r_1\right)\sum_{n\geq 1}\frac{r_1^n}{R^n r^n}\cdot E_n(\Phi_K^{-1}(rw))\\
&= \quad (A) \ + (B).
\end{aligned}$$

Since $r_1 < 1$,

$$\|(A)\| = \sup_{\theta\in[0,2\pi]}\left|\frac{e^{i\theta} - r_1}{1 - r_1 e^{i\theta}}\right| \leq 1.$$

For the second term, (7) gives for all $0 < r_1 < 1 < r_0 < r_0^{'} < r$, $R > 1$:

$$\begin{aligned}
\|(B)\| &\leq \left(\frac{1}{r_1} - r_1\right)\sum_{n\geq 1}\frac{r_1^n}{R^n r^n}\cdot\sup_{|w|=1}|E_n(\Phi_K^{-1}(rw))|\\
&\leq \left(\frac{1}{r_1} - r_1\right)\sum_{n\geq 1}\frac{r_1^n r_0^n}{R^n r^n}\cdot\frac{\mathtt{length}(\partial\Omega_{r_0}^K)}{2\pi\,\mathtt{dist}(\partial\Omega_{r_0}^K, \partial\Omega_r^K)}\\
&\leq \left(\frac{1}{r_1} - r_1\right)\cdot\frac{r_1 r_0}{Rr}\cdot\frac{\mathtt{length}(\partial\Omega_{r_0}^K)}{2\pi\,\mathtt{dist}(\partial\Omega_{r_0}^K, \partial\Omega_{r_0^{'}}^K)}\sum_{n\geq 0}\frac{r_1^n r_0^n}{R^n r^n}\\
&\leq \frac{1}{r}\left(\frac{1}{r_1} - r_1\right)\cdot\frac{r_1 r_0}{R}\cdot\frac{\mathtt{length}(\partial\Omega_{r_0}^K)}{2\pi\,\mathtt{dist}(\partial\Omega_{r_0}^K, \partial\Omega_{r_0^{'}}^K)}\cdot\frac{1}{1 - \frac{r_1 r_0}{rR}}\\
&\leq \frac{1}{r}\left(\frac{1}{r_1} - r_1\right)\cdot\frac{r_1 r_0}{R}\cdot\frac{\mathtt{length}(\partial\Omega_{r_0}^K)}{2\pi\,\mathtt{dist}\left(\partial\Omega_{r_0}^K, \partial\Omega_{r_0^{'}}^K\right)}\cdot\frac{1}{1 - \frac{r_0}{r_0^{'}R}}\\
&\leq \frac{1}{r}\left(\frac{1}{r_1} - r_1\right)\cdot\frac{r_0 r_0^{'}}{r_0^{'} - r_0}\cdot\frac{\mathtt{length}(\partial\Omega_{r_0}^K)}{2\pi\,\mathtt{dist}(\partial\Omega_{r_0}^K, \partial\Omega_{r_0^{'}}^K)}\\
&= \frac{M}{r}\left(\frac{1}{r_1} - r_1\right),
\end{aligned}$$

where $M > 0$ depends only on r_0 and $r_0^{'}$. Then

$$\sup_{z\in\partial\Omega_R^{\Omega_r^K}} |f_{r_1}(z)| := \|f_{r_1}\|_{\Omega_R^{\Omega_r^K}} \leq \|(A)\| + \|(B)\| \leq 1 + M\left(\frac{1}{r_1} - r_1\right)\cdot\frac{1}{r}$$

for all r, r_1, R such that $0 < r_1 < 1,\ r > r_0'$, and $R > 1$. □

Suppose now $R > B(\Omega_r^K)$. The function $f_{r_1}/\|f_{r_1}\|_{\Omega_R^{\Omega_r^K}}$ is holomorphic on $\Omega_R^{\Omega_r^K}$ with values in $\mathbb{D}$; hence

$$r_1 + \left(\frac{1}{r_1} - r_1\right)\sum_{n\geq 1}\frac{r_1^n}{R^n}\|F_{\Omega_r^K,n}\|_{\Omega_r^K} \leq \|f_{r_1}\|_{\Omega_R^{\Omega_r^K}} \leq 1 + M\left(\frac{1}{r_1} - r_1\right)\cdot\frac{1}{r};$$

i.e.,

$$\left(\frac{1}{r_1} - r_1\right)\sum_{n\geq 1}\frac{r_1^n}{R^n}\|F_{\Omega_r^K,n}\|_{\Omega_r^K} \leq 1 - r_1 + M\left(\frac{1}{r_1} - r_1\right)\cdot\frac{1}{r}.$$

Therefore

$$\sum_{n\geq 1}\frac{r_1^n}{R^n}\|F_{\Omega_r^K,n}\|_{\Omega_r^K} \leq \frac{r_1}{1+r_1} + \frac{M}{r}.$$

With (6) and (7), we can write for any r with $1 < r_0 < r$,

$$\begin{aligned}\|F_{\Omega_r^K,n}\|_{\Omega_r^K} &\geq 1 - \frac{\|E_n(\Phi_K^{-1}(rw))\|_{\partial\mathbb{D}}}{r^n}\\ &\geq 1 - \frac{r_0^n}{r^n}\cdot\frac{\mathtt{length}(\partial\Omega_{r_0}^K)}{2\pi\,\mathtt{dist}(\partial\Omega_{r_0}^K, \partial\Omega_r^K)} := 1 - \frac{r_0^n}{r^n}\cdot M'(r).\end{aligned}$$

In the same way, we have also an upper bound for $\|F_{\Omega_r,n}\|_{\Omega_r^K}$. So finally,

$$1 - \frac{r_0^n}{r^n}\cdot M'(r) \leq \|F_{\Omega_r^K,n}\|_{\Omega_r^K} \leq 1 + \frac{r_0^n}{r^n}\cdot M'(r). \tag{8}$$

Let r be such that $1 < r_0 < r_0' < r$. Then for any $R > B(\Omega_r^K)$, we have

$$\sum_{n\geq 1}\frac{r_1^n}{R^n}\left(1 - \frac{r_0^n}{r^n}\cdot M'(r)\right) \leq \sum_{n\geq 1}\frac{r_1^n}{R^n}\|F_{\Omega_r^K,n}\|_{\Omega_r^K} \leq \frac{r_1}{1+r_1} + \frac{M}{r}$$

for all $0 < r_1 < 1$. Thus

$$\begin{aligned}\frac{r_1}{R-r_1} &\le \frac{r_1}{1+r_1} + \frac{M}{r} + M'(r)\cdot\sum_{n\ge 1}\left(\frac{r_0 r_1}{Rr}\right)^n\\ &= \frac{r_1}{1+r_1} + \frac{M}{r} + M'(r)\cdot\frac{r_0 r_1}{Rr - r_0 r_1}\end{aligned}$$

for all $0 < r_1 < 1$. Now letting $r_1 \to 1$, we get

$$\frac{1}{R-1} \le \frac{1}{2} + \frac{M}{r} + M'(r)\cdot\frac{r_0}{Rr - r_0} = \frac{1}{2} + \varepsilon(r),$$

where $\lim_{r\to\infty}\varepsilon(r) = 0$ uniformly with respect to R larger than one.
Hence for $R > B(\Omega_r^K)$, we have

$$R \ge 3 - \varepsilon'(r),$$

and so

$$B(\Omega_r^K) \ge 3 - \varepsilon'(r),$$

where $\lim_{r\to\infty}\varepsilon'(r) = 0$. Note that, in particular,

$$\liminf_{r\to+\infty} B(\Omega_r^K) \ge 3. \tag{9}$$

Now let us look for an upper bound for $B(\Omega_r^K)$ when r is large. First observe that $(\Omega_r^K, (\Omega_{rR}^K)_{R>1}, (F_{\Omega_r^K,n})_n)$ is the condenser associated with Ω_r^K if $(K, (\Omega_r^K)_{r>1}, (F_{K,n})_n)$ is the condenser associated with K.

Let $f = \sum_n a_n F_{\Omega_r^K,n} \in \mathscr{O}(\Omega_R^{\Omega_r^K}) = \mathscr{O}(\Omega_{rR}^K)$ be a function such that $f(\Omega_{rR}^K) \subset \mathbb{D}$. The proof of Proposition 1 on the annulus $A(\frac{1}{r}, R)$ leads to

$$|a_n| \le \frac{2\mathrm{re}(a_0)}{R^n - r^{-n}}, \quad \forall n \in \mathbb{N}.$$

Assuming again $a_0 \ge 0$, the Bohr phenomenon will occur if

$$a_0 + 2(1-a_0)\sum_{n\ge 1}\frac{\left\|F_{\Omega_r^K,n}\right\|_{\Omega_r^K}}{R^n - r^{-n}} \le 1.$$

This implies

$$2\sum_{n\ge 1}\frac{\left\|F_{\Omega_r^K,n}\right\|_{\Omega_r^K}}{R^n - r^{-n}} \le 1.$$

From (8), it follows that $\left\| F_{\Omega_r^K,n} \right\|_{\Omega_r^K} \le 1 + \frac{r_0^n}{r^n} \cdot M'(r)$, and thus

$$2 \sum_{n \ge 1} \frac{\left\| F_{\Omega_r^K,n} \right\|_{\Omega_r^K}}{R^n - r^{-n}} \le 2 \sum_{n \ge 1} \frac{1 + r_0^n r^{-n} \cdot M'(r)}{R^n - r^{-n}}.$$

If $r > r_0 > 1$ is large enough, there exists a unique $R(r) > 1$ such that

$$2 \sum_{n \ge 1} \frac{1 + r_0^n r^{-n} \cdot M'(r)}{R(r)^n - r^{-n}} = 1.$$

If $R_\infty := \lim_{r \to \infty} R(r)$, we must have

$$2 \sum_{n \ge 1} R_\infty^{-n} = 1;$$

that is, $R_\infty = 3$. On the other hand, $R(r) \ge B(\Omega_r^K)$, which implies, for r large enough,

$$B(\Omega_r^K) \le 3 + \varepsilon(r). \tag{10}$$

Formulas (9) and (10) give

$$\lim_{r \to \infty} B(\Omega_r^K) = 3.$$

□

5 The Positive Class of Condenser and the Proof of Theorem 3

Let us consider a special class of Faber–Green condenser.

Definition 1 We say that K is in the positive class of Faber–Green condenser or positive class, if we have for the continuum K:

$$z = \Phi_K^{-1}(w) = \frac{w}{\gamma} + \beta_0 + \sum_{j=1}^{\infty} \frac{\beta_j}{w^j}, \text{ with } \beta_j \ge 0, \forall\, j \ge 1.$$

A continuum K with this property has been considered by Curtiss and Pommerenke [7,18]. All the disks, all the lines, all the ellipses, and all the m-cusped hypocycloids are in this class. If K is the closure of an analytic Jordan curve, then K being in the positive class implies that K is a starlike domain [7,18]. This class seems to be of some interest because we can evaluate precisely the sup-norm in K of the Faber polynomials.

Remark Clearly, the Bohr radius is invariant by the automorphisms of the complex plane. Hence, Theorem 3 is valid not only for the positive class but also for the pseudo-positive class: the orbit of the positive class by this group of automorphisms. Now the pseudo-positive class contains all the examples of continui considered by Eiermann and Varga [10].

5.1 Proof of Theorem 3

Consider, for r_1 close to 1, the family of functions

$$G_{r_1} = \frac{f_{r_1}}{||f_{r_1}||_{\Omega_3^K}} \text{ with } f_{r_1}(z) = -r_1 + \left(\frac{1}{r_1} - r_1\right) \sum_{n\geq 1} \frac{r_1^n}{3^n} F_{K,n}(z).$$

Clearly, G_{r_1} is holomorphic in $\overline{\Omega_3^K}$ and $\|G_{r_1}\|_{\Omega_3^K} \leq 1$. Suppose we have the Bohr property for G_{r_1}; i.e, $G_{r_1} := \sum a_n F_{K,n}$ and $\sum |a_n| \cdot \|F_{K,n}\|_K \leq 1$. This last inequality implies

$$r_1 + \left(\frac{1}{r_1} - r_1\right) \sum_{n\geq 1} \frac{r_1^n}{3^n} \|F_{K,n}\|_K \leq \sup_{|w|=3} \left| f_{r_1}(\phi_K^{-1}(w)) \right|. \tag{11}$$

• Estimate of $|f_{r_1}(\phi_K^{-1}(w))|$:

In w-coordinate, $F_{K,n}(\phi_K^{-1}(w)) = w^n + \sum_{j\geq 1} \frac{\alpha_j^{(n)}}{w^j}$ with $\sum_{j\geq 1} \frac{\alpha_j^{(n)}}{w^j}$ converges absolutely and uniformly on any compact set of $\{|w| > 1\}$. For $w = 3e^{i\theta}$, we have the following inequality:

$$\begin{aligned} |f_{r_1}(\phi_K^{-1}(w))| &\leq \left| \frac{e^{i\theta} - r_1}{1 - r_1 e^{i\theta}} + \left(\frac{1}{r_1} - r_1\right) \sum_{n\geq 1} \frac{r_1^n}{3^n} \left(\sum_{j\geq 1} \frac{\alpha_j^{(n)}}{3^j e^{i\theta j}} \right) \right| \\ &\leq 1 + \left(\frac{1}{r_1} - r_1\right) \sum_{n\geq 1} \frac{r_1^n}{3^n} \left(\sum_{j\geq 1} \frac{\alpha_j^{(n)}}{3^j} \right). \end{aligned}$$

To get the previous inequality, we use two facts: the double series converges absolutely, and $\alpha_j^{(n)}$ are nonnegative reals if K is in the positive class. This is the crucial result of [7]. • Estimate of $\|F_{K,n}\|_K$:

Even in the positive class, $\sum_{j\geq 1} \frac{\alpha_j^{(n)}}{w^j}$ is no longer absolutely convergent on $\{|w| = 1\}$. We therefore have to modify the previous approach. Anyway, we have the following equality:

$$\|F_{K,n}\|_K = \lim_{r\to 1^+} \sup_{|w|=r} \left| \frac{F_{K,n}(\phi_K^{-1}(w))}{w^n} \right|.$$

The term on the right is equal to $\lim_{r\to 1^+}(1+\frac{1}{r^n}\sup_{|w|=r}|\sum_{j\geq 1}\frac{\alpha_j^{(n)}}{w^j}|)$ because $\alpha_j^{(n)}$ are nonnegative reals by the theorem of Curtiss.[1] Finally,

$$||F_{K,n}||_K = 1 + \lim_{r\to 1^+}\sup_{|w|=r}\left|\sum_{j\geq 1}\frac{\alpha_j^{(n)}}{w^j}\right| \geq 1 + \sum_{j\geq 1}\frac{\alpha_j^{(n)}}{r^j}$$

for all $r > 1$. Choose $1 < r_0 < 3$, and suppose (11) is valid. Then we must have

$$\left(\frac{1}{r_1}-r_1\right)\sum_{n\geq 1}\frac{r_1^n}{3^n}\left(1+\sum_{j\geq 1}\frac{\alpha_j^{(n)}}{r_0^j}\right) \leq 1 - r_1 + \left(\frac{1}{r_1}-r_1\right)\sum_{n\geq 1}\frac{r_1^n}{3^n}\left(\sum_{j\geq 1}\frac{\alpha_j^{(n)}}{3^j}\right),$$

which implies the inequality

$$\frac{r_1}{3-r_1} \leq \frac{r_1}{1+r_1} + \sum_{n\geq 1}\frac{r_1^n}{3^n}\left(\sum_{j\geq 1}\alpha_j^{(n)}\left(\frac{1}{3^j}-\frac{1}{r_0^j}\right)\right).$$

If r_1 tends to 1, we obtain

$$0 \leq \sum_{n\geq 1}\frac{1}{3^n}\left(\sum_{j\geq 1}\alpha_j^{(n)}\left(\frac{1}{3^j}-\frac{1}{r_0^j}\right)\right).$$

Suppose K is not a disk. Then one of the $\alpha_j^{(n)}$ is strictly positive, and the last inequality is not true. We have proved: if K is not the disk, then $B(K) > 3$. We know by the classical Bohr theorem that the Bohr radius of disks is 3. The proof of Theorem 3 is now complete. □

As H_m is in the positive class, we have the corollary:

Corollary 3 *If H_m is the m-cusped hypocycloid, then $B(H_m) > 3$.*

6 Concluding Remarks

(1) We are not able to prove Theorem 3 in a larger class than the positive one. We suspect that the theorem is true at least for starlike domains, but the proof seems delicate. Furthermore, it could be very interesting to produce a counter-example for general continuum.

(2) For all convex continuum K, is it true or not that $B(K) \leq B([-1, 1])$? Theorem 1 gives $B(K) \lesssim 5.26$, and in [16] we proved that $B([-1, 1]) \simeq 5.1284$. The methods of Theorem 1 are far from giving such inequality.

[1] The hypothesis that K is in the positive class is crucial here to obtain a good lower bound for $||F_{K,n}||_K$.

(3) In general, it seems hopeless to compute the exact value of $B(K)$ for an arbitrary continuum K. But it seems possible to compute the exact value of the Bohr radius for the 3, 4-cusped hypocycloids H_3, H_4.

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