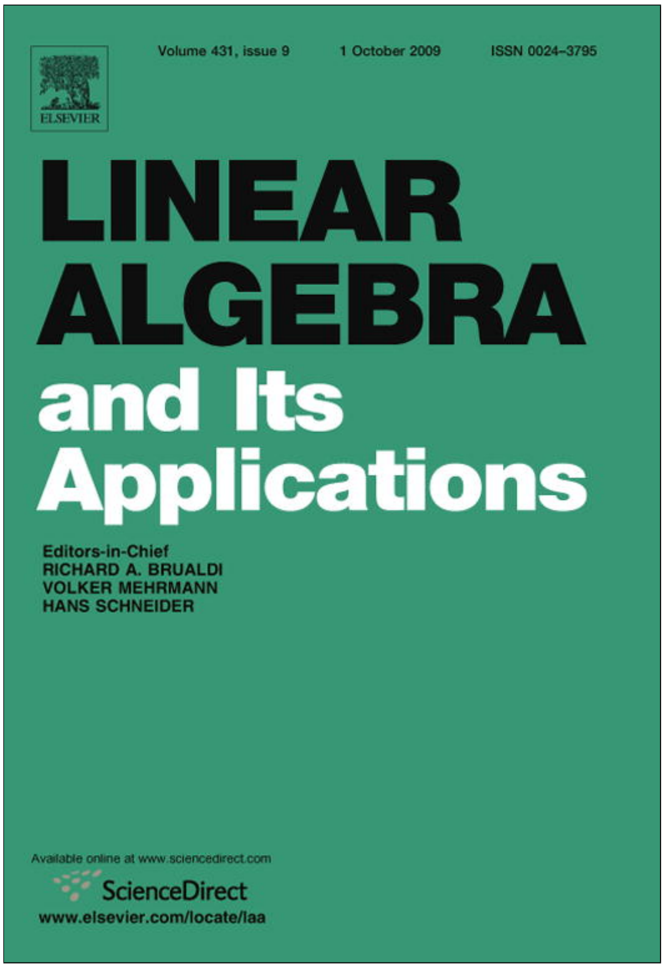
Volume 431, issue 9
1 October 2009
ISSN 0024-3795
LINEAR ALGEBRA and Its Applications
Editors-in-Chief
RICHARD A. BRUALDI
VOLKER MEHRMANN
HANS SCHNEIDER
Available online at www.sciencedirect.com
ScienceDirect
www.elsevier.com/locate/laa

Linear Algebra and its Applications 431 (2009) 1616–1619

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

One more simple proof of the Craig–Sakamoto theorem

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ARTICLE INFO

Article history:
Received 26 May 2008
Accepted 25 May 2009
Available online 9 July 2009

Submitted by C.K. Li

AMS classification:
15A15
15A18
15A24
15A48

Keywords:
Craig–Sakamoto's theorem
Linear and bilinear algebra

ABSTRACT

We give one more elementary proof of the Craig–Sakamoto's theorem: given $A, B \in \mathcal{S}_n(\mathbb{R})$ such that $\det(I_n - xA - yB) = \det(I_n - xA)\det(I_n - yB),\ \forall x, y \in \mathbb{R}$; then $AB = 0$.

1. Introduction

The Craig–Sakamoto theorem asserts that

Given $A, B \in \mathcal{S}_n(\mathbb{R})$ *then* $\det(I_n - xA - yB) = \det(I_n - xA)\det(I_n - yB),\ \forall x, y \in \mathbb{R}$ *if and only if* $AB = 0$.

For an historical viewpoint of this result coming from statistical-probabilities, the interested reader can look at [1,2,5,6]; since his first statement it has inspired many proofs (see [3,4,8–10], and the references listed in the previous papers). The purpose of this note is to give one more new (let us hope...) proof of this theorem using the "elementary" machinery of linear and bilinear algebra.

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0024-3795/$ - see front matter © 2009 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2009.05.031

2. The proof

The key of the proof is the following:

Property ($\bigstar$). *Let $A, B \in \mathcal{S}_n(\mathbb{R}) \setminus \{O_{M_n(\mathbb{R})}\}$ such that*

$$\det(I_n - xA - yB) = \det(I_n - xA)\det(I_n - yB), \quad \forall x, y \in \mathbb{R}.$$

Then B (or A, the same by symmetry) admits a nonzero eigenvalue λ such that

$$\ker(A) \cap \ker\left(I_n - \lambda^{-1}B\right) \neq \{0\}.$$

First, we are going to show how this property implies the Craig–Sakamoto's theorem.

2.1. Proof of the Craig–Sakamoto's theorem

We do it by induction on the size n of the matrices. For $n = 1$, the Craig–Sakamoto's theorem is clear, so let $n \geqslant 2$ and suppose it true up to rank $n-1$; let $A, B \in \mathcal{S}_n(\mathbb{R}) \setminus \{O_{M_n(\mathbb{R})}\}$ (we exclude the trivial case where one of the two matrices is zero) such that

$$\det(I_n - xA - yB) = \det(I_n - xA)\det(I_n - yB), \quad \forall x, y \in \mathbb{R}.$$

We have to prove that $AB = 0$.

Because of property ($\bigstar$) there exists a nonzero eigenvalue of B, say λ, such that $\ker(A) \cap \ker(I - \lambda B) \neq \{0\}$; so choose $e_\lambda \in \ker(A) \cap \ker(I - \lambda B) \setminus \{0\}$ and consider an orthogonal basis $\mathcal{B}$ of $\mathbb{R}^n$ with first term e_λ. Because of the choice of e_λ, the symmetrics matrices of A and B in the basis $\mathcal{B}$ have the respective shapes $\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & A' \end{array}\right)$ and $\left(\begin{array}{c|c} \lambda & 0 \\ \hline 0 & B' \end{array}\right)$ where the matrices A' and B' belong to $\mathcal{S}_{n-1}(\mathbb{R})$. An elementary computation gives

$$\det(I_n - xA - yB) = (1 - \lambda y)\det\left(I_{n-1} - xA' - yB'\right),$$
$$\det(I_n - xA)\det(I_n - yB) = (1 - \lambda y)\det\left(I_{n-1} - xA'\right)\det\left(I_{n-1} - yB'\right),$$

so

$$\det\left(I_{n-1} - xA' - yB'\right) = \det\left(I_{n-1} - xA'\right)\det\left(I_{n-1} - yB'\right), \quad \forall\, x, y \in \mathbb{R}.$$

Then, by the induction hypothesis $A'B' = 0$, and we have

$$AB = P^{-1}\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & A' \end{array}\right)\left(\begin{array}{c|c} \lambda & 0 \\ \hline 0 & B' \end{array}\right)P = P^{-1}\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & A'B' \end{array}\right)P = 0$$

and we are done. $\square$

2.2. Proof of property ($\bigstar$)

For it, we first need two lemmas.

Lemma 1. *Let $U = ((u_{ij}))$ a symmetric positive semi-definite matrix; if a diagonal coefficient u_{ii} $(1 \leqslant i \leqslant n)$ is equal to zero, then $u_{ij} = u_{ji} = 0$ for all $1 \leqslant j \leqslant n$.*

Proof of the lemma 1. This is classical and elementary (see [7, problem 20.1]) but for this demonstration to be self-contained we include the proof: let U such a matrix with $u_{ii} = 0$ and, by contradiction, suppose that there exists a coefficient $u_{ji} \neq 0$. Let $X_t = (x_k)_1^n$ the vector where $x_j = 1, x_i = tu_{ji}$, $t \in \mathbb{R}$ and where the other components are zero, then ${}^tX_tUX_t = u_{jj} + 2tu_{ji}^2$ change sign when t runs through $\mathbb{R}$ which is impossible. $\square$

Lemma 2. *Let $U, V \in \mathcal{S}_n(\mathbb{R})$. Suppose $U \geqslant 0$ and*

$$\forall\, t \in \mathbb{R}: \quad \det(U - tV) = 0.$$

Then

$$\ker(V) \cap \ker(U) \neq \{0\}.$$

Remark. Note that this lemma is obviously false without some symmetry hypothesis; for example, consider $U = \left(\begin{array}{c|c} I_{n-1} & 0 \\ \hline 0 & 0 \end{array}\right)$, $V = \left(\begin{array}{c|c} 0 & 1 \\ \hline O_{n-1} & 0 \end{array}\right)$.

Proof of the lemma 2. V is diagonalisable in an orthonormal basis:

$$\exists P \in \mathcal{O}_n(\mathbb{R}): \quad PV{}^tP = \mathrm{Diag}(\lambda_1, \ldots, \lambda_r, 0, \ldots, 0) = \left(\begin{array}{c|c} D_r & 0 \\ \hline 0 & 0 \end{array}\right), \quad \lambda_i \neq 0$$

(note that $r < n$ because $\det(U) = 0$) then we have:

$$PU{}^tP - tPV{}^tP = \left(\begin{array}{c|c} U_1 & U_2 \\ \hline {}^tU_2 & U_3 \end{array}\right) - t\left(\begin{array}{c|c} D_r & 0 \\ \hline 0 & 0 \end{array}\right) = \left(\begin{array}{c|c} U_1 - tD_r & U_2 \\ \hline {}^tU_2 & U_3 \end{array}\right).$$

And in fact, it is possible to choose P (at the expense of changing the $n - r$ lasts basis vectors) to have also U_3 diagonal: precisely, let $Q \in \mathcal{O}_{n-r}$ so that $QU_3{}^tQ$ is diagonal, then, in the new basis associated to the orthogonal matrix $\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & Q \end{array}\right)$ our matrix $PU{}^tP - tPV{}^tP$ is

$$\left(\begin{array}{c|c} U_1 - tD_r & U_2' \\ \hline {}^tU_2' & QU_3{}^tQ \end{array}\right)$$

and is in the required shape.

So let us consider such a P, because of the hypothesis, the polynomial

$$\mathbb{R} \ni t \mapsto \det(PU{}^tP - tD) = 0$$

is nul; the coefficient of t^r being (up to a sign) $\lambda_1 \ldots \lambda_r \det(U_3)$, we have $\det(U_3) = 0$. Now, $\det(U_3) = 0$ and U_3 diagonal implies that the positive symmetric matrix $PU{}^tP$ admits a diagonal element equal to zero, say the i-th ($i \in \{r+1, \ldots, n\}$): then by lemma 1 the i-th column in $PU{}^tP$ is also null, e.g. $PU{}^tPe_i = 0$; but, because $i \in \{r+1, \ldots, n\}$ we have also

$$PV{}^tPe_i = \left(\begin{array}{c|c} D_r & 0 \\ \hline 0 & 0 \end{array}\right) e_i = 0.$$

Consequently $U{}^tPe_i = V{}^tPe_i = 0$, and finally ${}^tPe_i \in \ker(U) \cap \ker(V)$. What we had to prove. $\square$

Now the proof of the property (★) is easy:

Proof of property (★). Let λ be a non zero eigenvalue of B, we have

$$\det\left(I_n - \lambda^{-1}B - xA\right) = 0, \quad \forall x \in \mathbb{R}$$

and we will be in the case of the lemma 2 with $V = A$ and $U = I_n - \lambda^{-1}B$ who will be positive semi-definite if we choose λ as the greatest nonzero eigenvalue of B. $\square$

Acknowledgments

It is our pleasure to thank our colleague Jean Baptiste Hiriart-Urruty (University of Toulouse) who first told us about the Craig–Sakamoto's theorem.

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