

Convexity of sublevel sets of plurisubharmonic extremal functions

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Abstract. Let X be a convex domain in \mathbb{C}^n and let E be a convex subset of X . The relative extremal function $u_{E,X}$ for E in X is the supremum of the class of plurisubharmonic functions $v < 0$ on X with $v \leq -1$ on E . We show that if E is either open or compact, then the sublevel sets of $u_{E,X}$ are convex. The proof uses the theory of envelopes of disc functionals and a new result on Blaschke products.

1. Introduction. If X is a complex manifold and E is a subset of X , then the *relative extremal function* for E in X is defined as

$$u_{E,X} = \sup\{v \in \text{PSH}(X) : v \leq -\chi_E\},$$

where $\text{PSH}(X)$ denotes the class of plurisubharmonic functions on X and χ_E is the characteristic function of E , i.e., $\chi_E = 1$ on E and $\chi_E = 0$ on $X \setminus E$. The main result of this paper is the following theorem.

THEOREM 1. *Let X be a convex domain in \mathbb{C}^n and let E be a convex subset of X . If E is either open or compact, then the sublevel sets*

$$\{\zeta \in X : u_{E,X}(\zeta) < \alpha\}$$

of the relative extremal function for E in X are convex for all $\alpha \in [-1, 0]$.

Convexity of sublevel sets of various extremal functions has been studied by several authors. If X is a domain in \mathbb{C} and $E \subset X$ is compact, then $u_{E,X}$ is harmonic in $X \setminus E$, and satisfies the boundary conditions $u_{E,X} = 0$ on ∂X and $u_{E,X} = -1$ on ∂E . The convexity of the level sets of the solution of the boundary value problem $\Delta u = 0$ on $X \setminus E$, $u = 0$ on ∂X , and $u = -1$ on

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∂E , where X is a bounded convex domain in \mathbb{R}^n and E is a compact convex subset of X , has been proved by several authors. See Papadimitrakis [1992], Rosay and Rudin [1989], and the references therein.

The pluricomplex Green function G of a domain X in \mathbb{C}^n with a pole at p in X is the supremum of the class of negative plurisubharmonic functions v on X such that $v - \log |\cdot - p|$ is bounded above near p . If X is bounded and convex, then the sublevel sets of G are convex. See Momm [1994].

The pluricomplex Green function of a subset E of \mathbb{C}^n with a logarithmic pole at infinity is defined by the formula $V_E = \sup\{v \in \mathcal{L} : v \leq 0 \text{ on } E\}$, where \mathcal{L} denotes the set of plurisubharmonic functions v on \mathbb{C}^n such that $v - \log^+ |\cdot|$ is bounded above. Lempert has proved that the sublevel sets of V_E are convex if E is convex, compact, and non-pluripolar. See Momm [1996].

The proof of Theorem 1 is based on the theory of envelopes of disc functionals, which we will discuss in Section 2, and the following result on Blaschke products, which is of independent interest and will be proved in Section 3.

THEOREM 2. *Let S_1 and S_2 be Borel subsets of the unit circle with $\lambda(S_1), \lambda(S_2) > \beta$. Then there are Blaschke products φ_1 and φ_2 preserving the origin with*

$$\lambda(\check{\varphi}_1^{-1}(S_1) \cap \check{\varphi}_2^{-1}(S_2)) > \beta.$$

Here, λ denotes the arc length measure on the unit circle \mathbb{T} . Also, for a bounded holomorphic function φ on the unit disc \mathbb{D} in \mathbb{C} , we denote by $\check{\varphi}$ the Borel measurable function on \mathbb{T} whose value at almost every $x \in \mathbb{T}$ is the non-tangential limit of φ at x .

Roughly speaking, Theorem 2 states that two Borel sets of equal length on \mathbb{T} can almost be expressed as the images of a single Borel set on \mathbb{T} by two Blaschke products preserving the origin.

The preprint of Edigarian and Poletsky [1997], which came to our attention after we completed this paper, essentially contains a proof of Theorem 2, very similar to ours.

After this paper was submitted for publication, we received a preprint from J. Duval, entitled *Sur la fonction extrémale plurisousharmonique relative à deux convexes*, containing a different proof of Theorem 1.

2. Extremal functions as envelopes of disc functionals. Let X be a domain in \mathbb{C}^n . An *analytic disc* in X is a holomorphic map $f : \mathbb{D} \rightarrow X$. We say that f is *closed* if f can be extended to a holomorphic map on some neighbourhood of the closure $\overline{\mathbb{D}}$ of \mathbb{D} with values in X , and we say that f is *bounded* if $f(\mathbb{D})$ is relatively compact in X . We let \mathcal{A}_X denote the set of

all closed analytic discs in X and \mathcal{B}_X denote the set of all bounded analytic discs in X . Clearly, $\mathcal{A}_X \subset \mathcal{B}_X$.

If $f \in \mathcal{B}_X$, then f has a non-tangential limit at almost every point in \mathbb{T} , and the limits define a Borel measurable function $\check{f} : \mathbb{T} \rightarrow X$. If $f \in \mathcal{A}_X$, then $\check{f} = g|_{\mathbb{T}}$, where g is a holomorphic extension of f to some neighbourhood of $\overline{\mathbb{D}}$.

For every upper semicontinuous function $\varphi : X \rightarrow \mathbb{R} \cup \{-\infty\}$ we define the *disc functional* $H_\varphi : \mathcal{B}_X \rightarrow \mathbb{R} \cup \{-\infty\}$ by the formula

$$H_\varphi(f) = \frac{1}{2\pi} \int_{\mathbb{T}} \varphi \circ \check{f} \, d\lambda, \quad f \in \mathcal{B}_X.$$

If $v \in \text{PSH}(X)$, $v \leq \varphi$, $f \in \mathcal{B}_X$, and $f(0) = \zeta$, then for $r \in (0, 1)$,

$$(2.1) \quad v(\zeta) \leq \frac{1}{2\pi} \int_0^{2\pi} v \circ f(re^{i\theta}) \, d\theta \leq \frac{1}{2\pi} \int_{\mathbb{T}} v \circ \check{f} \, d\lambda \leq H_\varphi(f).$$

The *envelope* of the functional H_φ is the function $EH_\varphi : X \rightarrow \mathbb{R} \cup \{-\infty\}$ defined by the formula

$$EH_\varphi(\zeta) = \inf\{H_\varphi(f) : f \in \mathcal{A}_X, f(0) = \zeta\}, \quad \zeta \in X.$$

The inequality (2.1) implies that $\sup\{v \in \text{PSH}(X) : v \leq \varphi\} \leq EH_\varphi$, and by taking for f the constant ζ , we see that $EH_\varphi \leq \varphi$. By a theorem of Poletsky [1991, 1993], EH_φ is plurisubharmonic for every upper semicontinuous function φ , so we have

$$\sup\{v \in \text{PSH}(X) : v \leq \varphi\} = EH_\varphi.$$

See Lárusson and Sigurdsson [1996] for an alternative proof of Poletsky's theorem with generalizations to manifolds.

If E is an open subset of X , then $\varphi = -\chi_E$ is upper semicontinuous and

$$\int_{\mathbb{T}} \chi_E \circ \check{f} \, d\lambda = \lambda(\check{f}^{-1}(E)),$$

so by (2.1) and Poletsky's theorem,

$$(2.2) \quad \begin{aligned} u_{E,X}(\zeta) &= -\frac{1}{2\pi} \sup\{\lambda(\check{f}^{-1}(E)) : f \in \mathcal{B}_X, f(0) = \zeta\} \\ &= -\frac{1}{2\pi} \sup\{\lambda(\check{f}^{-1}(E)) : f \in \mathcal{A}_X, f(0) = \zeta\}, \quad \zeta \in X. \end{aligned}$$

In particular, $u_{E,X}$ is plurisubharmonic on X . (When E is not open, $u_{E,X}$ may fail to be upper semicontinuous.)

The proof of Theorem 1 is based on the formula (2.2) for $u_{E,X}$. Namely, assume that X is a convex domain in \mathbb{C}^n and E is an open convex subset of X . Take two points ζ_1 and ζ_2 in X such that $u_{E,X}(\zeta_j) < \alpha$, $j = 1, 2$. By

(2.2), there are holomorphic maps $f_j : U \rightarrow X$, $j = 1, 2$, defined in some neighbourhood U of $\overline{\mathbb{D}}$, such that $f_j(0) = \zeta_j$ and $\lambda(\check{f}_j^{-1}(E)) > -2\pi\alpha$.

By Theorem 2, there are Blaschke products $\varphi_j : \mathbb{D} \rightarrow \mathbb{D}$, $j = 1, 2$, such that $\varphi_j(0) = 0$ and $S = \check{\varphi}_1^{-1}(\check{f}_1^{-1}(E)) \cap \check{\varphi}_2^{-1}(\check{f}_2^{-1}(E))$ has $\lambda(S) > -2\pi\alpha$. Let $0 < t < 1$. Since X is convex,

$$f = t f_1 \circ \varphi_1 + (1 - t) f_2 \circ \varphi_2$$

is a bounded analytic disc in X with $f(0) = t\zeta_1 + (1 - t)\zeta_2$. Since E is convex, $S \subset \check{f}^{-1}(E)$, so by (2.2),

$$u_{E,X}(t\zeta_1 + (1 - t)\zeta_2) \leq -\frac{1}{2\pi}\lambda(S) < \alpha.$$

This proves Theorem 1 in case E is open. Assume now that E is compact. Let (V_n) be a decreasing basis of convex open neighbourhoods for E . Then $E = \bigcap \overline{V}_n$, so by Klimek [1991, Proposition 4.5.10], $u_{\overline{V}_n,X} \nearrow u_{E,X}$, and hence $u_{V_n,X} \nearrow u_{E,X}$. Therefore,

$$u_{E,X}^{-1}[-1, \alpha) = \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} u_{V_n,X}^{-1}[-1, \alpha - 1/m)$$

is convex, and Theorem 1 is proved.

3. Boundary preimages under Blaschke products. Let us begin by observing that if (a, b) is an interval on the real axis, then the bounded harmonic function v in the upper half plane \mathbb{H} , which has boundary values equal to 1 on (a, b) and 0 on $\mathbb{R} \setminus [a, b]$, is given at $z \in \mathbb{H}$ as the angle between $b - z$ and $a - z$ divided by π , i.e.,

$$v(z) = \frac{1}{\pi} \operatorname{Arg} \frac{z - b}{z - a},$$

where $\operatorname{Arg} : \mathbb{C} \setminus (-\infty, 0] \rightarrow (-\pi, \pi)$ is the principal branch of the argument.

Let $A \neq \mathbb{T}$ be the union of a finite number of non-empty open arcs I_j , $j = 1, \dots, m$, $m \geq 1$, in \mathbb{T} , whose closures are mutually disjoint, with endpoints $a_j \neq b_j$ listed counterclockwise. Let $\sigma : \mathbb{D} \rightarrow \mathbb{H}$ be a biholomorphic map which sends a point in $\mathbb{T} \setminus \overline{A}$ to ∞ . Then σ maps the arcs I_j onto bounded open intervals in \mathbb{R} . From the above formula for v we see that the harmonic function u_A in \mathbb{D} having boundary values 1 on A and 0 on $\mathbb{T} \setminus \overline{A}$ is given by the formula

$$u_A(z) = \int_{\mathbb{T}} P(z, \cdot) \chi_A d\lambda = \frac{1}{\pi} \sum_{j=1}^m \operatorname{Arg} \frac{\sigma(z) - \sigma(b_j)}{\sigma(z) - \sigma(a_j)}, \quad z \in \mathbb{D},$$

where P denotes the Poisson kernel for \mathbb{D} . Observe that $u_A(0) = \lambda(A)/(2\pi)$.

If $z \in \mathbb{D}$, then the complex numbers

$$\frac{\sigma(z) - \sigma(b_j)}{\sigma(z) - \sigma(a_j)}, \quad j = 1, \dots, m,$$

are in \mathbb{H} and the sum of their arguments is less than π . Hence their product is also in \mathbb{H} , and we can write $u_A = \operatorname{Re} F_A$, where

$$F_A = \frac{1}{i\pi} \operatorname{Log} \left(c_A \prod_{j=1}^m \frac{\sigma - \sigma(b_j)}{\sigma - \sigma(a_j)} \right) \in \mathcal{O}(\mathbb{D}).$$

Here, Log denotes the principal branch of the logarithm, and we take

$$c_A = \left| \prod_{j=1}^m \frac{\sigma(0) - \sigma(a_j)}{\sigma(0) - \sigma(b_j)} \right| > 0,$$

so that $\operatorname{Im} F_A(0) = 0$. Note that F_A depends only on A , and not on σ .

The rational function

$$B_A = \sigma^{-1} \circ \left(c_A \prod_{j=1}^m \frac{\sigma - \sigma(b_j)}{\sigma - \sigma(a_j)} \right) = \sigma^{-1} \circ e^{i\pi F_A}$$

is a finite Blaschke product, for it maps \mathbb{D} into \mathbb{D} and \mathbb{T} into \mathbb{T} . Observe that

$$B_A(0) = \sigma^{-1}(e^{i\lambda(A)/2}),$$

and

$$A = (e^{i\pi F_A})^{-1}(-\infty, 0) = B_A^{-1}(J),$$

where J is the arc $\sigma^{-1}(-\infty, 0)$ in \mathbb{T} .

By Sard's theorem, there are automorphisms ψ of \mathbb{D} arbitrarily close to the identity such that $u_A(\psi(0))$ is a regular value for u_A . Then $u_A(\psi(0))$ is a regular value for $u_A \circ \psi = u_{\psi^{-1}(A)}$. This shows that for every $\varepsilon > 0$, there is a finite union A' of open arcs in \mathbb{T} as above with $\lambda(A \Delta A') < \varepsilon$, such that $u_{A'}(0)$ is a regular value for $u_{A'}$, and hence $B_{A'}(0)$ is a regular value for $B_{A'}$. Here, Δ denotes the symmetric difference.

Turning now to the proof of Theorem 2, let S_1 and S_2 be Borel sets in \mathbb{T} . Without loss of generality, we may assume that $\lambda(S_1) = \lambda(S_2)$. By regularity of the arc length measure, there are finite unions A_1 and A_2 of open arcs in \mathbb{T} as above such that $\lambda(S_j \Delta A_j)$, $j = 1, 2$, are arbitrarily small, and $\lambda(A_1) = \lambda(A_2)$. Let B_1 and B_2 be the finite Blaschke products associated with A_1 and A_2 as above. Since $\lambda(A_1) = \lambda(A_2)$, we have $B_1(0) = B_2(0)$. Let C be the set of critical values of B_1 in \mathbb{D} , which is a finite subset of \mathbb{D} . By the above, we may assume that $B_1(0) \notin C$.

Let $\varphi_2 : \mathbb{D} \rightarrow \mathbb{D} \setminus B_2^{-1}(C)$ be a universal covering map with $\varphi_2(0) = 0$. We claim that φ_2 is a Blaschke product. It is then an infinite Blaschke product, unless $C = \emptyset$, which means that A_2 consists of a single arc. First of all, $\varphi_2^{-1}(B_2^{-1}(C))$ is null, because φ_2 cannot be constant on a set of non-zero

measure. Let us show that the non-tangential boundary values of φ_2 lie in the boundary of $\mathbb{D} \setminus B_2^{-1}(C)$. Generally, let U be a bounded domain in \mathbb{C} and $\varphi : \mathbb{D} \rightarrow U$ be the universal covering map. Suppose $x \in \mathbb{T}$ has $\check{\varphi}(x) \in U$. Let V be a compact disc in U containing $\check{\varphi}(x)$. Then there is a line segment $L = x(1 - \varepsilon, 1)$, $\varepsilon > 0$, such that $\varphi(L) \subset V$ and $L \subset \varphi^{-1}(V)$. But $\varphi^{-1}(V)$ is a disjoint union of compact domains in \mathbb{D} , so it cannot contain L . Hence, all the non-tangential boundary values of φ lie in the boundary of U . This shows that φ_2 is an inner function. Also, 0 is not a non-tangential boundary value of φ_2 because $0 \notin B_2^{-1}(C)$. Hence, φ_2 is a Blaschke product by Frostman's theorem. See Noshiro [1960, p. 33].

Now the composition $B_2 \circ \varphi_2 : \mathbb{D} \rightarrow \mathbb{D} \setminus C$ lifts by the covering map $B_1 : \mathbb{D} \setminus B_1^{-1}(C) \rightarrow \mathbb{D} \setminus C$ to a holomorphic map $\varphi_1 : \mathbb{D} \rightarrow \mathbb{D} \setminus B_1^{-1}(C)$ with $\varphi_1(0) = 0$, as shown in the following diagram.

$$\begin{array}{ccc}
 \mathbb{D} & \xrightarrow{\varphi_1} & \mathbb{D} \setminus B_1^{-1}(C) \\
 \varphi_2 \downarrow & & \downarrow B_1 \\
 \mathbb{D} \setminus B_2^{-1}(C) & \xrightarrow{B_2} & \mathbb{D} \setminus C
 \end{array}$$

Since the non-tangential boundary values of $B_2 \circ \varphi_2$ lie in $\mathbb{T} \cup C$, the non-tangential boundary values of φ_1 lie in $\mathbb{T} \cup B_1^{-1}(C)$. Hence, φ_1 is inner, and 0 is not a non-tangential boundary value of φ_1 , so by Frostman's theorem, φ_1 is a Blaschke product. Furthermore,

$$\check{\varphi}_1^{-1}(A_1) = \check{\varphi}_1^{-1}(B_1^{-1}(J)) = \check{\varphi}_2^{-1}(B_2^{-1}(J)) = \check{\varphi}_2^{-1}(A_2).$$

To conclude the proof of Theorem 2, we shall show that if φ is an inner function with $\varphi(0) = 0$, and S is a Borel set in \mathbb{T} , then $\lambda(\check{\varphi}^{-1}(S)) = \lambda(S)$. In other words, $\check{\varphi}_* \lambda = \lambda$. For an L^∞ function g on \mathbb{T} , we let $H[g]$ denote the bounded harmonic function on \mathbb{D} with non-tangential limit $g(x)$ at almost every $x \in \mathbb{T}$.

First we consider the case when S is an arc. Suppose φ has a non-tangential limit in \mathbb{T} at $x \in \mathbb{T}$, and let $z_n \rightarrow x$ non-tangentially in \mathbb{D} . Then $\varphi(z_n) \rightarrow \check{\varphi}(x) \in \mathbb{T}$. If $\check{\varphi}(x)$ is not an endpoint of S , then χ_S is continuous at $\check{\varphi}(x)$, so

$$(H[\chi_S] \circ \varphi)(z_n) \rightarrow \chi_S(\check{\varphi}(x)) = \chi_{\check{\varphi}^{-1}(S)}(x).$$

Now $\check{\varphi}$ cannot be constant on a set of non-zero measure, so the preimage under $\check{\varphi}$ of the endpoints of S is a nullset. Therefore, the bounded harmonic function $H[\chi_S] \circ \varphi$ has non-tangential limit $\chi_{\check{\varphi}^{-1}(S)}(x)$ at almost every $x \in \mathbb{T}$, so $H[\chi_S] \circ \varphi = H[\chi_{\check{\varphi}^{-1}(S)}]$. Hence,

$$\frac{\lambda(S)}{2\pi} = H[\chi_S](0) = (H[\chi_S] \circ \varphi)(0) = H[\chi_{\check{\varphi}^{-1}(S)}](0) = \frac{\lambda(\check{\varphi}^{-1}(S))}{2\pi}.$$

This shows that $\lambda(\check{\varphi}^{-1}(S)) = \lambda(S)$ for open sets and hence for compact sets. In the general case,

$$\begin{aligned}\lambda(S) &= \sup \lambda(K) = \sup \lambda(\check{\varphi}^{-1}(K)) \\ &\leq \lambda(\check{\varphi}^{-1}(S)) \leq \inf \lambda(\check{\varphi}^{-1}(U)) = \inf \lambda(U) = \lambda(S),\end{aligned}$$

where the supremum is taken over compact subsets K of S , and the infimum is taken over open supersets U of S .

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