L1CUPGE - Gpe 4 - Semaine 10 TD4

le 12/04/2021

Départ 11h15

TD1 semaine 11 : Mardi
demain matin à 7h45

Feuille 10 - Exo 12

$$\begin{cases} u_0 = u_1 = 0, \ u_2 = 1 \\ u_{n+3} = 2u_{n+2} + u_{n+1} - 2u_n \end{cases}$$

$n=0 \quad u_3 = 2u_2 + u_1 - 2u_0 = 2.1 - 0 - 0 = 2$

$u_4 = 2u_3 + u_2 - 2u_1 = $ etc..

Que vaut u_n ?

$U_n = \begin{pmatrix} u_n \\ u_{n+1} \\ u_{n+2} \end{pmatrix}$ Trouver une matrice $A \in M_3(\mathbb{R})$

telle que $\quad U_{n+1} = A\,U_n$

$$U_{n+1} = \begin{pmatrix} u_{n+1} \\ u_{n+2} \\ u_{n+3} \end{pmatrix} = \begin{pmatrix} & ? & \end{pmatrix} \begin{pmatrix} u_n \\ u_{n+1} \\ u_{n+2} \end{pmatrix}$$

$$U_{n+1} = \begin{pmatrix} u_{n+1} \\ u_{n+2} \\ u_{n+3} \end{pmatrix} = \begin{pmatrix} & ? & \end{pmatrix} \begin{pmatrix} u_n \\ u_{n+1} \\ u_{n+2} \end{pmatrix}$$

$$A\,U_n = \underbrace{\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}}_{3 \times 3} \underbrace{\begin{pmatrix} u_n \\ u_{n+1} \\ u_{n+2} \end{pmatrix}}_{3 \times 1} = \begin{pmatrix} u_{n+1} \\ u_{n+2} \\ u_{n+3} = 2u_{n+2} + u_{n+1} - 2u_n \end{pmatrix}$$

$$= \begin{pmatrix} \overset{0}{a}\,u_n + \overset{1}{b}\,u_{n+1} + \overset{0}{c}\,u_{n+2} \\ \underset{0}{d}\,u_n + \underset{0}{e}\,u_{n+1} + \underset{1}{f}\,u_{n+2} \\ \underset{-2}{g}\,u_n + \underset{1}{h}\,u_{n+1} + i\,u_{n+2} \end{pmatrix} = \begin{pmatrix} u_{n+1} \\ u_{n+2} \\ 2u_{n+2} + u_{n+1} - 2u_n \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix} \qquad U_{n+1} = A U_n$$

(2) $P = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 4 & 1 & 1 \end{pmatrix}$ inversible & calculer P^{-1}

$$\left(\begin{array}{ccc|ccc} \boxed{1} & 1 & 1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ 4 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \qquad \begin{array}{l} L_2 \leftarrow L_2 - 2L_1 \\ L_3 \leftarrow L_3 - 4L_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -3 & -1 & -2 & 1 & 0 \\ 0 & -3 & -3 & -4 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -3 & -1 & -2 & 1 & 0 \\ 0 & -3 & -3 & -4 & 0 & 1 \end{array}\right) \qquad L_3 \leftarrow L_3 - L_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -3 & -1 & -2 & 1 & 0 \\ 0 & 0 & \boxed{-2} & -2 & -1 & 1 \end{array}\right) \qquad L_3 \leftarrow \frac{-L_3}{2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -3 & -1 & -2 & 1 & 0 \\ 0 & 0 & \underline{\underline{1}} & 1 & 1/2 & -1/2 \end{array}\right) \qquad \begin{array}{l} L_1 \leftarrow L_1 - L_3 \\ L_2 \leftarrow L_2 + L_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & -1/2 & +1/2 \\ 0 & -3 & 0 & -1 & 3/2 & -1/2 \\ 0 & 0 & 1 & 1 & 1/2 & -1/2 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & -1/2 & +1/2 \\ 0 & -3 & 0 & -1 & 3/2 & -1/2 \\ 0 & 0 & 1 & 1 & 1/2 & -1/2 \end{array}\right) \leftarrow -L_2/3$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & -1/2 & 1/2 \\ 0 & 1 & 0 & 1/3 & -1/2 & 1/6 \\ 0 & 0 & 1 & 1 & 1/2 & \cancel{1/4}\ -1/2 \end{array}\right) \quad L_1 \leftarrow L_1 - L_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/3 & 0 & 1/2 - 1/6 = 1/3 \\ 0 & 1 & 0 & 1/3 & -1/2 & 1/6 \\ 0 & 0 & 1 & 1 & 1/2 & \cancel{1/4}\ -1/2 \end{array}\right)$$

$$P^{-1} = \begin{pmatrix} -1/3 & 0 & 1/3 \\ 1/3 & -1/2 & 1/6 \\ 1 & 1/2 & \cancel{1/4} \\ & & -1/2 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} -1/3 & 0 & 1/3 \\ 1/3 & -1/2 & 1/6 \\ 1 & 1/2 & \cancel{1/4} \ -1/2 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} -2 & 0 & 2 \\ 2 & -3 & 1 \\ 6 & 3 & -3 \end{pmatrix}$$

$$P^{-1}P = \frac{1}{6}\begin{pmatrix} -2 & 0 & 2 \\ 2 & -3 & 1 \\ 6 & 3 & -3 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 4 & 1 & 1 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

$$P^{-1} = \frac{1}{6}\begin{pmatrix} -2 & 0 & 2 \\ 2 & -3 & 1 \\ 6 & 3 & -3 \end{pmatrix}$$

$$P^{-1} A P = \frac{1}{6}\begin{pmatrix} -2 & 0 & 2 \\ 2 & -3 & 1 \\ 6 & 3 & -3 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 4 & 1 & 1 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} -2 & 0 & 2 \\ 2 & -3 & 1 \\ 6 & 3 & -3 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ 4 & 1 & 1 \\ 8 & -1 & 1 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} 12 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = D$$

<u>diagonale</u>

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = D = P^{-1} A P$$

$u_n = ??$

$$U_{n+1} = \overset{1}{A} \underbrace{U_n}_{n} \rightarrow 1+n$$

$$= A (A U_{n-1})$$

$2+n-1 = n+1$

$$= A^{2} \underbrace{U_{n-1}} = A^{2} (A U_{n-2})$$

$$= A^{3} \underbrace{U_{n-2}} \rightarrow 3+n-2 = n+1$$

on descend

$\dots$

$$= A^{n+1} U_0 \rightarrow n+1$$

On a donc $\boxed{U_n = A^n U_0}$

$$U_n = \begin{pmatrix} u_n \\ u_{n+1} \\ u_{n+2} \end{pmatrix} = A^n \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix} = A^n \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Si on connaît A^n on en déduit u_n en f° de n

Tout revient à connaître A^n
ce qui semble sans espoir

Ici :

$$D = P^{-1} A P \Rightarrow A =$$

$$PD = \underbrace{PP^{-1}}_{I_n} A P = AP$$

$$PDP^{-1} = APP^{-1} \Rightarrow A = PDP^{-1}$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 2 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$ABAB = (AB)^2 \neq A^2 B^2$$

Donc $A^n = (PDP^{-1})^n = ??$

$$A^n = (PDP^{-1})^n$$

$$= (P D \underbrace{P^{-1})(P}_{I_n} D \underbrace{P^{-1})(P}_{I_n} D \underbrace{P^{-1})}_{I_n} \cdots (PDP^{-1})$$

$$= P D D \cdots D P^{-1}$$

$$= P D^n P^{-1}$$

$$A^2 = PD\underbrace{P^{-1} \cdot P}_{I_n}DP^{-1}$$

$$= P D^2 P^{-1}$$

$$A^n = \frac{1}{6}\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 4 & 1 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

$$\frac{1}{6}\begin{pmatrix} -2 & 0 & 2 \\ 2 & -3 & 1 \\ 6 & 3 & -3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix}^3 = \begin{pmatrix} 1^3 & & 0 \\ & 2^3 & \\ 0 & & 3^3 \end{pmatrix}$$

$$\frac{1}{6}\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 4 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2^n & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -2 & 0 & 2 \\ 2 & -3 & 1 \\ 6 & 3 & -3 \end{pmatrix}$$

$$U_n = \begin{pmatrix} u_n \\ u_{n+1} \\ u_{n+2} \end{pmatrix} = A^n U_0 = \frac{1}{6}\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 4 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2^n & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -2 & 0 & 2 \\ 2 & -3 & 1 \\ 6 & 3 & -3 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 4 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2^n & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 4 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2^{n+1} \\ (-1)^n \\ -3 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} 2^{n+1} + (-1)^n - 3 \\ \cdots \\ \cdots \end{pmatrix} = \begin{pmatrix} u_n \\ u_{n+1} \\ u_{n+2} \end{pmatrix}$$

$$\boxed{u_n = \frac{2^{n+1} + (-1)^n - 3}{6}}$$