L1 CUPGE – Semaine 11 – TD3

Gpe 4

le 15 Avril 2021

10h05

Exercice 14-F11

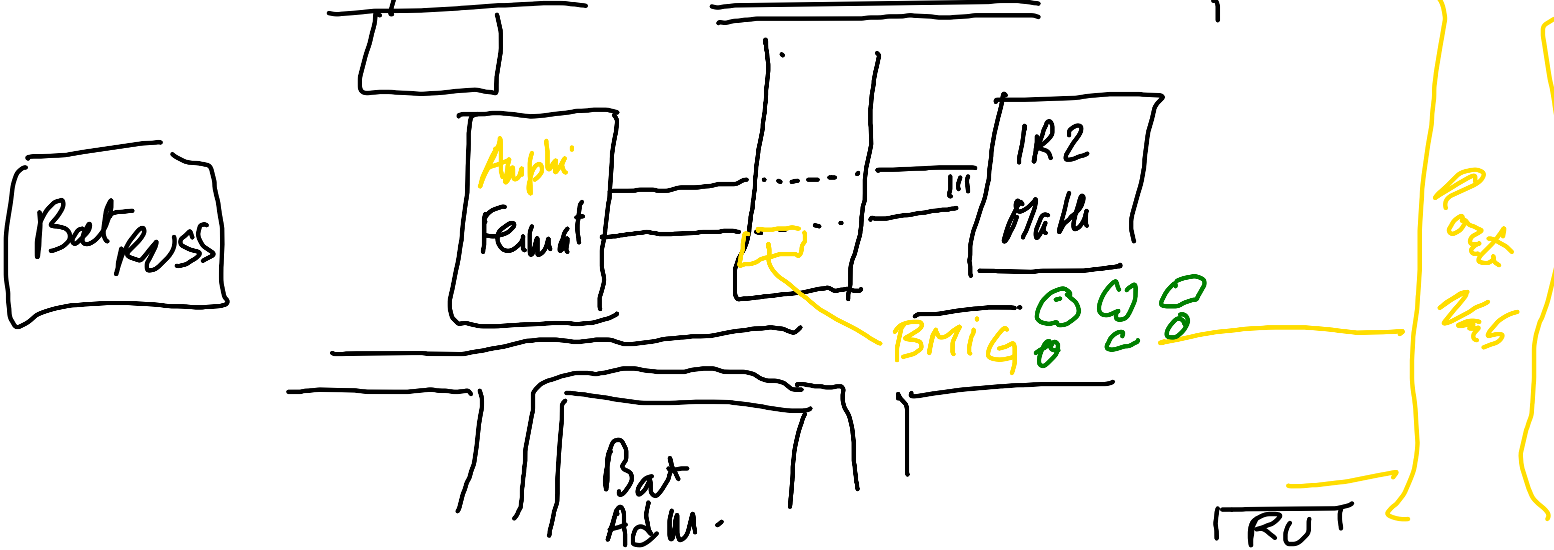
$$f : \mathbb{R}_2[X] \longrightarrow \mathbb{R}^3$$
$$P \longmapsto f(P) = \big(P(0), P'(0), P(1)\big)$$

① Montrer que f est un isomorphisme

U3

Copies CC2 : Salle BMIG : à partir de $13^{15} \to 17^{h}$

U4



Exercice 14-F11

$$f : \mathbb{R}_2[X] \longrightarrow \mathbb{R}^3$$
$$P \longmapsto f(P) = (P(0), P'(0), P(1))$$

(1) Montrer que f est un isomorphisme

ex $P(X) = X^2 + 1$ $\quad f(P) = (P(0), P'(0), P(1))$

$P'(X) = 2X$ $\quad = (1, 0, 2)$

. Il faut montrer que :

$\longrightarrow$ f est linéaire

$\longrightarrow$ f bijective.

$\boxed{f \text{ linéaire ?}}$ $\quad f(P + \lambda Q) = ((P+\lambda Q)(0), (P+\lambda Q)'(0), (P+\lambda)(1))$

$P, Q \in \mathbb{R}_2(X)$, $\lambda \in \mathbb{R}$ $\quad = (P(0) + \lambda Q(0), P'(0) + \lambda Q'(0), P(1) + \lambda Q(1))$

$= (P(0), P'(0), P(1)) + \lambda (Q(0), Q'(0), Q(1))$

$= f(P) + \lambda f(Q)$

$\rightarrow$ <u>f est bijective</u> $\Leftrightarrow$ f est <u>surjective</u> & <u>injective</u>

$\Leftrightarrow$ f injective

CAR $\dim \underbrace{\mathbb{R}_2[X]}_{\text{esp. départ}} = 3 = \dim \underbrace{\mathbb{R}^3}_{\text{esp. arrivée}}$

f <u>injective</u> $\Leftrightarrow$ $\mathrm{Ker}(f) = \{0_{\mathbb{R}_2(X)}\}$

Si $P \in \mathbb{R}_2[X]$ et $f(P) = 0_{\mathbb{R}^3} = (0,0,0)$ il faut montrer que $P = 0_{\mathbb{R}_2(X)}$

$$\underline{\underline{f(P) = (P(0), P'(0), P(1))}}$$

$P(x) = a + bx + cx^2 \in \mathbb{R}_2[X]$

$P'(x) = b + 2cx$

si $P \in \ker f$: $f(P) = (P(0), P'(0), P(1)) = (0,0,0)$

$\Leftrightarrow \boxed{(a, b, a+b+c)} = (0,0,0)$

$\Leftrightarrow \begin{cases} a = 0 \\ b = 0 \\ a+b+c = 0 \end{cases} \Rightarrow a = b = c = 0 \Rightarrow P = 0_{\mathbb{R}_2(X)}$

Donc f est $\underline{\text{injective}}$ $\underset{\text{car}}{\Leftrightarrow}$ f bijective

c'est donc un isomorphisme entre $\mathbb{R}_2(X)$ et $\mathbb{R}^3$

Remarque

$$g : P \in \overbrace{\mathbb{R}_3(X)}^{4} \longrightarrow g(P) = (\overbrace{\overbrace{P(0)}^{x/P}, P'(0)}^{x^2/P}, \overbrace{P(1)}^{x-1/P})$$

g n'est plus injective $\quad P(x) = x^2(x-1) \in \operatorname{Ker} P$

$\operatorname{Ker} P = \operatorname{Vect}\left(x^2(x-1)\right)$

2) $f : \mathbb{R}_2(X) \longrightarrow \mathbb{R}^3$ est bijective : $\exists f^{-1} : \mathbb{R}^3 \longrightarrow \mathbb{R}_2(X)$

$(a,b,c) \in \mathbb{R}^3 : f^{-1}(a,b,c) \overset{?}{=}$

$\operatorname{Mat}(f) \overset{?}{=}$

dans les bases canonique

$$\mathbb{R}_2(X) \longrightarrow \mathbb{R}^3$$

$\{1, X, X^2\}$ $\quad$ $\{e_1, e_2, e_3\}$

$e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$

$$\mathbb{R}_3(X) \longrightarrow \mathbb{R}^3$$

$$\{1, X, X^2\} \qquad \{e_1, e_2, e_3\}$$

$$e_1 = (1,0,0), \quad e_2 = (0,1,0), \quad e_3 = (0,0,1)$$

$$\left\{ \begin{array}{l} X^2 \\ X^2 + X \\ X^2 + X + 1 \end{array} \right.$$

$$m(f) = \begin{array}{c|c} \begin{array}{ccc} f(1) & f(X) & f(X^2) \end{array} & \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} & \begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array} \end{array}$$

$e_1 + e_2 =$

e ~

e

$$m_{B_1, B_2}(f)$$

$$f(1) = (P(0), P'(0), P(1)) = (1, 0, 1)$$

$$f(X) = (0, 1, 1)$$

$$f(X^2) = (0, 0, 1)$$

1

$$m(f) = \begin{array}{c|ccc|c} & f(1) & f(x) & f(x^2) & \\ \end{array}$$

$$m(f) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{matrix} e_1 \\ e_2 \\ e_3 \end{matrix}$$

$$m(f^{-1}) = m(f)^{-1}$$

$$f(P = a + bx + cx^2) = m(f) \begin{pmatrix} a \\ b \\ c \end{pmatrix}_{(1, x, x^2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ a+b+c \end{pmatrix}$$

$$f^{-1}(\alpha, \beta, \gamma) = \underbrace{m(f^{-1})}_{??} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

Si on trouve $\underline{m(f^{-1})}$ on connait f^{-1}

Comme $m(f^{-1}) = m(f)^{-1}$, inversons $m(f)$:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & \underline{1} & 1 & -1 & 0 & 1 \end{array}\right) \quad L_3 \leftarrow L_3 - L_1$$

$$\left(\underbrace{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}}_{I_3} \middle| \underbrace{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{array}}_{m(f)^{-1}}\right) \quad L_3 \leftarrow L_3 - L_2$$

Donc

$$m(f)^{-1} = \begin{pmatrix}1&0&0\\0&1&0\\-1&-1&1\end{pmatrix} = m(f^{-1})$$

Par conséquent

$$f^{-1}(\underbrace{\alpha,\beta,\gamma}_{\in\mathbb{R}^3}) = m(f^{-1})\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}$$

$$= \begin{pmatrix}1&0&0\\0&1&0\\-1&-1&1\end{pmatrix}\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix} = \begin{pmatrix}\alpha\\\beta\\-\alpha-\beta+\gamma\end{pmatrix}_{\mathbb{R}_2[X]}$$

$$\simeq \underbrace{\alpha+\beta X+(\gamma-\alpha-\beta)X^2}_{= P(X)}$$

$$f^{-1}(\alpha,\beta,\gamma) = P$$

$$\iff f(P) = (\alpha,\beta,\gamma)$$

Vérification

$$f(P) = (P(0), P'(0), P(1)) = (\alpha, \beta, \alpha+\beta+\gamma-\alpha-\beta) = (\alpha,\beta,\gamma)$$

Solution 2

$$f : \mathbb{R}_2(X) \longrightarrow \mathbb{R}^3$$

$$P = f^{-1}(\alpha,\beta,\gamma) \overset{?}{\longleftarrow} (\alpha,\beta,\gamma)$$

$\Leftrightarrow f(P)=(\alpha,\beta,\gamma)$ on cherche P, (α,β,γ sont donnés)

$$\Updownarrow$$

$$(a,b,a+b+c)=(\alpha,\beta,\gamma)$$

$$\begin{cases} \boxed{\begin{array}{l} a=\alpha \\ b=\beta \end{array}} \\ a+b+c=\gamma \end{cases} \longrightarrow c=\gamma-a-b \Leftrightarrow \boxed{\gamma-\alpha-\beta=c}$$

$a+bx+cx^2$

$f^{-1}(e_1)\ f^{-1}(e_2)\ f^{-1}(e_3)$

$$f^{-1}(\alpha,\beta,\gamma)=\alpha+\beta x+(\gamma-\alpha-\beta)x^2$$

$$M(f^{-1})=\begin{array}{c} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \end{array}\begin{array}{l} 1 \\ X \\ X^2 \end{array}$$

$f^{-1}(e_1)=f^{-1}(1,0,0)=1-X^2$

$f(0,1,0)=X-X^2$

$f(0,0,1)=X^2$

$\boxed{\text{Exercice 13 - F11}}$

$C \in M_2(\mathbb{R})$ fixée

$$f : M_2(\mathbb{R}) \longrightarrow M_2(\mathbb{R})$$
$$M \longmapsto f(M) = CM + MC \in M_2(\mathbb{R})$$

$\mathbb{R}^3$

$\mathbb{R}_2[X]$

1) mq f est un endomorphisme

un endomorphisme
$E \longrightarrow E$
linéaire
Automorphisme : endo. bijec
↳ isom. & endo.

Reste à prouver la linéarité de f : $M, N \in M_2(\mathbb{R})$
$\lambda \in \mathbb{R}$

mq $f(M+\lambda N) \overset{?}{=} f(M) + \lambda f(N)$

$$f(M+\lambda N) = C(M+\lambda N) + (M+\lambda N)C$$
$$= \underline{CM} + \lambda CN + \underline{MC} + \lambda NC$$
$$= CM + MC + \lambda(CN + NC)$$
$$= f(M) + \lambda f(N)$$

f est bien linéaire de $M_2(\mathbb{R})$ dans $M_2(\mathbb{R})$: c'est bien un endomorphisme

Donner la matrice de f dans la base canonique de $M_2(\mathbb{R})$: $\{E_{11}, E_{12}, E_{21}, E_{22}\} = \mathcal{B}$

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\quad E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\quad E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\quad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}}$$

$$M_{\mathcal{B}}(f) = \begin{array}{c} \begin{matrix} f(E_{11}) & f(E_{12}) & f(E_{21}) & f(E_{22}) \end{matrix} \\ \begin{pmatrix} 2a & c & b & 0 \\ b & d+a & 0 & b \\ c & 0 & a+d & c \\ 0 & c & b & 2d \end{pmatrix} \end{array} \begin{matrix} E_{11} \\ E_{12} \\ E_{21} \\ E_{22} \end{matrix}$$

3

$Tr(C) = a+d$

$Tr(M(f)) = 4a+4d$

$= 4\,tr(C)$

$$\begin{pmatrix} 2a \\ b \\ c \\ 0 \end{pmatrix}_{E_{ij}}$$

$$f(E_{11}) = CE_{11} + E_{11}C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2a & b \\ c & 0 \end{pmatrix} = 2a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= 2a\,E_{11} + b\,E_{12} + c\,E_{21}$$

Exercice 10-F11

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$(x,y,z) \longmapsto f(x,y,z) = (y-z, -2x-y+z, -2x-y+z)$$

1) f est linéaire (exercice) c'est donc un endo. de $\mathbb{R}^3$

2) $$\operatorname{Mat}_{\mathcal{E}(e_1,e_2,e_3)}(f) = \begin{array}{c c} \begin{array}{ccc} f(e_1) & f(e_2) & f(e_3) \end{array} & \\ \begin{pmatrix} 0 & 1 & -1 \\ -2 & -1 & 1 \\ -2 & -1 & 1 \end{pmatrix} & \begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array} \end{array}$$

$f(e_3) = -e_1 + e_2 + e_3$

$f(e_1) = (0, -2, -2)$
$= -2e_2 - 2e_3$

$f(e_2) = e_1 - e_2 - e_3$

3) $$\operatorname{mat}_{\mathcal{E}'(e_3,e_2,e_1)}(f) = \begin{array}{c c} \begin{array}{ccc} f(e_3) & f(e_2) & f(e_1) \end{array} & \\ \begin{pmatrix} 1 & -1 & -2 \\ 1 & -1 & -2 \\ -1 & 0 & 0 \end{pmatrix} & \begin{array}{c} e_3 \\ e_2 \\ e_1 \end{array} \end{array}$$

<u>exercice</u> Retrouver cette matrice via la formule :

$$m_{\mathcal{E}'}(f) = P^{-1} m_{\mathcal{E}}(f) P \quad \begin{array}{c c} \begin{array}{ccc} \mathcal{E}' e_3 & e_2 & e_1 \end{array} & \\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \begin{array}{c} \mathcal{E}\, e_1 \\ e_2 \\ e_3 \end{array} \end{array}$$

$$\mathcal{E}'' = \{ \underbrace{e_2}_{u_1}, \underbrace{e_1 - e_2 - e_3}_{u_2}, \underbrace{-2e_2 - 2e_3}_{u_3} \}$$

• Montrer que $\mathcal{E}''$ base de $\mathbb{R}^3$: On a 3 vecteurs dans $\mathbb{R}^3$ qui est de dimension 3 donc il suffit de montrer que cette famille est libre :

$$\lambda u_1 + \mu u_2 + \gamma u_3 = 0_{\mathbb{R}^3} \overset{?}{\Longrightarrow} \lambda = \mu = \gamma = 0$$

$$\lambda e_2 + \mu (e_1 - e_2 - e_3) + \gamma(-2e_2 - 2e_3)$$

$$= \mu e_1 + (\lambda - \mu - 2\gamma) e_2 + (-\mu - 2\gamma) e_3 = 0_{\mathbb{R}^3}$$

$$\Longrightarrow \begin{cases} \mu = 0 \\ \lambda - \mu - 2\gamma = 0 \\ -\mu - 2\gamma = 0 \end{cases}$$

$\{e_1, e_2, e_3\}$ libre (base)

$$\begin{cases} \mu = 0 \\ \lambda - \mu - 2\gamma = 0 \\ -\mu - 2\gamma = 0 \end{cases} \Rightarrow \begin{cases} \mu = 0 \\ \lambda - 2\gamma = 0 \rightarrow \lambda = 0 \\ \gamma = 0 \end{cases}$$

la famille est bien libre
c'est une base !!

3) Matrice de f dans la base $\mathcal{E}'' = (\underbrace{e_2}_{u_1}, \underbrace{e_1 - e_2 - e_3}_{u_2}, \underbrace{-2e_2 - 2e_3}_{u_3})$

$$u_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad u_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \quad u_3 = \begin{pmatrix} 0 \\ -2 \\ -2 \end{pmatrix}$$

$$\begin{array}{cccc} & f(u_1) & f(u_2) & f(u_3) \end{array}$$

$$\mathrm{M}_{\mathcal{E}''}(f) = \left(\begin{array}{ccc} \alpha = 0 & & \\ \beta = 1 & & \\ \gamma = 0 & & \end{array}\right) \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array}$$

méthode 1, à la main

$$= P^{-1} \, \mathrm{Mat}_{\mathcal{E}}(f) \, P \quad \text{où} \quad P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & -2 \\ 0 & -1 & -2 \end{pmatrix}$$

$$f(u_1) = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}_{\mathcal{E}} = \alpha u_1 + \beta u_2 + \gamma u_3 = u_2$$

$$\mathrm{M}_{\mathcal{E}}(f) = \begin{pmatrix} 0 & 1 & -1 \\ -2 & -1 & 1 \\ -2 & -1 & 1 \end{pmatrix}$$

L2 CUPGE - Gpe 4 - Semaine 11 - TD4

le 15/14/21

15h50

Copies à 12h45 Salle BMIG
Sauf contre ordre via moodle
dans la soirée

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & -2 \\ 0 & -1 & -2 \end{pmatrix}$$

$$M_{\mathcal{E}}(f) = \begin{pmatrix} 0 & 1 & -1 \\ -2 & -1 & 1 \\ -2 & -1 & 1 \end{pmatrix}$$

$3 = \underset{2}{\dim \operatorname{Im}} + \dim K$

$$\overbrace{M f}\; X = f(x)$$

$$f(u_2) = f(1,-1,-1) = \begin{pmatrix} 0 & 1 & -1 \\ -2 & -1 & 1 \\ -2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ -2 \end{pmatrix} = u_3$$

$$M_{f,\mathcal{E}'} = \begin{array}{c} \begin{array}{ccc} & f(u_2) & f(u_3) \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{array} \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array}$$

$$f(u_3) = f(0,-2,-2) = \begin{pmatrix} 0 & 1 & -1 \\ -2 & -1 & 1 \\ -2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Obs:

$\operatorname{Im} f = \operatorname{Vect}\{f(u_1), f(u_2), f(u_3)\}$
$= \operatorname{Vect}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\} \Rightarrow \boxed{\dim \operatorname{Im} f = 2}$

th. Rang

$\dim \operatorname{Ker} f = 1$

$f(u_3) = 0 \longrightarrow \operatorname{Ker} f = \operatorname{Vect} u_3$

$u_3 \in \operatorname{Ker} u$

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & -2 \\ 0 & -1 & -2 \end{pmatrix}$$

$$M_{\mathcal{E}}(f) = \begin{pmatrix} 0 & 1 & -1 \\ -2 & -1 & 1 \\ -2 & -1 & 1 \end{pmatrix}$$

$f(u_2) = f($[illegible]$) =$

$$M_{f,\mathcal{E}'} = \begin{array}{c} \\ \\ \end{array}\begin{array}{cccc} & & f(u_2) & f(u_3) \\ \left(\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right. & \begin{array}{c} 0 \\ 0 \\ 1 \end{array} & \boxed{\begin{array}{c} 0 \\ 0 \\ 0 \end{array}} & \left.\right) \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array} \end{array}$$

$$f(u_1) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \alpha u_1 + \beta u_2 + \gamma u_3$$

$$\begin{cases} 1 = \beta \\ 2 = 2\alpha - \beta - 2\gamma \\ 3 = -\beta - 2\gamma \end{cases}$$

Autre méthode :

$$M_{\mathcal{E}'}(f) = \underbrace{P^{-1}}_{} \underbrace{M_{\mathcal{E}}(f)}_{\text{connu}} \underbrace{P}_{\text{aussi}}$$

Si on connaît P^{-1} il n'a plus qu'à faire le produit

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & -2 & 0 & 1 & 0 \\ 0 & -1 & -2 & 0 & 0 & 1 \end{array}\right) \quad L_1 \leftrightarrow L_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & -2 & 0 & 1 & 0 \\ 0 & \boxed{1} & 0 & 1 & 0 & 0 \\ 0 & -1 & -2 & 0 & 0 & 1 \end{array}\right) \quad L_3 \leftarrow L_3 + L_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & -2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 & 1 \end{array}\right) \quad L_3 \leftarrow -\frac{L_3}{2}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & -2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/2 & 0 & -1/2 \end{array}\right) \quad L_1 \leftarrow L_1 + 2L_3$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/2 & 0 & -1/2 \end{array}\right) \quad L_1 \leftarrow L_1 + L_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/2 & 0 & -1/2 \end{array}\right) \quad L_1 \leftarrow L_1 + L_2$$

$$\left(\underbrace{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}}_{I_3} \middle| \underbrace{\begin{array}{ccc} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1/2 & 0 & -1/2 \end{array}}_{P^{-1}}\right)$$

$$m_{\mathcal{E}'}(f) = P^{-1} \, m_{\mathcal{E}}(f) \, P$$

$$= \begin{pmatrix} 0 & +1 & -1 \\ 1 & 0 & 0 \\ -1/2 & 0 & -1/2 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ -2 & -1 & 1 \\ -2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & -2 \\ 0 & -1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1/2 & 0 & -1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & -2 & 0 \\ -1 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = m_{\mathcal{E}'}(f)$$

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & -2 \\ 0 & -1 & -2 \end{pmatrix}$$

$$M_{\mathcal{E}}(f) = \begin{pmatrix} 0 & 1 & -1 \\ -2 & -1 & 1 \\ -2 & -1 & 1 \end{pmatrix}$$

$$3 = \underbrace{\dim \operatorname{Im}}_{2} + \dim K$$

$$\underbrace{M f}\; X = f(X)$$

$$f(u_2) = f(1,-1,-1) = \begin{pmatrix} 0 & 1 & -1 \\ -2 & -1 & 1 \\ -2 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ -2 \end{pmatrix} = u_3$$

$$\boxed{M_{f,\mathcal{E}'} = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} & \begin{array}{ccc} & f(u_2) & f(u_3) \end{array} \\ & \begin{pmatrix} 0 & 0 & \boxed{0} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array} \end{array}}$$

$$f(u_3) = f(0,-2,-2) = \begin{pmatrix} 0 & 1 & -1 \\ -2 & -1 & 1 \\ -2 & -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Obs:

$\operatorname{Im} f = \operatorname{Vect}\{f(u_1), f(u_2), f(u_3)\}$

$= \operatorname{Vect}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \Rightarrow \boxed{\dim \operatorname{Im} f = 2}$

th. Rang

$\dim \operatorname{Ker} f = 1$

$f(u_3) = 0 \longrightarrow \operatorname{Ker} f = \operatorname{Vect} u_3$

$u_3 \in \operatorname{Ker} u$

5) En déduire Im f . Ker f

$$M_{\mathcal{C}'}(f) = \begin{array}{cccc} f(u_1) & f(u_2) & f(u_3) & \\ \left(\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right. & \begin{array}{c} 0 \\ 0 \\ 1 \end{array} & \left.\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right) & \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array} \end{array}$$

Ker f sev $\mathbb{R}^3$

dim Ker f 0

1

2

3

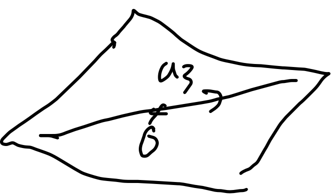
- $f(u_3) = 0_{\mathbb{R}^3} \Rightarrow \boxed{u_3 \in \operatorname{Ker} f} \Rightarrow \operatorname{Vect}(u_3) \subset \operatorname{Ker} f$

dim Ker $f \geq 1$

$u_3 = -2e_2 - 2e_3 = \begin{pmatrix} 0 \\ -2 \\ -2 \end{pmatrix}$

- $\operatorname{Im}(f) = \operatorname{Vect}\{f(e_1), f(e_2), f(e_3)\}$

$= \operatorname{Vect}\{f(u_1), f(u_2), f(u_3)\} = \operatorname{Vect}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right\}$



$= \operatorname{Vect}\left\{\underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\text{trivialement libre}}\right\}$

c'est une base

$$\boxed{rg(f) = \dim \operatorname{Im}(f) = 2}$$

$f \in \mathcal{L}(E,F)$ $\quad \dim E = \dim \ker f + \underbrace{\dim \mathrm{Im}(f)}_{= \mathrm{rang}(f)}$

$\boxed{\text{Exercice 16 - Feuille 11}}$

$f : \overbrace{\mathbb{R}_3[X]} \longrightarrow \overbrace{\mathbb{R}_3[X]}$

$P \longmapsto f(P) = \underline{P + P' + P''}$

f est un <u>automorphisme</u>

$\Leftrightarrow$ f $\underbrace{\text{linéaire}}_{\text{facile}}$ et $\underbrace{\text{bijective}}$

$\Leftrightarrow$ surj et inj

il faut faire les 2

car <u>$\dim \mathbb{R}(X) = \infty$</u>

<u>en dim finie</u>	<u>dim infinie</u>
bij	bij
$\Leftrightarrow$	$\Leftrightarrow$ inj <u>et</u> surj
inj	
$\Leftrightarrow$	
surj	

On regarde pour le moment :

$$f : \mathbb{R}_3(X) \longrightarrow \mathbb{R}_3(X)$$

c'est toujours un endomorphisme

f bijective $\Leftrightarrow$ f injective $\Leftrightarrow$ $\operatorname{Ker} f \overset{?}{=} \{0_{\mathbb{R}_3(X)}\}$

$$P = a + bx + cx^2 + dx^3$$

$$P' = b + 2cx + 3dx^2$$

$$P'' = 2c + 6dx$$

$$\alpha \cdot 1 + \beta x + \gamma x^2 + \delta x^3 = 0$$

$\Leftrightarrow \alpha = \beta = \gamma = \delta = 0$

car $\{1, x, x^2, x^3\}$ libre

$$f(P) = 0_{\mathbb{R}_3(X)} \Leftrightarrow P + P' + P'' = a + bx + cx^2 + dx^3 + b + 2cx + 3dx^2 + 2c + 6dx$$

$$= a+b+2c + x(b+2c+6d) + x^2(c+3d) + dx^3 = 0_{\mathbb{R}_3}$$

$$\Leftrightarrow \begin{cases} a+b+2c = 0 \\ b+2c+6d = 0 \\ c+3d = 0 \\ d = 0 \end{cases} \to d = 0 \quad \to \quad \begin{aligned} & a+b+2c = 0 \to a = 0 \\ & b+2c = 0 \\ & c = d = 0 \end{aligned}$$

donc $P = 0_{\mathbb{R}_3(X)}$

$\operatorname{Ker} f = \{0_{\mathbb{R}_3[X]}\}$

donc f est <u>injective</u> de $\mathbb{R}_3(X)$ dans lui même

donc elle est bijective (dim fini)

De même on montre que pour tout $n \in \mathbb{N}^*$

$$f : \mathbb{R}_n(X) \longrightarrow \mathbb{R}_n(X)$$

est encore bijective

$f : \mathbb{R}(X) \to \overline{\mathbb{R}(X) \text{ bijective}}$?

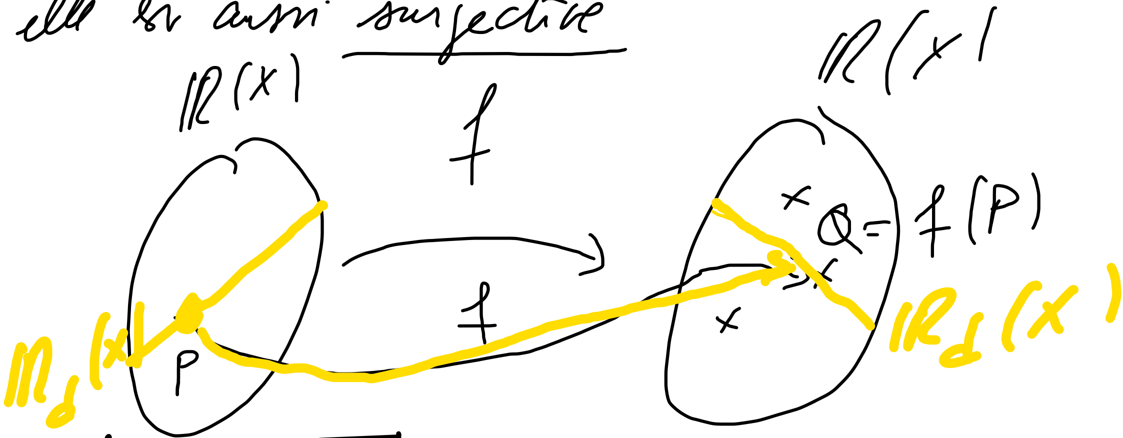
<u>Idée</u> $P \in \ker f$: $f(P) = 0_{\mathbb{R}(X)}$

$d = \deg(f)$ on sait que $f : \mathbb{R}_d(X) \to \mathbb{R}_d(X)$

est bijective : et $f(P) = 0 \iff \boxed{P = 0}$

$\ker f = \{0\}$ elle est <u>injective</u> : $\mathbb{R}(X) \to \mathbb{R}(X)$

∘ elle est aussi <u>surjective</u>



Soit $\boxed{Q \in \mathbb{R}(X)}$ $\qquad d = d^\circ Q$

On sait que $f : \mathbb{R}_d(X) \to \mathbb{R}_d(X)$ est <u>bijective</u>

donc on sait qu'il existe un unique $P \in \mathbb{R}_d(X) \subset \mathbb{R}(X)$

$$\underline{f(P) = Q}$$

$f : \mathbb{R}(X) \to \mathbb{R}(X)$ est bien <u>surjective</u>

$\boxed{\text{Feuille 11 - Exo 4}}$ E ev de dimension finie, $f \in \mathcal{L}(E,E)$

Montrer que : $\mathrm{Im} f = \mathrm{Ker} f \iff f^2 = 0_{\mathcal{L}(E)}$

et $n = 2\,\mathrm{rg}(f)$

$n = \dim E$.

n pair

$\Rightarrow$ On suppose que $\mathrm{Im} f = \mathrm{Ker} f$

Montrons que $\cdot$ $f^2 = f \circ f = 0_{\mathcal{L}(E)}$

$\cdot$ $n = 2\,\mathrm{rg}(f)$

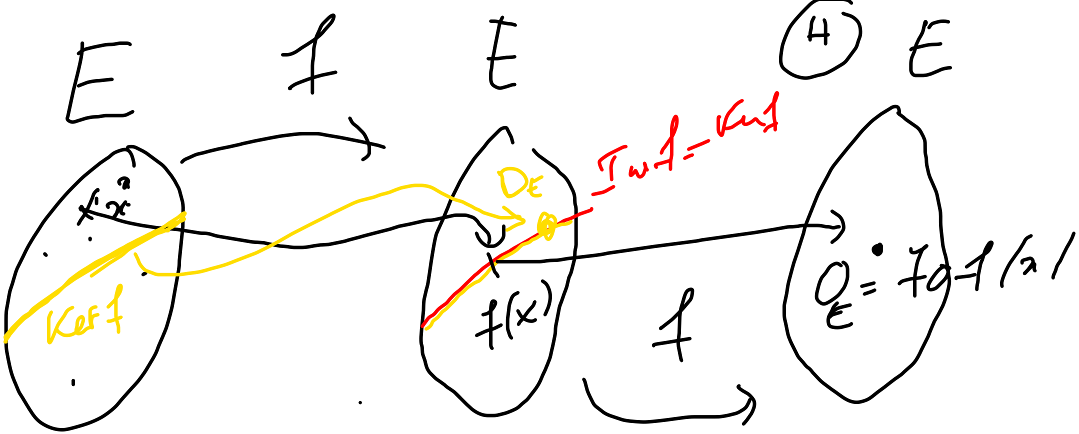
Hyp $\ker f = \operatorname{Im} f$ 　　　 But $f \circ f = 0$

$\boxed{n = 2\,rg(f)}$

$$f \circ f = 0_{\mathcal{L}(E)} \iff \underline{\forall x \in E} : \underline{f \circ f(x) = 0_E}$$

or $\underbrace{f \circ f}(x) = f\big(\underbrace{f}(x)\big) = 0$

$\in \operatorname{Im}(f) = \ker f$ donc $f(x) \in \ker f$



Il reste à prouver que $n = 2\,\mathrm{rg}(f)$:

Par le th. du rang :

$$n = \dim E = \dim \operatorname{Ker} f + \dim \operatorname{Im} f$$

mais $\operatorname{Im} f = \operatorname{Ker} f \Rightarrow \dim \operatorname{Ker} f = \dim \operatorname{Im} f$

donc
$$\begin{aligned} n &= \dim \operatorname{Ker} f + \dim \operatorname{Im} f \\ &= 2 \dim \operatorname{Im} f = 2\,\mathrm{rg}(f) \end{aligned}$$

$\Leftarrow$ Si $f \circ f = 0$ et $\underline{n = 2\,\mathrm{rg}(f)}$

$$\boxed{\operatorname{Im} f = \operatorname{Ker} f}$$

$f \circ f$: $\forall x \; f(f(x)) = 0 \Leftrightarrow \forall x,\ f(x) \in \operatorname{Ker} f$ (1) (2)

$\Leftrightarrow \boxed{\operatorname{Im} f \subset \operatorname{Ker} f}$

Th Rang $n = \dim \operatorname{Ker} f + \mathrm{rg}(f) = 2\,\mathrm{rg}(f) \Rightarrow \underline{\dim \operatorname{Ker} f = \dim \operatorname{Im} f = \mathrm{rg}(f)}$

donc par le th. du rang : $\dim \operatorname{Ker} f = 1$

$\left(\text{car } 3 = \dim \overbrace{\operatorname{Ker} f}^{1} + \overbrace{rg(f)}^{2} \right)$

Comme $\left.\begin{array}{l} \operatorname{Vect}(u_3) \subset \operatorname{Ker} f \\ \dim \operatorname{Ker} u = 1 \end{array}\right\} \operatorname{Ker} f = \operatorname{Vect}\{u_3\}$

Sinon : je cherche Ker f : $X = (x,y,z) \in \operatorname{Ker} f$: $f(X) = 0_{\mathbb{R}^3}$

clairement

$$\begin{cases} y - z = 0 \\ -2x - y + z = 0 \\ -2x - y + z = 0 \end{cases} \iff \begin{cases} y = z \\ 2x + y - z = 0 \end{cases}$$

$$\iff \begin{cases} \boxed{z = y} \\ 2x + y - y = 0 \end{cases} \Rightarrow \boxed{\cancel{}} \ \boxed{x = 0}$$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ y \end{pmatrix} = y \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} : \operatorname{Ker} f = \operatorname{Vect}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \operatorname{Vect}(e_2 + e_3) = \operatorname{Vect}(-2e_2 - 2e_3) = \operatorname{Vect}(u_3)$$

$$f(E_{12}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} + \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} c & d+a \\ 0 & c \end{pmatrix}$$

$$=$$

$$f(E_{21}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} b & 0 \\ d & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} = \begin{pmatrix} b & 0 \\ a+d & b \end{pmatrix}$$

$$f(E_{22}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & b \\ c & 2d \end{pmatrix}$$