L1CUPGE - Gpe 4 - Semaine 11

TD 1

le 13/04/21

départ à 7^h45

Feuille 11 - Exercice 1

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$
$$(x,y,z) \longmapsto f(x,y,z) = (x+y, y+2z)$$

1) $f \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$ exercice

1') Quelle est la matrice de f dans la base canonique de $\mathbb{R}^3$ $(e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1))$
$\mathbb{R}^2$ $(\tilde{e}_1 = (1,0), \tilde{e}_2 (0,1))$

$f(e_1) = (1,0) = \tilde{e}_1$ $\quad x = 1$, $y = z = 0$

$f(e_2) = (1,1) = \tilde{e}_1 + \tilde{e}_2$ $\quad x = z = 0$, $y = 1$

$f(e_3) = (0,2) = 2\tilde{e}_2$

$$\boxed{M_{\mathcal{B}_C}(f) = \begin{array}{c} \begin{array}{ccc} f(e_1) & f(e_2) & f(e_3) \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \end{array} \begin{array}{c} \tilde{e}_1 \\ \tilde{e}_2 \end{array}}$$

$$\boxed{M_{\mathcal{BC}}(f) = \begin{array}{c} \begin{array}{ccc} f(e_1) & f(e_2) & f(e_3) \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \end{array} \begin{array}{l} \tilde{e}_1 \\ \tilde{e}_2 \end{array}}$$

$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\mathcal{BC}}$: $f(X) = M_{\mathcal{BC}}(f) X$

$$= \underset{2\times \underline{\underline{3}}}{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}} \underset{\underline{\underline{3}}\times 1}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \underset{2\times 1}{\begin{pmatrix} x+y \\ y+2z \end{pmatrix}} = \underset{f(X)}{f(x,y,z)}$$

(2) Montrer que f est <u>surjective</u> $\Leftrightarrow \operatorname{Im}(f) = f(\mathbb{R}^3) = \mathbb{R}^2$

$\Leftrightarrow \dim \operatorname{Im}(f) = 2$

<u>Solution 1</u> : Revenir à la définition : $\forall \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2, \exists \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$

$f(x,y,z) = (a,b)$.

$\Leftrightarrow \forall (a,b) \in \mathbb{R} \quad \exists (x,y,z) \in \mathbb{R}^3$:

$$f(x,y,z) = (x+y \,, y+2z) = (a,b)$$

$\Leftrightarrow$ le système $\begin{cases} x+y=a \\ y+2z=b \end{cases}$ d'inconnues x,y,z admet toujours au moins une solution

$$\begin{cases} y = a - x \\ y + 2z = a - x + 2z = b \Rightarrow 2z = b + x - a \end{cases}$$

$$z = \frac{b+x-a}{2}$$

$\mathrm{Vect}\begin{pmatrix} 1 \\ -1 \\ 1/2 \end{pmatrix} = \mathrm{Vect}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

Soit $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ a-x \\ \frac{b+x-a}{2} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ a \\ \frac{b-a}{2} \end{pmatrix}} + x \underbrace{\begin{pmatrix} 1 \\ -1 \\ 1/2 \end{pmatrix}}_{\text{le noyau de } f}$

$f\left(0, a, \frac{b-a}{2}\right)$
$= \left(0+a, a+2 \cdot \frac{b-a}{2}\right) = (a,b) \longrightarrow f$ est bien surjective

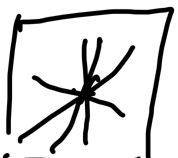
$\mathrm{Ker}(u) = \mathrm{Vect}\, 2u$

Solution 2 (+ simple)

On sait (cours) que $\{f(e_1), f(e_2), f(e_3)\}$ est une famille génératrice de $\mathrm{Im}\, f$

$$\Big[\ X \in \mathrm{Im}\, f \ : \ \exists (x,y,z) \in \mathbb{R}^3 \ : \ f(x,y,z) = X$$

$$\Leftrightarrow f(xe_1 + ye_2 + ze_3) \underset{fl}{=} x f(e_1) + y f(e_2) + z f(e_3) \in \mathrm{Vect}\{f(e_1), f(e_2), f(e_3)\}\ \Big]$$



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$$\mathrm{Im}\, f = \mathrm{Vect}\left\{ \begin{pmatrix}1\\0\end{pmatrix} \begin{pmatrix}1\\1\end{pmatrix} \begin{pmatrix}0\\2\end{pmatrix} \right\} \subset \underline{\underline{\mathbb{R}^2}}$$

$\underline{\dim \geq 2}$ ← sont libres

$$\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\2\end{pmatrix}$$

les 3 sont liés

$\mathrm{Im}\, f \subset \mathbb{R}^2$

$\dim \mathrm{Im}\, f \leq 2$

$'' = 2 \underset{si}{\Longleftrightarrow} \mathrm{Im}\, f = \mathbb{R}^2$

$$= \mathrm{Vect}\left\{ \begin{pmatrix}1\\0\end{pmatrix} \begin{pmatrix}0\\2\end{pmatrix} \right\}$$

$$= \mathbb{R}^2$$

2') f est-elle **injective** ? $\iff \operatorname{Ker} f = \{0_{\mathbb{R}^3}\}$

Théorème du rang nous dit que :

$$\dim \mathbb{R}^3 = 3 = \dim \operatorname{Ker} f + \underbrace{\dim \operatorname{Im} f}_{=2}$$

donc $\boxed{\dim \operatorname{Ker} f = 1} \Rightarrow f$ n'est pas **injective**

3) **Montrer que** $(2,-2,1) \in \operatorname{Ker} f$

$$f(2,-2,1) = (2-2\ ,\ -2+2.1) = (0,0)$$

donc $(2,-2,1) \in \operatorname{Ker} f$

Comme $\dim \operatorname{Ker} f = 1$

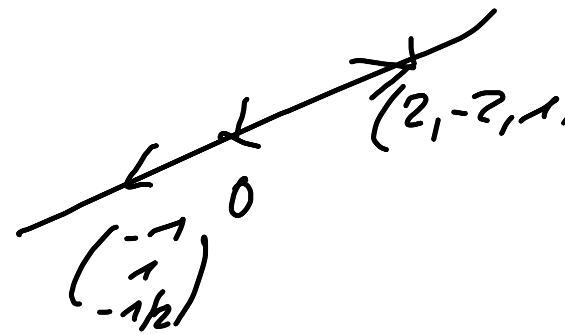
$$\boxed{\operatorname{Ker} f = \operatorname{Vect}(2,-2,1)}$$

Dans le cas général :

$$X = (x, y, z) \in \operatorname{Ker} f \; : \; f(x, y, z) = (0, 0)$$

$$\Leftrightarrow \begin{cases} x + y = 0 \\ y + 2z = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = -y \\ y = y \\ z = -y/2 \end{cases}$$



$$\Leftrightarrow X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ y \\ -y/2 \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \\ -1/2 \end{pmatrix}$$

$$= -\frac{y}{2} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

$$\in \operatorname{Vect}\left\{ \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\} = \operatorname{Ker} f$$

$$M_f = \begin{pmatrix} 1 & 1 & 0 \\ \underbrace{0 \quad 1}_{} & & 2 \end{pmatrix}$$

libre en $\operatorname{Im} f = \mathbb{R}^2$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$f(e_1) + \frac{1}{2} f(e_3) = f(e_2)$$

$$f\left(e_1 - e_2 + \frac{1}{2} e_3\right) = 0_{\mathbb{R}^2}$$

$$\begin{pmatrix} 1 \\ -1 \\ 1/2 \end{pmatrix} = e_1 - e_2 + \frac{e_3}{2} \in \underbrace{\operatorname{Ker} f}_{\dim 1} \longrightarrow \operatorname{Ker} f = \operatorname{Vect}\begin{pmatrix} 1 \\ -1 \\ 1/2 \end{pmatrix}$$

Feuille 11 - exo 12

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$(x,y,z) \longmapsto (-4x - 6y,\ 3x + 5y,\ 3x + 6y + 5z)$$

1) Matrice de f dans ~~cette~~ base canonique

$$A = \mathrm{mat}_{BC}(f) = \begin{array}{c} \begin{array}{ccc} f(e_1) & f(e_2) & f(e_3) \end{array} \\ \begin{pmatrix} -4 & -6 & 0 \\ 3 & 5 & 0 \\ 3 & 6 & 5 \end{pmatrix} \end{array} \begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array} \quad 3$$

$$3$$

$$= x \underbrace{\begin{pmatrix} -4 \\ 3 \\ 3 \end{pmatrix}}_{f(e_1)} + y \underbrace{\begin{pmatrix} -6 \\ 5 \\ 6 \end{pmatrix}}_{f(e_2)} + z \underbrace{\begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}}_{f(e_3)}$$

$f(x,y,z)$

$f(xe_1 + ye_2 + ze_3)$

$= x f(e_1) + y$

$f(1)$

$f(X)$

$f(X^2)$

$$f(e_1) = (-4, 3, 3) = -4e_1 + 3e_2 + 3e_3$$

$$\| $$

$$f(\underset{x}{1}, \underset{y}{0}, \underset{z}{0})$$

$$f(e_2) = (-6, 5, 6)$$

$$f(e_3) = (0, 0, 5)$$

$$\begin{pmatrix} -4 & -6 & 0 \\ 3 & 5 & 0 \\ 3 & 6 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4x-6y \\ 3x+5y \\ 3x+6y+5z \end{pmatrix}$$

ça marche !

$f(u) \; f(v) \; f(w)$

$$\Big(\qquad \Big)_{u,v,w} = M_{\mathcal{B}}(f) \qquad f(x,y,z)$$

② Montrer $\underset{u}{(-2,1,0)}$ $\underset{v}{(-1,1,-1)}$ $\underset{w}{(0,0,1)}$ base $\mathcal{B}$ de $\mathbb{R}^3$

$\mathbb{R}^3$ étant de dim 3 il suffit de montrer que $\{u,v,w\}$ est libre : $\alpha u + \beta v + \gamma w = 0_{\mathbb{R}^3}$ $\alpha,\beta,\gamma \in \mathbb{R} \rightarrow \alpha=\beta=\gamma=0$

$$\alpha\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + \gamma\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} -2\alpha - \beta = 0 \rightarrow \beta = 2\alpha \\ \alpha + \beta = 0 \rightarrow \beta = -\alpha \\ -\beta + \gamma = 0 \end{cases}$$

$\boxed{\alpha = 0}$ $\boxed{\beta = 0}$

$\hookrightarrow \gamma = \beta$ $\quad \gamma = 0$

la famille est bien libre !!

$\mathrm{Im}\, f \subset \mathbb{R}^2$

$$\begin{cases} \dim \mathrm{Im}\, f = 0 & \mathrm{Im}\, f = \{0_{\mathbb{R}^2}\} \quad f \equiv 0 \\ \dim \mathrm{Im}\, f = 1 & \mathrm{Im}\, f \ \underline{\text{droite}} \\ \dim \mathrm{Im}\, f = 2 & \longrightarrow \mathrm{Im}\, f = \mathbb{R}^2 \end{cases}$$