L1 CUPGE - Gpe 4 - Semaine 11 - Td 2

14 Avril 2021

13^h30

Feuille 11 - Exo 12 (suite et fin).

1) $f : \mathbb{R}^3 \to \mathbb{R}^3$ $(x,y,z) \mapsto f(x,y,z) = (-4x-6y, 3x+5y, 3x+6y+5z)$

$$M_{\mathcal{BC}}(f) = \begin{pmatrix} -4 & -6 & 0 \\ 3 & 5 & 0 \\ 3 & 6 & 5 \end{pmatrix} \begin{matrix} e_1 \\ e_2 \\ e_3 \end{matrix}$$ (first column: $f(e_1)$)

$f(x,y,z) = M_{\mathcal{BC}}(f) \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

2) $\mathcal{B} = \{ \underbrace{(-2,1,0)}_{u_1}, \underbrace{(-1,1,-1)}_{u_2}, \underbrace{(0,0,1)}_{u_3} \}$ base de $\mathbb{R}^3$ (coordinates x, y, z)

3) Matrice de f dans la base $\mathcal{B}$.

$$Mat_{\mathcal{B}}(f) = \begin{pmatrix} a & \vdots & ? \\ b & & \vdots \\ c & \vdots & \end{pmatrix} \begin{matrix} u_1 \\ u_2 \\ u_3 \end{matrix}$$ (columns: $f(u_1)$, $f(u_2)$, $f(u_3)$)

$f(u_1) = a u_1 + b u_2 + c u_3$

$$f(u_1) = \begin{pmatrix} -4 & -6 & 0 \\ 3 & 5 & 0 \\ 3 & 6 & 5 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}_{\mathcal{BC}} = 2e_1 - e_2 = -u_1$$

$$\mathrm{Mat}_{\mathcal{B}}(f) = \begin{pmatrix} a & \vdots & ? \\ b & \vdots & \\ c & \vdots & \end{pmatrix} \begin{matrix} u_1 \\ u_2 \\ u_3 \end{matrix}$$

(columns: $f(u_1)$, $f(u_2)$, $f(u_3)$)

$f(u_1) = a u_1 + b$

$$f(u_1) = \begin{pmatrix} -4 & -6 & 0 \\ 3 & 5 & 0 \\ 3 & 6 & 5 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}_{BC} = 2e_1 - e_2 = -u_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}_{\{u_1, u_2, u_3\}}$$

$$f(u_2) = \begin{pmatrix} -4 & -6 & 0 \\ 3 & 5 & 0 \\ 3 & 6 & 5 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix} = 2 \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = 2u_2$$

$$f(u_3) = \begin{pmatrix} -4 & -6 & 0 \\ 3 & 5 & 0 \\ 3 & 6 & 5 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} = 5u_3$$

$$M_{\{u_1, u_2, u_3\}}(f) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{matrix} u_1 \\ u_2 \\ u_3 \end{matrix}$$

(columns: $f(u_1)$, $f(u_2)$, $f(u_3)$)

$$M_{\{u_1,u_2,u_3\}}(f) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

(columns: $f(u_1)$, $f(u_2)$, $f(u_3)$; rows: u_1, u_2, u_3)

En déduire pour tout $n \in \mathbb{N}$ l'expression de $f^n = f \circ f \ldots \circ f$

$$M_{f \circ \ldots \circ f} = (M_f)^n \qquad M_{BC}(f^n) = \begin{pmatrix} -4 & -6 & 0 \\ 3 & 5 & 0 \\ 3 & 6 & 5 \end{pmatrix}^n$$

$$M_{BC}(f) = \begin{pmatrix} -4 & -6 & 0 \\ 3 & 5 & 0 \\ 3 & 6 & 5 \end{pmatrix} = P^{-1} M_{B=\{u_1 u_2 u_3\}}(f)\, P$$

$$= P^{-1} \underbrace{\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}}_{D} P$$

$$P = \begin{pmatrix} -2 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

(columns: u_1, u_2, u_3; rows: e_1, e_2, e_3)

matrice de passage

$$P = \begin{pmatrix} -2 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$
(columns $u_1\ u_2\ u_3$, rows e_1, e_2, e_3)

Calcul de P^{-1}

$$\left(\begin{array}{ccc|ccc} -2 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\underset{\mathcal{B}_C \to \mathcal{B}}{P} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$
(columns $u_1 \cdots u_3$ / $\mathcal{B}$, rows e_1, ~~$\mathcal{B}_C$~~ e_2, e_3)

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$$\mathrm{Mat}_{\mathcal{B}'}(f) = P^{-1} \underset{\mathcal{B}_C}{M}(f)\, P$$

$$A' = P^{-1} A P$$

$$\left(\begin{array}{ccc|ccc} -2 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\vdots$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 \end{array}\right)$$

$$\underbrace{}_{I_3} \qquad \underbrace{}_{P^{-1}}$$

$$\overset{P}{\begin{pmatrix} -2 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}} \overset{P^{-1}}{\begin{pmatrix} -1 & -1 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_{BC}(f) = P^{-1} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix} P = \begin{pmatrix} -4 & -6 & 0 \\ 3 & 5 & 0 \\ 3 & 6 & 5 \end{pmatrix}$$

$$Tr(A) = \sum \text{elts diag} \qquad 6 = 6$$

$$M_{BC}(f) = P^{-1}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix} P$$

$$\underbrace{M_{BC}(f)}_{A} = \underbrace{P^{-1}}_{P}\ \underbrace{D}_{A'}\ \underbrace{P}_{P^{-1}}$$

$$M_{BC}(f^n) = M_{BC}(f)^n = \left[P^{-1} D P\right]^n$$

$$= P^{-1} D P P^{-1} D P P^{-1} D P \ldots P^{-1} D P$$

$$= P^{-1} D \ldots D P = P^{-1} D^n P$$

$$f^n(x,y,z) = M_{BC}(f^n)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = P^{-1} D^n P \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$f^n(x,y,z) = M_{BC}(f^n)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = P^{-1} D^n P \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= P^{-1} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}^n P \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= P^{-1} \begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 5^n \end{pmatrix} P \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -1 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 5^n \end{pmatrix} \begin{pmatrix} -2 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -1 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} -2(-1)^n & (-1)^{n+1} & 0 \\ 2^n & 2^n & 0 \\ 0 & -5^n & 5^n \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -1 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} -2(-1)^n & (-1)^{n+1} & 0 \\ 2^n & 2^n & 0 \\ 0 & -5^n & 5^n \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 2(-1)^n - 2^n & (-1)^n - 2^n & 0 \\ -2(-1)^n + 2^{n+1} & (-1)^{n+1} - 2^{n+1} & 0 \\ -2(-1)^n + 2^{n+1} & (-1)^{n+1} - 2^{n+1} - 5^n & 5^n \end{pmatrix}}_{M_{BC}(f^n)} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} (2(-1)^n - 2^n)x + ((-1)^n - 2^n)y \\ (2^{n+1} - 2(-1)^n)x + ((-1)^{n+1} - 2^{n+1})y \\ (2^{n+1} - 2(-1)^n)x + ((-1)^{n+1} - 2^{n+1} - 5^n)y + 5^n z \end{pmatrix} = f^n(x,y,z)$$

$$f^{2021}(x,y,z) = \left((-2 - 2^{2021})x + (-1 - 2^{2021})y, \ \ldots \right)$$

A matrice de f dans la BC

A' ——————— B'

~~$A = P^{-1} A' P$~~ où P = matrice de passage de B à B'

Attention : il y a une petite erreur !

$A' = P^{-1} A P$ (cours)

$$A = P A' P^{-1}$$

$$A = \mathrm{mat}(f, B) = P\, \mathrm{mat}(f, B')\, P^{-1}$$

Exercice 10 – F11

$f: \mathbb{R}^3 \to \mathbb{R}^3$
$(x,y,z) \mapsto f(x,y,z) = (y-z, -2x-y+z, -2x-y+z)$

1) $M_{BC}(f) = \begin{pmatrix} 0 & 1 & -1 \\ -2 & -1 & 1 \\ -2 & -1 & 1 \end{pmatrix}$

colonnes $= f(e_1)$, $f(e_2)$, $f(e_3)$

$f(e_1)$, $f(e_2)$ → 2 vecteurs libres

$3 = \dim \ker f + \dim \operatorname{Im} f$

~~0~~	~~3~~
1	2)
2	1)
3	0)

$f(e_3) = -f(e_2)$
$f(e_3) + f(e_2) = 0$
$f(e_3 + e_2) = 0$
$\Leftrightarrow e_2 + e_3 \in \ker f$, $e_2 + e_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$\operatorname{Im} f = \operatorname{Vect}\{f(e_1), f(e_2), f(e_3)\}$ $\Rightarrow$ $\dim \ker f \geq 1$
($f(e_1), f(e_2)$ libre)

⇓

$\dim \operatorname{Im} f \leq 2$

car $3 = \dim \ker f + \dim \operatorname{Im} f$ (≥, ≤)

$\operatorname{Im} f$ ≤ 2

$\operatorname{Im} f = \operatorname{Vect}\{f(e_1), f(e_2)\}$ base
dim 2
$\Rightarrow \ker f$ dim 1
$\ker f = \operatorname{Vect}\{e_2 + e_3\}$

2) $\underset{\mathcal{E}'}{\mathrm{mat}}(f) = \begin{pmatrix} 1 & -1 & -2 \\ 1 & -1 & -2 \\ -1 & 1 & 0 \end{pmatrix}$ with columns $f(e_3)$, $f(e_2)$, $f(e_1)$ and rows e_3, e_2, e_1

$\mathcal{E}' = \{e_3, e_2, e_1\}$

$f(e_3) = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}_{\mathcal{E}} = -e_1 + e_2 + e_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}_{\mathcal{E}'}$ (rows e_3, e_2, e_1)

$f(e_2) = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}_{\mathcal{E}} = e_1 - e_2 - e_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}_{\mathcal{E}'}$

$f(e_1) = \begin{pmatrix} 0 \\ -2 \\ -2 \end{pmatrix} = 0 \cdot e_1 - 2e_2 - 2e_3 = \begin{pmatrix} -2 \\ -2 \\ 0 \end{pmatrix}$

Autre méthode

trop de calculs !

$\mathcal{E}' = \{e_3, e_2, e_1\}$

$$\text{mat}_{\mathcal{E}'}(f) = P^{-1}\,\text{mat}_{\mathcal{E}}(f)\,P$$

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$\mathcal{E}' = (e'_1, e'_2, e'_3)$, with $e'_1 = e_3$, $e'_2 = e_2$, $e'_3 = e_1$

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{matrix} e_1 \\ e_2 \\ e_3 \end{matrix}$$

$$\boxed{P^{-1}\,\text{mat}_{\mathcal{E}}(f)\,P = \text{mat}_{\mathcal{E}'}(f)}$$

$$P^{-1} = \begin{pmatrix} ? \end{pmatrix} = P$$

$$P^{-1} A P = PAP = \dots$$

Exercice

$$\hat{A'} = P^{-1} \hat{A} P$$

$$A = P A' P^{-1}$$

$$\boxed{A^n = P A'^n P^{-1}}$$

Exercice 15

$f : \mathbb{R}_3(X) \longrightarrow \mathbb{R}_3(X)$

$P \longmapsto f(P) = P'' \in \mathbb{R}_3(X)$ $\qquad B = \{1, X, X^2, X^3\}$

1) f est linéaire : $P, Q \in \mathbb{R}_3(X)$, $\lambda \in \mathbb{R}$

$$f(P + \lambda Q) = (P + \lambda Q)'' = P'' + \lambda Q'' = f(P) + \lambda f(Q)$$

f est bien linéaire : c'est un endomorphisme de $\mathbb{R}^3$

2) Image et noyau de f

• Noyau $P \in \operatorname{Ker} f$: $P = a + bx + cx^2 + dx^3 \in \mathbb{R}_3(X)$

$$f(P) = P'' = 0_{\mathbb{R}_3(X)} = 2c + 6dx$$

$\Downarrow$

$\boxed{c = d = 0}$

donc

$\operatorname{Ker} f = \{a + bx \; ; \; a, b \in \mathbb{R}\}$

donc $\ker f = \underline{\mathrm{Vect}\{1, x\}}$ sev de dim 2 dans $\mathbb{R}_3(X)$
il admet pour base $\underline{\underline{\{1, X\}}}$ (on sait que c'est
generateur et cm libre car extrait de la bc)

Donc $\mathrm{Im}(f)$ sera de dimension $4-2=2$ par le th. du rang

Déterminons $\mathrm{Im}(f)$:

$$\mathrm{Im}\, f = \{ f(P) = P'' \ , \ P \in \mathbb{R}_3(X) \} \qquad P = a + bx + cx^2 + dx^3$$

$$= \{ 2c + 6dx \ , \ \text{où } c, d \in \mathbb{R} \}$$

$$= \underline{\mathrm{Vect}\{1, X\}} = \mathbb{R}_1(X) \text{ sev des pol de } d^\circ \leq 1$$

le th. du rang est bien vérifié $4 = \dim \mathbb{R}_4(X) = 2 + 2 = \dim \ker f + \dim \mathrm{Im}(f)$

③ Matrice de f dans la base $\mathcal{B} = \{1, X, X^2, X^3\}$

$$\mathcal{M}_{\mathcal{B}}(f) = \begin{array}{cc} \begin{array}{cccc} f(1) & f(X) & f(X^2) & f(X^3) \end{array} & \\ \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{array}{c} 1 \\ X \\ X^2 \\ X^3 \end{array} \end{array}$$

$f(1) = (1)'' = 0$

$f(X) = (X)'' = 0$

$f(X^2) = \underline{\underline{2}}$

$f(X^3) = 6X$

④ Matrice de $f \circ f$ de 2 manière $\neq$.

$f \circ f = f^2$

$$\mathcal{M}_{\mathcal{B}}(f^2) = \mathcal{M}_{\mathcal{B}}(f)^2 = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$M_B(f^2) = M_B(f)^2 = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad f^2 = 0_{\mathcal{L}(\mathbb{R}_3(x))}$$

$$P = a + bx + cx^2 + dx^3 = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}_B$$

$$f(P) = P'' = 2c + 6dx + 0x^2 + 0x^3$$

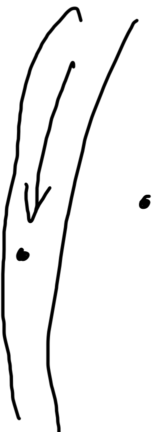
$$f(P) = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}_B = \begin{pmatrix} 2c \\ 6d \\ 0 \\ 0 \end{pmatrix}_B \rightarrow 2c \cdot 1 + 6dx + 0x^2 + 0x^3 = 2c + 6dx$$

Autre démonstration

$p \in \mathbb{R}_3(X)$

$$f^2(p) = f \circ f(p)$$
$$= f(f(p))$$
$$= f(p'') = (p'')'' = p^{(4)}$$
$$= 0$$

↑ $d^\circ p \leq 3$



$$f^2 = 0$$

CQFD