L1 CUPGE ~ Semaine 9 ~ TD3 | Gpe 4

1 Avril 2021

Dèpart 10h

Exercice 4-4

$$x + x^3 + \overset{\overset{0}{\shortparallel}}{o(x^5)}$$

$$\shortparallel$$

$$DL_{5,0} \text{ de } f(x) = \frac{x+x^3}{1+x+x^2} = \underline{\underline{(x+x^3)}} \cdot \frac{1}{1+x+x^2}$$

$$x \sim \cancel{x} \qquad \shortparallel$$

$$DL_5$$

DL_4 suffit

$$\frac{1}{1+x+x^2} = \frac{1}{1+u} \quad \text{avec } u = x + x^2 \xrightarrow[x \to 0]{} 0 \quad \boxed{u \underset{0}{\sim} x}$$

$$= 1 - u + u^2 - u^3 + u^4 + o(u^4)$$

$$= 1 - (x+x^2) + (x+x^2)^2 - (x+x^2)^3 + (x+x^2)^4 + o(x^4)$$

$$= 1 - x - \underline{x^2} + \underline{x^2} + 2x^3 + \underline{\underline{x^4}} - (x^3 + 3x^4 + o(x^4)) + \underline{\underline{x^4}} + o(x^4)$$

$$(a+b)^3 = a^3 + 3a^2b + 3b^2a + b^3$$

$$= 1 - x + x^3 - x^4 + o(x^4) \text{ on reporte}$$

$$f(x) = (x+x^3)(1 - x + x^3 - x^4 + o(x^4))$$

$$= x - x^2 + \cancel{x^4} + x^3 - \cancel{x^4} + o(x^4) = x - x^2 + x^3 + o(x^4)$$

$$f(x) = (x + x^3)(1 - x + x^3 - x^4 + o(x^4))$$

$$= x - x^2 + \cancel{x^4} + x^3 - \cancel{x^4} + o(x^4) = x - x^2 + x^3 + o(x^4)$$

$$= x - x^2 + o(x^2) \quad DL_2$$

$$= x - x^2 + x^3 + o(x^3) \quad DL_3$$

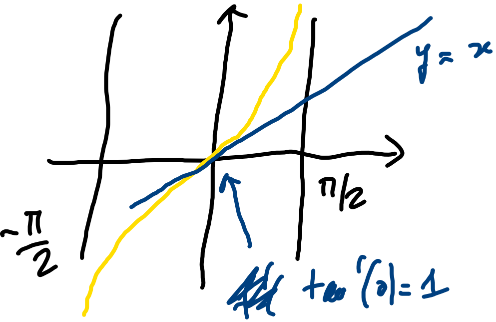
DL_5

$$= x - \underline{x^2} + \cancel{x^4} - x^5 + \underline{x^3} - \cancel{x^4} + o(x^5)$$

$$= \underbrace{x - x^2 + x^3}_{DL_4} \underbrace{- x^5 + o(x^5)}_{o(x^4)} \quad \underline{DL_{5,0}}$$

Exercice 7 - F9

$$f : \left]-\pi/2, +\pi/2\right[\longrightarrow \mathbb{R}$$
$$x \longmapsto f(x) = e^x \tan(x)$$



$$(\tan)' = \frac{1}{\cos^2} = \frac{\cos^2+\sin^2}{\cos^2} = 1+\frac{\sin^2}{\cos^2} = 1+\tan^2$$

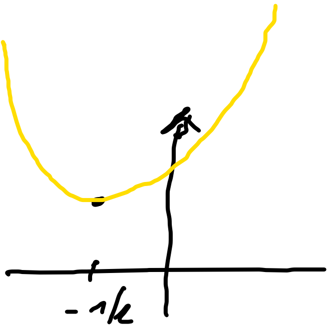
① Montrer que f est bijective $\Leftrightarrow$ f injective & surj : $f\left(\left]-\frac{\pi}{2}, +\frac{\pi}{2}\right[\right) = \mathbb{R}$

Etudions f :

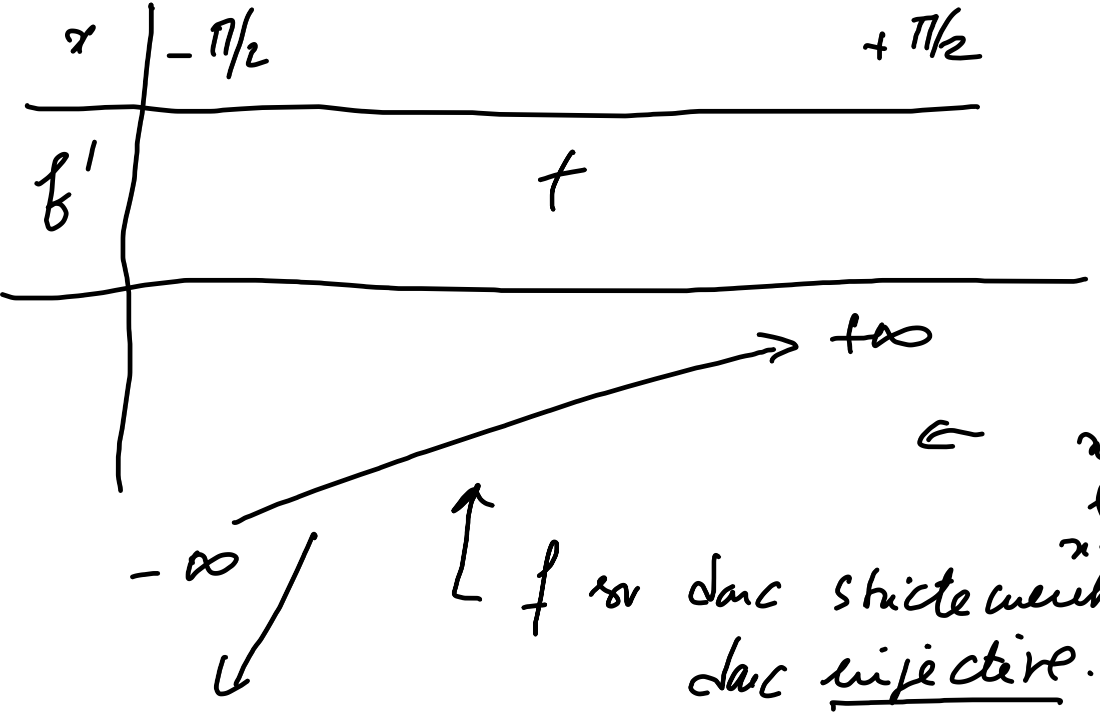
$$f'(x) = e^x \tan(x) + e^x (\tan x)'$$
$$= e^x \tan(x) + e^x (1+\tan^2 x)$$
$$= e^x (\underbrace{\tan^2 x + \tan x + 1}_{>0\ \forall x \in \mathbb{R}})$$
$$= e^x P(\tan x) \text{ où } P(u) = u^2+u+1$$

donc $f'(x) = 0 \Leftrightarrow P(\tan x) = 0 \rightarrow P(u) > 0 \ \forall u \in \mathbb{R}$

$\Delta = 1-4 = -3$ pas de racines réelles



Donc $f'(x) > 0$ pour tout $x \in]-\pi/2, +\pi/2[$



$$f(x) = \overbrace{\underbrace{e^x}_{e^{\pi/2}}}^{e^{-\pi/2}} \overbrace{\underbrace{\tan x}_{\to +\infty}}^{-\infty} \quad \text{ou } -\infty$$

$$\downarrow$$

$$\lim_{x \to \pi/2-} f(x) = +\infty$$

$\leftarrow$

$$\lim_{x \to -\pi/2+} f(x)$$

f est donc strictement croissante sur $]-\pi/2 + \pi/2[$ donc <u>injective</u>.

$f(]-\pi/2, +\pi/2[) = \mathbb{R}$: f est <u>surjective</u> donc <u>bijective</u>

$$\begin{tikzcd}
\left]-\pi/2 , +\pi/2\right[\arrow[r, shift left, "f"] & \mathbb{R} \arrow[l, shift left, "f^{-1}"]
\end{tikzcd}$$

f^{-1} application <u>réciproque</u>

$$\boxed{f^{-1}(f(x)) = \underline{\underline{x}}}$$

$y = e^x \tan x = f(x)$

$\downarrow$

$x = f^{-1}(y)$

semble impossible d'expliciter f^{-1}

2) <u>Donner</u> le $DL_{3,0}$ de f^{-1}

f^{-1} admet un $DL_{3,0}$ (admettons le)

$f'(0) \neq 0$

$$f^{-1}(x) = \alpha + \beta x + \gamma x^2 + \delta x^3 + o(x^3) \longleftarrow$$

on cherche $\alpha, \beta, \gamma, \delta$

• Cherchons une $DL_{3,0}$ de $f(x) = e^{x} \tan(x)$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)\right)\left(\right.$$

$$\tan x = \frac{\sin(x)}{\cos(x)} = \left(\underline{\underline{x}} - x^3/6 + o(x^3)\right) \cdot \frac{1}{1 - \frac{x^2}{2} + o(x^2)} \longleftarrow \underbrace{DL_{2,0}}$$

$$\frac{1}{1+u} = 1 - u + o(u) = 1 + \frac{x^2}{2} + o(x^2)$$

$$u \sim -x^2/2$$

$$= \left(x - \frac{x^3}{6} + o(x^3)\right)\left(1 + \frac{x^2}{2} + o(x^2)\right)$$

$$= x + \frac{x^3}{2} - \frac{x^3}{6} \qquad + o(x^3)$$

$$\frac{1}{2} - \frac{1}{6} \qquad \frac{3-1}{6}$$

$$= x + \frac{x^3}{3} + o(x^3)$$

$$f(x) = \tan(x) = \underset{\underset{0}{\|}}{f(0)} + \underbrace{f'(0)}_{1} x + \underbrace{f''(0)}_{0} \frac{x^2}{2} + \underbrace{f'''(0)}_{2} \cdot \frac{x^3}{3!} + o(x^3)$$

$$= x + \frac{x^3}{3} + o(x^3)$$

$$(\tan x)' = 1 + \tan^2(x) \longrightarrow f'(0) = 1$$

$$(q^\alpha)' = \alpha q' q^{\alpha - 1}$$

$$(\tan x)'' = 2(1 + \tan^2 x) \cdot \tan x \longrightarrow \underline{f''(0) = 0}$$

$$(\tan x)''' = [2 \tan x + 2 \tan^3 x]' \qquad (\tan^3)' = 3(\tan)' \tan^2$$

$$= 2(1 + \tan^2 x) + 2 \cdot \underbrace{3(1 + \tan^2 x)}_{} \underline{\underline{\tan^2}} x$$

$$(\tan)^{(3)}(0) = 2$$

Retour au plan

$$f(x) = e^{x}\tan(x)$$

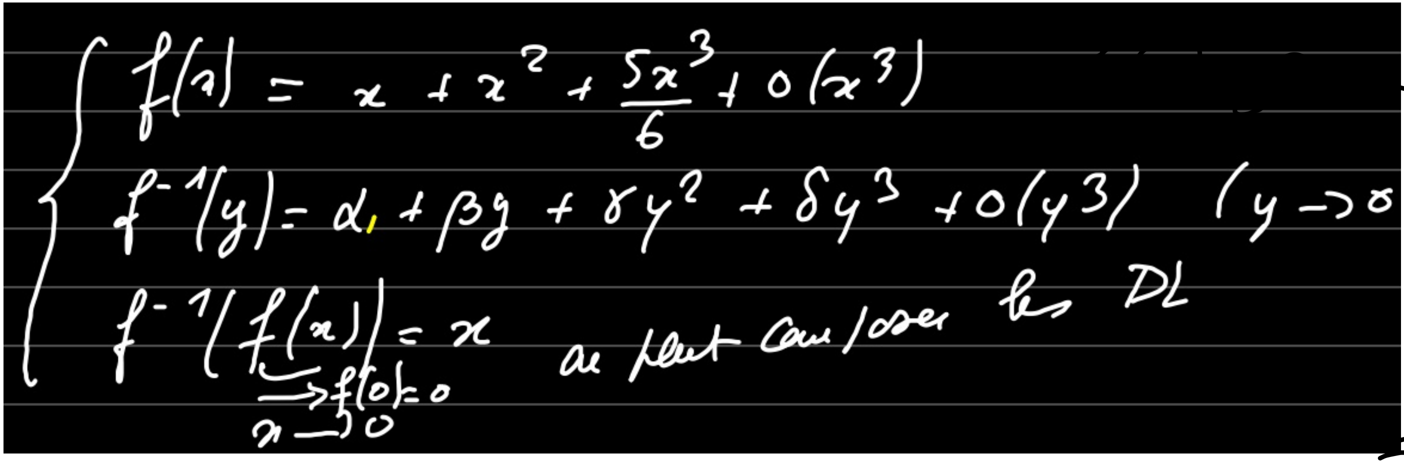
$$= \Big(\underbrace{1 + x + \frac{x^2}{2}}\ \overbrace{\underbrace{+ \frac{x^3}{6} + o(x^3)}_{\text{inutile}}}^{o(x^2)}\Big)\cdot\Big(\underbrace{\underline{x} + \frac{x^3}{3}}\ + o(x^3)\Big)$$

$$= x + \frac{x^3}{3} + x^2 + \frac{x^3}{2} + o(x^3)$$

$$\frac{1}{3} + \frac{1}{2} = \frac{2+}{6}$$

$$\begin{cases} f(x) = x + x^2 + \frac{5x^3}{6} + o(x^3) \\ f^{-1}(y) = \alpha + \beta y + \gamma y^2 + \delta y^3 + o(y^3) \quad (y \to 0) \\ f^{-1}(\underbrace{f(x)}_{\xrightarrow[x\to 0]{} f(0)=0}) = x \end{cases}$$

on peut composer les DL



$$\begin{cases} f(x) = x + x^2 + \frac{5x^3}{6} + o(x^3) \\ f^{-1}(y) = \alpha + \beta y + \gamma y^2 + \delta y^3 + o(y^3) \quad (y \to 0 \\ f^{-1}(\underbrace{f(x)}_{\xrightarrow[x \to 0]{} f(0)=0}) = x \quad \text{on peut composer les DL} \end{cases}$$

$= o(x^3)$

$$x = f^{-1}(f(\underbrace{x})) = \alpha + \beta f(x) + \gamma f^2(x) + \delta f^3(x) + o(f(x)^3)$$

$$= \alpha + \beta\left(x + x^2 + \frac{5x^3}{6} + o(x^3)\right) + \gamma\left(\underline{x} + \underline{x}^2 + \frac{5x^3}{6} + o(x^3)\right)^2$$

$$+ \delta(\underline{\underline{\underline{x}}} + x^2 + \text{''} + \text{''})^3 + \underline{\underline{o(x^3)}}$$

$$= \alpha + \beta x + \beta x^2 + \frac{5\beta}{6}x^3 + \gamma x^2 + 2\gamma x^3 + \delta x^3 + o(x^3)$$

$$x = \alpha + \beta x + \beta x^2 + \underline{\underline{\frac{5\beta}{6}}} x^3 + \gamma x^2 + 2\gamma x^3 + \delta x^3 + o(x^3)$$

$$\Longleftrightarrow$$

$$\underbrace{x + o(x^3)} = \underline{\underline{x}} = \underline{\alpha + \underline{\beta x} + x^2(\beta+\gamma) + x^3\left(\frac{5\beta}{6} + 2\gamma + \delta\right)} + o(x^3)$$

$\Big\downarrow$ par unicité de la partie principale d'un DL

$$\begin{cases} \boxed{\alpha = 0} \\ \boxed{\beta = 1} \\ \beta + \gamma = 0 \end{cases} \longrightarrow \boxed{\gamma = -1}$$

$$\frac{5\beta}{6} + 2\gamma + \delta = 0 \Rightarrow \frac{5}{6} - 2 + \delta = 0$$

$$\boxed{\delta = 7/6}$$

$f(0) = 0$

$\downarrow$

$f^{-1}(0) = 0$

$$\boxed{f^{-1}(y) = y - y^2 + \frac{7}{6}y^3 + o(y^3)}$$

Exercice 10 - F9

$$f(\underline{x}) = \sum_{0}^{N} \frac{x^n}{n!} f^{(n)}(0) \boxed{+ o(x^N) \quad \underline{TY}}$$

Young

$$= \sum_{0}^{N} \frac{x^n}{n!} f^{(n)}(\underline{0}) + \frac{x^{N+1}}{(N+1)!} f^{(N+1)}(\xi)$$

où $\xi \in (0, x)$

$\uparrow$

$n=0$ (TAF)

Lagrange

$$= \sum_{0}^{N} \frac{(x-a)^n}{n!} f^{(n)}(a) + \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$$

$\underline{\underline{TL}}$ $\xi \in]\cancel{x}, a[$

$\swarrow$ $n=0$

$$f(x) = f(a) + (x-a) f'(\xi) \quad + \underline{\underline{o((x-a)^n)}} \,]\, Y$$

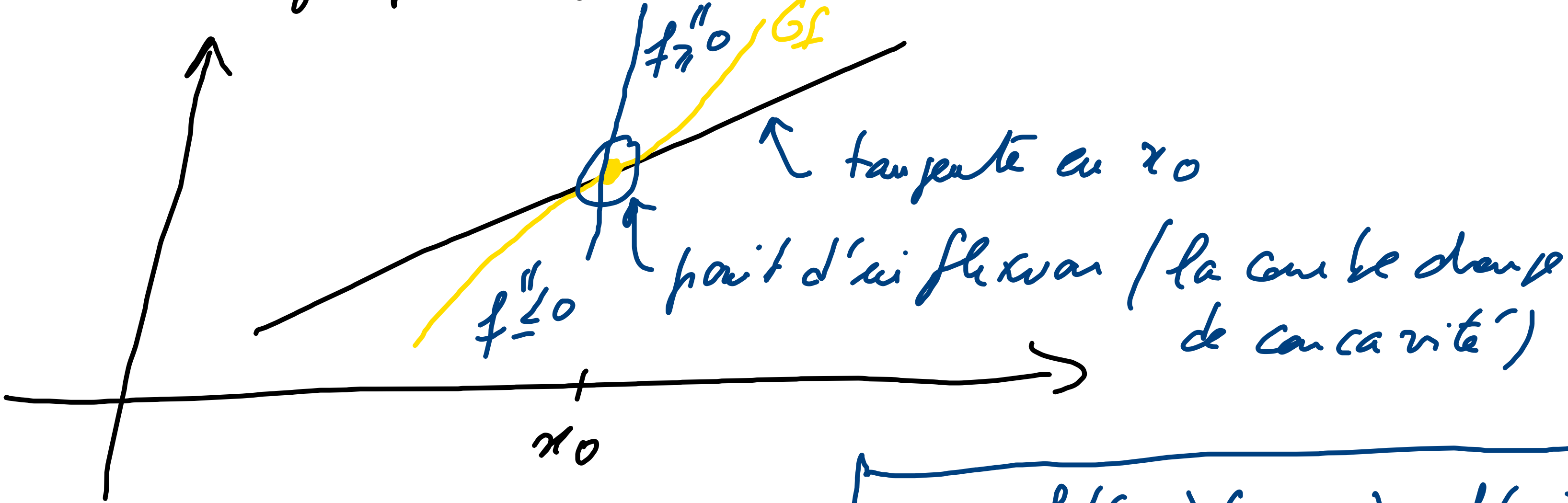
$$\frac{f(x) - f(a)}{x-a} = f'(\xi) \quad \text{où } \xi \in]a, x[$$

TAF

$f : I \longrightarrow \mathbb{R}$, I intervalle de $\mathbb{R}$

f est 2 fois dérivable.

1) Montrer que si $f''(x_0) = 0$ en changeant de signe alors le graphe de f traverse sa tangente en x_0



tangente en x_0

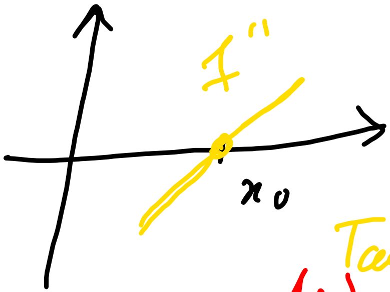
point d'inflexion (la courbe change de concavité)

Equation de la tangente en x_0 : $\boxed{y = f'(x_0)(x - x_0) + f(x_0)}$

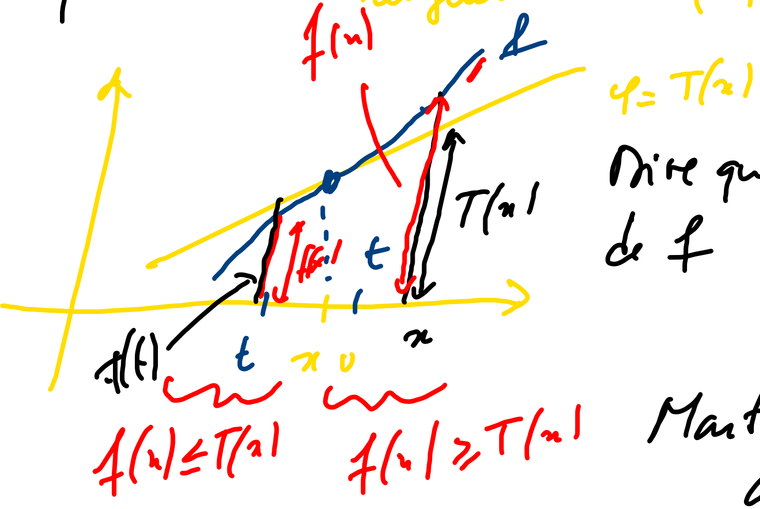
On sait que $f''(x_0) = 0$ en changeant de signe

$f''(t) > 0$ si $t > x_0$

$f''(t) < 0$ si $t < x_0$



Tangente : $T(x) = y = f(x_0) + (x - x_0) f'(x_0)$



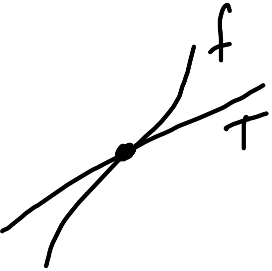
Dire que la tangente traverse le graphe de f c'est dire que

$\varphi(x) = f(x) - T(x)$ change de signe

Montrons que φ change de signe au voisinage de x_0

$$\varphi(x) = f(x) - f(x_0) - f'(x_0)(x - x_0)$$

$$\varphi(x_0) = f(x_0) - f(x_0) - f'(x_0)(x_0 - x_0) = 0$$



$$\varphi(x) = f(x) - f(x_0) - f'(x_0)(x - x_0)$$

DL_{1,x_0} de f : $f(x) \overset{+\varphi}{=} f(x_0) + f'(x_0)(x - x_0) + \underbrace{o(x - x_0)}$

$$\Rightarrow \quad = T(x) + o(x - x_0) \Rightarrow \varphi(x) = o(x - x_0)$$

↑ impossible de voir que φ change de signe

<u>On fait</u> TL

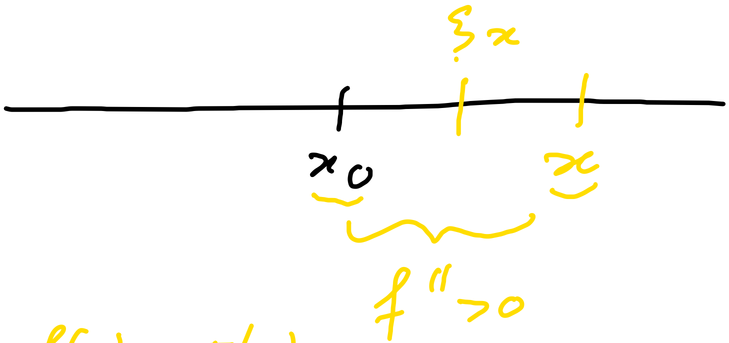
$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{(x - x_0)^2}{2!} f''(\xi_x)$$

où $\xi_x \in (x, x_0)$

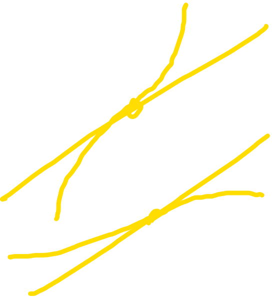
Donc

$$\varphi(x) = f(x) - T(x) \overset{TL}{=} \underbrace{\frac{(x-x_0)^2}{2}}_{>0} \underbrace{f''(\xi_x)}_{>0} > 0$$

si $x > x_0$

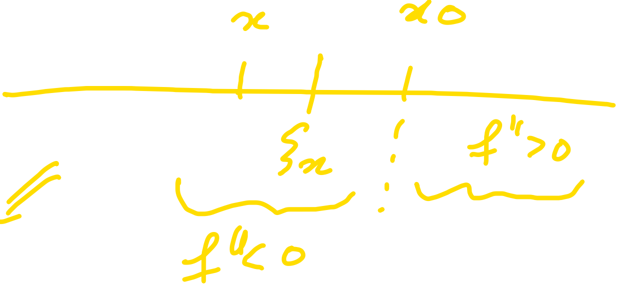


donc $f''(\xi_x) > 0$



ie $f(x) \geq T(x)$ si $x \geq x_0$: f au-dessus de sa tangente lorsque $x \geq x_0$

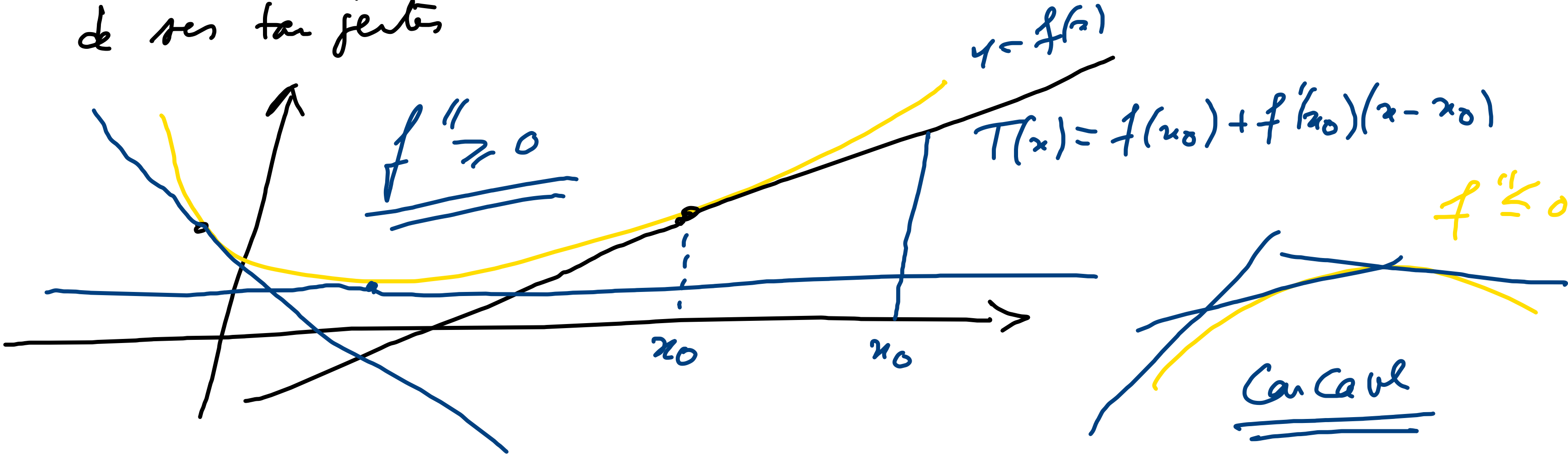
si $x < x_0$ alors :



$$\varphi(x) = \underbrace{\frac{(x-x_0)^2}{2}}_{>0} \underbrace{f''(\xi_x)}_{<0} < 0$$

$$\boxed{\begin{array}{c} f(x) \leq T(x) \\ x < x_0 \end{array}}$$

2) On suppose $\underline{\underline{f'' \geq 0}}$: Montrer que f est au dessus de ses tangentes

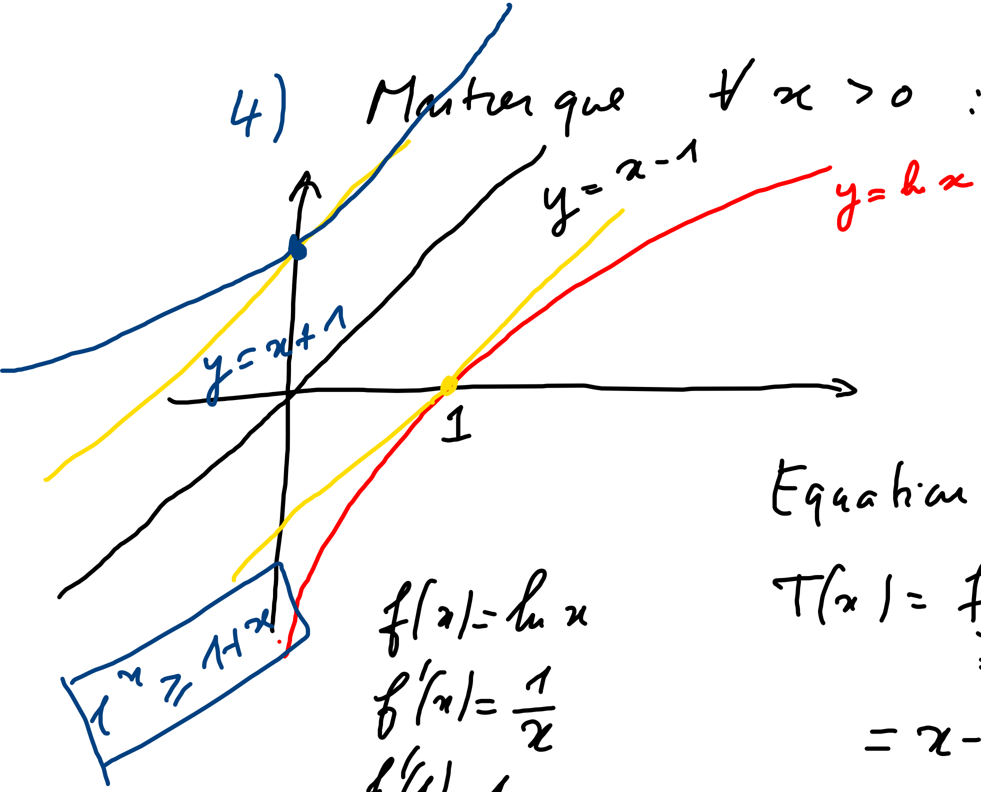


Il faut montrer que $f(x) \geq T(x) \quad \forall x \in \mathbb{R}$

$$\varphi(x) = f(x) - T(x) = f(x) - f(x_0) - f'(x_0)(x - x_0)$$

$$\overset{TL}{=} \frac{(x-x_0)^2}{2} \underbrace{f''(\xi_x)}_{\geq 0} \underset{hyp}{\geq 0} \longrightarrow \underline{f(x) \geq T(x)}$$

4) Montrer que $\forall x > 0 : f(x) = \ln(x) \leq x - 1$



$\varphi(x) = \ln x - x + 1 \leq 0$

NON Spontif

Equation de la tangente de f en x = 1

$$T(x) = \underbrace{f(1)}_{=0} + \underbrace{f'(1)}_{=1}(x-1)$$

$$= x - 1$$

$f(x) = \ln x$

$f'(x) = \frac{1}{x}$

$f'(1) = 1$

$f''(x) = -\frac{1}{x^2} \leq 0$ donc vu ce qui précède f est concave donc sous ses tangentes $f(x) \leq T(x)$ $\underline{\forall x > 0}$

Soit

Exercice 5-5

$$\lim_{\underline{\underline{x \to 1+}}} \frac{\boxed{x^x - 1}}{\underbrace{\ln\left(1+\sqrt{x^2-1}\right)}_{\to \ln 1 = 0}} \to \text{du type } \frac{0}{0}$$

$$\bullet \quad \underline{x^x - 1} = e^{x \ln x} - 1 \qquad u = x \ln x \xrightarrow[x \to 1]{} 0$$

$$= e^u - 1 = \cancel{1} + u + o(u) - \cancel{1}$$

$$= x \ln x + \underline{o(x \ln x)}$$

$$\to \; = x - 1 + o(x-1)$$

$$x \to 1+$$

$$x = 1+h, \quad h \to 0_+$$

$$h = x - 1$$

$$\underline{x \ln x} = (1+h)\ln(1+h)$$

$$= (1+h)(h + o(h)) = h + h^2 + o(h^2) \to \text{~~co coef~~}$$

$$= x - 1 + (x-1)^2 + o(x-1)^2$$

$$= x - 1 + o(x-1)$$

$$x \ln x \sim x - 1$$

$$x^{x-1} = x-1 + o(x-1) \sim x-1$$

$$\frac{x^x - 1}{\sqrt{x-1}\,\ln(1-\sqrt{x^2-1})}$$

$$\sqrt{x-1}\,\ln(1-\underbrace{\sqrt{x^2-1}}_{\xrightarrow[x\to 1_+]{} 0}) = \ln(1-u) \sim -\underline{\underline{u}} = -\sqrt{x^2-1}$$

$u \to 0$

$\to 0$

$$\overline{x^2-1} = \sqrt{x-1}\cdot\sqrt{x-1}$$

$$\downarrow$$

$$\simeq \frac{-1}{\sqrt{x+1}} \to -1/\sqrt{2}$$

$\sqrt{x-1}$

Dac

$$\frac{\boxed{x^x - 1}}{\ln(1-\sqrt{x^2-1})}$$

$$\| $$

$$\sim -\sqrt{\frac{x-1}{2}}$$

$$\sim \frac{x-1}{\boxed{1+(-\sqrt{x^2-1})}} = \frac{x-1}{-\sqrt{x-1}\cdot\underbrace{\sqrt{x+1}}_{\to\sqrt{2}}}$$

$$= -\frac{\sqrt{x-1}}{\sqrt{2}} \xrightarrow[x\to 1]{} 0$$