L2 Analyse — Vendredi 25/09 — TDA1

Exercice 1 : • $DL_{3,0}$ de $f(x) = \cos(x)\sin(x)$

$$\sin(x) = x - \frac{x^3}{6} + o(x^3) \quad DL_3$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + o(x^7) \quad DL_7$$

$$\cos(x) = 1 - \frac{x^2}{2} + o(x^2) \quad DL_2$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^4) \quad DL_4$$

$$= 1 - x^2/2 + 0.x^3 + o(x^3) = 1 - x^2/2 + o(x^3)$$

$$f(x) = \sin(x)\cos(x)$$

$$\sin(x) = x - x^3/6 + o(x^3)$$

$$\cos(x) = 1 - x^2/2 + o(x^3)$$

$$\frac{x^5}{x^3} = x^2 \xrightarrow[x \to 0]{} 0$$

donc $x^5 = o(x^3)$

Donc $f(x) = \left(x - \frac{x^3}{6} + o(x^3)\right)\left(1 - \frac{x^2}{2} + o(x^3)\right)$

$$= x - \frac{x^3}{2} - \frac{x^3}{6} + o(x^3)$$

$$\frac{1}{2} + \frac{1}{6} = \frac{4}{6} =$$

$$= x - \left(\frac{1}{2} + \frac{1}{6}\right)x^3 + o(x^3)$$

$o(x^3)$

$= x^3 \varepsilon(x)$

ou $\varepsilon(x) \xrightarrow[x \to 0]{} 0$

$$\boxed{= x - \frac{2x^3}{3} + o(x^3)} = f(0) + x f'(0) + \frac{x^2}{2!} \underbrace{f''(0)}_{=0} + \frac{x^3}{3!} f^{(3)}(0) + o(x^3)$$

$$O(x^3) = x^3\varepsilon(x)$$

$$\frac{O(x^3)}{x^3} = \frac{x^3\varepsilon(x)}{x^3} = \varepsilon(x) \xrightarrow[x\to 0]{} 0$$

- **Ou bien** $f(x) = \sin(x)\cos(x) = \frac{\boxed{2\sin(x)\cos(x)}}{2} = \frac{\sin(\overbrace{2x}^{U})}{2}$

$$\boxed{\begin{aligned}\cos(2a) &= \cos^2(a) - \sin^2(a)\\ \sin(2a) &= 2\sin a\cos a\end{aligned}}$$

On sait que si $U \to 0$ $\underbrace{\quad}_{o(U^3)}$

$$\sin U = U - \frac{U^3}{6} + U^3\varepsilon(U) \quad (u \to 0)$$

Donc $f(x) = \frac{\sin(2x)}{2} = \frac{1}{2}\left[2X - \frac{\overset{8x^3}{\overset{\shortparallel}{(2X)^3}}}{6} + \underset{\underset{x^3\varepsilon(x)}{\shortparallel}}{(2X)^3\varepsilon(2X)}\right] = \boxed{X - \frac{2x^3}{3} + x^3\varepsilon(X)}$

2) $DL_{3,0}$ de $\sin^{\circ k}(x) = \sin(\sin(\sin \ldots (\sin x)))$
k sinus !

ex

$k=1$ $\sin^{\circ 1}(x) = \sin(x) = x - \frac{x^3}{6} + x^3\varepsilon(x)$

$k=2$ $\sin^{\circ 2}(x) = \sin(\sin x) = ?$

$k=3$ $\sin^{\circ 3}(x) = \sin(\sin(\sin x)) = ?$

Exercice 1
seconde question

$k=2$ Cherchons le $DL_{3,0}$ de

$$f(x) = \sin(\sin(x))$$

$$U = x - \frac{x^3}{6} + x^3\varepsilon(x) \qquad = \sin\Big(\underbrace{x - \frac{x^3}{6} + x^3\varepsilon(x)}_{U \xrightarrow[x\to 0]{} 0}\Big) = U - \frac{U^3}{6} + \underset{\substack{\parallel \\ x^3\varepsilon(x)}}{U^3\varepsilon(U)}$$

$\sim_0 x$

$$\sin(\sin x) = \sin\Big(\overbrace{x - \frac{x^3}{6} + x^3\varepsilon(x)}^{U \underset{0}{\sim} x}\Big) = \overbrace{U - \frac{U^3}{6}}^{-\frac{1}{6}\times U^3} + U^3\varepsilon(U)$$

$$= x - \frac{x^3}{6} + \underbrace{x^3\varepsilon(x)} \underbrace{- \frac{1}{6}}_{-\frac{1}{6}} \underbrace{\Big(\overbrace{x - \frac{x^3}{6} + x^3\varepsilon(x)}^{U^3}\Big)^3}_{x^3 + \text{terme de degré} > 3} + \underbrace{x^3\varepsilon(x)}$$

$$= x - \frac{x^3}{6} - \frac{x^3}{6} + x^3\varepsilon(x)$$

$$\sin(\sin x) = \boxed{x - \frac{2x^3}{6} + x^3\varepsilon(x) = x - \frac{x^3}{3} + x^3\varepsilon(x)}$$

$$f_1(x) = \sin(x) = \boxed{x - \frac{x^3}{6} + x^3\varepsilon(x)}$$

$$f_2(x) = \sin(\sin x) = \boxed{x - \frac{2x^3}{6} + x^3\varepsilon(x)}$$

$$f_3(x) = \sin(\underbrace{\sin(\sin x)}_{f_2(x)}) = ?$$

DL de $\sin(U)$ en U

$$= \sin\Big(\underbrace{x - \frac{2x^3}{6}}_{U \to 0,\ U \sim x} + x^3\varepsilon(x)\Big) = \sin(\underset{\to 0}{U}) = U - \frac{1}{6}U^3 + U^3\varepsilon(U)$$

$$= x - \frac{2x^3}{6} - \frac{1}{6}\underbrace{\left(x - \frac{2x^3}{6} + x^3\varepsilon(x)\right)^3}_{= x^3 + x^3\varepsilon(x)} + x^3\varepsilon(x)$$

$$= x - \frac{2x^3}{6} - \frac{x^3}{6} + x^3\varepsilon(x) = \boxed{x - \frac{3x^3}{6} + x^3\varepsilon(x)}$$

Ceci suggère que $f_4(x) = x - \frac{4x^3}{6} + x^3\varepsilon(x)$ ect..

et plus généralement $\boxed{f_k(x) = x - \frac{kx^3}{6} + x^3\varepsilon(x)}$

Il faut le démontrer. On procède par récurrence sur $k \geq 1$

- On vient de voir que la propriété est vraie pour $k = 1, 2, 3$
- <u>HR</u> : On suppose la propriété vraie jusqu'au rang k

Montrons-la au rang $k+1$

$$f_{k+1}(x) = \underbrace{\sin(\sin \ldots \sin x))}_{k+1} = \sin\Big[\underbrace{f_k(x)}_{\overset{HR}{=}\, x - \frac{kx^3}{6} + x^3\varepsilon(x)}\Big]$$

$$f_{k+1}(x) = \sin(f_k(x)) = \sin\Big(\underbrace{x - \frac{kx^3}{6} + x^3\varepsilon(x)}_{U \to 0,\ U \sim x}\Big)$$

$$= U - \frac{U^3}{6} + U^3\varepsilon(U)$$

$$= x - \frac{kx^3}{6} - \frac{1}{6}\Big(\underbrace{x - \frac{kx^3}{6} + x^3\varepsilon(x)}\Big)^3 + x^3\varepsilon(x)$$

$$= x^3 + x^3\varepsilon(x)$$

$$= x - \frac{kx^3}{6} - \frac{x^3}{6} + x^3\varepsilon(x)$$

$$= x - \frac{(k+1)x^3}{6} + x^3\varepsilon(x)$$ la ppte au rang $k+1$ est bien vérifiée

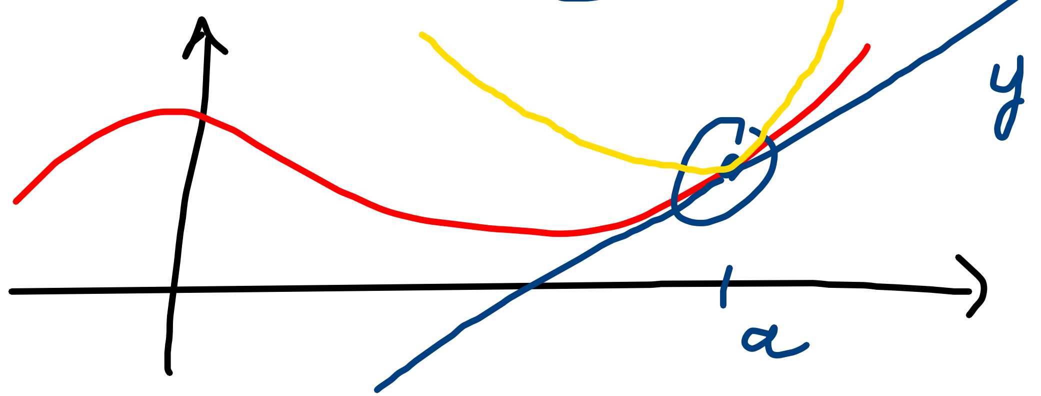
<u>Conclusion</u> $\forall k \geq 1$ le $DL_{3,0}$ de f_k est $f_k(x) = x - kx^3/6 + x^3\varepsilon(x)$

3) Déterminer le $DL_{3,1}$ de $f(x) = e^{\sqrt{x}}$

$DL_{3,1}$: $f(x) = \underbrace{f(1) + (x-1) f'(1)}_{} + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1) + (x-1)^3 \varepsilon(x-1)$, $x \to 1$

$DL_{3,0}$ $f(x) = f(0) + x f'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{3} f^{(3)}(0) + x^3 \varepsilon(x)$, $x \to 0$

$DL_{2,a}$ $f(x) = \underbrace{f(a) + (x-a) f'(a)}_{} + \frac{(x-a)^2}{2!} f''(a) + (x-a)^2 \varepsilon(x-a)$, $x \to a$



$y = f(a) + (x-a) f'(a)$

$DL_{3,1}$ de $f(x) = e^{\sqrt{x}}$

Technique : on se ramène à chercher un DL en 0 pour cela, comme $x \longrightarrow 1$ on peut l'écrire sous la forme : $\boxed{x = 1 + h \text{ avec } h \to 0} \longrightarrow \boxed{h = x - 1}$

Donc $f(x) = f(1+h) = e^{\sqrt{1+h}} = g(h)$

on va cherche le $DL_{3,0}$ de g en la variable "h"

Cherchons le $DL_{3,0,h}$ de $e^{\sqrt{1+h}}$

On commence par chercher le $DL_{3,0}$ de $\sqrt{1+h}$

$$\sqrt{1+h} = ? = (1+h)^{1/2}$$

Cours sur les DL $\alpha \neq 0$: $(1+h)^\alpha = 1 + \alpha h + \frac{\alpha(\alpha-1)}{2!}h^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3! \to 6}h^3 + h^3\varepsilon(h)$ $(h \to 0)$

$\frac{1}{2}\left(-\frac{1}{2}\right)$ $\quad$ $\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)$

ici $\alpha = 1/2$ donc

$$(1+h)^{1/2} = 1 + \frac{h}{2} - \frac{h^2}{8} + \frac{3}{2\cancel{6}.8}h^3 + h^3\varepsilon(h)$$

$$= \boxed{1 + \frac{h}{2} - \frac{h^2}{8} + \frac{h^3}{16} + h^3\varepsilon(h)}$$

Donc $g(h) = \exp\left(\sqrt{1+h}\right) = \exp\left(1 + \underbrace{\frac{h}{2} - \frac{h^2}{8} + \frac{h^3}{16} + h^3\varepsilon(h)}\right)$

$U \xrightarrow[h \to 0]{?} 1$ 0

On connaît le $DL_{3,0}$ de l'exponentielle

$$\exp(U) = 1 + U + \frac{U^2}{2} + \frac{U^3}{6} + o(U^3) \quad (U \to 0)$$

$e^{a+b} = e^a \cdot e^b$ donc

$$g(h) = e^1 \cdot e^{\underbrace{\frac{h}{2} - \frac{h^2}{8} + \frac{h^3}{16} + h^3\varepsilon(h)}}$$

$= e \cdot e^U$ où $U \xrightarrow[h \to 0]{} 0$ $\quad U \sim \frac{h}{2}$

$= e\left\{1 + U + \frac{U^2}{2} + \frac{U^3}{6} + \underbrace{U^3\varepsilon(U)}_{= h^3\varepsilon(h)}\right.$

$=$ on remplace

$$\exp\sqrt{1+h} = e\cdot\left(1+u+\frac{u^2}{2}+\frac{u^3}{6}+u^3\varepsilon(u)\right) \text{ où } u=\frac{h}{2}-\frac{h^2}{8}+\frac{h^3}{16}+h^3\varepsilon(h)$$

$$= e\left(1+\frac{h}{2}-\frac{h^2}{8}+\frac{h^3}{16}+\frac{1}{2}\left(\frac{h}{2}-\frac{h^2}{8}+\frac{h^3}{16}\right)^2+\frac{1}{6}\left(\frac{h}{2}-\frac{h^2}{8}+\frac{h^3}{16}\right)^3+h^3\varepsilon(h)\right)$$

$$= e\left(1+\frac{h}{2}-\frac{h^2}{8}+\frac{h^3}{16}+\frac{1}{2}\left(\frac{h^2}{4}\underbrace{-2\frac{h}{2}\cdot\frac{h^2}{8}}_{-h^3/8}\right)+\frac{1}{6}\left(\frac{h^3}{8}\right)+h^3\varepsilon(h)\right)$$

$$(a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc$$

$$(\quad)^3 = a^3+b^3+c^3+3a^2b+3a^2c+3$$

$$= e\left(1+\frac{h}{2}+h^2\underbrace{\left(-\frac{1}{8}+\frac{1}{8}\right)}_{=0}+h^3\underbrace{\left(\frac{1}{16}-\frac{1}{16}+\frac{1}{48}\right)}_{1/48}+h^3\varepsilon(h)\right)= e\cdot\left(1+\frac{h}{2}+\frac{h^3}{48}+h^3\varepsilon(h)\right)$$

Donc $\boxed{\exp\sqrt{1+h} = e+\frac{eh}{2}+\frac{eh^3}{48}+h^3\varepsilon(h)}$ $(h\to 0)$

Comme $h = x-1$ on a finalement

$$e^{\sqrt{x}} = e + \frac{eh}{2} + \frac{eh^3}{48} + h^3\varepsilon(h) \quad h \to 0$$

$$= \boxed{e + \frac{e(x-1)}{2} + \frac{e}{48}(x-1)^3 + (x-1)^3\varepsilon(x-1)} \quad (x \to 1)$$

$f(1) + (x-1)\underbrace{f'(1)}_{= e/2} + x$

$$\left(e^{\sqrt{x}}\right)' = \frac{1}{2\sqrt{x}}\, e^{\sqrt{x}} = \frac{e}{2}$$

$x = 1$

DL en 2 $\quad x = 2+h \quad \sqrt{x} = \sqrt{2+h} = \sqrt{2}\sqrt{1+\frac{h}{2}} = \sqrt{2}\left(1+\underbrace{\frac{h}{2}}\right)^{1/2}$

DL

Exercice 2 : c'est le cours .

Exercice 3 : Etudier la convergence de la suite

$$u_n = \underbrace{\sqrt{n^2+n+1}}_{\longrightarrow +\infty} - \underbrace{\sqrt{n}}_{\longrightarrow +\infty} = +\infty - \infty$$

indéterminé

- $$u_n = \sqrt{n}\sqrt{n+1+\frac{1}{n}} - \sqrt{n}$$

$$= \underbrace{\underbrace{\sqrt{n}}_{\longrightarrow +\infty}\left\{ \underbrace{\sqrt{\underbrace{n+1+\frac{1}{n}}_{\longrightarrow +\infty}} - 1}_{\longrightarrow +\infty} \right\}} \longrightarrow +\infty$$

- $u_n = \sqrt{n^2+n+1} - \sqrt{n}$

$= n\sqrt{1+\frac{n+1}{n^2}} - \sqrt{n} = n\left\{\underbrace{\overbrace{\sqrt{1+\frac{n+1}{n^2}}}^{\to 1} - \overbrace{\frac{1}{\sqrt{n}}}^{\to 0}}_{\to 1}\right\} \longrightarrow +\infty$

$=$

$\underset{n}{\sim} n$

$a_n \underset{n}{\sim} b_n \Leftrightarrow \frac{a_n}{b_n} \longrightarrow 1$

$\frac{u_n}{n} = \sqrt{1+\frac{n+1}{n^2}} - \frac{1}{\sqrt{n}} \longrightarrow 1 \quad \boxed{u_n \sim n}$

$\frac{\sin x}{x} = \frac{x}{x} + \frac{o(x)}{x} \Longrightarrow \frac{\sin x}{x} \sim 1 \quad \sin x \sim x$

$$u_n = \sqrt{n^2+n+1} - \sqrt{n}$$

$(1+u)^{1/2}$ avec $u = \frac{n}{n^2+1} \longrightarrow 0$

$$= n\sqrt{1+\frac{n+1}{n^2}} - \sqrt{n} = n\left(1+\frac{n+1}{n^2}\right)^{1/2} - \sqrt{n}$$

$$\text{"}\ \frac{1}{n} + \underbrace{\frac{1}{n^2}}_{} = \frac{1}{n} + \frac{1}{n}\varepsilon(1/n)$$

$$o\left(\frac{1}{n}\right) = \frac{1}{n}\varepsilon\left(\frac{1}{n}\right)$$

$$= n\left(1+\frac{1}{n}+o\left(\frac{1}{n}\right)\right)^{1/2} - \sqrt{n}$$

$$(1+u)^\alpha = 1+\alpha u + o(u)$$

$$= n\left[1+\frac{1}{2n}+o\left(\frac{1}{n}\right)\right] - \sqrt{n}$$

ou bien $o(1/n) = \frac{1}{n}\varepsilon\left(\frac{1}{n}\right)$

$$= n + \frac{1}{2} + \underbrace{n\,o\left(\frac{1}{n}\right)}_{\longrightarrow 0} - \sqrt{n}$$

$\longrightarrow 0$ car $n\,o(1/n) = \frac{o(1/n)}{1/n} \longrightarrow 0$

$$\boxed{u_n = n - \sqrt{n} + \frac{1}{2} + \varepsilon(1/n)}$$

$$u_n \sim n$$

$$(\sqrt{a} - \sqrt{b}) \frac{\sqrt{a} + \sqrt{b}}{\sqrt{a} + \sqrt{b}} = \frac{a-b}{\sqrt{a} + \sqrt{b}}$$

- On $u_n = \sqrt{n^2+n+1} - \sqrt{n} \times \frac{(\sqrt{n^2+n+1} + \sqrt{n})}{\sqrt{} + \sqrt{}}$

$$= \frac{n^2 + \cancel{n} + 1 \cancel{- n}}{\sqrt{n^2+n+1} + \sqrt{n}} = \frac{n^2 + 1}{\sqrt{n^2+n+1} + \sqrt{n}}$$

$$n^2 = o(e^n)$$

$$\sqrt{n} = o(n)$$

$$\frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} \longrightarrow 0$$

$$a_n = o(b_n) \iff \frac{a_n}{b_n} \to 0.$$

$$a_n \sim b_n \qquad a_n/b_n \to 1$$

$$= \frac{n^2 + 1}{n\left\{\sqrt{1 + \frac{n+1}{n^2}} + \frac{1}{\sqrt{n}}\right\}} = \frac{n + 1/n}{\sqrt{1 + \frac{n+1}{n^2}} + \frac{1}{\sqrt{n}}}$$

$$\longrightarrow 1$$

$$\sim n + \frac{1}{n} \underset{\infty}{\sim} n$$

Pour la prochaine séance :

exo 3 3 —— 7