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Matrices aléatoires, processus stochastiques et groupes de réflexions

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Just in case Neil and Wolfgang are more comfortable with English, here is a translation of the passages concerning them.

Kenilworth is one of these damp villages that English countryside grows in the middle of nowhere. Though courteous to the foreigner, Kenilworth seldom offers the opportunity of a successful social integration. I entirely owe it to my friendship with Neil O'Connell that I keep such wonderful memories of my stay there at his invitation. He co-supervised my thesis with precious care and enthusiasm. I've been deeply moved by the tokens of his confidence in me. I can't describe the tremendous pleasure I had talking maths with him and benefitting from his scientific fire. His beautiful ideas around the reflexion principle pervade all this thesis. And, let us not forget that he prepares the best Irish coffee in the world, which makes me even more grateful to him.

I wish to thank Philippe Biane and Wolfgang König warmly for shouldering the painful roles of referees. Thanks to their careful reading and precious comments, some of my mistakes have been surgically removed and relevant viewpoints have been included. I consider it a privilege that they are part of my jury.

Liste de publications

- Doumerc, Y., O'Connell, N.
  Exit problems associated with finite reflection groups
  A paraître dans Probab. Theor. Relat. Fields.
- Donati-Martin, C., Doumerc, Y., Matsumoto, H., Yor, M.
  Some properties of the Wishart processes and a matrix extension of the Hartman-Watson laws.
  Publ. Res. Inst. Math. Sci. 40 (2004), no. 4, 1385–1412.
- Doumerc, Y.
  A note on representations of eigenvalues of classical Gaussian matrices.
  Séminaire de Probabilités XXXVII, 370–384, Lecture Notes in Math., 1832, Springer, Berlin, 2003.

Table des matières

IV Brownian motion and reflection groups 153

Première partie

Introduction

Chapitre 1

Considérations générales

1.1 Un mot de présentation

L'univers des matrices aléatoires est, selon la formule consacrée dans un autre usage, en expansion. Sont présentées dans cette thèse quelques-unes de nos modestes tentatives pour pénétrer cet univers par diverses portes, dérobées pour certaines. Les récentes années ont vu fleurir une abondante littérature sur le sujet en question, touchant à des branches très diverses des mathématiques, parfois inattendues dans ce contexte, renouvelant ainsi un intérêt vieux d'une cinquantaine d'années. Nos travaux, bien que ne se situant pas au coeur des préoccupations classiques de la théorie des matrices aléatoires, ont été largement inspirés par l'effervescence actuelle de cette dernière et les grandes avancées qu'elle vient de connaître. Si elles gravitent toutes sur une même orbite des mathématiques, les études ici rassemblées ne poursuivent pas, de manière directe, un objectif commun et les points de vue propres à chacune sont assez différents. Le ciment qui les unit est d'en appeler à un même cercle d'objets, d'outils et d'idées ayant tous partie liée à la très vaste théorie que nous avons évoquée.

1.2 Organisation de ce document

Cette thèse est divisée en cinq parties. La partie I correspond à l'introduction, dans laquelle nous sommes. La partie II comprend l'article publié [Dou03] (chapitre 5) ainsi que deux prépublications (chapitres 6 et 7). Elle aborde les liens qui existent entre valeurs propres de certaines matrices gaussiennes, processus sans collision et combinatoire des tableaux de Young. Dans la partie III, nous regroupons l'article [DMDMY04] (chapitre 8), écrit en collaboration avec C. Donati-Martin, H. Matsumoto et M. Yor, ainsi qu'un texte non-publié (chapitre 9). Nous y examinons des extensions aux matrices symétriques de processus stochastiques classiques en dimension un, les carrés de Bessel d'une part et les processus de Jacobi d'autre part. Ensuite, notre partie IV fait figurer l'article [DO04] (chapitre 10), en collaboration avec N. O'Connell, ainsi qu'une note non-publiée (chapitre 11) qui en constitue une suite naturelle. Nous y discutons le temps de sortie du mouvement brownien de régions de l'espace euclidien qui sont des domaines fondamentaux pour l'action de groupes de réflexions, finis ou affines. Enfin, la partie V est un appendice consacré aux propriétés du déterminant de Vandermonde vis-à-vis des diffusions usuelles ainsi qu'à un rappel, en français, sur la combinatoire des tableaux de Young et l'algorithme RSK.

Nous avons souhaité diviser notre introduction (partie I) en plusieurs chapitres (2, 3 et 4), chacun correspondant à une des parties (II, III et IV) pour présenter le contexte dans lequel ces dernières se situent et décrire brièvement les résultats qu'elles contiennent. En particulier, lorsqu'un résultat de la thèse est annoncé en introduction, nous l'accompagnons de la référence au théorème ou à la proposition correspondants dans les parties II, III et IV.

Dans l'espoir de faciliter la lecture, chaque chapitre de cette thèse possède sa propre bibliographie, exception faite des chapitres d'introduction (partie I) qui partagent la même bibliographie. Une bibliographie générale est aussi rassemblée en fin de document.

1.3 A propos des matrices aléatoires

Tout d'abord, nous voudrions dire quelques mots de ce tentaculaire monde des matrices aléatoires auquel notre travail a le sentiment d'appartenir. Voici la question fondamentale, informellement exprimée : comment sont distribuées les valeurs propres d'une matrice dont les coefficients sont des variables aléatoires ? En termes mathématiques, si M est une variable aléatoire de loi μ_n sur l'ensemble $\mathcal{M}_n(\mathbb{C})$ des matrices $n \times n$ complexes, quelle est la loi ν_n de l'ensemble de ses valeurs propres ? On peut faire remonter l'intérêt pour une telle question aux travaux du statisticien J. Wishart dans les années 30 ([Wis28], [Wis55]) puis, indépendamment, à ceux du physicien E. Wigner dans les années 50 ([Wig51], [Wig55], [Wig57]).

Le premier moment de l'étude a consisté à définir les lois μ_n en question : leur support et leurs invariances devaient correspondre aux données du problème (physique ou statistique) étudié. Ainsi, les statistiques multivariées ont été conduites à considérer des matrices de covariance empirique, donnant plus tard naissance aux lois dites « de Wishart » (voir [Jam60], [Jam64], [Mui82]). La modélisation de hamiltoniens en mécanique quantique a amené Wigner à introduire les ensembles de matrices hermitiennes (resp. symétriques) unitairement (resp. orthogonalement) invariants très étudiés par la suite et ses conjectures d'universalité ont fait porter l'intérêt sur des matrices hermitiennes ou symétriques dites « de Wigner » (ie dont les coefficients sont indépendants et identiquement distribués). Il existe d'autres lois de probabilité sur des espaces de matrices qui font de nos jours l'objet de nombreux travaux, par exemple les matrices distribuées selon la mesure de Haar sur un sous-groupe de $GL_n(\mathbb{C})$ ([Joh97], [Rai98]), les matrices de bande ([KK02], [Shl98]), les matrices non-hermitiennes ([Gin65], [Ede97]), les matrices faiblement non-hermitiennes ([Gir95a], [Gir95b]), les matrices asymétriques tri-diagonales ([GK00]), etc.

Une fois la loi μ_n définie, il est légitime de chercher à obtenir, à n fixé, la loi ν_n des valeurs propres $\lambda_1, \dots, \lambda_n$. Ceci n'est explicitement réalisable que lorsque μ_n possède assez d'invariance, par exemple pour la mesure de Haar sur $\mathcal{U}(n)$ (formule de H. Weyl) ou pour des matrices gaussiennes (les premiers résultats sont obtenus indépendamment, pour des matrices de covariance empirique, dans [Fis39], [Gir39], [Hsu39]).

Ensuite, l'on s'est demandé comment renormaliser les valeurs propres $\lambda_1, \dots, \lambda_n$ en $\tilde{\lambda}_1, \dots, \tilde{\lambda}_n$ de telle manière que la mesure spectrale empirique $\frac{1}{n} \sum_i \delta_{\tilde{\lambda}_i}$ converge quand $n \to \infty$? Lorsque cette convergence a lieu, on a cherché à en préciser la nature (convergence des moments, convergence faible presque-sûre ou en moyenne) ainsi qu'à identifier la mesure limite. La recherche de telles « lois de grands nombres » constitue l'étude du « régime global ». Depuis le célèbre théorème de la loi du demi-cercle de Wigner, de nombreux résultats ont été obtenus dans cette direction, par un vaste éventail de techniques : combinatoire et méthode des moments, transformée de Stieltjes, polynômes orthogonaux, théorie du potentiel et mesures d'équilibre, etc.

On peut alors chercher à accompagner ces « lois de grands nombres » de résultats plus précis sur les fluctuations ([CD01], [SS98]), les vitesses de convergence ([GT03], [GT04]) ou les grandes déviations ([Gui04]) associées.

On peut aussi choisir de s'intéresser à une valeur propre particulière, la plus grande (ou la plus petite) par exemple, si les valeurs propres sont rélles. Des théorèmes sont obtenus concernant sa limite presque-sûre, dont il est, par exemple, pertinent de se demander si elle coïncide avec le bord du support de la mesure spectrale limite ([FK81],[BS04]). Les fluctuations de cette valeur propre sont aussi d'un grand intérêt : leur ordre de grandeur et leur nature précise ont fait l'objet des résultats récents les plus marquants ([TW94], [BBAP04]).

Enfin, un autre régime d'intérêt, dit « régime local », concerne l'étude, dans une autre échelle, des interactions entre valeurs propres voisines, en particulier l'espacement entre deux d'entre elles consécutives ([DKM$^+$99], [Joh01c]). Les comportements que l'on observe à cette échelle exhibent de mystérieuses similarités avec ceux des racines de la fonction zêta de Riemann, ce qui motive une intense activité en lien avec la théorie des nombres ([KS99], [KS03]).

D'importantes contributions ([BOO00], [BO00], [Oko00], [Oko01], [Joh00], [Joh02], [Joh01a], [PS02] entre autres) mettent en évidence de surprenantes analogies, notamment au niveau des comportements asymptotiques, entre les matrices aléatoires et des problèmes mathématiques apparemment très éloignés (mesures provenant des représentations de groupes ou de la combinatoire, modèles de croissance issus de la physique). Leur point commun est de partager la structure de processus ponctuels déterminantaux (cf [Sos00]) dont la théorie est un outil majeur pour l'analyse asymptotique. Nous reviendrons, au chapitre 2, sur les similarités entre ces différents problèmes mais en insistant sur les identités non-asymptotiques.

Concernant l'aspect asymptotique de toutes ces questions, les défis actuels sont doubles. D'une part, il s'agit, pour des modèles « intégrables » (ie dont la structure se prête à des calculs exacts et explicites, comme par exemple les ensembles de matrices invariants, les modèles de percolation de dernier passage avec variables géométriques, la plus longue sous-suite croissante, etc) d'analyser leur comportement de manière de plus en plus fine. D'autre part, il s'agit aussi de prouver les « conjectures d'universalité », c'est-à-dire de démontrer rigoureusement la validité de résultats connus seulement pour certains modèles « intégrables » et dont on s'attend à ce qu'ils soient vrais en toute généralité. Une avancée majeure dans ce domaine est réalisée dans [Sos99].

Les matrices aléatoires jouent aussi un rôle important du côté de la géométrie des convexes en grande dimension ou de la géométrie des espaces de Banach (voir [DS01], [LPR$^+$04] et les références qu'ils contiennent), des algèbres d'opérateurs ([Haa02], [HT99]) et des probabilités libres. Pour ces dernières, les matrices aléatoires fournissent, asymptotiquement, des prototypes de variables libres et les lois qui apparaissent comme limites spectrales de grandes matrices s'interprètent naturellement au sein des probabilités libres. Pour une introduction à ces dernières, on consultera les passionnants exposés [Voi00] et [Bia03]. On trouvera dans [Bia98] un lien supplémentaire et remarquable entre probabilités libres et représentations des (grands) groupes symétriques.

Nous voudrions enfin signaler les interventions que les matrices aléatoires ont faites récemment dans des problèmes très divers et où il n'est a priori question d'aucune matrice aléatoire ! Par souci de brièveté et manque de compétence, nous nous bornons à un recensement partiel et sans détail. Mentionnons donc l'étude de modèles de physique théorique et de mécanique statistique (on pourra consulter [Kaz01], [KSW96], [GM04], [Eyn00] et leurs références), d'énumération de graphes ([DF01], [Zvo97]), des questions de théorie des noeuds ou des cordes ([AKK03], [ZJZ00]) et encore des problèmes en théorie de l'information ([TV04], [Kha05]).

Pour terminer, soulignons quelques éléments bibliographiques. La référence fondamentale est l'ouvrage [Meh91] de M.L. Mehta, qui présente, du point de vue de leurs applications en physique, les ensembles de matrices aléatoires les plus courants et contient un nombre considérable de calculs et formules (densité des valeurs propres, fonctions de corrélation, etc). Le livre [Dei99] de P. Deift permet à la fois un retour clair et rigoureux sur des résultats et techniques classiques (polynômes orthogonaux, notamment) ainsi qu'une introduction à l'utilisation des méthodes de Riemann-Hilbert dans ce contexte. Dans [For], P. Forrester offre un traitement analytique très complet, inspiré à la fois par les préoccupations originelles de la physique et par la théorie des systèmes intégrables. On y trouve, en particulier, de riches discussions autour de l'intégrale de Selberg et des équations de Painlevé. D'une veine très différente, l'article [Bai99] de Z.D. Bai insiste sur l'aspect méthodologique de la discipline et présente les techniques utilisées dans l'obtension des résultats les plus importants du régime global (méthode des moments et transformée de Stieltjes). Les articles [Joh01b] de K. Johansson et [O'C03c] de N. O'Connell constituent de très agréables lectures autour des liens entre modèles de croissance, files d'attente, processus sans collision et matrices aléatoires. Enfin, un remarquable article de survol est le récent [Kon04], qui dresse un vaste panorama du domaine, choisissant le point de vue des gas de Coulomb comme fil d'Ariane et exposant les résultats connus, les méthodes employées et les questions ouvertes.

Chapitre 2

Matrices aléatoires et combinatoire

Cette partie de notre travail discute quelques-uns des liens qui existent entre valeurs propres de matrices aléatoires, processus sans collision et un objet combinatoire appelé correspondance de Robinson-Schensted-Knuth (RSK en abrégé).

2.1 Contexte

2.1.1 La loi de Tracy-Widom

Rappelons tout d'abord le plus spectaculaire de ces liens. Soient $(X_{i,j})_{1\leq i<j\leq N}$ (respectivement $(X_{i,i})_{1\leq i\leq N}$) des variables aléatoires gaussiennes standards complexes (respectivement réelles), ie $\mathbb{E}(X_{i,j}) = 0$, $\mathbb{E}(|X_{i,j}|^2) = 1$. On suppose les $(X_{i,j}, 1 \leq i \leq j \leq N)$ indépendantes, on pose $X_{i,j} = \overline{X_{j,i}}$ pour $i > j$ et $X^N = (X_{i,j})_{1\leq i,j\leq N}$. X^N est une matrice aléatoire dite du GUE(N). Elle induit, sur l'espace $\mathcal{H}_N$ des matrices hermitiennes $N \times N$, la loi suivante

$$P_N(dH) = Z_N^{-1} \exp\Big(-\frac{1}{2}\operatorname{Tr}(H^2)\Big) dH, \tag{2.1}$$

où dH est la mesure de Lebesgue sur $\mathcal{H}_N \simeq \mathbb{R}^{N^2}$. Notons $\lambda_1^N > \cdots > \lambda_N^N$ les valeurs propres de X^N. Alors, on a la convergence suivante

$$N^{2/3}\left(\frac{\lambda_1^N}{2N^{1/2}} - 1\right) \xrightarrow[N\to\infty]{d} TW, \tag{2.2}$$

où TW désigne la loi de Tracy-Widom définie à partir du déterminant de Fredholm d'opérateurs intégraux associés au noyau d'Airy (cf [TW94]). Il convient de remarquer que la normalistation et la loi limite dans (2.2) diffèrent de celles que l'on trouve dans le théorème central-limite classique. C'est, avec le résultat (2.2), la première fois qu'apparaît la loi de Tracy-Widom. Maintenant, soit σ une permutation aléatoire de loi uniforme sur $\mathfrak{S}_N$ et $L^N := \max\{k\,;\ \exists i_1 < \cdots < i_k,\ \sigma(i_1) < \cdots < \sigma(i_k)\}$ la taille de sa plus longue sous-suite croissante. Alors, il est prouvé dans [BDJ99] que

$$N^{1/3}\left(\frac{L^N}{2N^{1/2}} - 1\right) \xrightarrow[N\to\infty]{d} TW. \tag{2.3}$$

On aperçoit l'exacte similarité entre les deux comportements asymptotiques (2.2) et (2.3), tant pour le type de normalisation que pour la loi limite elle-même. En réalité, ces identités ne concernent pas uniquement la valeur propre maximale et s'étendent aux autres valeurs propres de la manière suivante. Si l'on note

$$\Lambda_N = \{l \in \mathbb{N}^N;\ l_1 \geq \cdots \geq l_N,\ \sum_i l_i = N\},$$

f^l la dimension de la représentation irréductible de $\mathfrak{S}_N$ indicée par $l \in \Lambda_N$ (égale au nombre de tableaux de Young standards de forme l) et

$$P_N(l) := \frac{(f^l)^2}{N!}, \tag{2.4}$$

alors P_N est une mesure de probabilité sur Λ_N, appelée mesure de Plancherel. Si l^N est une variable aléatoire de loi P_N, sa 1ère composante l_1^N a la loi de L^N. On définit $y_i^N := N^{1/3}\left(\frac{l_i^N}{2N^{1/2}} - 1\right)$ ainsi que les quantités analogues pour les valeurs propres du GUE(N) : $x_i^N := N^{1/3}\left(\frac{\lambda_i^N}{2N^{1/2}} - 1\right)$. Alors, pour k fixé, $(x_1^N, \ldots, x_k^N)$ et $(y_1^N, \ldots, y_k^N)$ ont la même limite en loi lorsque $N \to \infty$ (cf [Oko00], [BOO00], [BDJ00]). Il apparaît, ainsi, une similarité de comportements asymptotiques pour deux problèmes complètement différents a priori.

2.1.2 Les gas de Coulomb

La question se pose naturellement de savoir si l'on peut observer un lien non-asymptotique (ie à N fixé) entre les deux problèmes ? Un premier élément de réponse est que les lois de $\lambda_1^N > \cdots > \lambda_N^N$ et $l_1^N \geq \cdots \geq l_N^N$ partagent la même structure de gas de Coulomb de paramètre $\beta = 2$.

Definition 2.1.1. *On appelle gas de Coulomb de paramètre β toute mesure de probabilité de la forme*

$$\mu_{N,\beta}(dx) = Z_{N,\beta}^{-1}\, h(x)^\beta\, \mu^{\otimes N}(dx),\ x \in W = \{x \in \mathbb{R}^N :\ x_1 > \cdots > x_N\}, \tag{2.5}$$

où $h(x) = \prod_{1\leq i<j\leq N}(x_i - x_j)$ est la fonction de Vandermonde, μ est une mesure de probabilité sur $\mathbb{R}$ (éventuellement discrète, concentrée sur $\mathbb{N}$ ou $\mathbb{Z}$) et $Z_{N,\beta}$ est la constante de normalisation.

La loi des valeurs propres $\lambda_1^N > \cdots > \lambda_N^N$ du GUE(N) est un gas de Coulomb de paramètre $\beta = 2$ associé à la mesure gaussienne standard sur $\mathbb{R}$. Si $l_1^N \geq \cdots \geq l_N^N$ est distribuée selon la mesure de Plancherel sur Λ_N et si on note $h_i^N = l_i^N + N - i$, la loi des $h_1^N > \cdots > h_N^N$ est un gas de Coulomb de paramètre $\beta = 2$ associé à la mesure (non-normalisée) $\mu(m) = 1/(m!)^2$, $m \in \mathbb{N}$ (cf [Joh01a]).

Il existe des méthodes pour calculer les fonctions de corrélation de telles mesures et analyser leur comportement asymptotique. On a recours aux polynômes orthogonaux de μ, ce qui explique le nom de « orthogonal polynomial ensembles » qu'on attribue aussi à ces mesures (cf [TW98], [Kon04]).

2.1.3 Des identités en loi

On peut approfondir cette analogie en mentionnant une remarquable identité (due à [Bar01], [GTW01]) pour la valeur propre maximale du GUE(N) :

$$\lambda_1^N \stackrel{d}{=} \mathcal{L}^N(B) := \sup_{0=t_0 \leq \cdots \leq t_N = 1} \sum_{i=1}^{N} (B_i(t_i) - B_i(t_{i-1})), \tag{2.6}$$

où $(B_i)_{1 \leq i \leq N}$ est un mouvement brownien standard N-dimensionnel. On observe que la fonctionnelle $\mathcal{L}^N$ d'une fonction continue est très analogue à la fonctionnelle L^N d'une permutation. En fait, des identités similaires à (2.6) ont récemment été obtenues pour toutes les valeurs propres ([OY02], [BJ02], [O'C03b], [BBO04]). Elles apparaissent comme les marginales à temps fixe d'identités valables pour des processus stochastiques. Précisément, si $\mathcal{D}_0(\mathbb{R}_+)$ est l'espace des fonctions $f : \mathbb{R}_+ \to \mathbb{R}$, cadlag, nulles en 0, on définit

$$(f \otimes g)(t) = \inf_{0 \leq s \leq t} (f(s) + g(t) - g(s)) \quad \text{et} \quad (f \odot g)(t) = \sup_{0 \leq s \leq t} (f(s) + g(t) - g(s)),$$

puis $\Gamma^{(N)} : \mathcal{D}_0(\mathbb{R}_+)^N \to \mathcal{D}_0(\mathbb{R}_+)^N$ par récurrence :

$$\Gamma^{(2)}(f, g) = (f \otimes g, g \odot f)$$

et pour $N > 2$, si $f = (f_1, \ldots, f_N)$,

$$\Gamma^{(N)}(f) = \left(f_1 \otimes \cdots \otimes f_N, \Gamma^{(N-1)}(f_2 \odot f_1, f_3 \odot (f_1 \otimes f_2), \ldots, f_N \odot (f_1 \otimes \cdots \otimes f_{N-1}))\right).$$

Le résultat fondamental de [OY02] est :

$$\lambda^N \stackrel{d}{=} \Gamma^{(N)}(B), \tag{2.7}$$

où B est le mouvement brownien standard dans $\mathbb{R}^n$ et $\lambda^{(N)}$ la trajectoire des valeurs propres, rangées par ordre croissant, d'un mouvement brownien hermitien (défini à la remarque 2.2.1). L'identité (2.6) correspond, modulo les égalités $B \stackrel{d}{=} -B$ et $\lambda_{max}^N \stackrel{d}{=} -\lambda_{min}^N$, à la première composante de l'identité (2.7).

2.2 Une note sur les représentations de valeurs propres de matrices gaussiennes

Nous avons considéré une égalité en loi analogue à (2.6) pour un autre ensemble de matrices aléatoires, le $LUE(N, M)$, $M \geq N$. Celui-ci est composé des matrices $Y^{N,M} := A\,\overline{A}^{\top}$, où A est une matrice $N \times M$ dont les coefficients sont des gaussiennes standards, complexes et indépendantes. De manière équivalente, $LUE(N, M)$ est la loi suivante sur $\mathcal{H}_N$:

$$P_{N,M}(dH) = Z_{N,M}^{-1}(\det H)^{M-N} \exp(-\operatorname{Tr} H)\mathbf{1}_{H\geq 0}dH\ . \tag{2.8}$$

Si $\mu_1^{N,M} > \cdots > \mu_N^{N,M} \geq 0$ désignent les valeurs propres de $Y^{N,M}$, alors Johansson ([Joh00]) a montré que

$$\mu_1^{N,M} \stackrel{d}{=} H(M, N) := \max\Big\{ \sum_{(i,j)\in\pi} w_{i,j}\ ;\ \pi \in \mathcal{P}(M, N)\Big\}, \tag{2.9}$$

où les $(w_{i,j}\,,\,(i, j) \in (\mathbb{N} \setminus \{0\})^2)$ sont des variables exponentielles i.i.d. de paramètre 1 et $\mathcal{P}(M, N)$ est l'ensemble des chemins π effectuant des pas $(0, 1)$ ou $(1, 0)$ dans le rectangle $\{1, \ldots , M\} \times \{1, \ldots , N\}$.

2.2.1 Un théorème central-limite

Notre première observation est l'existence d'un théorème central-limite qui fait apparaître GUE(N) comme une certaine limite de LUE(N, M) :

Théorème 2.2.1 (cf Th 5.2.1). *Soient $Y^{N,M}$ et X^N des matrices respectivement du LUE(N, M) et du GUE(N). Alors*

$$\frac{Y^{N,M} - M\,\mathrm{Id}_N}{\sqrt{M}} \xrightarrow[M\to\infty]{d} X^N\ . \tag{2.10}$$

Remarque 2.2.1. *En réalité, nous démontrons une telle convergence au niveau des processus stochastiques. Précisément, si l'on remplace les variables aléatoires gaussiennes utilisées pour définir LUE(N, M) et GUE(N) par des mouvements browniens, on obtient des processus $\{Y^{N,M}(t),\ t \geq 0\}$ et $\{X^N(t),\ t \geq 0\}$, appelés processus de Laguerre et mouvement brownien hermitien, qui vérifient le*

Théorème 2.2.2 (cf Th 5.2.2).

$$\Big(\frac{Y^{N,M}(t) - Mt\,\mathrm{Id}_N}{\sqrt{M}}\Big)_{t\geq 0} \xrightarrow[M\to\infty]{d} (X^N(t^2))_{t\geq 0} \tag{2.11}$$

au sens de la convergence faible sur $\mathcal{C}(\mathbb{R}_+, \mathcal{H}_N)$.

2.2.2 La plus grande valeur propre

Les valeurs propres d'une matrice hermitienne étant des fonctions continues de cette matrice, on a donc

$$\frac{\mu^{N,M} - M}{\sqrt{M}} \xrightarrow[M\to\infty]{d} \lambda_1^N .$$

Ceci, joint à (2.9) et au principe d'invariance suivant, dû à Glynn-Whitt [GW91],

$$\frac{H(M,N) - M}{\sqrt{M}} \xrightarrow[M\to\infty]{d} \sup_{0=t_0\leq\ldots\leq t_N=1} \sum_{i=1}^{N} (B_i(t_i) - B_i(t_{i-1})), \tag{2.12}$$

redémontre (2.6) trivialement.

2.2.3 Les autres valeurs propres

Notre deuxième remarque est que le raisonnement précédent se généralise aux autres valeurs propres. Précisément, pour $1 \leq k \leq N$, si l'on définit :

$$H_k(M,N) := \max \sum_{(i,j)\in\pi_1\cup\cdots\cup\pi_k} w_{i,j}, \tag{2.13}$$

où le max est pris sur l'ensemble des chemins disjoints $\pi_1, \ldots, \pi_k \in \mathcal{P}(M,N)$, alors

$$(H_k(M,N))_{1\leq k\leq N} \stackrel{d}{=} \left(\mu_1^{N,M} + \mu_2^{N,M} + \cdots + \mu_k^{N,M}\right)_{1\leq k\leq N} . \tag{2.14}$$

Ensuite, nous établissons un principe d'invariance pour $H_k(M,N)$ (cf Eq (5.19)) :

$$\frac{H_k(M,N) - kM}{\sqrt{M}} \xrightarrow[M\to\infty]{d} \Omega_k^{(N)} := \sup \sum_{j=1}^{N} \sum_{p=1}^{k} (\, B_j(s_{j-p+1}^p) - B_j(s_{j-p}^p)\,), \tag{2.15}$$

où le sup est pris sur toutes les subdivisions (s_i^p) de $[0,1]$ de la forme :

$$s_i^p \in [0,1] \ , \ s_i^{p+1} \leq s_i^p \leq s_{i+1}^p \ , \ s_i^p = 0 \text{ pour } i \leq 0 \text{ et } s_i^p = 1 \text{ pour } i \geq N-k+1.$$

Nous obtenons donc la représentation suivante pour les valeurs propres du GUE(N) :

$$\left(\Omega_k^{(N)}\right)_{1\leq k\leq N} \stackrel{d}{=} \left(\lambda_1^N + \lambda_2^N + \cdots + \lambda_k^N\right)_{1\leq k\leq N} . \tag{2.16}$$

Remarque 2.2.2. *Ce résultat implique que*

$$\left(\Omega_k^{(N)}\right)_{1\leq k\leq N} \stackrel{d}{=} \left(\Gamma_N^{(N)}(B)(1) + \Gamma_{N-1}^{(N)}(B)(1) + \cdots + \Gamma_{N-k+1}^{(N)}(B)(1)\right)_{1\leq k\leq N} ,$$

ce qui est en accord avec l'équivalence, obtenue dans [O'C03b], entre la fonctionnelle Γ *et la correspondance RSK. La formule précédente apparaît comme un analogue, dans ce contexte continu, des formules de Greene exprimant la somme des tailles des premières lignes du diagramme obtenu en appliquant RSK à une permutation en fonction des sous-suites croissantes disjointes de cette dernière.*

L'ambition que suscitent naturellement de telles observations serait d'obtenir une représentation analogue à (2.7) pour les trajectoires des valeurs propres du processus de Laguerre dont on sait qu'elles ont la loi de carrés de Bessel conditionnés à ne pas s'intersecter (cf [KO01]). Nous n'avons pas été capable d'obtenir une telle représentation mais, dans le but d'imiter l'approche de [OY02], nous avons été amené à considérer une version discrète des carrés de Bessel et à définir le conditionnement associé.

2.3 Processus sans collision et l'ensemble de Meixner

Si, dans (2.13), les $(w_{i,j}\,,\,(i,j)\in(\mathbb{N}\setminus\{0\})^2)$ sont remplacées par des variables géométriques i.i.d. de paramètre q, la version discrète de (2.14) s'écrit :

$$(H_k(M,N))_{1\le k\le N} \overset{d}{=} (\nu_1+\nu_2+\cdots+\nu_k)_{1\le k\le N}\ , \tag{2.17}$$

où $\nu_1+N-1>\nu_2+N-2>\cdots>\nu_N$ a la loi du gas de Coulomb suivant, appelé ensemble de Meixner,

$$\mathbf{Me}_{N,\theta,q}(y)=(Z_{N,\theta,q})^{-1}\,h(y)^2\prod_{j=1}^{N}w_q^\theta(y_j),\quad y\in W:=\{y\in\mathbb{N}^N;\ y_1>\cdots>y_N\},$$

où $\theta=M-N+1$, $w_q^\theta(y)=\binom{y+\theta-1}{y}q^y$ pour $y\in\mathbb{N}$ et $Z_{N,\theta,q}$ est une constante de normalisation telle que $\mathbf{Me}_{N,\theta,q}$ soit une mesure de probabilité sur W et $h(y)$ est le déterminant de Vandermonde

$$h(y)=\prod_{1\le i<j\le N}(y_i-y_j)\,.$$

Notre but est de faire apparaître cet ensemble comme la marginale à temps fixe de processus stochastiques discrets conditionnés à ne pas s'intersecter.

2.3.1 Harmonicité du déterminant de Vandermonde

Précisément, nous établissons la

Proposition 2.3.1 (cf Prop 6.2.1). *Soient* $X_1, \dots, X_N$, *N copies indépendantes d'un processus de vie et de mort dans* $\mathbb{N}$, *de taux de vie* $\beta(x) = ax^2 + bx + c$ *et de taux de mort* $\delta(x) = \overline{a}x^2 + \overline{b}x + \overline{c}$, $x \in \mathbb{N}$. *h est une fonction propre pour le générateur du processus* $\mathbf{X} = (X_1, \dots, X_N)$ *si et seulement si* $a = \overline{a}$ *et la valeur propre correspondante est* $a\frac{N(N-1)(N-2)}{3} + (b - \overline{b})\frac{N(N-1)}{2}$. *En particulier, h est harmonique si et seulement si* $a = \overline{a}$ *et* $b - \overline{b} = -\frac{2}{3}a(N-2)$.

2.3.2 Processus de Yule sans collision

Si $X_1, \dots, X_N$ sont des processus de Yule ($\beta(x) = x + \theta$, $\theta > 0$ et $\delta(x) = 0$) indépendants, h est harmonique pour le processus $\mathbf{X} = (X_1, \dots, X_N)$ tué au temps $T \wedge \tau$, où $T = \inf\{t > 0\,;\, \mathbf{X}(t) \notin W\}$ et τ est un temps exponentiel de paramètre $\lambda = \frac{N(N-1)}{2}$, indépendant de $\mathbf{X}$. On peut alors définir la h-transformée suivante :

$$\mathbb{P}_x^h(\mathbf{X}(t) = y) = e^{-\lambda t}\frac{h(y)}{h(x)}\,\mathbb{P}_x(\mathbf{X}(t) = y,\, T > t), \tag{2.18}$$

pour $x, y \in W$. On peut voir ce nouveau processus comme le processus de départ conditionné à ne jamais quitter W. On peut alors montrer la

Proposition 2.3.2 (cf Prop 6.3.1). *Posons* $x^* = (N-1, N-2, \dots, 0)$. *Alors, pour tout* $y \in W$ *et tout* $t > 0$,

$$\mathbb{P}_{x^*}^h(\mathbf{X}(t) = y) = \mathbf{Me}_{N,\theta,1-e^{-t}}(y) = C_t\, h(y)^2\, \mathbb{P}_0(\mathbf{X}(t) = y)\,.$$

2.3.3 Processus de vie et de mort linéaires sans collision

Si $Y_1, \dots, Y_N$ sont N copies indépendantes d'un processus de vie et de mort avec $\beta(x) = x + \theta$, $\theta > 0$ et $\delta(x) = x$, alors h est harmonique pour $\mathbf{Y} = (Y_1, \dots, Y_N)$ et l'on définit :

$$\mathbb{P}_x^h(\mathbf{Y}(t) = y) = \frac{h(y)}{h(x)}\,\mathbb{P}_x(\mathbf{Y}(t) = y,\, T > t) \tag{2.19}$$

pour $x, y \in W$ et $T = \inf\{t > 0\,;\, \mathbf{X}(t) \notin W\}$. Alors

Proposition 2.3.3 (cf Prop 6.4.1). *Si* $x^* = (N-1, N-2, \dots, 0)$, $y \in W$ *et* $t > 0$, *on a :*

$$\mathbb{P}_{x^*}^h(\mathbf{Y}(t) = y) = \mathbf{Me}_{N,\theta,t/(1+t)}(y) = D_t\, h(y)^2\, \mathbb{P}_0(\mathbf{Y}(t) = y)\,.$$

2.3.4 Frontière de Martin

On peut facilement analyser l'asymptotique du noyau de Martin et montrer que la compactification de Martin de $\mathbf{X}$ tué au temps $T \wedge \tau$ est $MC = W \cup \Sigma$, où

$$\Sigma := \{p \in [0,1]^N \,|\, p_1 \geq \ldots \geq p_N,\ |p| = \sum_i p_i = 1\}$$

et une suite $(y_n) \in W^{\mathbb{N}}$ converge vers $p \in \Sigma$ si et seulement si $|y_n| \to \infty$ et $y_n/|y_n| \to p$. Le noyau de Martin (basé en x^*) associé au point $p \in \Sigma$ est

$$M(x,p) = \prod_{i=1}^{N} \frac{(\theta)_{N-i}}{(\theta)_{x_i}} \frac{\Gamma(N\theta + \lambda + |x|)}{\Gamma(N\theta + \lambda + |x^*|)} \mathrm{Schur}_x(p),$$

où

$$\mathrm{Schur}_x(p) = \frac{\det\left(p_j^{x_i}\right)_{1 \leq i,j \leq N}}{h(p)}.$$

Nous remarquons que h est une fonction harmonique pour $\mathbf{L}^\lambda$ mais n'est pas extrêmale, ce qui diffère de la situation des marches aléatoires (cf [KOR02], [O'C03b] et [O'C03a]). Il serait intéressant de trouver une mesure de mélange μ_h (a priori, nous devons dire « une », puisque nous n'avons pas déterminé la partie minimale de la frontière) telle que :

$$h(x) = h(x^*) \prod_{i=1}^{N} \frac{(\theta)_{N-i}}{(\theta)_{x_i}} \frac{\Gamma(N\theta + |x| + \lambda)}{\Gamma(N\theta + |x^*| + \lambda)} N^{|x^*|-|x|} \int_\Sigma \mathrm{Schur}_x(Np)\, \mu_h(dp).$$

2.4 L'algorithme RSK appliqué à un mot échangeable

Si ξ est la marche aléatoire simple symétrique sur $\mathbb{Z}$ démarrant en 0 et $\overline{\xi}$ est le processus de son maximum passé alors une version discrète du théorème de Pitman ([Pit75]) affirme deux choses : d'abord que $2\overline{\xi} - \xi$ est une chaîne de Markov et ensuite que $2\overline{\xi} - \xi$ a la loi de la chaîne ξ conditionnée à rester éternellement positive ou nulle. De récents travaux ([OY02], [BJ02], [O'C03b], [BBO04]) ont étendu ce résultat dans des cadres multi-dimensionnels. La correspondance RSK est un algorithme combinatoire qui fournit l'analogue multi-dimensionnel F de la transformation $f : \xi \to 2\overline{\xi} - \xi$. Nous avons examiné les deux affirmations du théorème de Pitman lorsque l'on remplace f par F et ξ par X, le processus des types d'un mot infini échangeable.

2.4.1 Le processus des formes

Précisément, nous considérons $\eta = (\eta_n)_{n \geq 1}$, une suite échangeable de variables aléatoires à valeurs dans $[k] := \{1, \ldots, k\}$ et définissons le processus $X \in (\mathbb{N}^k)^{\mathbb{N}}$ par

$$X_i(n) = |\{m \leq n \,|\, \eta_m = i\}|, \quad 1 \leq i \leq k,\ n \geq 0.$$

On voit facilement X est une chaîne de Markov de transitions

$$P_X(\alpha, \beta) = \frac{q(\beta)}{q(\alpha)} \mathbf{1}_{\alpha \nearrow \beta}, \tag{2.20}$$

où $\alpha \nearrow \beta$ signifie que $\beta - \alpha$ est un vecteur de la base canonique de $\mathbb{R}^k$ et q est une fonction qui détermine la loi de η. Si $\widetilde{X}(n)$ est la forme des tableaux obtenus en appliquant l'algorithme RSK au mot $(\eta_1, \dots, \eta_n)$, alors

Théorème 2.4.1 (cf Th 7.3.1). *$\widetilde{X}$ est une chaîne de Markov sur l'ensemble $\Omega = \{\lambda \in \mathbb{N}^k;\ \lambda_1 \geq \cdots \geq \lambda_k\}$ et ses transitions sont données par :*

$$P_{\widetilde{X}}(\mu, \lambda) = \frac{f(\lambda)}{f(\mu)} \mathbf{1}_{\mu \nearrow \lambda}, \tag{2.21}$$

où la fonction f est définie par

$$f(\lambda) = \sum_{\alpha} K_{\lambda\alpha}\, q(\alpha) \tag{2.22}$$

et $K_{\lambda\alpha}$ est le nombre (dit de Kostka) de tableaux semi-standards de forme λ et de type α (pour les définitions combinatoires, une référence est [Ful97] mais on trouvera un résumé dans la section 7.2 et le chapitre 13 de cette thèse).

Remarque 2.4.1. *Le processus $\widetilde{X}$ est une fonction déterministe de X, $\widetilde{X} = F^k(X)$, où la fonctionnelle F^k a été décrite par O'Connell dans [O'C03b]. Le théorème 2.4.1 apparaît ainsi comme un analogue multi-dimensionnel de la première partie du théorème de Pitman.*

Le théorème de de Finetti affirme, dans ce contexte, que $\frac{X(n)}{n}$ converge presque-sûrement vers une variable X_∞ de loi $d\rho$ sur $S_k = \{p \in [0,1]^k\ ,\ \sum p_i = 1\}$. On a alors

$$f(\lambda) = \int s_\lambda(p)\, d\rho(p),$$

où $s_\lambda(x) = \frac{\det\left(x_i^{\lambda_j + k - j}\right)_{1 \leq i,j \leq k}}{\prod_{i<j}(x_i - x_j)}$ est la fonction de Schur d'indice λ. Nous montrons que :

Proposition 2.4.1 (cf Prop 7.3.3).

$$\lim_{n \to \infty} \frac{\widetilde{X}(n)}{n} = \widetilde{X}_\infty \ p.s., \tag{2.23}$$

où $\widetilde{X}_\infty$, de loi $d\tilde{\rho}$, est le réarrangement décroissant de X_∞.

2.4.2 Le processus conditionné

Soit η' un autre mot infini échangeable auquel on associe comme précédemment le processus des types X' et la « loi de mélange » $d\rho'$. Si l'on suppose

$$\rho'(\{p \in S_k;\ p_1 > \cdots > p_k\}) > 0,$$

on peut conditionner X' à rester éternellement dans Ω au sens usuel et le processus $\widehat{X'}$ ainsi obtenu est une chaîne de Markov dont les transitions sont données par :

$$P_{\widehat{X'}}(\mu, \lambda) = \frac{g(\lambda)}{g(\mu)} P_{Z,\Omega}(\mu, \lambda), \tag{2.24}$$

où

$$g(\lambda) = \int s_\lambda(kp)\, d\widehat{\rho'}(p)$$

et $d\widehat{\rho'}$ est la mesure de probabilité donnée par

$$d\widehat{\rho'}(p) = \frac{1}{C_{\rho'}} p^{-\delta} \prod_{i<j} (p_i - p_j) \mathbf{1}_W(p)\, d\rho'(p). \tag{2.25}$$

2.4.3 Le lien entre conditionnement et RSK

Proposition 2.4.2 (cf Prop 7.4.2). *$\widetilde{X}$ a la même loi que $\widehat{X'}$ si et seulement si $\tilde{\rho} = \widehat{\rho'}$.*

Un corollaire de cette proposition est la CNS $\tilde{\rho} = \widehat{\rho}$ pour que la deuxième partie du théorème de Pitman soit vraie, c'est-à-dire pour que processus des formes $\widetilde{X}$ et processus conditionné $\widehat{X}$ associés à un même mot η aient la même loi. Cette condition est évidemment vérifiée lorsque η est une suite de variables i.i.d. En revanche, elle n'est pas vérifiée lorsque η code la couleur des boules ajoutées aux instants successifs dans une urne de Polya.

2.4.4 Une réciproque « à la Rogers » du théorème de Pitman

Nous souhaiterions caractériser les mots infinis η tels que les processus associés X et $\widetilde{X}$ soient markoviens. Nous ne savons pas résoudre ce problème en toute généralité mais pouvons en traiter un cas facile. Si (R, S) sont les tableaux obtenus en appliquant l'algorithme RSK au mot w, nous définissons F par

$$\mathbb{P}[(\eta_1, \ldots, \eta_n) = w] = F(R, S).$$

Alors, on a la

Proposition 2.4.3 (cf Prop 7.4.4). *Si $F(R, S) = F(R)$ alors $\widetilde{X}$ est automatiquement une chaîne de Markov. Si, en outre, X est aussi markovien, alors η est échangeable.*

Chapitre 3

Diffusions à valeurs matricielles

3.1 Contexte

3.1.1 Des variables aléatoires à valeurs matricielles

Les principales familles de lois de probabilité sur $\mathbb{R}$ admettent des généralisations naturelles sur les espaces de matrices classiques. Par exemple, sur l'espace $\mathcal{S}_n(\mathbb{R})$ des matrices symétriques de taille n, on peut définir les mesures de probabilité suivantes :

1. le GOE(n) de densité $(2\pi)^{n(n-1)/4}\, e^{-\mathrm{tr} A^2}\, dA$,
2. le LOE(n, p), $p > n-1$, de densité
$$2^{-np/2}\Gamma_m(p/2)^{-1}\, (\det A)^{(p-n-1)/2} e^{-\mathrm{tr} A/2}\, \mathbf{1}_{A>0}\, dA,$$
3. le JOE(n, p, q), $p > n-1$, $q > n-1$, de densité
$$\frac{\Gamma_m((p+q)/2)}{\Gamma_m(p/2)\Gamma_m(q/2)}\, (\det A)^{(p-n-1)/2} (\det(1_n - A))^{(q-n-1)/2}\, \mathbf{1}_{0_n<A<1_n}\, dA,$$

où $dA = \prod_{i\leq j} dA_{ij}$ est la mesure de Lebesgue sur $\mathcal{S}_n(\mathbb{R})$ et Γ_m est la fonction Gamma multivariée (cf [Mui82] ou bien section 8.5). Les GOE(n), LOE(n, p) et JOE(n, p, q) sont respectivement les analogues de la loi gaussienne, de la loi Gamma et de la loi Bêta sur $\mathbb{R}$. Une fois définies ces « matrices aléatoires », il est naturel de se demander si elles partagent les propriétés classiques des variables aléatoires réelles dont elles sont les pendants. Par exemple, on doit à Lévy ([Lév48]) l'ancienne et remarquable observation que, malgré le bon comportement des lois LOE(n, p) sous la convolution

$$\mathrm{LOE}(n, p) * \mathrm{LOE}(n, p') = \mathrm{LOE}(n, p + p'),$$

LOE(n, p) n'est pas infiniment divisible, ceci à cause de l'appartenance de p à l'ensemble dit de Gindikin $\{1, \ldots, n-1\} \cup]n-1, \infty[$. On trouve dans la littérature, essentiellement statistique, de très nombreux travaux concernés par cette étude. L'introduction des polynômes zonaux et des fonctions hypergéométriques matricielles par

A. T. James et A. G. Constantine ([Jam61], [Jam68], [Con63]) fut un progrès majeur, permettant l'écriture de certaines intégrales sur des variétés de Stiefel qui interviennent dans les distributions non-centrales en statistique multivariée. On doit à C.S. Herz la généralisation aux matrices symétriques des fonctions de Bessel ([Her55]) et [Con66], [JC74] ont considéré les polynômes orthogonaux de mesures matricielles. De remarquables apports à l'étude des propriétés des lois de Wishart se trouvent dans les contributions de G. Letac et coauteurs, notamment leur extension aux cônes symétriques, leur caractérisation et le calcul de leurs moments ([CL96], [LW00], [GLM03]).

3.1.2 Des processus à valeurs matricielles

L'idée remonte déjà à Dyson ([Dys62]) de proposer une généralisation supplémentaire en considérant, non plus des variables aléatoires, mais des processus stochastiques à valeurs matricielles comme le mouvement brownien hermitien. Cette vision dynamique lui a permis de mettre clairement en évidence les répulsions qui existent entre les valeurs propres de telles matrices. En effet, il est possible de décrire explicitement les trajectoires suivies par les valeurs propres et l'on constate que l'équation différentielle stochastique qui les dirige met en jeu une force de répulsion électrostatique exercée sur chaque valeur propre λ_i par toutes les autres λ_j, $j \neq i$ et de manière inversement proportionnelle à la distance qui les sépare, ie en $1/(\lambda_i - \lambda_j)$. De telles études ont été menées dans [McK69], [Dyn61], [NRW86], [Ken90], etc. Elles peuvent être considérées comme faisant partie des préoccupations plus vastes de la géométrie différentielle stochastique (voir [RW00] pour une très lisible introduction). On y établit par exemple ([PR88]) des résultats de skew-products généraux pour le mouvement brownien sur une variété riemannienne M lorsque celle-ci admet une décomposition $M \approx \Lambda \times U$. D'agréables exemples de tels skew-products sont notamment fournis par les diffusions matricielles quand on regarde la décomposition $\mathcal{S}_n(\mathbb{R}) \approx \mathbb{R}^n \times \mathcal{O}_n(\mathbb{R})$ correspondant à la diagonalisation. Un intérêt spécifique fut aussi porté aux trajectoires des valeurs propres en elles-mêmes, en lien avec la théorie des non-colliding processes ([CL01], [KT03a], [KT03b], [Gra99], [Gil03]). Mentionnons encore l'usage que l'on peut faire d'un calcul stochastique matriciel tel qu'il a été développé dans [CDG01] et [BCG03]. Inspiré par le calcul stochastique libre ([Bia97], [BS98]), il a permis d'établir des principes de grande déviation pour les processus de mesures spectrales empiriques ainsi que des inégalités concernant l'entropie libre de Voiculescu. Enfin, signalons qu'il est possible de faire tendre vers l'infini la taille du processus de Wishart dont nous allons parler dans 3.2 pour obtenir un processus de Wishart « libre » ([CDM03]).

3.2 Quelques propriétés des processus de Wishart et une extension matricielle des lois de Hartman-Watson

Parmi ces études de processus de diffusion à valeurs matricielles, nous avons porté notre attention sur un travail de M.F. Bru ([Bru89c], [Bru91], [Bru89a], [Bru89b]), décrivant des généralisations matricielles des processus uni-dimensionnels que sont les carrés de Bessel.

3.2.1 Le carrés de Bessel

Rappelons d'abord, brièvement, les caractéristiques de ces derniers.

(i) Equation différentielle stochastique
Pour $\delta \geqq 0$, l'équation différentielle stochastique

$$dX_t = 2\sqrt{X_t}\, dB_t + \delta\, dt, \quad X_0 = x \geq 0, \tag{3.1}$$

admet une unique solution forte. On appelle ce processus un « carré de Bessel » de dimension δ et l'on note Q_x^δ sa loi.
(ii) Propriété d'additivité

$$Q_x^\delta * Q_{x'}^{\delta'} = Q_{x+x'}^{\delta+\delta'}$$

(iii) Relation d'absolue d'absolue continuité et loi de Hartman-Watson
Si $\mathcal{F}_t = \sigma\{X_s, s \leqq t\}$, $\delta = 2(1+\nu)$ et $Q^{(\nu)} = Q^\delta$, on a

$$Q_x^{(\nu)}|_{\mathcal{F}_t} = \left(\frac{X_t}{x}\right)^{\nu/2} \exp\left(-\frac{\nu^2}{2}\int_0^t \frac{ds}{X_s}\right) \cdot Q_x^{(0)}|_{\mathcal{F}_t},$$

ce qui permet de déduire que la loi $Q_x^{(0)}$-conditionnelle de $\int_0^t (X_s)^{-1}\, ds$ sachant $X_t = y$ est la loi de Hartman-Watson caractérisée par

$$Q_x^{(0)}\left[\exp\left(-\frac{\nu^2}{2}\int_0^t \frac{ds}{X_s}\right)|X_t = y\right] = \frac{I_\nu(r)}{I_0(r)}, \tag{3.2}$$

où $r = \sqrt{xy}/t$.
(iv) Inversion du temps
Si X est distribué selon Q_x^δ, $\{t^2 X(1/t),\, t \geq 0\}$ est un carré de Bessel généralisé de drift $\sqrt{x}$, démarrant en 0 (voir [Wat75] ou [PY81] pour une définition).

(v) Entrelacement
Si $Q_t^\delta(x, dy)$ désigne le semi-groupe de Q^δ, alors

$$Q_t^{\delta+\delta'} \Lambda_{\delta,\delta'} = \Lambda_{\delta,\delta'} Q_t^\delta, \tag{3.3}$$

où $\Lambda_{\delta,\delta'}$ est l'opérateur de multiplication associé à $\beta_{\delta/2,\delta'/2}$, une variable Bêta de paramètre $(\delta/2, \delta'/2)$, i.e.,

$$\Lambda_{\delta,\delta'} f(x) = E[f(x\beta_{\delta/2,\delta'/2})].$$

3.2.2 Les processus de Wishart

Aidé du récapitulatif précédent, il est aisé de présenter, pour les processus de Wishart, les résultats de M.F. Bru ([Bru89c], [Bru91], [Bru89a], [Bru89b]) ainsi que les nôtres.

(i) Equation différentielle stochastique
L'équation (3.1) est remplacée par

$$dX_t = \sqrt{X_t}\, dB_t + dB_t' \sqrt{X_t} + \delta I_m\, dt, \quad X_0 = x, \tag{3.4}$$

où X est à valeurs dans l'espace $\mathcal{S}_m^+$ des matrices symétriques positives de taille m, B est une matrice brownienne $m \times m$ et B' est sa transposée. On appelle processus de Wishart de dimension δ une solution de (3.4) et on note par $\mathbf{Q}_x^\delta$ sa loi, qui est bien définie grâce au

Théorème 3.2.1 ([Bru91]). (i) *Si* $\delta \in (m-1, m+1)$, (3.4) *possède une unique solution en loi.*
(ii) *Si* $\delta \geqq m+1$, (3.4) *possède une unique solution qui reste définie positive en tous temps.*

(ii) Propriété d'additivité
Alors que

$$\mathbf{Q}_x^\delta * \mathbf{Q}_{x'}^{\delta'} = \mathbf{Q}_{x+x'}^{\delta+\delta'} \tag{3.5}$$

reste vraie, la loi $\mathbf{Q}_x^\delta$ n'est pas infiniment divisible à cause de l'appartenance de δ à l'ensemble de Gindikin $\Lambda_m = \{1, \dots, m-1\} \cup (m-1, \infty)$.
(iii) Relation d'absolue d'absolue continuité et loi de Hartman-Watson
Le résultat principal de notre travail est

Théorème 3.2.2 (cf Th 8.1.2). *Si* $\nu \geqq 0$, *on a :*

$$\mathbf{Q}_x^{m+1+2\nu}|_{\mathcal{F}_t} = \left(\frac{\det(X_t)}{\det(x)}\right)^{\nu/2} \exp\left(-\frac{\nu^2}{2}\int_0^t \mathrm{tr}(X_s^{-1})\, ds\right) \cdot \mathbf{Q}_x^{m+1}|_{\mathcal{F}_t}. \tag{3.6}$$

Ceci permet de définir un analogue matriciel de la loi de Hartman-Watson, caractérisé par

$$\mathbf{Q}_x^{m+1}\left[\exp\left(-\frac{\nu^2}{2}\int_0^t \operatorname{tr}(X_s^{-1})\,ds\right)|X_t=y\right]=\frac{\widetilde{\mathbf{I}}_\nu(xy/4t^2)}{\widetilde{\mathbf{I}}_0(xy/4t^2)},\tag{3.7}$$

où

$$\widetilde{\mathbf{I}}_\nu(z)=\frac{(\det(z))^{\nu/2}}{\mathbf{\Gamma}_m((m+1)/2+\nu)}\,{}_0\mathbf{F}_1((m+1)/2+\nu;z)\tag{3.8}$$

est un analogue matriciel des fonctions de Bessel et ${}_0\mathbf{F}_1$ est une fonction hypergéométrique matricielle définie en 8.5. Nous pouvons aussi écrire une relation d'absolue continuité pour une dimension $m+1+2\nu$ avec $-1<\nu\leq 0$, ce qui permet d'expliciter la loi du premier temps où $\det X$ est nul.

(iv) Inversion du temps

Nous définissons des processus de Wishart généralisés avec drift $\in \mathcal{S}_m^+$ et montrons qu'ils s'obtiennent aussi par inversion du temps à partir de processus de Wishart classiques.

(v) Entrelacement

$$\mathbf{Q}_t^{\delta+\delta'}\Lambda_{\delta,\delta'}=\Lambda_{\delta,\delta'}\mathbf{Q}_t^{\delta},\tag{3.9}$$

où $\Lambda_{\delta,\delta'}$ est l'opérateur de multiplication associé à une variable Bêta $\beta_{\delta/2,\delta'/2}$ sur $\mathcal{S}_m$, de paramètre $(\delta/2,\delta'/2)$ (définie à la Def.3.3.2 dans [Mui82]) i.e.,

$$\Lambda_{\delta,\delta'}f(x)=E[f(\sqrt{x}\beta_{\delta/2,\delta'/2}\sqrt{x})].$$

3.3 Les processus de Jacobi matriciels

Ici, nous discutons les généralisations aux matrices symétriques d'une autre classe de processus réels, les processus de Jacobi. En dimension 1, ces derniers sont des diffusions sur l'intervalle $(0,1)$ associées au générateur $2x(1-x)\partial^2+(p-(p+q)x)\,\partial$. Leur mesure réversible est la loi Bêta

$$\frac{\Gamma((p+q)/2)}{\Gamma(p/2)\Gamma(q/2)}\,x^{p/2-1}(1-x)^{q/2-1}\,\mathbf{1}_{(0,1)}(x)\,dx.$$

Ces processus apparaissent, pour des dimensions p,q entières, comme des projections du mouvement brownien sur la sphère de $\mathbb{R}^n$, $n=p+q$ (voir [Bak96]) et interviennent dans un skew-product attaché au quotient de deux carrés de Bessel ([WY]). On trouvera dans [EK86] et [KT81] (resp.[DS]) des études de ces processus motivés par des modèles de génétique (resp. de mathématique financière). Disons enfin que les processus de Jacobi jouent un rôle fondamental dans la classification des diffusions réelles associées

à une famille de polynômes orthogonaux ([Maz97]).
Notre objectif est de proposer une définition de processus de Jacobi matriciels à travers l'étude d'une équation différentielle stochastique matricielle et de donner les premières propriétés de tels processus (interprétation géométrique pour des dimensions entières, relations d'absolue continuité, mesures réversibles et trajectoires des valeurs propres). Le point de départ de notre travail est l'équivalence entre deux constructions des lois Bêta matricielles, l'une, classique, à partir de quotients de lois de Wishart et l'autre consistant à prélever un coin rectangulaire d'une matrice distribuée selon la mesure de Haar sur $\mathcal{SO}_n(\mathbb{R})$ (cf [Col03]).

3.3.1 Le cas de dimensions entières

Nous partons du mouvement brownien Θ sur $\mathcal{SO}_n(\mathbb{R})$ et considérons la matrice X formée des m premières lignes et p premières colonnes de Θ.

Théorème 3.3.1 (cf Th 9.2.3). *Le processus $J := XX^*$ est une diffusion sur l'espace $\bar{\mathcal{S}}_m = \{x \in \mathcal{S}_m :\ 0_m \leq x \leq 1_m\}$. Si $p \geq m+1$ et $q \geq m+1$, alors J est solution de l'équation différentielle stochastique suivante :*

$$dJ = \sqrt{J} dB \sqrt{1_m - J} + \sqrt{1_m - J} dB^* \sqrt{J} + (pI - (p+q)J)\, dt, \tag{3.10}$$

où B est une matrice brownienne $m \times m$ et $q = n - p$. J sera appelé processus de Jacobi de dimensions (p, q).

3.3.2 Etude de l'EDS pour des dimensions non-entières

Les coefficients de l'équation multi-dimensionnelle (3.10) n'étant pas lipschitziens mais seulement hölderiens d'indice 1/2, l'étude des solutions de celle-ci demande quelques précautions. Nos arguments sont proches de ceux de [Bru91] mais l'usage de temps locaux permet de simplifier le traitement de l'existence en loi ((ii) du théorème suivant).

Théorème 3.3.2 (cf Th 9.3.1). (i) *Si $p \wedge q \geq m+1$ et $J_0 = x$ a ses valeurs propres dans $]0,1[$, (3.10) a une unique solution forte dont les valeurs propres restent éternellement dans $]0,1[$.*

(ii) *Si $p \wedge q > m-1$ et $J_0 = x$ a ses valeurs propres distinctes et dans $[0,1]$, (3.10) a une unique solution en loi.*

(iii) *Si elles sont distinctes au départ, les valeurs propres de J le restent éternellement et peuvent être numérotées $\lambda_1 > \cdots > \lambda_m$. Elles vérifient l'EDS suivante :*

$$d\lambda_i = 2\sqrt{\lambda_i(1-\lambda_i)}\, db_i + \Big\{ (p - (p+q)\lambda_i) + \sum_{j\,(\neq i)} \frac{\lambda_i(1-\lambda_j) + \lambda_j(1-\lambda_i)}{\lambda_i - \lambda_j} \Big\} dt, \tag{3.11}$$

pour $1 \leq i \leq m$ et des mouvements browniens réels indépendants $b_1, \ldots, b_m$.

3.3.3 Propriétés du processus de Jacobi

On peut expliciter la mesure réversible du processus de Jacobi matriciel.

Proposition 3.3.1 (cf Prop 9.4.2). *Si $p > m-1$ et $q > m-1$, la mesure réversible du processus de Jacobi de dimensions (p,q) est la loi Bêta matricielle $\mu_{p,q}$ sur $\mathcal{S}_m$ définie par :*

$$\mu_{p,q}(dx) = \frac{\Gamma_m\left((p+q)/2\right)}{\Gamma_m(p/2)\Gamma_m(q/2)} \det(x)^{(p-m-1)/2} \det(1_m - x)^{(q-m-1)/2} \mathbf{1}_{0_m \leq x \leq 1_m}\, dx\,.$$

Si l'on note $\mathbb{P}_x^{p,q}$ la loi d'un processus de Jacobi de dimensions (p,q), les relations d'absolue continuité s'écrivent :

Théorème 3.3.3 (cf Th 9.4.3). *Si $\mathcal{F}_t = \sigma(J_s,\ s \leq t)$ et $T = \inf\{t \mid \det J_t(1_m - J_t) = 0\}$, nous avons :*

$$\mathbb{P}_x^{p',q'}|_{\mathcal{F}_t \cap \{T>t\}} = \left(\frac{\det J_t}{\det j}\right)^{\alpha} \left(\frac{\det(1_m - J_t)}{\det(1_m - j)}\right)^{\beta}$$

$$\exp\left(-\int_0^t ds\left(c + u\,\mathrm{tr}\big(J_s^{-1}\big) + v\,\mathrm{tr}\big((1_m - J_s)^{-1}\big)\right)\right) \mathbb{P}_x^{p,q}|_{\mathcal{F}_t \cap \{T>t\}},$$

où $\alpha = (p'-p)/4$, $\beta = (q'-q)/4$, $u = \frac{p'-p}{4}\left(\frac{p'+p}{2} - m - 1\right)$, $v = \frac{q'-q}{4}\left(\frac{q'+q}{2} - m - 1\right)$, $c = m\left(\frac{p'+q'-p-q}{4}\right)\left(m + 1 - \frac{p'+q'+p+q}{2}\right)$.

Ceci permet d'écrire des relations d'absolue continuité entre dimensions (p,q) et (q,p) ainsi qu'entre dimensions $(m+1+2\mu, m+1+2\nu)$ et $(m+1-2\mu, m+1-2\nu)$. Si on note $\mathbb{P}^{(\mu,\nu)} = \mathbb{P}^{m+1+2\mu, m+1+2\nu}$, on peut en déduire la loi de T :

Corollary 3.3.2 (cf Cor 9.4.6). *Pour $0 \leq \mu,\ \nu < 1$,*

$$\mathbb{P}_x^{(-\mu,-\nu)}\,(T > t) = \mathbb{E}_x^{(\mu,\nu)}\left[\left(\frac{\det J_t}{\det x}\right)^{-\mu} \left(\frac{\det(1_m - J_t)}{\det(1_m - x)}\right)^{-\nu}\right].$$

Enfin, il est possible de définir, de manière exactement analogue, un processus de Jacobi hermitien (en remplaçant les matrices symétriques par des matrices hermitiennes) de taille m et l'on peut caractériser les trajectoires de ses valeurs propres.

Proposition 3.3.3 (cf Prop 9.4.7). *Les valeurs propres d'un processus de Jacobi hermitien de dimensions (p,q) ont les trajectoires de m processus de Jacobi uni-dimensionnels de dimensions $(2(p-m+1), 2(q-m+1))$ conditionnés (au sens de Doob) à ne jamais entrer en collision.*

Chapitre 4

Mouvement brownien et groupes de réflexions

4.1 Contexte

Les études qui suivent ont été inspirées par l'intérêt récemment porté, en relation avec les matrices aléatoires, au mouvement brownien dans les chambres de Weyl ([BJ02], [O'C03b], [BBO04]). Une observation qui remonte à Dyson ([Dys62]) est que le processus des valeurs propres $\lambda_1(t) > \cdots > \lambda_n(t)$ d'un mouvement brownien hermitien démarrant en 0 a même loi qu'un mouvement brownien dans $\mathbb{R}^n$ démarrant en 0 et conditionné à ne jamais quitter $C = \{x \in \mathbb{R}^n : \; x_1 > \cdots > x_n\}$. Ce dernier processus admet une représentation en loi comme fonctionnelle du mouvement brownien standard dans $\mathbb{R}^n$ ([OY02],[BJ02], [O'C03b], [BBO04]). Cette représentation est un analogue multi-dimensionnel du théorème classique de Pitman ([Pit75]) affirmant que si B est un mouvement brownien réel issu de 0 et si M est son maximum passé, $2M - B$ a la même loi que B conditionné à rester éternellement positif. Il se trouve que la région C est un domaine fondamental pour le pavage de l'espace $\mathbb{R}^n$ par le groupe symétrique $\mathfrak{S}_n$. Ce dernier est un exemple de groupe fini engendré par des réflexions euclidiennes (ie, de manière équivalente, de groupe de Coxeter fini). De tels groupes font depuis longtemps l'objet de nombreuses recherches en algèbre, géométrie et combinatoire. En particulier, ils ont été entièrement classifiés et leur liste est codée par des diagrammes de Dynkin (cf [Hum90] pour une introduction). [BBO04] prouve que des théorèmes de Pitman peuvent être donnés pour les chambres C associées à ces groupes. Cette contribution majeure met en évidence les liens qu'entretient la « fonctionnelle de Pitman » avec les chemins de Littelmann et la théorie des représentations. Pour différentes et bien plus modestes qu'elles soient, les questions auxquelles nous nous intéressons dans les travaux qui suivent ont en commun avec [BBO04] de faire appel, dans un contexte « brownien », au cadre algébraïco-géométrique des groupes de réflexions et de leurs systèmes de racines.

4.2 Problèmes de sortie associés à des groupes de réflexions finis

Parmi les temps d'atteinte associés au mouvement brownien, le plus simple et le plus fondamental est, sans doute, le temps d'atteinte de 0 d'un mouvement brownien réel issu de $x > 0$, $T = \inf\{t \geq 0,\ B_t = 0\}$. Sa loi peut être obtenue par un principe de réflexion. Celui-ci exprime le semi-groupe $p_t^*(x, y)$ du mouvement brownien tué en 0 en fonction du semi-groupe $p_t(x, y)$ du mouvement brownien standard :

$$p_t^*(x, y) = p_t(x, y) - p_t(x, -y). \tag{4.1}$$

En intégrant sur $y > 0$, on obtient :

$$\mathbb{P}_x(T > t) = \mathbb{P}_x(B_t > 0) - \mathbb{P}_x(B_t < 0) = \mathbb{P}_0(|B_t| \leq x). \tag{4.2}$$

L'argument essentiel est l'invariance de la loi brownienne par la réflexion $x \to -x$. De manière générale, la loi d'un mouvement brownien dans $\mathbb{R}^n$ est invariante par tout sous-groupe de $\mathcal{O}_n(\mathbb{R})$, en particulier par tout groupe fini W engendré par des réflexions. On peut alors chercher les analogues de (4.1) et (4.2) en remplaçant $\{x > 0\}$ par le domaine fondamental C associé au pavage de $\mathbb{R}^n$ par W et T par le temps de sortie de C. La formule (4.1) se généralise en

$$p_t^*(x, y) = \sum_{w \in W} \varepsilon(w) p_t(x, w(y)), \tag{4.3}$$

où $\varepsilon(w) = \det(w)$ ([GZ92],[Bia92]). Dans le cas où $W = \mathfrak{S}_n$, on a $C = \{x_1 > \cdots > x_n\}$ et (4.3) s'écrit :

$$p_t^*(x, y) = \det\left(p_t(x_i, y_j)\right)_{1 \leq i,j \leq n}, \tag{4.4}$$

formule que l'on doit à [KM59]. L'intégration de (4.3) donne

$$\mathbb{P}_x(T > t) = \sum_{w \in W} \varepsilon(w) \mathbb{P}_x(B_t \in w(C)). \tag{4.5}$$

La formule précédente fait figurer une somme alternée de $|W|$ termes, qui sont chacun des intégrales multi-dimensionnelles délicates à calculer. Notre propos est d'obtenir, par une approche directe, des formules alternatives comportant moins de termes et ne faisant intervenir que des intégrales de dimension un ou deux.

4.2.1 Le résultat principal

Le cadre est celui d'un système (fini) de racines Φ dans un espace euclidien V, avec système positif Π et système simple Δ. W est le groupe (fini) associé à Φ et la chambre est :

$$C = \{x \in V : \ \forall \alpha \in \Pi, \ (\alpha, x) > 0\} = \{x \in V : \ \forall \alpha \in \Delta, \ (\alpha, x) > 0\}.$$

Nous définissons, dans la section 10.2.1, la notion de « consistance » pour un ensemble $I \subset \Pi$ et notre résultat principal est le suivant :

Proposition 4.2.1 (cf Prop 10.2.3). *Si I est consistant, on introduit*

$$\mathcal{I} = \{A = wI : \ w \in W, \ wI \subset \Pi\}$$

et l'on peut définir sans ambiguïté $\varepsilon_A = \varepsilon(w)$ pour $A = wI \in \mathcal{I}$. Alors

$$\mathbb{P}_x(T > t) = \sum_{A \in \mathcal{I}} \varepsilon_A \, \mathbb{P}_x(T_A > t), \tag{4.6}$$

où $T_A = \inf\{t : \ \exists \alpha \in A, \ (B_t, \alpha) = 0\}$ est le temps de sortie de l'orthant associé à A.

Cette formule est particulièrement agréable lorsque I est orthogonal (ie que ses éléments sont deux à deux orthogonaux), auquel cas elle s'écrit

$$\mathbb{P}_x(T > t) = \sum_{A \in \mathcal{I}} \varepsilon_A \prod_{\alpha \in A} \gamma\left(\widehat{\alpha}(x)/\sqrt{t}\right), \tag{4.7}$$

où $\widehat{\alpha}(x) = (\alpha, x)/|\alpha|$ et $\gamma(a) = \sqrt{\frac{2}{\pi}} \int_0^a e^{-y^2/2} dy$. Dans ce cas, on peut même établir une formule « duale » pour $\mathbb{P}_x(T \leq t)$. Cette dernière fait intervenir une « action » des racines simples $\alpha \in \Delta$ sur les ensembles orthogonaux $B \subset \Pi$ définie par :

$$\alpha.B = \begin{cases} B & \text{si } \alpha \in B; \\ \{\alpha\} \cup B & \text{si } \alpha \in B^\perp; \\ s_\alpha B & \text{sinon.} \end{cases}$$

On peut alors définir la « longueur » $l(B)$ de B par :

$$l(B) = \inf\{l \in \mathbb{N} : \ \exists \, \alpha_1, \alpha_2, \ldots, \alpha_l \in \Delta, \ B = \alpha_l \ldots \alpha_2.\alpha_1.\emptyset\} \tag{4.8}$$

et l'on a :

$$\mathbb{P}_x(T \leq t) = \sum_B (-1)^{l(B)-1} \, \mathbb{P}_x[\forall \beta \in B, \ T_\beta \leq t], \tag{4.9}$$

où la somme porte sur les ensembles orthogonaux $B \subset \Pi$ et $\neq \emptyset$.

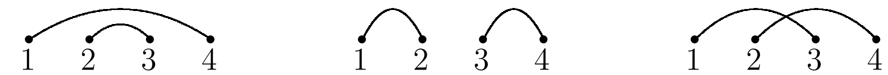
4.2.2 Consistance et application au mouvement brownien

Pour la majorité des groupes finis de réflexions, nous exhibons un ensemble I dont nous vérifions la consistance et pour lequel nous identifions $\mathcal{I}$. En conséquence, nous pouvons appliquer le résultat principal 4.2.1 dans les différents cas.

Citons le cas de $W = \mathfrak{S}_n$ pour lequel $C = \{x_1 > \cdots > x_n\}$ et

$$\mathbb{P}_x(T > t) = \sum_{\pi \in P_2(n)} (-1)^{c(\pi)} \prod_{\{i<j\} \in \pi} p_{ij}, \tag{4.10}$$

où $P_2(n)$ est l'ensemble des partitions de $\{1, \ldots, n\}$ en paires (avec éventuellement un singleton si n est impair), $c(\pi)$ est le nombre de croisements dans la représentation graphique de $\pi \in P_2(n)$ (cf figure 4.1) et $p_{ij} = \mathbb{P}_x(T_{e_i - e_j} > t) = \gamma\left(\frac{x_i - x_j}{\sqrt{2t}}\right)$.



$\pi = \{\{1,4\},\{2,3\}\}$ $\quad$ $\pi = \{\{1,2\},\{3,4\}\}$ $\quad$ $\pi = \{\{1,3\},\{2,4\}\}$

$c(\pi) = 0$ $\quad$ $c(\pi) = 0$ $\quad$ $c(\pi) = 1$

FIG. 4.1 – Partitions en paires et leurs signes pour $n = 4$

On peut traduire (4.10) en termes de pfaffien (cf 10.8.2 pour une définition) :

$$\mathbb{P}_x(T > t) = \begin{cases} \operatorname{Pf}\,(p_{ij})_{i,j \in [n]} & \text{si } n \text{ est pair },\\ \sum_{l=1}^{n} (-1)^{l+1} \operatorname{Pf}\,(p_{ij})_{i,j \in [n]\setminus\{l\}} & \text{si } n \text{ est impair}, \end{cases} \tag{4.11}$$

avec la convention que $p_{ji} = -p_{ij}$ pour $i \leq j$.

Signalons aussi le cas du groupe dihédral $I_2(m)$ des symétries d'un polygone régulier à m côtés. T est le temps de sortie d'un cône d'angle π/m :

$$C = \{re^{i\theta} :\ r \geq 0,\ 0 < \theta < \pi/m\} \subset \mathbb{C} \simeq \mathbb{R}^2.$$

Si l'on note $\alpha_l = e^{i\pi(l/m - 1/2)}$ et $\alpha_l' = e^{i\pi/2}\alpha_l$, le résultat s'écrit

$$\mathbb{P}_x(T > t) = \begin{cases} \sum_{i=1}^{m} (-1)^{i-1} \mathbb{P}_x(T_{\alpha_i} > t) & \text{si } m \text{ est impair },\\ \sum_{i=1}^{m} (-1)^{i-1} \mathbb{P}_x(T_{\{\alpha_i, \alpha_i'\}} > t) & \text{si } m \equiv 2 \ (\bmod\ 4). \end{cases} \tag{4.12}$$

Si m est multiple de 4, la proposition 4.2.1 ne s'applique pas mais nous pouvons tout de même écrire des formules d'allure un peu différente (cf section 10.4.5).

4.2.3 Calculs d'asymptotiques et de moyennes

Le résultat (4.6) permet d'analyser l'asymptotique de $\mathbb{P}_x(T > t)$ selon que t est grand ou petit. Par exemple, nous démontrons la

Proposition 4.2.2 (cf Prop 10.4.1). *Si I est consistant, on a le développement suivant :*

$$\mathbb{P}_x(T > t) = h(x) \sum_{q \geq 0} E_q(x)\, t^{-(q+n/2)}, \tag{4.13}$$

où $n = |\Pi|$, $h(x) = \prod_{\alpha \in \Pi}(x, \alpha)$, $E_q(x)$ est un polynôme W-invariant de degré $2q$. En particulier, il existe une constante κ telle que :

$$\mathbb{P}_x(T > t) \sim \frac{\kappa\, h(x)}{t^{n/2}} \text{ as } t \to \infty. \tag{4.14}$$

Nous étudions aussi le cas où t est petit et où x est à égale distance de tous les murs de la chambre (cf section 10.4.6).

Dans certains cas, il est aussi possible de calculer l'espérance de T. Mentionnons l'exemple du groupe symétrique pour lequel $C = \{x_1 > \cdots > x_n\}$. On a $\mathbb{E}_x(T) = (x_1 - x_2)(x_2 - x_3)$ pour $n = 3$ et, pour $p = \lfloor n/2 \rfloor \geq 2$,

$$\mathbb{E}_x(T) = \sum_{\pi \in P_2(n)} (-1)^{c(\pi)} F_p(x_\pi), \tag{4.15}$$

où $x_\pi = (x_i - x_j)_{\{i<j\} \in \pi} \in \mathbb{R}_+^p$ et

$$F_p(y_1, \ldots, y_p) = \frac{2^{p+1}\Gamma(p/2)}{\pi^{p/2}(p-2)} \int_0^{y_1} \cdots \int_0^{y_p} \frac{dz_1 \ldots dz_p}{(z_1^2 + \cdots + z_p^2)^{p/2-1}}. \tag{4.16}$$

.

4.2.4 Formules de de Bruijn et combinatoire

Ensuite, nous remarquons, pour le cas $W = \mathfrak{S}_n$ (n pair), que l'intégration de (4.4) sur $C = \{x_1 > \cdots > x_n\}$ donne une expression alternative pour $\mathbb{P}_x(T > t)$, ce qui implique que

$$\int_C \det\left(f_i(y_j)\right)_{1 \leq i,j \leq n} dy = \mathrm{Pf}\left(I(f_i, f_j)\right)_{1 \leq i,j \leq n},$$

où $f_i = p_t(x_i, .)$ et $I(f, g) = \int_{y>z} (f(y)g(z) - f(z)g(y))\, dydz$. Cette formule s'étend par linéarité et densité à des fonctions suffisamment régulières et intégrables. Elle a été démontrée pour la première fois par de Bruijn ([dB55]) par des méthodes tout à fait différentes. Notre approche nous permet d'en donner une version dans le cadre général de la proposition 4.2.1 que l'on peut traduire explicitement dans chaque cas particulier (cf section 10.5).

Enfin, en appliquant la proposition 4.2.1 à des marches aléatoires, nous retrouvons des résultats de Gordon et Gessel sur le dénombrement des tableaux de Young de hauteur bornée (cf 10.6).

4.3 Problèmes de sortie associés à des groupes de réflexions affines

Les domaines dont nous avons examiné le temps de sortie dans la section 4.2 sont des régions non-bornées, bordées par des hyperplans qui passent tous par l'origine. Le groupe de transformations associé à de tels domaines est fini et constitué d'isométries linéaires, qui laissent invariante la loi du mouvement brownien. Mais celle-ci est aussi invariante par translation, ce qui permet d'adapter les idées précédentes à certains groupes infinis engendrés par des réflexions affines. Le domaine fondamental, appelé « alcôve », attaché à de tels groupes apparaîtra comme une région bornée de l'espace euclidien, bordée d'hyperplans vectoriels ou affines. L'exemple le plus simple est celui de l'intervalle $(0,1) \subset \mathbb{R}$, pour lequel le groupe associé est $W_a = \{x \to \pm x + 2l,\ l \in \mathbb{Z}\}$, engendré par les réflexions $x \to -x$ et $x \to 2-x$ par rapport à 0 et 1.

Le propos du chapitre 11 est de présenter les formules pour la loi du temps de sortie de telles alcôves ainsi que de décrire un langage très commode, celui des systèmes de racines affines, qui permet de transposer, sans effort, les preuves données au chapitre 10.

4.3.1 Le cadre géométrique

Dans un espace euclidien V, on se donne un système (irréductible) de racines Φ, avec système simple associé Δ, système positif associé Φ^+ et groupe associé W. On suppose que Φ est crystallographique, ce qui signifie en substance que W stabilise un réseau. On définit le groupe W_a associé à Φ comme le groupe engendré par les réflexions affines par rapport aux hyperplans $H_{\alpha,n} = \{x \in V :\ (x,\alpha) = n\}$, $\alpha \in \Phi$, $n \in \mathbb{Z}$. De manière équivalente, les éléments de W_a s'écrivent, de façon unique, sous la forme $\tau(l)w$, où $w \in W$, L est le $\mathbb{Z}$-module engendré par $\Phi^\vee = \{\alpha^\vee = 2\alpha/(\alpha,\alpha),\ \alpha \in \Phi\}$ et $\tau(l)$ est la translation de $l \in L$. L'alcôve fondamentale est

$$\mathcal{A} = \{x \in V :\ \forall \alpha \in \Phi^+,\, 0 < (x,\alpha) < 1\}.$$

Il est très pratique d'introduire aussi le langage suivant. On définit le système de racines affines comme $\Phi_a := \Phi \times \mathbb{Z}$, le système de racines affines positives $\Phi_a^+ := \{(\alpha,n) :\ (n = 0 \text{ et } \alpha \in \Phi^+) \text{ ou } n \leq -1\}$ et le système de racines affines simples $\Delta_a := \{(\alpha,0),\, \alpha \in \Delta;\ (-\tilde{\alpha},-1)\}$. Si $\lambda = (\alpha,n) \in \Phi_a$, on pose $\lambda(x) := (\alpha,x) - n$ et $H_\lambda := \{x \in V :\ \lambda(x) = 0\} = H_{\alpha,n}$. On peut enfin définir l'action de $w_a = \tau(l)w \in W_a$

sur une racine affine $\lambda = (\alpha, n) \in \Phi_a$ par $w_a(\lambda) = (w\alpha, n + (w\alpha, l)) \in \Phi_a$. De cette façon, on a $w_a H_\lambda = H_{w_a(\lambda)}$ et l'alcôve fondamentale peut être décrite ainsi

$$\mathcal{A} = \{x \in V : \ \forall \lambda \in \Phi_a^+, \ \lambda(x) > 0\} = \{x \in V : \ \forall \lambda \in \Delta_a, \ \lambda(x) > 0\}.$$

4.3.2 Le résultat principal

Nous définissons la notion de consistance pour un ensemble $I_a \subset \Phi_a^+$ (cf section 11.3.1) et prouvons la

Proposition 4.3.1 (cf Prop 11.3.2). *Si I_a est consistant, on peut définir sans ambiguïté $\varepsilon_A = \det(w_a)$ si $A = w_a I_a \in \mathcal{I}_a := \{A = w_a I_a : \ w_a \in W_a, \ w_a I_a \subset \Phi_a^+\}$. Alors, la loi du temps T de sortie de $\mathcal{A}$ pour le mouvement brownien B est donnée par*

$$\mathbb{P}_x(T > t) = \sum_{A \in \mathcal{I}_a} \varepsilon_A \, \mathbb{P}_x(T_A > t), \tag{4.17}$$

où $T_A = \inf\{t \geq 0 : \ \exists \lambda \in A, \ \lambda(B_t) = 0\}$.

Nous voulons mentionner l'exemple de $\tilde{A}_{n-1}$ qui correspond à la chambre

$$\mathcal{A} = \{x \in V : \ 1 + x_n > x_1 > \cdots > x_n\}$$

dans $\mathbb{R}^n$ (ou dans $\{x_1 + \cdots + x_n = 0\}$). Si $n = 2p$ est pair, $I_a = \{(e_{2i-1} - e_{2i}, 0), \ (-e_{2i-1} + e_{2i}, -1) \, ; \, 1 \leq i \leq p\}$ est consistant et $\mathcal{I}_a$ s'identifie avec l'ensemble $P_2(n)$ des partitions de $[n]$ en paires. Le signe correspond à la parité du nombre de croisements de la partition (cf figure 11.1). Ainsi, (4.17) s'écrit

$$\mathbb{P}_x(T > t) = \sum_{\pi \in P_2(n)} (-1)^{c(\pi)} \prod_{\{i<j\} \in \pi} \tilde{p}_{ij} = \mathrm{Pf} \, (\tilde{p}_{ij})_{i,j \in [k]} \tag{4.18}$$

où $\tilde{p}_{ij} = \mathbb{P}_x(T_{(e_i - e_j, 0), (-e_i + e_j, -1)} > t) = \mathbb{P}_x(\forall s \leq t, \, 0 < X_s^i - X_s^j < 1) = \phi(x_i - x_j, 2t)$ et $\phi(x, t) = \mathbb{P}_x(\forall s \leq t, 0 < B_s < 1)$ pour un mouvement brownien standard B_s.
Lorsque n est impair, notre approche échoue : l'ensemble I_a n'est plus consistant. Cette différence peut être vue directement au niveau des partitions : l'échange de 1 et n dans les blocs de $\pi \in P_2(n)$, qui correspond à l'action de la réflexion par rapport au mur affine $\{x_1 - x_n = 1\}$ de l'alcôve fondamentale, n'altère le signe de π que lorsque n est pair. Lorsque n est impair, la conservation du signe par cette opération signifie que les éléments de $\{w_a : \ w_a I_a = I_a\}$ ne sont pas tous de déterminant 1, ce qui est contraire à la définition de consistance. Le cas du triangle équilateral, correspondant malheureusement à $n = 3$, n'est pas justiciable de la formule (4.17) ! Il y a là un phénomène que nous comprenons mal, d'autant que l'espérance de T est explicitement connue dans ce cas (cf [AFR]).

Citons aussi le cas de $\widetilde{G}_2$ dont l'alcôve fondamentale est un triangle $\mathcal{T}$ d'angles $(\pi/2, \pi/3, \pi/6)$. Notre résultat s'applique à la loi du temps de sortie $T_{\mathcal{T}}$ de $\mathcal{T}$:

$$\mathbb{P}_x(T_{\mathcal{T}} > T) = \mathbb{P}_x(T_{\mathcal{R}_1} > T) - \mathbb{P}_x(T_{\mathcal{R}_2} > T) + \mathbb{P}_x(T_{\mathcal{R}_3} > T),$$

où les $T_{\mathcal{R}_i}$ sont les temps de sortie de trois rectangles issus du pavage du plan par le groupe affine associé (cf Eq (11.12) et Fig 11.2).

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Deuxième partie

Random matrices and combinatorics

Chapitre 5

A note on representations of eigenvalues of classical Gaussian matrices

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Abstract : We use a matrix central-limit theorem which makes the Gaussian Unitary Ensemble appear as a limit of the Laguerre Unitary Ensemble together with an observation due to Johansson in order to derive new representations for the eigenvalues of GUE. For instance, it is possible to recover the celebrated equality in distribution between the maximal eigenvalue of GUE and a last-passage time in some directed Brownian percolation. Similar identities for the other eigenvalues of GUE also appear.

5.1 Introduction

The most famous ensembles of Hermitian random matrices are undoubtedly the Gaussian Unitary Ensemble (GUE) and the Laguerre Unitary Ensemble (LUE). Let $(X_{i,j})_{1\leq i<j\leq N}$ (respectively $(X_{i,i})_{1\leq i\leq N}$) be complex (respectively real) standard independent Gaussian variables ($\mathbb{E}(X_{i,j})=0$, $\mathbb{E}(|X_{i,j}|^2)=1$) and let $X_{i,j}=\overline{X_{j,i}}$ for $i>j$. The GUE(N) is defined to be the random matrix $X^N=(X_{i,j})_{1\leq i,j\leq N}$. It induces the following probability measure on the space $\mathcal{H}_N$ of $N\times N$ Hermitian matrices :

$$P_N(dH)=Z_N^{-1}\exp\Big(-\frac{1}{2}\operatorname{Tr}(H^2)\Big)dH \tag{5.1}$$

where dH is Lebesgue measure on $\mathcal{H}_N$. In the same way, if $M \geq N$ and $A^{N,M}$ is a $N \times M$ matrix whose entries are complex standard independent Gaussian variables, then LUE(N, M) is defined to be the random $N \times N$ matrix $Y^{N,M} = A^{N,M}(A^{N,M})^*$ where $*$ stands for the conjugate of the transposed matrix. Alternatively, LUE(N, M) corresponds to the following measure on $\mathcal{H}_N$:

$$P_{N,M}(dH) = Z_{N,M}^{-1}(\det H)^{M-N} \exp(-\operatorname{Tr} H) 1\!\!1_{H \geq 0} dH \ . \tag{5.2}$$

A central-limit theorem which already appeared in the Introduction of [Jon82] asserts that GUE(N) is the limit in distribution of LUE(N, M) as $M \to \infty$ in the following asymptotic regime :

$$\frac{Y^{N,M} - M \operatorname{Id}_N}{\sqrt{M}} \xrightarrow[M \to \infty]{d} X^N \ . \tag{5.3}$$

For connections with this result, see Theorem 2.5 of [Det01] and a note in Section 5 of [OY01]. We also state a process-level version of the previous convergence when the Gaussian entries of the matrices are replaced by Brownian motions. The convergence takes place for the trajectories of the eigenvalues.

Next, we make use of this matrix central-limit theorem together with an observation due to Johansson [Joh00] and an invariance principle for a last-passsage time due to Glynn and Whitt [GW91] in order to recover the following celebrated equality in distribution between the maximal eigenvalue $\lambda_{\max}^N$ of GUE(N) and some functional of standard N-dimensional Brownian motion $(B_i)_{1 \leq i \leq N}$ as

$$\lambda_{\max}^N \stackrel{d}{=} \sup_{0=t_0 \leq \cdots \leq t_N = 1} \sum_{i=1}^{N} (B_i(t_i) - B_i(t_{i-1})) \ . \tag{5.4}$$

The right-hand side of (5.4) can be thought of as a last-passage time in an oriented Brownian percolation. Its discrete analogue for an oriented percolation on the sites of $\mathbb{N}^2$ is the object of Johansson's remark. The identity (5.4) first appeared in [Bar01] and [GTW01]. Very recently, O'Connell and Yor shed a remarkable light on this result in [OY02]. Their work involves a representation similar to (5.4) for all the eigenvalues of GUE(N). We notice here that analogous formula can be written for all the eigenvalues of LUE(N, M). On the one hand, seeing the particular expression of these formula, a central-limit theorem can be established for them and the limit variable Ω is identified in terms of Brownian functionals. On the other hand, the previous formulas for eigenvalues of LUE(N, M) converge, in the limit given by (5.3), to the representation found in [OY02] for GUE(N) in terms of some path-transformation Γ of brownian motion. It is not immediately obvious to us that functionals Γ and Ω coincide. In particular, is this identity true pathwise or only in distribution ?

The matrix central-limit theorem is presented in Section 5.2 and its proof is postponed to the last section. In section 5.3, we described the consequences to eigenvalues representations and the connection with the O'Connell-Yor approach.

5.2 The central-limit theorem

Here is the basic form of the matrix-central limit theorem :

Theorem 5.2.1. *Let $Y^{N,M}$ and X^N be taken respectively from $LUE(N, M)$ and $GUE(N)$. Then*

$$\frac{Y^{N,M} - M\,\mathrm{Id}_N}{\sqrt{M}} \xrightarrow[M\to\infty]{d} X^N . \tag{5.5}$$

We turn to the process version of the previous result. Let $A^{N,M} = (A_{i,j})$ be a $N \times M$ matrix whose entries are independent standard complex Brownian motions. The Laguerre process is defined to be $Y^{N,M} = A^{N,M}(A^{N,M})^*$. It is built in exactly the same way as LUE(N, M) but with Brownian motions instead of Gaussian variables. Similarly, we can define the Hermitian Brownian motion X^N as the process extension of GUE(N).

Theorem 5.2.2. *If $Y^{N,M}$ is the Laguerre process and $(X^N(t))_{t\geq 0}$ is Hermitian Brownian motion, then :*

$$\Big(\frac{Y^{N,M}(t) - Mt\,\mathrm{Id}_N}{\sqrt{M}}\Big)_{t\geq 0} \xrightarrow[M\to\infty]{d} (X^N(t^2))_{t\geq 0} \tag{5.6}$$

in the sense of weak convergence in $\mathcal{C}(\mathbb{R}_+, \mathcal{H}_N)$.

As announced, the proofs of the previous theorems are postponed up to section 5.4. Theorem 5.2.1 is an easy consequence of the usual multi-dimensional central-limit theorem. For Theorem 5.2.2, our central-limit convergence is shown to follow from a law of large numbers at the level of quadratic variations.

Let us mention the straightforward consequence of Theorems 5.2.1 and 5.2.2 on the convergence of eigenvalues. If $H \in \mathcal{H}_N$, let us denote by $l_1(H) \leq \cdots \leq l_N(H)$ its (real) eigenvalues and $l(H) = (l_1(H), \ldots, l_N(H))$. Using the min-max formulas, it is not difficult to see that each l_i is 1-Lipschitz for the Euclidean norm on $\mathcal{H}_N$. Thus, l is continuous on $\mathcal{H}_N$. Therefore, if we set $\mu^{N,M} = l(Y^{N,M})$ and $\lambda^N = l(X^N)$

$$\Big(\frac{\mu_i^{N,M} - M}{\sqrt{M}}\Big)_{1\leq i\leq N} \xrightarrow[M\to\infty]{d} (\lambda_i^N)_{1\leq i\leq N} \tag{5.7}$$

With the obvious notations, the process version also takes place :

$$\Big(\Big(\frac{\mu_i^{N,M}(t) - Mt}{\sqrt{M}}\Big)_{1\leq i\leq N}\Big)_{t\geq 0} \xrightarrow[M\to\infty]{d} \big((\lambda_i^N(t^2))_{1\leq i\leq N}\big)_{t\geq 0} \tag{5.8}$$

Analogous results hold in the real case of GOE and LOE and they can be proved with the same arguments. To our knowledge, the process version had not been considered in the existing literature.

5.3 Consequences on representations for eigenvalues

5.3.1 The largest eigenvalue

Let us first indicate how to recover from (5.7) the identity

$$\lambda_{\max}^N \stackrel{d}{=} \sup_{0=t_0\leq\ldots\leq t_N=1} \sum_{i=1}^N (B_i(t_i) - B_i(t_{i-1})) \tag{5.9}$$

where $\lambda_{\max}^N = \lambda_N^N$ is the maximal eigenvalue of GUE(N) and $(B_i\,,\,1 \leq i \leq N)$ is a standard N-dimensional Brownian motion. If $(w_{i,j}\,,\,(i,j) \in (\mathbb{N} \setminus \{0\})^2)$ are i.i.d. exponential variables with parameter one, define

$$H(M,N) = \max\Big\{ \sum_{(i,j)\in\pi} w_{i,j} \;;\; \pi \in \mathcal{P}(M,N)\Big\} \tag{5.10}$$

where $\mathcal{P}(M,N)$ is the set of all paths π taking only unit steps in the north-east direction in the rectangle $\{1,\ldots,M\} \times \{1,\ldots,N\}$. In [Joh00], it is noticed that

$$H(M,N) \stackrel{d}{=} \mu_{\max}^{N,M} \tag{5.11}$$

where $\mu_{\max}^{N,M} = \mu_N^{N,M}$ is the largest eigenvalue of LUE(N,M). Now an invariance principle due to Glynn and Whitt in [GW91] shows that

$$\frac{H(M,N) - M}{\sqrt{M}} \xrightarrow[M\to\infty]{d} \sup_{0=t_0\leq\cdots\leq t_N=1} \sum_{i=1}^N (B_i(t_i) - B_i(t_{i-1}))\,. \tag{5.12}$$

On the other hand, by (5.7)

$$\frac{\mu_{\max}^{N,M} - M}{\sqrt{M}} \xrightarrow[M\to\infty]{d} \lambda_{\max}^N\,. \tag{5.13}$$

Comparing (5.11), (5.12) and (5.13), we get (5.9) for free.

In the next section, we will give proofs of more general statements than (5.11) and (5.12).

5.3.2 The other eigenvalues

In fact, Johansson's observation involves all the eigenvalues of LUE(N,M) and not only the largest one. Although it does not appear exactly like that in [Joh00], it takes the following form. First, we need to extend definition (5.10) as follows : for each k, $1 \leq k \leq N$, set

$$H_k(M,N) = \max \left\{ \sum_{(i,j)\in\pi_1\cup\cdots\cup\pi_k} w_{i,j} \ ; \ \pi_1, \ldots, \pi_k \in \mathcal{P}(M,N)\,,\ \pi_1, \ldots, \pi_k \text{ all disjoint} \right\}. \tag{5.14}$$

Then, the link, analogous to (5.11), with the eigenvalues of LUE(N, M) is expressed by

$$H_k(M,N) \stackrel{d}{=} \mu_N^{N,M} + \mu_{N-1}^{N,M} + \cdots + \mu_{N-k+1}^{N,M} \,. \tag{5.15}$$

In fact, the previous equality in distribution is also valid for the vector $(H_k(M,N))_{1\leq k\leq N}$ and the corresponding sums of eigenvalues, which gives a representation for all the eigenvalues of LUE(N, M).

Proof of (5.15). The arguments and notations are taken from Section 2.1 in [Joh00]. Denote by $\mathcal{M}_{M,N}$ the set of $M \times N$ matrices $A = (a_{ij})$ with non-negative integer entries and by $\mathcal{M}^s_{M,N}$ the subset of $A \in \mathcal{M}_{M,N}$ such that $\Sigma(A) = \sum a_{ij} = s$. Let us recall that the Robinson-Schensted-Knuth (RSK) correspondence is a one-to-one mapping from $\mathcal{M}^s_{M,N}$ to the set of pairs (P, Q) of semi-standard Young tableaux of the same shape λ which is a partition of s, where P has elements in $\{1, \ldots, N\}$ and Q has elements in $\{1, \ldots, M\}$. Since $M \geq N$ and since the numbers are strictly increasing down the columns of P, the number of rows of λ is at most N. We will denote by RSK(A) the pair of Young tableaux associated to a matrix A by the RSK correspondence and by $\lambda(\text{RSK}(A))$ their commun shape. The crucial fact about this correspondence is the combinatorial property that, if $\lambda = \lambda(\text{RSK}(A))$, then for all k, $1 \leq k \leq N$,

$$\lambda_1+\lambda_2+\cdots+\lambda_k = \max \left\{ \sum_{(i,j)\in\pi_1\cup\cdots\cup\pi_k} a_{i,j} \ ; \ \pi_1, \ldots, \pi_k \in \mathcal{P}(M,N)\,,\ \pi_1, \ldots, \pi_k \text{ all disjoint} \right\}. \tag{5.16}$$

Now consider a random $M \times N$ matrix X whose entries (x_{ij}) are i.i.d. geometric variables with parameter q. Then for any λ^0 partition of an integer s, we have

$$\mathbb{P}(\lambda(\text{RSK}(X)) = \lambda^0\,) = \sum_{A\in\mathcal{M}^s_{M,N}\,,\ \lambda(\text{RSK}(A))=\lambda^0} \mathbb{P}(X = A)\,.$$

But for $A \in \mathcal{M}^s_{M,N}$, $\mathbb{P}(X = A) = (1-q)^{MN} q^s$ is independent of A, which implies

$$\mathbb{P}(\lambda(\text{RSK}(X) = \lambda^0)) = (1-q)^{MN} q^{\sum \lambda^0_i}\, L(\lambda^0, M, N)$$

where $L(\lambda^0, M, N) = \sharp\{A \in \mathcal{M}_{M,N}\,,\ \lambda(\text{RSK}(A)) = \lambda^0\}$. Since the RSK mapping is one-to-one

$$L(\lambda^0, M, N) = Y(\lambda^0, M)\, Y(\lambda^0, N)$$

where $Y(\lambda^0, K)$ is just the number of semi-standard Young tableaux of shape λ^0 with elements in $\{1, \dots, K\}$. This cardinal is well-known in combinatorics and finally

$$L(\lambda^0, M, N) = c_{MN}^{-1} \prod_{1 \le i < j \le N} (h_j^0 - h_i^0)^2 \prod_{1 \le i \le N} \frac{(h_i^0 + M - N)!}{h_i^0!}$$

where $c_{MN} = \prod_{0 \le i \le N-1} j\,!\,(M - N + j)\,!$ and $h_i^0 = \lambda_i^0 + N - i$ such that $h_1 > h_2 > \cdots > h_N \ge 0$. With the same correspondence as before between h and λ, we can write

$$\begin{aligned} \mathbb{P}(h(\mathrm{RSK}(X)) = h^0) &= c_{MN}^{-1} \frac{(1-q)^{MN}}{q^{N(N-1)/2}} \prod_{1 \le i < j \le N} (h_j^0 - h_i^0)^2 \prod_{1 \le i \le N} \frac{(h_i^0 + M - N)!}{h_i^0!} \\ &\stackrel{\mathrm{def}}{=} \rho_{(M,N,q)}(h^0)\,. \end{aligned}$$

Now set $q = 1 - L^{-1}$ and use the notation X_L instead of X to recall the dependence of the distribution on L. An easy asymptotic expansion shows that

$$L^N \rho_{(M,N,1-L^{-1})}(\lfloor Lx \rfloor) \underset{L \to \infty}{\longrightarrow} d_{MN}^{-1} \prod_{1 \le i < j \le N} (x_j - x_i)^2 \prod_{1 \le i \le N} x_i^{M-N} e^{-x_i} = \rho_{\mathrm{LUE}(N,M)}(x)$$

where $\rho_{\mathrm{LUE}(N,M)}$ is the joint density of the ordered eigenvalues of $\mathrm{LUE}(N, M)$. This can be used to prove that

$$\frac{1}{L} h(\mathrm{RSK}(X_L)) \xrightarrow[L \to \infty]{d} (\mu_N^{MN}, \mu_{N-1}^{MN}, \dots, \mu_1^{MN})\,. \tag{5.17}$$

On the other hand, if x_L is a geometric variable with parameter $1 - L^{-1}$, then x_L/L converges in ditribution, when $L \to \infty$, to an exponential variable of parameter one. Therefore, using the link between h and λ together with (5.16), we have

$$\frac{1}{L} \Big(\sum_{i=1}^{k} h_i(\mathrm{RSK}(X_L)) \Big)_{1 \le k \le N} \xrightarrow[L \to \infty]{d} (H_k(M, N))_{1 \le k \le N}\,.$$

Comparing with (5.17), we get the result. □

Now, let us try to adapt what we previously did with $H(M, N)$ and $\mu_{\max}^{M,N}$ to the new quantities $H_k(M, N)$. First, we would like to have an analogue of the Glynn-Whitt invariance principle (5.12). To avoid cumbersome notations, let us first look at the case $k = 2$, $N = 3$. In this case, the geometry involved in the $H_2(M, 3)$ is simple : we are trying to pick up the largest possible weight by using two north-east disjoint paths in the rectangle $\{1, \dots, M\} \times \{1, 2, 3\}$. The most favourable configuration corresponds to one path (the bottom one) starting at $(1, 1)$ and first going right. Then it jumps to some point of $\{2, \dots, M\} \times \{2\}$ and goes horizontally up to $(M, 2)$. The upper path

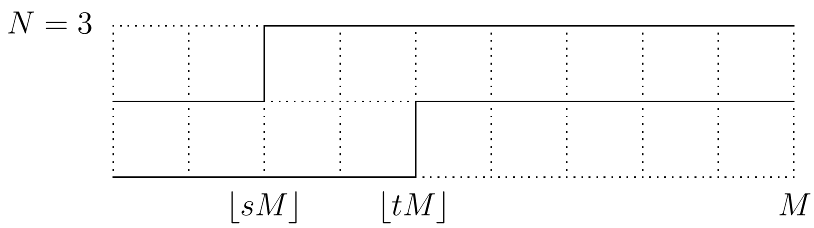

FIG. 5.1 – Configuration of paths in the case $k = 2$ and $N = 3$

starts at $(1,2)$, will also jump and go right up to $(M,3)$. The constraint that our paths must be disjoint forces the x-coordinate of the jump of the bottom path to be larger than that of the jump of the upper path. This corresponds to the obvious figure 5.1.

This figure suggests that in the Donsker limit of random walks converging to Brownian motion, we will have

$$\frac{H_2(M,3)-2M}{\sqrt{M}} \xrightarrow[M\to\infty]{d} \Omega_2^{(3)} \stackrel{\text{def}}{=} \sup_{0\le s\le t\le 1} \left(B_1(t)+B_2(s)+B_2(1)-B_2(t)+B_3(1)-B_3(s) \right)$$

where (B_1, B_2, B_3) is standard 3-dimensional Brownian motion.

For the case of $k = 2$ and general N, we have the same configuration except that the number of jumps for each path will be $N-2$ so that

$$\frac{H_2(M,N)-2M}{\sqrt{M}} \xrightarrow[M\to\infty]{d} \Omega_2^{(N)} \stackrel{\text{def}}{=} \sup \sum_{j=1}^{N} \left(B_j(s_{j-1})-B_j(s_{j-2})+B_j(t_j)-B_j(t_{j-1}) \right) \tag{5.18}$$

where $(B_j)_{1\le j\le N}$ is a standard N-dimensional Brownian motion and the sup is taken over all subdivisions of $[0,1]$ of the following form :

$$0 = s_{-1} = s_0 = t_0 \le s_1 \le t_1 \le s_2 \le t_2 \le \cdots \le t_{N-2} \le s_{N-1} = t_{N-1} = s_N = t_N = 1\ .$$

Proof of limit (5.18). Let us first consider the case of $H_2(M,N)$:

$$H_2(M,N) = \max\{ \sum_{(i,j)\in\pi_1\cup\pi_2} w_{i,j}\ ;\ \pi_1,\ \pi_2 \in \mathcal{P}(M,N)\ ;\ \pi_1,\ \pi_2 \text{ disjoint}\}$$

Since our paths are disjoint, one (say π_1) is always lower than the other (say π_2) : for all $i \in \{1,\ldots,M\}$, $\max\{j\,;\,(i,j)\in\pi_1\} < \min\{j\,;\,(i,j)\in\pi_2\}$. We will denote this by $\pi_1 < \pi_2$. Then, it is not difficult to see on a picture that, for any two paths $\pi_1 < \pi_2 \in \mathcal{P}(M,N)$, one can always find paths $\pi_1' < \pi_2' \in \mathcal{P}(M,N)$ such that $\pi_1\cup\pi_2 \subset \pi_1'\cup\pi_2'$, π_1' starts from $(1,1)$, visits $(2,1)$ then finishes in $(M,N-1)$ and π_2'

starts from $(2,1)$ and goes up to (M,N). Let us call $\mathcal{P}(M,N)'$ the set of pairs of such paths (π_1',π_2'). Thus

$$H_2(M,N) = \max\{ \sum_{(i,j)\in\pi_1\cup\pi_2} w_{i,j} \ ; \ (\pi_1,\pi_2)\in\mathcal{P}(M,N)' \} \, .$$

Now two paths $(\pi_1,\pi_2) \in \mathcal{P}(M,N)'$ are uniquely determined by the non-decreasing sequences of their $N-2$ vertical jumps, namely $0 \le t_1 \le \ldots \le t_{N-2} \le 1$ for π_1 and $0 \le s_1 \le \ldots \le s_{N-2} \le 1$ for π_2 such that :

- π_1 is horizontal on $[\,\lfloor t_{i-1}M\rfloor, \lfloor t_iM\rfloor\,] \times \{i\}$ and vertical on $\{\lfloor t_iM\rfloor\} \times [i,i+1]$,
- π_2 is horizontal on $[\,\lfloor s_{i-1}M\rfloor, \lfloor s_iM\rfloor\,]\times\{i+1\}$ and vertical on $\{\lfloor s_iM\rfloor\}\times[i+1,i+2]$,
- $s_i < t_i$ for all $i \in \{1,\ldots,N-2\}$, this constraint being equivalent to the fact that $\pi_1 < \pi_2$.

The weight picked up by two such paths coded by (t_i) and (s_i) is

- $w_{1,1} + w_{2,1} + \cdots + w_{\lfloor t_1M\rfloor,1}$ on the first floor,
- $w_{1,2} + \cdots + w_{\lfloor s_1M\rfloor,2} \ + \ w_{\lfloor t_1M\rfloor,2} + \cdots + w_{\lfloor t_2M\rfloor,2}$ on the second floor,
- $w_{\lfloor s_1M\rfloor,3} + \cdots + w_{\lfloor s_2M\rfloor,3} \ + \ w_{\lfloor t_2M\rfloor,3} + \cdots + w_{\lfloor t_3M\rfloor,3}$ on the third floor,
- and so on, up to floor N for which the contribution is $w_{\lfloor s_{N-2}M\rfloor,N} + \cdots + w_{M,N}$.

This yields

$$H_2(M,N) = \sup \sum_{j=1}^{N} \Big(\sum_{i=\lfloor s_{j-2}M\rfloor}^{\lfloor s_{j-1}M\rfloor} w_{i,j} \ + \sum_{i=\lfloor t_{j-1}M\rfloor}^{\lfloor t_jM\rfloor} w_{i,j} \Big) .$$

Hence,

$$\frac{H_2(M,N)-2M}{\sqrt{M}} = \sup \sum_{j=1}^{N} \Bigg(\frac{\sum_{i=\lfloor s_{j-2}M\rfloor}^{\lfloor s_{j-1}M\rfloor} w_{i,j} - (s_{j-1}-s_{j-2})M}{\sqrt{M}} + \frac{\sum_{i=\lfloor t_{j-1}M\rfloor}^{\lfloor t_jM\rfloor} w_{i,j} - (t_j - t_{j-1})M}{\sqrt{M}} \Bigg) .$$

Donsker's principle states that

$$\left(\frac{\sum_{i=1}^{\lfloor sM\rfloor} w_{ij} - sM}{\sqrt{M}} \right)_{1\le j\le N} \xrightarrow[M\to\infty]{d} (B_j(s))_{1\le j\le N}$$

where the convergence takes place in the space of cadlag trajectories of the variable $s \in \mathbb{R}_+$ equipped with the Skorohod topology. This allows us to conclude (see [GW91] for a detailed account on the continuity of our mappings in the Skorohod topology). $\square$

For general k and N, the same pattern works with k disjoint paths having each $N-k$ jumps. This yields the following central-limit behaviour :

$$\frac{H_k(M,N)-kM}{\sqrt{M}} \xrightarrow[M\to\infty]{d} \Omega_k^{(N)} \stackrel{\text{def}}{=} \sup \sum_{j=1}^{N} \sum_{p=1}^{k} (\, B_j(s^p_{j-p+1}) - B_j(s^p_{j-p})\,), \tag{5.19}$$

where the sup is taken over all subdivisions (s^p_i) of $[0,1]$ of the following form :

$$s^p_i \in [0,1] \ , \ s^{p+1}_i \leq s^p_i \leq s^p_{i+1} \ , \ s^p_i = 0 \text{ for } i \leq 0 \text{ and } s^p_i = 1 \text{ for } i \geq N-k+1.$$

Now, imitating the argument for the $\lambda^N_{\max}$, we obtain that

$$\Omega_k^{(N)} \stackrel{d}{=} \lambda^N_N + \lambda^N_{N-1} + \cdots + \lambda^N_{N-k+1}, \tag{5.20}$$

where we recall that $\lambda^N_1 \leq \cdots \leq \lambda^N_N$ are the eigenvalues of GUE(N). In fact, the previous equality is also true when considering the vector $(\Omega_k^{(N)})_{1\leq k\leq N}$ and the corresponding sums of eigenvalues, which yields a representation for all the eigenvalues of GUE(N).

A representation for the eigenvalues of GUE(N) was already obtained in [OY02]. Let us compare both representations. Denote by $\mathcal{D}_0(\mathbb{R}_+)$ the space of cadlag paths $f : \mathbb{R}_+ \to \mathbb{R}$ with $f(0) = 0$ and for $f, g \in \mathcal{D}_0(\mathbb{R}_+)$, define $f \otimes g \in \mathcal{D}_0(\mathbb{R}_+)$ and $f \odot g \in \mathcal{D}_0(\mathbb{R}_+)$ by

$$f \otimes g(t) = \inf_{0\leq s\leq t} (f(s) + g(t) - g(s)) \quad \text{and} \quad f \odot g(t) = \sup_{0\leq s\leq t} (f(s) + g(t) - g(s))\,.$$

By induction on N, define $\Gamma^{(N)} : \mathcal{D}_0(\mathbb{R}_+)^N \to \mathcal{D}_0(\mathbb{R}_+)^N$ by

$$\Gamma^{(2)}(f,g) = (f \otimes g, g \odot f)$$

and for $N > 2$ and $f = (f_1, \ldots, f_N)$

$$\Gamma^{(N)}(f) = \big(f_1 \otimes \cdots \otimes f_N, \Gamma^{(N-1)}(f_2 \odot f_1, f_3 \odot (f_1 \otimes f_2), \ldots, f_N \odot (f_1 \otimes \cdots \otimes f_{N-1}))\big)\,.$$

Then the main result in [OY02] is :

$$\lambda^N \stackrel{d}{=} \Gamma^{(N)}(B)(1) \tag{5.21}$$

where $B = (B_i)_{1\leq i\leq N}$ is standard N-dimensional Brownian motion and λ^N is the vector of eigenvalues of GUE(N). In fact, it is proved in [OY02] that identity (5.21) is true for the whole processes and not only their marginals at time 1.

Thus

$$\lambda_N^N + \lambda_{N-1}^N + \cdots + \lambda_{N-k+1}^N \stackrel{d}{=} \Gamma_N^{(N)}(B)(1) + \Gamma_{N-1}^{(N)}(B)(1) + \cdots + \Gamma_{N-k+1}^{(N)}(B)(1)\,.$$

Comparison with (5.20) gives

$$\Omega_k^{(N)} \stackrel{d}{=} \Gamma_N^{(N)}(B)(1) + \Gamma_{N-1}^{(N)}(B)(1) + \cdots + \Gamma_{N-k+1}^{(N)}(B)(1)\,. \tag{5.22}$$

This equality in distribution also holds for the N-vector $(\Omega_k^{(N)})_{1\le k\le N}$.

Now let us remark that the definition of the components $\Gamma_k^{(N)}$ of $\Gamma^{(N)}$ is quite intricate : it involves a sequence of nested "inf" and "sup". On the contrary, $\Omega_k^{(N)}$ is only defined by one "sup" but over a complicated sequence of nested subdivisions. We ignore whether these identities are : trivial and uninteresting ; already well-known ; true for the deterministic formulas (ie true when replacing independent Brownian motions by continuous functions) or true only in distribution.

Our concern raises the question about the link between the $\Gamma^{(N)}$ introduced in [OY02] and the Robinson-Schensted-Knuth correspondence that gave birth to our $\Omega^{(N)}$. Very interesting results in this direction are obtained by O'Connell in [O'C03].

Finally, let us notice that the heart of our arguments to get the previous representations is the identity (5.14). The proof presented here is taken from [Joh00] and is organized in two steps : first the computation of the joint density for $(H_k(M,N))_{1\le k\le N}$ by combinatorial means and second the observation that this density coincides with the eigenvalue density of LUE(N,M). It would be tempting to get a deeper understanding of this result. This would all amount to obtaining a representation for non-colliding squared Bessel processes.

5.4 Proofs

Proof of Theorem 5.2.1. Let us denote by $Z^{(j)}$ the matrix $(A_{k,j}A_{l,j})_{1\le k,j\le N}$ so that $Y^{N,M} = \sum_{j=1}^N Z^{(j)}$. Since $(Z^{(j)})_{j\ge1}$ are independent $\mathbb{L}^2$ random variables with commun law Z, the multi-dimensional central-limit theorem states that :

$$\frac{1}{\sqrt{M}}\Big(\sum_{j=1}^M Z^{(j)} - M\,\mathrm{Id}_N\Big) \xrightarrow[M\to\infty]{d} \mathcal{N}_{\mathcal{H}_N}(0,\mathbb{C}\mathrm{ov}Z)\,.$$

Thus, we just need to check that the covariance structure of Z coincides with that of X taken from GUE(N). In this case, $\mathbb{C}\mathrm{ov}(X_{a,b},X_{c,d}) = \delta_{a,d}\delta_{c,b}$ for $1\le a,b,c,d\le N$. For Z, $\mathbb{C}\mathrm{ov}(Z_{a,b},Z_{c,d}) = \mathbb{E}(A_{a,1}\overline{A_{b,1}}A_{c,1}\overline{A_{d,1}}) - \delta_{a,b}\delta_{c,d}$. We have to distinguish three cases to compute $e = \mathbb{E}(A_{a,1}\overline{A_{b,1}}A_{c,1}\overline{A_{d,1}})$: either all indexes are equal ($a=b=c=d$) and $e = \mathbb{E}(|A_{a,1}|^4) = 2$, or else one index is different from the three others and $e=0$, or else they are equal by pairs, which gives rise to three more situations : $a = b \neq c = d$ for which $e = \mathbb{E}(|A_{a,1}|^2)\mathbb{E}(|A_{c,1}|^2) = 1$, $a = c \neq b = d$ for which $e = \mathbb{E}(A_{a,1}^2)\mathbb{E}(A_{b,1}^2) = 0$ and $a = d \neq b = c$ for which $e = \mathbb{E}(A_{a,1}^2)\mathbb{E}(A_{b,1}^2) = 1$. In each case, $e - \delta_{a,b}\delta_{c,d} = \delta_{a,d}\delta_{c,b}$ which is our result. □

Remark 5.4.1. *In fact, one can also give an elementary proof by direct computation on the density of* $Y^{N,M}$ *just using Stirling's formula and the following asymptotic expansion* $\log\det(\mathrm{Id}_N + \epsilon H) = \epsilon \operatorname{Tr} H - \frac{\epsilon^2}{2} \operatorname{Tr} H^2 + \mathcal{O}(\epsilon^3)$ *for small* ϵ.

Proof of Theorem 5.2.2. We will write A instead of $A^{M,N}$. For $1 \leq i \leq N$, $1 \leq j \leq M$, the superscript ij when applied to a matrix stands for its entry at line i and column j. The value at time t of any process x will be denoted either $x(t)$ or x_t. Let us set

$$Z_M(t) = \frac{Y^{N,M}(t) - Mt\,\mathrm{Id}_N}{\sqrt{M}} = \frac{AA^*(t) - Mt\,\mathrm{Id}_N}{\sqrt{M}}\,.$$

Then

$$Z_M^{ij} = \frac{1}{\sqrt{M}}(\sum_{k=1}^{M} A^{ik}\overline{A}^{jk} - Mt\delta_{ij}) \quad , \quad dZ_M^{ij} = \frac{1}{\sqrt{M}}\sum_{k=1}^{M}(A^{ik}d\overline{A}^{jk} + \overline{A}^{jk}dA^{ik})\,,$$

which implies

$$dZ_M^{ij} \cdot dZ_M^{i'j'} = \frac{1}{M}\sum_{k=1}^{M}(A^{ik}\overline{A}^{j'k}\delta_{i'j} + \overline{A}^{jk}A^{i'k}\delta_{ij'})\,dt\,.$$

The quadratic variation follows to be :

$$\langle Z_M^{ij}, Z_M^{i'j'}\rangle_t = \frac{1}{M}\sum_{k=1}^{M}\int_0^t (A_s^{ik}\overline{A}_s^{j'k}\delta_{i'j} + \overline{A}_s^{jk}A_s^{i'k}\delta_{ij'})\,ds\,.$$

By the classical law of large numbers, we get that this converges almost surely to :

$$\int_0^t \left(\mathbb{E}(A_s^{i1}\overline{A}_s^{j'1})\delta_{i'j} + \mathbb{E}(\overline{A}_s^{j1}A_s^{i'1})\delta_{ij'}\right)\,ds \;=\; \int_0^t \delta_{ij'}\delta_{i'j}2s\,ds \;=\; t^2\delta_{ij'}\delta_{i'j}\,.$$

Note that he previous formula shows that, in the limit, the quadratic variation is 0 if $i \neq j'$ and $i' \neq j$, which is obvious even for finite M without calculations. However, if for instance $i = j'$ and $i' \neq j$, then the quadratic variation is not 0 for finite M and only becomes null in the limit. This is some form of asymptotic independence.

First, let us prove tightness of the process Z_M on any fixed finite interval of time $[0, T]$. It is sufficient to prove tightness for every component, let us do so for Z_M^{11} for example (Z_M^{11} is real). We will apply Aldous' criterion (see [KL99]). Since $Z_M^{11}(0) = 0$ for all M, it is enough to check that, for all $\epsilon > 0$,

$$\lim_{\delta\to 0} \limsup_{M\to\infty} \sup_{\tau\,,\,0\le\theta\le\delta} \mathbb{P}(\,|Z_M^{11}(\tau+\theta) - Z_M^{11}(\tau)| \ge \epsilon\,) \;=\; 0 \tag{5.23}$$

where the sup is taken over all stopping times τ bounded by T. For τ such a stopping time, $\epsilon > 0$ and $0 \le \theta \le \delta \le 1$, we have

$$\begin{aligned}
\mathbb{P}(\,|Z_M^{11}(\tau+\theta) - Z_M^{11}(\tau)| \ge \epsilon\,) \;&\le\; \frac{1}{\epsilon^2}\mathbb{E}((Z_M^{11}(\tau+\theta) - Z_M^{11}(\tau))^2) \\
&=\; \frac{1}{\epsilon^2}\mathbb{E}(\int_{\tau}^{\tau+\theta} d\langle Z_M^{11}, Z_M^{11}\rangle_t) \\
&=\; \frac{2}{M\epsilon^2}\sum_{k=1}^{M}\mathbb{E}(\int_{\tau}^{\tau+\theta} |A_s^{1k}|^2\,ds) \\
&\le\; \frac{2}{M\epsilon^2}\sum_{k=1}^{M}\mathbb{E}(\theta \sup_{0\le s\le T+1} |A_s^{1k}|^2) \\
&=\; \frac{2\theta}{\epsilon^2}\mathbb{E}(\sup_{0\le s\le T+1} |A_s^{11}|^2)
\end{aligned}$$

Since $c_T = \mathbb{E}(\sup_{0\le s\le T+1} |A_s^{11}|^2) < \infty$, then

$$\limsup_{M\to\infty} \sup_{\tau\,,\,0\le\theta\le\delta} \mathbb{P}(\,|Z_M^{11}(\tau+\theta) - Z_M^{11}(\tau)| \ge \epsilon\,) \;\le\; \frac{2\delta\, c_T}{\epsilon^2}\,.$$

This last line obviously proves (5.23).

Let us now see that the finite-dimensional distributions converge to the appropriate limit. Let us first fix i, j and look at the component $Z_M^{ij} = \frac{x_M + \sqrt{-1}y_M}{\sqrt{2}}$. We can write

$$\langle x_M, y_M\rangle_t = 0 \quad , \quad \langle x_M, x_M\rangle_t = \langle y_M, y_M\rangle_t = \frac{1}{M}\sum_{k=1}^{M}\int_0^t \alpha_s^k\,ds \tag{5.24}$$

where $\alpha_s^k = |A_s^{ik}|^2 + |A_s^{jk}|^2$. We are going to consider x_M. Let us fix $T \ge 0$. For any $(\nu_1, \ldots, \nu_n) \in [-T, T]^n$ and any $0 = t_0 < t_1 < \ldots < t_n \le T$, we have to prove that

$$\mathbb{E}\Bigg(\exp\Big(i\sum_{j=1}^{n}\nu_j(x_M(t_j) - x_M(t_{j-1}))\Big)\Bigg) \underset{M\to\infty}{\longrightarrow} \exp\Big(\sum_{j=1}^{n}\frac{\nu_j^2}{2}(t_j^2 - t_{j-1}^2)\Big)\,. \tag{5.25}$$

We can always suppose $|t_j - t_{j-1}| \le \delta$ where δ will be chosen later and will only depend on T (and not on n). We will prove property (5.25) by induction on n. For

$n = 0$, there is nothing to prove. Suppose it is true for $n-1$. Denote by $(\mathcal{F}_t)_{t\geq 0}$ the filtration associated to the process A. Then write :

$$\mathbb{E}\Big(e^{i\sum_{j=1}^{n}\nu_j(x_M(t_j)-x_M(t_{j-1}))}\Big) = \mathbb{E}\Big(e^{i\sum_{j=1}^{n-1}\nu_j(x_M(t_j)-x_M(t_{j-1}))}\,\mathbb{E}\big(e^{i(x_M(t_n)-x_M(t_{n-1}))}|\mathcal{F}_{t_{n-1}}\big)\Big). \tag{5.26}$$

We define the martingale $\mathcal{M}_t = e^{i\nu_n x_M(t)-\frac{\nu_n^2}{2}\langle x_M,x_M\rangle_t}$. Hence

$$\mathbb{E}\big(e^{i\nu_n(x_M(t_n)-x_M(t_{n-1}))}|\mathcal{F}_{t_{n-1}}\big) = \mathbb{E}\left(\frac{\mathcal{M}_{t_n}}{\mathcal{M}_{t_{n-1}}}e^{\frac{\nu_n^2}{2}\langle x_M,x_M\rangle_{t_{n-1}}^{t_n}} \mid \mathcal{F}_{t_{n-1}}\right)$$

with the notation $\langle x,x\rangle_s^t = \langle x,x\rangle_t - \langle x,x\rangle_s$. This yields

$$e^{-\frac{\nu_n^2}{2}(t_n^2-t_{n-1}^2)}\mathbb{E}\big(e^{i\nu_n(x_M(t_n)-x_M(t_{n-1}))} \mid \mathcal{F}_{t_{n-1}}\big) - 1 = \mathbb{E}\left(\frac{\mathcal{M}_{t_n}}{\mathcal{M}_{t_{n-1}}}\zeta_M \mid \mathcal{F}_{t_{n-1}}\right) \tag{5.27}$$

where we set $\zeta_M = e^{\frac{\nu_n^2}{2}(\langle x_M,x_M\rangle_{t_{n-1}}^{t_n}-(t_n^2-t_{n-1}^2))} - 1$. Using that $|e^z-1| \leq |z|e^{|z|}$, we deduce that

$$|\zeta_M| \leq K\,|\langle x_M,x_M\rangle_{t_{n-1}}^{t_n} - (t_n^2-t_{n-1}^2)|\,e^{\frac{\nu_n^2}{2}\langle x_M,x_M\rangle_{t_{n-1}}^{t_n}}$$

where $K = \nu_n^2/2$. The Cauchy-Schwarz inequality implies that

$$\mathbb{E}(|\zeta_M|) \leq K\,\left(\mathbb{E}\left(\langle x_M,x_M\rangle_{t_{n-1}}^{t_n} - (t_n^2-t_{n-1}^2)\right)^2\right)^{1/2}\left(\mathbb{E}\left(e^{\nu_n^2\langle x_M,x_M\rangle_{t_{n-1}}^{t_n}}\right)\right)^{1/2}.$$

By convexity of the function $x \to e^x$:

$$e^{\nu_n^2\langle x_M,x_M\rangle_{t_{n-1}}^{t_n}} = \exp\left(\frac{1}{M}\sum_{k=1}^{M}\nu_n^2\int_{t_{n-1}}^{t_n}\alpha_u^k\,du\right) \leq \frac{1}{M}\sum_{k=1}^{M}e^{\nu_n^2\,(t_n-t_{n-1})\sup_{0\leq u\leq t_n}\alpha_u^k}$$

and thus

$$\mathbb{E}\left(e^{\nu_n^2\langle x_M,x_M\rangle_{t_{n-1}}^{t_n}}\right) \leq \frac{1}{M}\sum_{k=1}^{M}\mathbb{E}\left(e^{\nu_n^2\,(t_n-t_{n-1})\sup_{0\leq u\leq t_n}\alpha_u^k}\right) = \mathbb{E}\left(e^{\nu_n^2(t_n-t_{n-1})\sup_{0\leq u\leq t_n}\alpha_u^1}\right).$$

Now let us recall that $\alpha_u^1 = |A_u^{i1}|^2 + |A_u^{j1}|^2$, which means that α^1 has the same law as a sum of squares of four independent Brownian motions. It is then easy to see that there exists $\delta > 0$ (depending only on T) such that $\mathbb{E}(\exp(T^2\delta\sup_{0\leq u\leq T}\alpha_u^1)) < \infty$. With this choice of δ, $K' = \mathbb{E}(e^{\nu_n^2(t_n-t_{n-1})\sup_{0\leq u\leq t_n}\alpha_u^1}) < \infty$ and thus :

$$\mathbb{E}(|\zeta_M|) \leq K\,K'\,\left(\mathbb{E}\left(\langle x_M,x_M\rangle_{t_{n-1}}^{t_n} - (t_n^2-t_{n-1}^2)\right)^2\right)^{1/2} \underset{M\to\infty}{\longrightarrow} 0$$

(by the law of large numbers for square-integrable independent variables). Since $|\frac{\mathcal{M}_{t_n}}{\mathcal{M}_{t_{n-1}}}| \leq 1$, we also have

$$\frac{\mathcal{M}_{t_n}}{\mathcal{M}_{t_{n-1}}} \zeta_M \xrightarrow[M\to\infty]{\mathbb{L}^1} 0\,.$$

Therefore

$$\mathbb{E}(\frac{\mathcal{M}_{t_n}}{\mathcal{M}_{t_{n-1}}} \zeta_M \,|\, \mathcal{F}_{t_{n-1}}) \xrightarrow[M\to\infty]{\mathbb{L}^1} 0\,. \tag{5.28}$$

In turn, by looking at (5.27), this means that

$$\mathbb{E}(e^{i\nu_n(x_M(t_n)-x_M(t_{n-1}))} \,|\, \mathcal{F}_{t_{n-1}}) \xrightarrow[M\to\infty]{\mathbb{L}^1} e^{\frac{\nu_n^2}{2}(t_n^2-t_{n-1}^2)}\,.$$

Now, plug this convergence and the induction hypothesis for $n-1$ into (5.26) to get the result for n.

The same is true for y_M. To check that the finite-dimensional distributions of Z_M^{ij} have the right convergence, we would have to prove that :

$$\mathbb{E}\Bigg(\exp\Big(i\sum_{i=1}^n \nu_i(x_M(t_i)-x_M(t_{i-1})) + \mu_i(y_M(t_i)-y_M(t_{i-1}))\Big)\Bigg)$$

$$\xrightarrow[M\to\infty]{} \exp\left(\sum_{i=1}^n \frac{\nu_i^2+\mu_i^2}{2}\,(t_i^2-t_{i-1}^2)\right)\,. \tag{5.29}$$

But since $\langle x_M, y_M\rangle = 0$,

$$\mathcal{M}_t = \exp\left(i(\nu_n x_M(t)+\mu_n y_M(t)) - \frac{\nu_n^2}{2}\langle x_M, x_M\rangle_t - \frac{\mu_n^2}{2}\langle y_M, y_M\rangle_t\right)$$

is a martingale and the reasoning is exactly the same as the previous one.

Finally, let us look at the asymptotic independence. For the sake of simplicity, let us take only two entries. Set for example $x_M = Z_M^{11}$ and $y_M = \sqrt{2}\,\mathrm{Re}(Z_M^{12})$. Then we have to prove (5.29) for our new x_M, y_M. Since $\langle x_M, y_M\rangle \neq 0$, $\mathcal{M}_t$ previously defined is no more a martingale. But

$$\mathcal{N}_t = \exp\left(i(\nu_n x_M(t)+\mu_n y_M(t)) - \frac{\nu_n^2}{2}\langle x_M, x_M\rangle_t - \frac{\mu_n^2}{2}\langle y_M, y_M\rangle_t - \nu_n\mu_n\langle x_M, y_M\rangle_t\right)$$

is a martingale and the fact that $\langle x_M, y_M\rangle_t \xrightarrow[M\to\infty]{\mathbb{L}^2} 0$ allows us to go along the same lines as before.

□

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Chapitre 6

Non-colliding processes and the Meixner ensemble

Abstract : We identify a class of birth-and-death processes X on $\mathbb{N}$ such that the Vandermonde determinant h is an eigenfunction for the generator of N independent copies of X. In two particular cases, we define the h-transform of the process killed when exiting the Weyl chamber $W = \{x \in \mathbb{N}^N \,;\, x_1 > x_2 > \cdots > x_N\}$ and prove that its fixed-time marginals are distributed according to the Meixner ensemble. We also include an analysis of the Martin boundary.

6.1 Introduction

Let $\mathbb{N}$ be the set of non-negative integers, $N \in \mathbb{N} \setminus \{0, 1\}$ and $W = \{x \in \mathbb{N}^N \,;\, x_1 > x_2 > \cdots > x_N\}$. For $q \in]0, 1[$ and $\theta > 0$, the Meixner ensemble (with $\beta = 2$) is defined to be

$$\mathbf{Me}_{N,\theta,q}(y) = (Z_{N,\theta,q})^{-1}\, h(y)^2 \prod_{j=1}^{N} w_q^\theta(y_j), \qquad y \in W,$$

where $w_q^\theta(y) = \binom{y+\theta-1}{y} q^y = \frac{\Gamma(y+\theta)}{\Gamma(\theta)\Gamma(y+1)}\, q^y$ for $y \in \mathbb{N}$, $Z_{N,\theta,q}$ is a normalisation constant such that $\mathbf{Me}_{N,\theta,q}$ is a probability measure on W and $h(y)$ is the Vandermonde function

$$h(y) = \prod_{1 \le i < j \le N} (y_i - y_j)\,.$$

Such ensembles associated with different kinds of weights w, either discrete or continuous, are classical in various contexts. They appear as the joint distribution of the eigenvalues of usual random matrices as well as measures on integer partitions (with at most N parts) arising from combinatorial or group-theoretic considerations (see [Joh00], [Joh02], [Joh01], [BO00]). From a slightly different point of view, they prove to be the fixed-time distribution of some non-colliding processes (see [OY02], [KOR02], [KO01]). The Meixner ensemble exhibits those two aspects : on the one hand, [Joh00] has shown that it is the law of the shape of the tableaux obtained by applying the RSK correspondance to a matrix filled in with iid geometric entries. On the other hand, our goal was to establish that the Meixner ensemble can also be described as the fixed-time marginal of some non-colliding processes.

We will exhibit a pure birth (resp. birth and death) process X (resp. Y) on $\mathbb{N}$ such that the Meixner ensemble turns out to be the fixed-time distribution of N independent copies of X (resp. Y) conditioned never to collide.

In view of this, section 6.2 identifies some birth and death generators with polynomial rates for which h is an eigenfunction. In sections 6.3 and 6.4, we define the associated Doob h-transforms and the Meixner ensemble is shown to appear as their fixed-time marginals. In section 6.5, we give a description of the Martin boundary for the Yule processes case.

Notation : For $x \in \mathbb{N}^N$, we write $|x| = \sum_{i=1}^N x_i$.

6.2 Harmonicity of h for birth and death processes

Let us denote by ∇_i the discrete gradient operator in the direction of the ith coordinate : $\nabla_i f(x) = f(x+e_i) - f(x)$ where (e_i) is the canonical basis of $\mathbb{R}^N$. Consider the operator L defined by :

$$Lf(x) = \sum_{i=1}^{N} P(x_i) \nabla_i f(x)$$

where $P \in \mathbb{R}[X]$.

Lemma 6.2.1. *If h is the Vandermonde function $h(x) = \prod_{i<j}(x_i - x_j)$ in $\mathbb{R}^N$ then $Lh(x)$ is a skew-symmetric polynomial in $x_1, \dots, x_N$. This implies that $\frac{Lh(x)}{h(x)}$ is a symmetric polynomial in $x_1, \dots, x_N$.*

Proof. If the action of $\mathfrak{S}_N$ on functions is given by

$$(\sigma.f)(x) = f(x_{\sigma(1)}, \dots, x_{\sigma(N)}),$$

then we easily check that $L(\sigma.f) = \sigma.Lf$ (when $P(x)\nabla$ is the generator of a Markov process of law μ, the previous commutation just expresses the $\mathfrak{S}_N$-invariance of $\mu^{\otimes N}$). Thus, L preserves skew-symmetric functions. Now, any skew-symmetric polynomial cancels when $x_i = x_j$ and is therefore divisible by h, the result of this division being symmetric. □

Lemma 6.2.2. *1.* $\sum_{i=1}^N \nabla_i h(x) = 0$

2. $\sum_{i=1}^N x_i \nabla_i h(x) = \frac{N(N-1)}{2} h(x)$

3. $\sum_{i=1}^N x_i^2 \nabla_i h(x) = \{(N-1)(x_1 + \cdots + x_N) + \frac{N(N-1)(N-2)}{6}\} h(x)$

Proof. Lemma 6.2.1 asserts that $S(x) = \frac{Lh(x)}{h(x)}$ is a symmetric polynomial. In the case where $P(x) = 1$, the degree of Lh is strictly less than that of h, hence $S = 0$. For $P(x) = x$, degree considerations guarantee that S is a constant, which is computable by evaluating Q at $x^* = (N-1, N-2, \ldots, 0)$. For $P(x) = x^2$, S is of degree at most 1 and symmetric. Thus $S(x) = \lambda(x_1 + \cdots + x_N) + \mu$. λ is found to be $N-1$ by comparing the coefficients of x_N^N. The value of μ is then obtained by evaluating at x^*. □

Remark 6.2.1. *The same line of reasonning shows that if ν is an exchangeable distribution on $\mathbb{N}^N$ then $\sum_{y \in \mathbb{N}^N} (h(x+y) - h(x))\, \nu(y) = 0$ (the left-hand side is a skew-symmetric polynomial in $x_1, \ldots, x_N$, of degree $< N(N-1)/2$). This means that h is harmonic for the random walk with increment distribution ν. This result was first proven in [KOR02].*

Now, define $\overline{\nabla}_i$ by $\overline{\nabla}_i f(x) := f(x - e_i) - f(x)$. We remark that $\overline{\nabla}_i f = \overline{\nabla_i \overline{f}}$, where $\overline{f}(x) := f(-x)$, which allows us to deduce from Lemma 6.2.2 similar identities with $\overline{\nabla}_i$. We finally get the

Proposition 6.2.1. *If we define the operator* $\mathbf{L}$ *by*

$$\mathbf{L}f = \sum_{i=1}^N \{(a x_i^2 + b x_i + c)\nabla_i f + (\overline{a} x_i^2 + \overline{b} x_i + \overline{c})\overline{\nabla}_i f\},$$

then

$$\mathbf{L}h = \left\{ (a - \overline{a})(N-1) \sum_{i=1}^N x_i + (a + \overline{a}) \frac{N(N-1)(N-2)}{6} + (b - \overline{b}) \frac{N(N-1)}{2} \right\} h. \tag{6.1}$$

Thus, h is an eigenfunction of $\mathbf{L}$ if and only if $a = \overline{a}$ and the corresponding eigenvalue is $a\frac{N(N-1)(N-2)}{3} + (b - \overline{b})\frac{N(N-1)}{2}$. In particular, h is harmonic if and only if $a = \overline{a}$ and $b - \overline{b} = -\frac{2}{3} a(N-2)$.

6.3 Non-colliding Yule processes

Let $\mathbf{X} = (X_1, \ldots, X_N)$ be N independent copies of a one-dimensional Yule process with branching rate 1 and immigration $\theta \geq 0$ and let $\mathbf{L}$ be its generator :

$$\mathbf{L}f = \sum_{i=1}^N (x_i + \theta)\nabla_i f,$$

for $f : \mathbb{N}^N \to \mathbb{R}$. Then, Proposition 6.2.1 states that $\mathbf{L}^\lambda h = 0$ where $\lambda = \frac{N(N-1)}{2}$ and $\mathbf{L}^\lambda f = \mathbf{L}f - \lambda f$. In other words, if τ is a random time independent of $\mathbf{X}$ with an exponential distribution of parameter λ, then h is harmonic for the process $\mathbf{X}'$, which is $\mathbf{X}$ killed at time τ. Since the components of $\mathbf{X}$ can't jump simultaneously, $\mathbf{X}'$ has no transition from W to $\overline{W}^c$. Therefore, the restriction of h to W is a strictly positive harmonic function for the process $\mathbf{X}'$ killed when exiting W. Thus, one can consider the following Doob h-transform :

$$\mathbb{P}_x^h(\mathbf{X}(t) = y) = \frac{h(y)}{h(x)}\,\mathbb{P}_x(\mathbf{X}(t) = y,\, T \wedge \tau > t) = e^{-\lambda t}\frac{h(y)}{h(x)}\,\mathbb{P}_x(\mathbf{X}(t) = y,\, T > t), \quad (6.2)$$

where $x, y \in W$ and $T = \inf\{t > 0\,;\, \mathbf{X}(t) \notin W\}$. This new process can be thought of as the original one conditioned to stay in W forever.

Proposition 6.3.1. *Suppose $\theta > 0$ and set $x^* = (N-1, N-2, \ldots, 0)$. Then for any $y \in W$ and any $t > 0$,*

$$\mathbb{P}_{x^*}^h(\mathbf{X}(t) = y) = \mathbf{Me}_{N,\theta,1-e^{-t}}(y) = C_t\, h(y)^2\, \mathbb{P}_0(\mathbf{X}(t) = y)\,.$$

Proof. Denote by $p_t^\theta(i,j)$ the transition probability for the one-dimensional (unconditioned) process : $p_t^\theta(i,j) = \mathbb{P}_i(X(t) = j)$, $j \geq i$. We know that

$$p_t^\theta(i,j) = (1-q_t)^{\theta+i}\binom{j+\theta-1}{j-i} q_t^{j-i}, \quad (6.3)$$

where $q_t = 1 - e^{-t}$ and $\binom{n}{m} = 0$ if $m \leq -1$. It is convenient to notice that

$$\binom{j+\theta-1}{j-i} = \binom{j+\theta-1}{j}\frac{P_i(j)}{(\theta)_i},$$

where $P_i(X) = \prod_{l=1}^{i}(X - i + l)$ and $(\theta)_i = \theta(\theta+1)\cdots(\theta+i-1)$ for $i \geq 1$, $P_0 = 1$, $(\theta)_0 = 1$. Indeed, $P_i(j) = 0$ if $j \in \mathbb{N}$ and $j < i$. Thus,

$$p_t^\theta(i,j) = \frac{(1-q_t)^{\theta+i}}{(\theta)_i}\binom{j+\theta-1}{j} q_t^{j-i}\, P_i(j),$$

For $x, y \in W$, the Karlin-McGregor formula ([KM59]) asserts that

$$\mathbb{P}_x(\mathbf{X}(t) = y,\, T > t) = \det(p_t^\theta(x_i, y_j))_{1\leq i,j\leq N}. \quad (6.4)$$

Factorizing along lines and columns, one obtains that

$$\begin{aligned}\mathbb{P}_x(\mathbf{X}(t) = y,\, T \wedge \tau > t) \;&=\; (1-q_t)^{N\theta+|x|+\lambda} q_t^{|y|-|x|}\prod_{i=1}^{N}\frac{1}{(\theta)_{x_i}} \\ &\quad \prod_{j=1}^{N}\binom{y_j+\theta-1}{y_j}\det\left(P_{x_i}(y_j)\right)_{1\leq i,j\leq N}.\end{aligned}$$

Since P_m is a polynomial of degree m and leading coefficient 1, the matrix $(P_{N-i}(y_j))_{1\le i,j\le N}$ is equal to the product AB where $B=(y_j^{N-i})_{1\le i,j\le N}$ and A is some upper-triangular matrix whose diagonal coefficients are 1s. Therefore, $\det\left(P_{N-i}(y_j)\right)_{1\le i,j\le N}=h(y)$. It follows that

$$\mathbb{P}_{x^*}(\mathbf{X}(t)=y,\,T\wedge\tau>t)=C(t,\theta,x^*)\,h(y)\prod_{j=1}^{N}\binom{\theta-1+y_j}{y_j}q_t^{y_j}. \tag{6.5}$$

Now, plug this into (6.2) with $x=x*$ to get the result. □

Proposition 6.3.2. *If $\theta=0$, we set $\mathbf{1}=(1,1,\dots,1)$. Then, for any $y\in W$ and $t>0$, we have*

$$\mathbb{P}^h_{x^*+\mathbf{1}}(\mathbf{X}(t)=y+\mathbf{1})=\mathbf{Me}_{N,1,1-e^{-t}}(y)=C'_t\,h(y)^2\,\mathbb{P}_{\mathbf{1}}(\mathbf{X}(t)=y+\mathbf{1})\,.$$

Proof. If X_t is the one-dimensional Yule process with $\theta=0$ on $\mathbb{N}\backslash\{0\}$, then formula (6.3) shows that X_t-1 has the law of a one-dimensional Yule process with immigration $\theta=1$, which concludes the proof seeing proposition 6.3.1. □

6.4 Non-colliding linear birth and death processes

Now, let us consider $\mathbf{Y}=(Y_1,\dots,Y_N)$, N independent copies of a one-dimensional birth and death process on $\mathbb{N}$ with death rate $\delta(x)=x$ and birth rate $\beta(x)=x+\theta$ where $\theta>0$. The generator of $\mathbf{Y}$ is

$$\mathbf{L}f=\sum_{i=1}^{N}\left\{(x_i+\theta)\nabla_i f+x_i\overline{\nabla}_i f\right\},$$

for $f:\mathbb{N}^N\to\mathbb{R}$ and Proposition 6.2.1 guarantees that h is harmonic for $\mathbf{Y}$. Since the components of $\mathbf{Y}$ are independent and have only jumps in $\{\pm1\}$, $\mathbf{Y}$ has no transition from W to $\overline{W}^c$. In the same way as in Section 6.3, we can consider the Doob h-transform of the original process defined by

$$\mathbb{P}^h_x(\mathbf{Y}(t)=y)=\frac{h(y)}{h(x)}\,\mathbb{P}_x(\mathbf{Y}(t)=y,\,T>t) \tag{6.6}$$

where $x,y\in W$ and $T=\inf\{t>0\,;\,\mathbf{Y}(t)\notin W\}$.

Proposition 6.4.1. *If $x^*=(N-1,N-2,\dots,0)$, $y\in W$ and $t>0$,*

$$\mathbb{P}^h_{x^*}(\mathbf{Y}(t)=y)=\mathbf{Me}_{N,\theta,t/(1+t)}(y)=D_t\,h(y)^2\,\mathbb{P}_0(\mathbf{Y}(t)=y)\,.$$

Proof. In [KM58], an explicit formula is given for the generating function of the one-dimensional (unconditioned) transition probability $p_t(i,j)$:

$$\sum_{j\geq 0} p_t(i,j)s^j = \frac{t^i}{(1+t)^{\theta+i}} \frac{(1+rs)^i}{(1-qs)^{\theta+i}}$$

where $r = \frac{1-t}{t}$ and $q = \frac{t}{1+t}$. It is easily deduced that

$$p_t(i,y_j) = a_i\, b_{y_j} \sum_{l=0}^{\min(i,y_j)} \binom{i}{l} u^l \frac{(\theta+i)_{y_j-l}}{(y_j-l)!} \tag{6.7}$$

where $a_i = \frac{t^i}{(1+t)^{\alpha+i+1}}$, $b_{y_j} = q^{y_j}$, $u = \frac{r}{q}$, $(\beta)_p = \beta(\beta+1)\cdots(\beta+p-1)$ for $p \geq 1$ and $(\beta)_0 = 1$.
Hence $p_t(0,y_j) = a_0\binom{\theta-1+y_j}{y_j}q^{y_j}$, which yields

$$\mathbb{P}_0(\mathbf{X}(t) = y) = a_0^N \prod_{j=1}^{N} \binom{\theta-1+y_j}{y_j} q^{y_j} \tag{6.8}$$

for $y \in \mathbb{N}^N$.

Now, we can write

$$\frac{(\theta+i)_{y_j-l}}{(y_j-l)!} = \binom{\theta-1+y_j}{y_j} \frac{\prod_{m=1}^{i-l}(\theta-1+y_j+m)\prod_{n=1}^{l}(y_j-l+n)}{(\theta)_i}$$

with the convention that empty products are 1. Define

$$P_i(y) = \frac{1}{(\theta)_i} \sum_{l=0}^{i} \binom{i}{l} u^l \prod_{m=1}^{i-l}(\theta-1+y_j+m) \prod_{n=1}^{l}(y_j-l+n)$$

Remark that if $l > y_j$ then $\prod_{n=1}^{l}(y_j-l+n) = 0$. Thus the restriction in the sum (6.7) can be forgotten to result in

$$p_t(i,y_j) = a_i\, b_{y_j} \binom{\theta-1+y_j}{y_j} P_i(y_j).$$

The use of the Karlin-MacGregor formula and the computation of the determinant are exactly the same as in the proof of Proposition 6.3.1. □

Remark 6.4.1. *It is interesting to remark that the representations of the Meixner ensemble presented in Sections 6.3 and 6.4 are quite different from the one in [O'C03a] obtained by conditioning random walks with geometric increments to stay ordered forever (this conditioning is shown to be equivalent to the application of the RSK algorithm).*

6.5 Martin boundary for Yule processes

We recall the definition of the Green kernel of $\mathbf{X}$ killed at time $T \wedge \tau$:

$$G(x,y) = \int_0^\infty \mathbb{P}_x(\mathbf{X}_t = y, T \wedge \tau > t)\, dt,$$

and that of the Martin kernel based at x^* :

$$M(x,y) = \frac{G(x,y)}{G(x^*,y)}.$$

We also need the definition of the Schur function with index $x \in W$:

$$\mathrm{Schur}_x(p) = \frac{\det(x_i^{p_j})_{1\le i,j\le N}}{h(x)}.$$

Proposition 6.5.1. *We have*

$$M(x,y) \to \prod_{i=1}^N \frac{(\theta)_{N-i}}{(\theta)_{x_i}} \frac{\Gamma(N\theta+\lambda+|x|)}{\Gamma(N\theta+\lambda+|x^*|)} \mathrm{Schur}_x(p),$$

as $|y| \to \infty$ and $y/|y| \to p$.

In other words, the Martin compactification of $\mathbf{X}$ killed at time $T \wedge \tau$ (with base point x^*) is $MC = W \cup \Sigma$, where

$$\Sigma := \{p \in [0,1]^N \,|\, p_1 \geq \ldots \geq p_N,\ |p| = 1\},$$

and the topology on MC is given by usual neighbourhoods for points of W and the following system of neighbourhoods for $p \in \Sigma$:

$$V_{\epsilon,\eta,M}(p) = \{q \in \Sigma \,|\, \|q-p\| < \epsilon\} \cup \{y \in W \,|\, |y| > M,\ \|\frac{y}{|y|} - p\| < \eta\}.$$

Alternatively, a sequence $(y_n) \in W^{\mathbb{N}}$ converges to $p \in \Sigma$ if and only if $|y_n| \to \infty$ and $y_n/|y_n| \to p$. The Martin kernel associated with $p \in \Sigma$ is

$$M(x,p) = \prod_{i=1}^N \frac{(\theta)_{N-i}}{(\theta)_{x_i}} \frac{\Gamma(N\theta+\lambda+|x|)}{\Gamma(N\theta+\lambda+|x^*|)} \mathrm{Schur}_x(p).$$

Proof. If $C(y) = \prod_{j=1}^N \binom{y_j+\theta-1}{y_j}$, recall that

$$\mathbb{P}_x(\mathbf{X}(t) = y,\ T\wedge\tau > t) = C(y)\,(1-q_t)^{N\theta+|x|+\lambda} q_t^{|y|-|x|} \prod_{i=1}^N \frac{1}{(\theta)_{x_i}} \det\left(P_{x_i}(y_j)\right),$$

so that, after changing variables $u = e^{-t}$ in the integral,

$$G(x,y) = C(y) \prod_{i=1}^{N} \frac{1}{(\theta)_{x_i}} \det\left(P_{x_i}(y_j)\right) \, B(N\theta + |x| + \lambda, |y| - |x| + 1),$$

where B is the Beta function. Thus,

$$M(x,y) = \prod_{i=1}^{N} \frac{(\theta)_{N-i}}{(\theta)_{x_i}} \frac{\det\left(P_{x_i}(y_j)\right)}{h(y)} \frac{B(N\theta + |x| + \lambda, |y| - |x| + 1)}{B(N\theta + |x^*| + \lambda, |y| - |x^*| + 1)}.$$

Now, using the facts that $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$, $\Gamma(a+c)/\Gamma(a+c^*) \sim a^{c-c^*}$ as $a \to \infty$ and $\det\left(P_{x_i}(y_j)\right) \sim \det\left(y_j^{x_i}\right)$ as $y \to \infty$, we get

$$M(x,y) \sim \prod_{i=1}^{N} \frac{(\theta)_{N-i}}{(\theta)_{x_i}} \frac{\Gamma(N\theta + |x| + \lambda)}{\Gamma(N\theta + |x^*| + \lambda)} \, \mathrm{Schur}_x(y) |y|^{|x^*|-|x|},$$

which concludes the proof if we remember that Schur_x is a homogeneous polynomial of degree $|x| - |x^*|$. □

Remark 6.5.1. *Define* $\phi(x) = \prod_{i=1}^{N} \frac{(\theta)_{N-i}}{(\theta)_{x_i}} \frac{\Gamma(N\theta+|x|+\lambda)}{\Gamma(N\theta+|x^*|+\lambda)} N^{|x^*|-|x|}$. *Then, it is easy to check that*

$$\mathbf{L}^\lambda f(x) = \frac{(N\theta + |x| + \lambda)\phi(x)}{N} \mathbf{G}(f/\phi)(x),$$

where $\mathbf{G}g(x) = \sum_{i=1}^{N} \left(g(x+e_i) - g(x)\right)$ *is the generator of* N *independent Poisson processes. Thus, the correspondance* $f \to f/\phi$ *is a bijection between* $\mathbf{L}^\lambda$*-harmonic functions and* $\mathbf{G}$*-harmonic functions preserving positivity and minimality. Therefore, the relation*

$$\frac{M(x,p)}{\phi(x)} = \mathrm{Schur}_x(Np)$$

is consistent with the Martin boundary analysis of Poisson processes killed when exiting W *performed in [KOR02].*

In conclusion, h turns out to be a harmonic function for $\mathbf{L}^\lambda$ but not an extremal one, which is different from the random walks situation (see [KOR02], [O'C03b] and [O'C03a]). It would be interesting to find a mixing measure (a priori we have to say "a" since we haven't determined the minimal part of the boundary) μ_h such that :

$$h(x) = h(x^*) \prod_{i=1}^{N} \frac{(\theta)_{N-i}}{(\theta)_{x_i}} \frac{\Gamma(N\theta + |x| + \lambda)}{\Gamma(N\theta + |x^*| + \lambda)} N^{|x^*|-|x|} \int_\Sigma \mathrm{Schur}_x(Np) \, \mu_h(dp).$$

Remark 6.5.2. *As P. Biane kindly pointed it to us, the processes investigated here are very close to those studied in the recent preprint [BO04]. Indeed, in the notations of formula (4.6) in section 4.3 of [BO04], our (one-dimensional) processes* X *and* Y *respectively have the rates* $\alpha_1(t) = 1$*,* $\beta_1(t) = 0$ *and* $\alpha_2(t) = \beta_2(t) = 1$*. If we set* $\xi_1(t) = 1 - e^{-t}$ *and* $\xi_2(t) = \frac{t}{1+t}$*, the equation (4.7) in [BO04], which is*

$$\frac{\xi_i'}{\xi_i(1-\xi_i)} = \frac{\alpha_i}{\xi_i} - \beta_i,$$

is verified for $i = 1, 2$*, which proves that* $\pi_{\theta,\xi_i(s)} P_i(s,t) = \pi_{\theta,\xi_i(t)}$*, where* $P_i(s,t)$ *is the semigroup of the process between times* s *and* t *and*

$$\pi_{\theta,q}(n) = (1-q)^n \frac{\Gamma(\theta+n)}{\Gamma(\theta)\Gamma(n+1)} q^n$$

is the negative binomial distribution on $\mathbb{N}$*. If we notice that* $\xi_i(0) = 0$ *and that* $\pi_{\theta,0} = \delta_0$*, we have that* $\pi_{\theta,\xi_i(t)}, (i = 1, 2)$ *are the distributions at time* t *of our (one-dimensional) processes* X *and* Y *starting from* 0*. This fact already appeared in our proofs (in fact, all the transition probabilities* $p_{\cdot}(x, y)$*, not only for* $x = 0$*, are needed for us and given in formulae (6.3) and (6.7)). However, our curves* ξ_1*,* ξ_2 *are not admissible in the terminology of [BO04]. At the cost of losing time-homogeneity, we can change time in order to match their constraints : set*

$$\widetilde{\xi}_1(\tau) = e^{2\tau}, \quad \widetilde{\xi}_2(\tau) = \frac{-1+\sqrt{1+e^{2\tau}}}{1+\sqrt{1+e^{2\tau}}}$$

which are admissible curves and call $N_{\theta,\widetilde{\xi}_1}$*,* $N_{\theta,\widetilde{\xi}_2}$ *the associated birth-and-death processes as in [BO04]. Then, for* $t \geq 0$*, we have*

$$\left(N_{\theta,\widetilde{\xi}_1}\left(\frac{1}{2}\log(1-e^{-t})\right),\ t \geq 0\right) \stackrel{d}{=} (X_t,\ t \geq 0)$$

and

$$\left(N_{\theta,\widetilde{\xi}_2}\left(\frac{1}{2}\log(t+t^2)\right),\ t \geq 0\right) \stackrel{d}{=} (Y_t,\ t \geq 0)\,.$$

Now, the partitions-valued processes $\Lambda_{N,N+\theta-1,\widetilde{\xi}_i}$ *defined in [BO04] are related to* $\mathbf{X}$ *and* $\mathbf{Y}$ *by the same time-change*

$$\left(\Lambda'_{N,N+\theta-1,\widetilde{\xi}_1}\left(\frac{1}{2}\log(1-e^{-t})\right),\ t \geq 0\right) \stackrel{d}{=} \left(\mathbf{X}^h_t,\ t \geq 0\right)$$

and

$$\left(\Lambda'_{N,N+\theta-1,\widetilde{\xi}_2}\left(\frac{1}{2}\log(t+t^2)\right),\ t \geq 0\right) \stackrel{d}{=} \left(\mathbf{Y}^h_t,\ t \geq 0\right),$$

where λ' *is defined by* $\lambda'_i = \lambda_i + N - i$ *for a partition* $\lambda_1 \geq \cdots \geq \lambda_N$ *and* $\mathbf{X}^h$*,* $\mathbf{Y}^h$ *have the* $\mathbb{P}^h$*-law of* $\mathbf{X}$*,* $\mathbf{Y}$*.*

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Chapitre 7

The RSK algorithm with exchangeable data

Abstract : On the one hand, we show that the shape evolution of the tableaux obtained by applying the RSK algorithm to an infinite exchangeable word is Markovian. On the other hand, we relate this shape evolution to the conditioning of the walk driven by another infinite exchangeable word. A necessary and sufficient condition is given for Pitman's $2M - X$ theorem to hold in this context. The example of Polya's urn is discussed as well as a partial version of Roger's result (characterizing diffusions X such that $2M - X$ is a diffusion) in this discrete multi-dimensional context.

7.1 Introduction

Suppose ξ is the simple symmetric random walk on $\mathbb{Z}$ starting at 0 and $\overline{\xi}$ is its past maximum process, $\overline{\xi}(n) = \max\{\xi(m),\ 0 \leq m \leq n\}$. Then, a discrete version of Pitman's theorem states two things : first, $2\overline{\xi} - \xi$ is a Markov chain and second, $2\overline{\xi} - \xi$ has the law of ξ conditioned to stay non-negative forever. This theorem dates back to [Pit75] and, since then, there has been an intensive literature concerning its reverberations and refinements in various contexts (see, for example, [RP81], [HMO01], [Ber92], [Bia94], [MY99a], [MY99b]).

Recent works ([OY02], [O'C03b], [O'C03a], [BJ02], [BBO04]) have extended the result to a multi-dimensional setting. The RSK correspondence is a combinatorial algorithm which plays a key-role in these discussions and provides a functional Φ on paths which is the relevant generalisation of the one-dimensional transform $\xi \to 2\overline{\xi} - \xi$.

The main result of our work is that, when X is the type of an exchangeable random word, the first part of Pitman's theorem still holds ($\Phi(X)$ is a Markov chain). We establish a necessary and sufficient condition for the second part of Pitman's theorem to be true in this case. This condition appears to be very special and rarely verified.

The example of Polya's urn is mentioned in connection with Yule branching processes (linear pure birth processes). We also discuss a partial converse of Pitman's theorem looking for all Markov chains X such that $\Phi(X)$ still has the Markov property.

7.2 Some preliminary combinatorics

In this section we recall some definitions and properties of integer partitons, tableaux, the RSK algorithm and Schur functions. The exposition here very closely follows that of [O'C03a] (with kind permission of the author). For more detailed accounts, see the books by Fulton [Ful97], Stanley [Sta99] and Macdonald [Mac79].

7.2.1 Words, integer partitions and tableaux

$[k]$ is the *alphabet* $\{1,2,\ldots,k\}$. A *word* $w=(w_1,\ldots,w_n)$ with n *letters* from $[k]$ is an element of $[k]^n$. If $\alpha_i = |\{j;\ w_j = i\}|$, the vector $\alpha\in\mathbb{N}^k$ will be called the *type* of w and written $\alpha=\mathrm{type}(w)$. If $(e_1,\ldots,e_k)$ is the canonical basis of $\mathbb{R}^k$, then $\alpha = e_{w_1}+\cdots+e_{w_n}$. It is convenient to write $|\alpha| = \sum_i \alpha_i = n$.

Let $\mathcal{P}$ denote the set of integer partitions

$$\{\lambda_1 \geq \lambda_2 \geq \cdots \geq 0 :\ |\lambda| = \sum_i \lambda_i < \infty\}.$$

If $|\lambda| = n$, we write $\lambda \vdash n$. The *parts* of λ are its non-zero components. It will be convenient to identify the set of integer partitions, with at most k parts, with the set

$$\Omega = \{\alpha\in\mathbb{N}^k \,|\, \alpha_1\geq\cdots\geq\alpha_k\}\,.$$

In this identification, the empty partition ϕ corresponds to the origin $0\in\mathbb{N}^k$.

The *diagram* of a partition λ is a left-justified array with λ_i boxes in the i-th row.

Call $\mathcal{T}_k$ the set of *tableaux* with entries from $[k]$, ie of diagrams filled in with numbers from $[k]$ in such a way that the entries are weakly increasing from left to right along rows, and strictly increasing down the columns. If $T\in\mathcal{T}_k$, its shape, denoted by $\mathrm{sh}\,T$, is the partition corresponding to the diagram of T and $\mathrm{type}(T)$ is the vector $\alpha\in\mathbb{N}^k$ where α_i be the number of i's in T. Elements of $\mathcal{T}_k$ are sometimes called *semistandard tableaux.*

A tableau with shape $\lambda\vdash n$ is *standard* if its entries (from $[n]$) are distinct. Let $\mathcal{S}_n$ denote the set of standard tableaux, with entries from $[n]$ and let f^λ denote the number of standard tableaux with shape λ. .

For $\alpha,\beta\in\mathbb{N}^k$, we write $\alpha\nearrow\beta$ when $\beta-\alpha\in\{e_1,\ldots,e_k\}$. The knowledge of a word $w\in[k]^n$ is equivalent to the knowledge of the sequence $0\nearrow\alpha^1\nearrow\cdots\nearrow\alpha^n$ where $\alpha^i = \mathrm{type}(w_1,\ldots,w_i)$. We denote by $T: w\to(\alpha^1,\ldots,\alpha^n)$ the induced bijection.

For integer partitions, $\phi = \lambda^0 \nearrow \lambda^1 \nearrow \cdots \nearrow \lambda^n = \lambda$ means that the diagram of λ^i is obtained from that of λ^{i-1} by adding a single box. If $S \in \mathcal{S}_n$, we can define integer partitions $\lambda^1, \ldots, \lambda^n$ as follows : λ^n is the shape of S, λ^{n-1} is the shape of the tableau obtained from S by removing the box containing n, and so on. This procedure gives a bijection $L : S \to (\lambda^1, \ldots, \lambda^n)$ between standard tableaux S with shape $\lambda \vdash n$ and sequences

$$\phi \nearrow \lambda^1 \nearrow \cdots \nearrow \lambda^n = \lambda.$$

7.2.2 The Robinson-Schensted correspondence

The Robinson-Schensted correspondence is a bijective mapping from the set of 'words' $[k]^n$ to the set

$$\{(P, Q) \in \mathcal{T}_k \times \mathcal{S}_n : \ \mathrm{sh}\, P = \mathrm{sh}\, Q\}.$$

Suppose $T \in \mathcal{T}_k$ and $i \in [k]$. We define a new tableau by inserting i in T as follows. If i is at least as large as all the entries in the first row of T, simply add a box labelled i to the end of the first row of T. Otherwise, browsing the entries of the first row from left to right, we replace the first number, say j, which is strictly larger than i by i. Then we repeat the same procedure to insert j in the second row, and so on. The tableau $T \leftarrow i$ we eventually obtain by this *row-insertion* operation has the entries of T together with i.

The Robinson-Schensted mapping is now defined as follows. Let (P, Q) denote the image of a word $w = w_1 \ldots w_n \in [k]^n$. Let $P^{(1)}$ be the tableau with the single entry w_1 and, for $m < n$, let $P^{(m+1)} = P^{(m)} \leftarrow w_{m+1}$. Then $P = P^{(n)}$ and Q is the standard tableau corresponding to the nested sequence $\phi \nearrow \mathrm{sh}\, P^{(1)} \nearrow \cdots \nearrow \mathrm{sh}\, P^{(n)}$. We call RSK this algorithm (K stands for Knuth who defined an extension of it to integer matrices instead of words) and we write $(P, Q) = \mathcal{RSK}(w)$.

7.2.3 Schur functions

Set $\delta = (k-1, k-2, \ldots, 0) \in \Omega$. For $\mu \in \Omega$ and a set of variables $x = (x_1, \ldots, x_k)$, set

$$a_\mu(x) = \det \left(x_i^{\mu_j}\right)_{1 \le i,j \le k}. \tag{7.1}$$

Then $a_\delta(x) = \det \left(x_i^{k-j}\right)_{1 \le i,j \le k} = \prod_{i<j\le k}(x_i - x_j)$ is the Vandermonde function. For $\lambda \in \Omega$, the Schur function s_λ is defined by

$$s_\lambda = a_{\lambda+\delta}/a_\delta.$$

There is an equivalent combinatorial definition given in terms of tableaux. For $\alpha \in \mathbb{N}^k$ we write $x^\alpha = x_1^{\alpha_1} \cdots x_k^{\alpha_k}$. Then

$$s_\lambda(x) = \sum x^{\mathrm{type}\ (T)}, \tag{7.2}$$

where the sum is over all tableaux with shape λ and entries from $[k]$. Alternatively, the *Kostka number* $K_{\lambda\alpha}$ is the number of tableaux with shape λ and type α and we can write

$$s_\lambda(x) = \sum_\alpha K_{\lambda\alpha} x^\alpha. \tag{7.3}$$

Schur functions have the important property that they are symmetric.

7.3 The shape process

7.3.1 Markov property of the shape evolution

We will consider a sequence $\eta = (\eta_n)_{n\geq 1}$ of random variables with values in $[k]$. $X = \{X(n)\,,, n \geq 0\}$ will be the “type process” of the infinite word η :

$$X(0) = 0 \quad \text{and} \quad X(n) = \text{type}(\eta_1, \ldots, \eta_n) = e_{\eta_1} + \cdots + e_{\eta_n} \text{ for } n \geq 1. \tag{7.4}$$

In words, X takes values in $\mathbb{N}^k$ and its ith coordinate increases by one at time n if $\eta_n = i$: $X_i(n) = |\{m \leq n \,|\, \eta_m = i\}|$. We remark that : $|X(n)| = n$.

Our crucial assumption is the exchangeability of the sequence $\eta = (\eta_n)_{n\geq 1}$. This property is equivalent to the fact that, for any word $w = (w_1, \ldots, w_n)$ with letters in $[k]$, $\mathbb{P}[\eta_1 = w_1, \ldots, \eta_n = w_n]$ just depends on w through its type. We will denote by q the function such that :

$$\mathbb{P}[\eta_1 = w_1, \ldots, \eta_n = w_n] = q(\text{type}(w))\,.$$

If $0 \nearrow \alpha^1 \nearrow \cdots \nearrow \alpha^n$, recalling the bijection T, we can compute :

$$\begin{aligned} &\mathbb{P}[X(1) = \alpha^1, \ldots, X(n) = \alpha^n] = \mathbb{P}[(\eta_1, \ldots, \eta_n) = T^{-1}(\alpha^1, \ldots, \alpha^n)] \\ &= q[\text{ type}(T^{-1}(\alpha^1, \ldots, \alpha^n))\,] = q(\alpha^n). \end{aligned} \tag{7.5}$$

This proves that X is a Markov chain on $\mathbb{N}^k$ with transitions given by :

$$P_X(\alpha, \beta) = \frac{q(\beta)}{q(\alpha)}\, \mathbf{1}_{\alpha\nearrow\beta}. \tag{7.6}$$

The fixed-time marginals of X can be expressed :

$$\mathbb{P}[X(n) = \alpha] = \sum_{w,\, \text{type}(w)=\alpha} q(\alpha) = \binom{n}{\alpha}\, q(\alpha)\, \mathbf{1}_{|\alpha|=n}.$$

Then let (R^n, S^n) be the image of the word $(\eta_1, \ldots, \eta_n)$ by the Robinson-Schensted correspondance and define $\widetilde{X}(n)$ to be the common shape of R^n and S^n.

Theorem 7.3.1. *$\widetilde{X}$ is a Markov chain on the set of partitions with at most k parts and its transition probabilities are given by :*

$$P_{\widetilde{X}}(\mu,\lambda) = \frac{f(\lambda)}{f(\mu)}\,\mathbf{1}_{\mu\nearrow\lambda}, \tag{7.7}$$

where the function f is defined on partitions with at most k parts by

$$f(\lambda) = \sum_{R\in\mathcal{T}_k\,,\,\mathrm{sh}R=\lambda} q(\mathrm{type}(R)) = \sum_{\alpha} K_{\lambda\alpha}\,q(\alpha). \tag{7.8}$$

Proof. Let $(R,S)\in\mathcal{T}_k\times\mathcal{S}_n$. Then

$$\begin{aligned}&\mathbb{P}[(R^{(n)},S^{(n)})=(R,S)] = \mathbb{P}[(\eta_1,\dots,\eta_n)=\mathcal{RSK}^{-1}(R,S)]\\ &= q(\mathrm{type}(\mathcal{RSK}^{-1}(R,S)))\,\mathbf{1}_{\mathrm{sh}R=\mathrm{sh}S} = q(\mathrm{type}(R))\,\mathbf{1}_{\mathrm{sh}R=\mathrm{sh}S}.\end{aligned}$$

Suppose $\phi\nearrow\lambda^1\nearrow\cdots\nearrow\lambda^n$. Recalling the bijection L introduced at the end of section 7.2.1 and defining $S=L^{-1}(\lambda^1,\dots,\lambda^n)$:

$$\begin{aligned}&\mathbb{P}[N(1)=\lambda^1,\dots,N(n)=\lambda^n] = \mathbb{P}[S^n=S]\\ &= \textstyle\sum_{\mathrm{sh}R=\lambda^n}\mathbb{P}[(R^n,S^n)=(R,S)] = \sum_{\mathrm{sh}R=\lambda^n} q(\mathrm{type}(R)) = f(\lambda^n).\end{aligned} \tag{7.9}$$

Equality (7.9) proves the theorem. ☐

In [O'C03b], O'Connell gave explicit formulae for the shape of the tableaux obtained by applying RSK with *column* insertion to a word. We recall here how those formulae are expressed when considering RSK with *row* insertion. Let Λ_k be the paths $x:\mathbb{N}\to\mathbb{N}^k$ such that $x(0)=0$ and $x(n)-x(n-1)\in\{0,e_1,\dots,e_k\}$ for $n>0$. For $x_1,x_2\in\Lambda_1$, we define $x_1\vartriangle x_2\in\Lambda_1$ and $x_1\triangledown x_2\in\Lambda_1$ by

$$(x_1\vartriangle x_2)(n) = \min_{0\le m\le n}[x_1(m)+x_2(n)-x_2(m)] \tag{7.10}$$

$$(x_1\triangledown x_2)(n) = \max_{0\le m\le n}[x_1(m)+x_2(n)-x_2(m)]. \tag{7.11}$$

Those operations are not associative and, if not specified, their composition has to be read from left to right. For instance, $x_1\vartriangle x_2\vartriangle x_3$ means $(x_1\vartriangle x_2)\vartriangle x_3$. We then define $F^k:\Lambda_k\to\Lambda_k$ by

$$F^2(x) = (x_1\triangledown x_2, x_2\vartriangle x_1) \tag{7.12}$$

and

$$F^k(x) = \left(x_1\triangledown x_2\triangledown\dots\triangledown x_k, F^{k-1}\left(\tau^k(x)\right)\right), \tag{7.13}$$

where

$$\tau^k(x) = (x_2\vartriangle x_1, x_3\vartriangle(x_1\triangledown x_2),\dots,x_k\vartriangle(x_1\triangledown\dots\triangledown x_{k-1})). \tag{7.14}$$

If $x(m)=\mathrm{type}(w_1,\dots,w_m)$ and $\mathcal{RSK}$ denotes the RSK algorithm with *row* insertion, then

$$F^k(x)(n) = \mathrm{sh}\left(\mathcal{RSK}(w_1,\dots,w_n)\right). \tag{7.15}$$

Remark 7.3.1. *The functional G^k described in [O'C03b] is such that $G^k(x')(n)^* = \mathrm{sh}\,(\mathcal{RSK}'(w'_1,\ldots,w'_n))$ where $\mathcal{RSK}'$ denotes the RSK algorithm with column insertion, $x'(i) = \mathrm{type}\,(w'_1,\ldots,w'_i)$ and $y^* = (y_{k-i+1})_{1\le i\le k}$. Definitions of G^k and F^k differ by the inversion of roles of up-triangles and down-triangles, which reflects the difference between row and column insertions. The relation between the two functionals is as follows. Fix $n \ge 0$ and set $x'(m) = x(n) - x(n-m)$ for $0 \le m \le n$. Then,*

$$F^k(x)(n) = G^k(x')(n)^*. \tag{7.16}$$

Seeing that $x'(m) = \mathrm{type}(w'_1,\ldots,w'_m)$ with $w' = (w_n,\ldots,w_1)$, (7.16) is consistent with the fact that the R-tableaux of $\mathcal{RSK}(w)$ and $\mathcal{RSK}'(w')$ coincide.

Corollary 7.3.2. *If X is the type process of an infinite exchangeable word with letters in $[k]$ then $F^k(X)$ is a Markov chain on the set Ω. In particular, when $k = 2$, $\xi = X_1 - X_2$ and $\overline{\xi}(n) = \max\{\xi(m)\,,\,0 \le m \le n\}$, then ξ and $2\overline{\xi} - \xi$ are Markov chains.*

Proof. The first part is a rephrasing of theorem 7.3.1 since $\widetilde{X} = F^k(X)$. The second part consists in noticing that the Markov property of $(F_1^k(X), F_2^k(X))$ implies that of $F_1^k(X) - F_2^k(X) = 2\overline{\xi} - \xi$, since $(F_1^k(X) + F_2^k(X))(n) = n$. □

Remark 7.3.2. *In general, ξ and $2\overline{\xi} - \xi$ are not time-homogeneous.*

Remark 7.3.3. *When X is just a random walk, ξ is the simple symmetric one-dimensional random walk and the last part of the corollary is the first part of the classical Pitman's theorem in this discrete context.*

Remark 7.3.4. *We can easily express the fixed-time marginals of N by summing (7.9) over $S \in \mathcal{S}_n$ with shape λ :*

$$\mathbb{P}[N(n) = \lambda] = f^\lambda\, f(\lambda)\, \mathbf{1}_{\lambda\vdash n}. \tag{7.17}$$

Remark 7.3.5. *It is also possible to compute the following conditional expectation :*

$$\Lambda(\lambda,\alpha) = \mathbb{P}[X(n) = \alpha \mid N(m), m \le n\,; N(n) = \lambda] = \frac{K_{\lambda\alpha} q(\alpha)}{f(\lambda)} \tag{7.18}$$

This can be checked by observing the equality between the following sigma-algebras : $\sigma(N(m), m \le n) = \sigma(S^n)$.

Remark 7.3.6. *The following intertwining result holds :*

$$P_{\widetilde{X}}\Lambda = \Lambda P_X. \tag{7.19}$$

This is standard in the context of Markov functions (see, for example, [RP81] and [CPY98]).

Example 7.3.1. *The fundamental example is of course a sequence η of iid random variables of law p : $\mathbb{P}[\eta_1 = l] = p_l$ for $1 \leq l \leq k$. The process $X = Z_p$ defined by (7.4) is just a random walk and its behaviour under the Robinson-Schensted algorithm is studied by O'Connell ([O'C03b], [O'C03a]). In this case,*

- $q(\alpha) = p_1^{\alpha_1} \dots p_k^{\alpha_k} = p^\alpha$
- f *is a Schur function :* $f(\lambda) = s_\lambda(p)$

We will write Z for the simple random walk corresponding to $p = (\frac{1}{k}, \dots, \frac{1}{k})$.

7.3.2 Consequences of De Finetti's theorem

Define the simplex $S_k = \{p \in [0,1]^k \ , \ \sum p_i = 1\}$ which can be identified with the set of probability measures on $[k]$. A version of De Finetti's theorem in this context states that $\frac{X(n)}{n}$ converges almost surely to a random variable $X_\infty \in S_k$ and that, conditionally on $X_\infty = p$, η is an iid sequence of law p. We will denote by $d\rho$ the law of X_∞. From this result, we can deduce that :

$$q(\alpha) = \int \mathbb{P}[\eta_1 = w_1, \dots, \eta_n = w_n | X_\infty = p] \, d\rho(p) = \int p^\alpha \, d\rho(p) = \mathbb{E}[X_\infty^\alpha] \tag{7.20}$$

and hence

$$f(\lambda) = \sum_\alpha K_{\lambda\alpha} \, \mathbb{E}[X_\infty^\alpha] = \mathbb{E}[s_\lambda(X_\infty)] = \int s_\lambda(p) \, d\rho(p) \, . \tag{7.21}$$

Thanks to the symmetry of Schur functions, it is clear that, for any permutation $\sigma \in \mathfrak{S}_k$, the words $(\eta_n)_{n\geq 1}$ and $(\sigma(\eta_n))_{n\geq 1}$ give rise to the same law of the shape evolution $\widetilde{X}$.

If $P_{Z,\Omega}(\mu, \lambda) = \frac{1}{k}\mathbf{1}_{\mu \nearrow \lambda}$ is the transition kernel of the simple random walk Z killed when exiting Ω, then (7.7) shows that

$$P_{\widetilde{X}}(\mu, \lambda) = \frac{k^{|\lambda|} f(\lambda)}{k^{|\mu|} f(\mu)} \, P_{Z,\Omega}(\mu, \lambda), \tag{7.22}$$

which makes $P_{\widetilde{X}}$ appear as a Doob-transform of $P_{Z,\Omega}$ by the function $\lambda \to \int s_\lambda(kp) \, d\rho(p)$. The analysis of the Martin boundary of $P_{Z,\Omega}$ (see [KOR02], [O'C03b] and [O'C03a]) indicates that

$$\mathbb{P}[\lim_{n\to\infty} \frac{\widetilde{X}(n)}{n} = \widetilde{p} \,|\, X_\infty = p] = 1,$$

where $\widetilde{p}$ is the decreasing reordering of p. Thus, we deduce the :

Proposition 7.3.3.

$$\lim_{n\to\infty} \frac{\widetilde{X}(n)}{n} = \widetilde{X}_\infty \ a.s. \, , \tag{7.23}$$

where $\widetilde{X}_\infty$ *has the law* $\widetilde{\rho}$ *of the order statistics (ie decreasing reordering) of* X_∞.

In particular, if X_∞ takes values in $\{p \in S_k\,;\, p_1 \geq p_2 \geq \cdots \geq p_k\}$, then $\widetilde{X}_\infty = X_\infty$.
In fact, (7.23) could be seen directly from the explicit formula of the functional F^k at a deterministic level. Indeed, we record here a property of the RSK algorithm :

Proposition 7.3.4. *If* $x(n)/n \to p$ *as* $n \to \infty$ *then* $F^k(x)(n)/n \to \widetilde{p}$*, where* $\widetilde{p}$ *is the decreasing reordering of* p.

Proof. We use the notation $x \rightsquigarrow p$ to mean that $x(n)/n \to p$ and $\max(q) = \max(q_1, \ldots, q_l)$ if $q \in \mathbb{R}^l$. First, it is an easy check that if $(x_1, x_2) \rightsquigarrow (p_1, p_2)$ then $(x_1 \triangledown x_2, x_2 \vartriangle x_1) \rightsquigarrow (\max(p_1, p_2), \min(p_1, p_2))$. Thus, we deduce that, if $x \rightsquigarrow p \in \mathbb{R}^k$, $\tau^k(x) \rightsquigarrow \theta^k(p) \in \mathbb{R}^{k-1}$, where τ^k is defined in (7.14) and

$$\theta^k(p) = \left(\min(p_2, p_1), \min\left(p_3, \max(p_1, p_2)\right), \ldots, \min\left(p_k, \max(p_1, \ldots, p_{k-1})\right)\right).$$

Now, we set $\delta(y) = (y_1 \triangledown y_2 \triangledown \ldots \triangledown y_l)$ for $y \in \Lambda_l$, which has the property that $\delta(y) \rightsquigarrow \max(q)$ if $y \rightsquigarrow q$. The definition of F^k is equivalent to

$$F_i^k(x) = \delta\left(\tau^{k-i+2} \circ \ldots \circ \tau^{k-1} \circ \tau^k(x)\right),$$

so that, if $x \rightsquigarrow p$,

$$F_i^k(x) \rightsquigarrow \max\left(\theta^{k-i+2} \circ \ldots \circ \theta^{k-1} \circ \theta^k(p)\right). \tag{7.24}$$

If we prove that

$$\mathcal{P} \;:\; \theta^k(p) \in \mathbb{R}^{k-1} \text{ is a permutation of the vector } (\widetilde{p}_2, \ldots, \widetilde{p}_k),$$

then, by iteration, $\theta^{k-i+2} \circ \ldots \circ \theta^{k-1} \circ \theta^k(p)$ is a permutation of $(\widetilde{p}_i, \ldots, \widetilde{p}_k)$ and

$$\max\left(\theta^{k-i+2} \circ \ldots \circ \theta^{k-1} \circ \theta^k(p)\right) = \widetilde{p}_i,$$

which, seeing (7.24), is the result. Let us show $\mathcal{P}$ when all the components of p are distinct, which is enough by continuity and density. Then, if $\widetilde{p}_1 = p_i$, we have

$$\min\left(p_{j+1}, \max(p_1, \ldots, p_j)\right) = \min(p_{j+1}, p_i) = p_i = \widetilde{p}_1 \text{ for} j \geq i$$

and

$$\min\left(p_{j+1}, \max(p_1, \ldots, p_j)\right) \leq \max(p_1, \ldots, p_j) < p_i = \widetilde{p}_1 \text{ for } j < i,$$

which proves that $\theta^k(p)$ does not contain $\widetilde{p}_1$. □

7.3.3 Polya urn example

Let us describe a version of Polya's urn with a parameter $a \in (\mathbb{R}_+)^k$. We will have k possible colours of balls, numbered from 1 to k. We say that the urn is of type $x \in \mathbb{N}^k$ if it contains x_i balls of colour i. The dynamics is the following : if the urn has type x at time n then we add a ball of colour i with probability $(a_i + x_i)/(|a| + |x|)$, so that its type at time $n+1$ becomes $x + e_i$. If we define $\xi(n)$ to be the type of the urn at time n, then :

$$\mathbb{P}[\xi(n) = x + e_i \mid \xi(n-1) = x] = \frac{a_i + x_i}{|a| + |x|}.$$

We will denote by $\mathbb{P}_x^a$ the law induced by the chain ξ starting at x, with parameter a. We define

$$X(n) = \xi(n) - \xi(0) = e_{\eta_1} + \cdots + e_{\eta_n},$$

where $\eta_i \in [k]$ to be the colour of the ball added between time $i-1$ and time i.

It is a well-known and fundamental fact that η is an exchangeable sequence and more precisely :

$$\mathbb{P}_x^a[(\eta_1, \ldots, \eta_n) = w] = q(\text{type}(w)) \quad \text{where} \quad q(\alpha) = \frac{(a+x)_\alpha}{(|a| + |x|)_{|\alpha|}}, \tag{7.25}$$

with the notation $(y)_\alpha = \prod_{i=1}^k (y_i)(y_i+1)\ldots(y_i+\alpha_i-1)$ for $y \in \mathbb{R}^k, \alpha \in \mathbb{N}^k$.

The law of X_∞ under $\mathbb{P}_x^a$ is known as Dirichlet-multinomial. It is described as folllows : let $\Gamma_1, \ldots, \Gamma_k$ be independent Gamma random variables with respective parameters $b_1 = a_1 + x_1, \ldots, b_k = a_k + x_k$, then

$$X_\infty \stackrel{d}{=} \frac{1}{\Gamma_1 + \cdots + \Gamma_k}(\Gamma_1, \ldots, \Gamma_k),$$

and the explicit expression of the law of the previous random variable is given by

$$\rho(dp) = \frac{\Gamma(b_1 + \ldots + b_k)}{\Gamma(b_1)\ldots\Gamma(b_k)} p_1^{b_1 - 1} \ldots p_k^{b_k - 1} \mathbf{1}_{p \in S_k} \, dp_1 \ldots dp_{k-1}. \tag{7.26}$$

Now take k independent continuous-time Yule processes $Y_1, \ldots, Y_k$ with branching rates 1, immigration rates $a_1, \ldots, a_k$ and starting from 0. The generator of $Y = (Y_1, \ldots, Y_k)$ is

$$Lf(y) = \sum_{i=1}^k (x_i + a_i)\left(f(x+e_i) - f(x)\right),$$

and the embedded discrete-time chain is the process X previously described. We can apply the RSK correspondance to the word whose letters record the coordinates of the successive jumps of Y, like in the discrete-time setting, and we denote by $\widetilde{Y}$ the resulting continuous-time shape process. If M_t is the number of jumps of Y before time t, the process M is a (one-dimensional) Yule process with branching rate 1 and immigration rate $|a| = \sum_i a_i$. We have $\widetilde{Y}_t = \widetilde{X}(M_t)$. We can define Φ^k, the continuous-time analogue of F^k, by the recursive equations (7.12) and (7.13) with the triangle operations now defined by

$$(f_1 \triangle f_2)(t) = \inf_{0\leq s\leq t}[f_1(s) + f_2(t) - f_2(s)] \tag{7.27}$$

$$(f_1 \triangle f_2)(t) = \sup_{0\leq s\leq t}[f_1(s) + f_2(t) - f_2(s)]. \tag{7.28}$$

Proposition 7.3.5. *$\widetilde{Y} = \Phi^k(Y)$ is a (continuous-time) Markov process with values in Ω.*

Proof. Once we notice that $M_t = |\widetilde{Y}_t|$, it is easy to describe the Markov evolution of $\widetilde{Y}$: if $\widetilde{Y}_t = \mu$ then $\widetilde{Y}_t$ waits for an exponential time of parameter $|a| + |\mu|$ and jumps to λ with probability $P_{\widetilde{X}}(\mu, \lambda)$. □

Remark 7.3.7. *For $k = 2$, Proposition 7.3.5 means that*

$$\left(Y_t^2 + \sup_{s\leq t}(Y_s^1 - Y_s^2), Y_t^1 - \sup_{s\leq t}(Y_s^1 - Y_s^2)\right)_{t\geq 0}$$

is a Markov process. However, $(Z_t = Y_t^1 - Y_t^2)_{t\geq 0}$ and $(2\sup_{s\leq t} Z_s - Z_t)_{t\geq 0}$ no longer are since $Y^1 + Y^2$ is not trivial, unlike in the discrete-time case.

7.4 The conditioned process

7.4.1 Presentation

Let us now consider the type process X' of another infinite exchangeable word η'. X'_∞ will be the almost-sure limit of $\frac{X'(n)}{n}$ and $d\rho'$ the law of X'_∞. In the sequel, for any process V, we will abbreviate the event $\{\forall\, n \geq 0,\, V(n) \in \Omega\}$ in $\{V \in \Omega\}$. Our goal is to condition the process X' on the event $\{X' \in \Omega\}$ that it stays in Ω forever. For this purpose, we recall the following result obtained in [O'C03b] and [O'C03a] about the random walk Z_p :

$$\mathbb{P}[Z_p \in \Omega \mid Z_p(0) = \lambda] = p^{-\lambda-\delta} a_{\lambda+\delta}(p) \mathbf{1}_W(p), \tag{7.29}$$

where $W = \{p \in S_k;\ p_1 > \cdots > p_k\}$.Thus, we can compute :

$$C_{\rho'} := \mathbb{P}[X' \in \Omega] = \int \mathbb{P}[Z_p \in \Omega]\, d\rho'(p) = \int p^{-\delta} a_\delta(p) \mathbf{1}_W(p)\, d\rho'(p) \tag{7.30}$$

We will suppose that $\rho'(W) > 0$, which makes sure that $\mathbb{P}[X' \in \Omega] > 0$ and allows us to perform the conditioning in the classical sense. More precisely, if $\phi \nearrow \lambda^1 \nearrow \cdots \nearrow \lambda^n$, we get,using (7.29) :

$$\begin{aligned}
&\mathbb{P}[X'(1) = \lambda^1, \ldots, X'(n) = \lambda^n \,; X' \in \Omega] \\
= &\int \mathbb{P}[Z_p(1) = \lambda^1, \ldots, Z_p(n) = \lambda^n \,; Z_p \in \Omega]\, d\rho'(p) \\
= &\int \mathbb{P}[Z_p(1) = \lambda^1, \ldots, Z_p(n) = \lambda^n]\, \mathbb{P}[Z_p \in \Omega \,|\, Z_p(0) = \lambda^n]\, d\rho'(p) \\
= &\int p^{-\delta} a_{\lambda^n+\delta}(p) \mathbf{1}_W(p)\, d\rho'(p).
\end{aligned}$$

Hence, the law of X' under the probability $\mathbb{P}[\,\cdot \mid X' \in \Omega]$ is the law of the Markov chain $\widehat{X'}$ whose transition probabilities $P_{\widehat{X'}}$ appear as Doob-transforms of $P_{Z,\Omega}$:

$$P_{\widehat{X'}}(\mu, \lambda) = \frac{g(\lambda)}{g(\mu)}\, P_{Z,\Omega}(\mu, \lambda), \tag{7.31}$$

where $g(\lambda) = k^{|\lambda|} \int p^{-\delta} a_{\lambda+\delta}(p) \mathbf{1}_W(p)\, d\rho'(p)$.

Recalling that $a_{\lambda+\delta} = a_\delta\, s_\lambda$, we obtain that

$$g(\lambda) = \int s_\lambda(kp)\, p^{-\delta} a_\delta(p) \mathbf{1}_{p_1 > \cdots > p_k}\, d\rho'(p).$$

The Martin boundary analysis of $P_{Z,\Omega}$ (see [KOR02], [O'C03b], [O'C03a]) proves the

Proposition 7.4.1.

$$\lim_{n\to\infty} \frac{\widehat{X'}(n)}{n} = \widehat{X'}_\infty \;\; a.s.\,, \tag{7.32}$$

where $\widehat{X'}_\infty$ has the law $\widehat{\rho'}$ given by

$$d\widehat{\rho'}(p) = \frac{1}{C_{\rho'}}\, p^{-\delta} a_\delta(p) \mathbf{1}_W(p)\, d\rho'(p). \tag{7.33}$$

Remark 7.4.1. *The "almost-sure" in proposition 7.4.1 is not precise since we have only defined the law of $\widehat{X'}$. We mean that we can find an almost-sure version of this convergence on some probability space.*

7.4.2 Connection with RSK and Pitman's theorem

Let X and X' be the processes defined in sections 7.3.1 and 7.4.1 with corresponding mixing measures ρ and ρ'.

Our previous analysis shows the

Proposition 7.4.2. *$\widetilde{X}$ has the same law as $\widehat{X'}$ if and only if $\widetilde{\rho} = \widehat{\rho'}$.*

Hence, starting from a process X' with corresponding measure ρ' with $\rho'(W) > 0$, Proposition 7.4.2 gives us a way of realizing the law of the conditioned process $\widehat{X'}$: construct an infinite word η with mixing measure $\frac{1}{C_{\rho'}} p^{-\delta} a_\delta(p) \mathbf{1}_W(p)\, d\rho'(p)$ and apply RSK to it, then the resulting shape process has the law of $\widehat{X'}$.

Corollary 7.4.3. *Suppose that $\rho(W) > 0$. Then $\widetilde{X}$ has the same law as $\widehat{X}$ if and only if*

$$\widetilde{\rho} = \widehat{\rho}. \tag{7.34}$$

In particular, if ρ is supported on W, (7.34) is verified if and only if ρ is supported on a level set of the function $p \mapsto p^{-\delta} a_\delta(p)$.

Proof. Just use the fact that, for a function h, $h(p)\, d\rho(p)$ is null if and only if $\rho\{h = 0\} = 1$. □

Example 7.4.1. *The case of a point mass $\rho = \delta_q$ with $q \in W$ is covered by corollary 7.4.3, which is the second part of Pitman's theorem for random walks.*

Example 7.4.2. *The Dirichlet-multinomial distribution ρ defined in (7.26) does not verify (7.34) so that $\widetilde{X}$ and $\widehat{X}$ don't have the same distribution. In fact, $\widetilde{X}$ does not have the law of any process $\widehat{X'}$ since $\int_W p^\delta a_\delta(p)^{-1} \widetilde{\rho}(dp) = \infty$. However, we can realize $\widehat{X}$ by applying RSK to an exchangeable word with mixing measure $\frac{1}{C_\rho} p^{-\delta} a_\delta(p) \mathbf{1}_W(p)\, d\rho(p)$ and looking at the induced shape process. The latter has the law of the composition of a Polya's urn conditioned to have forever more balls of colour 1 than of colour 2, more balls of colour 2 than of colour 3, etc.*

7.4.3 In search for a Rogers' type converse to Pitman's theorem

Take η an infinite random word, X its type process and $\widetilde{X}$ the shape process when applying RSK to η. We would like to characterize the possible laws of η such that X and $\widetilde{X}$ are (autonomous) Markov chains. This would be a multi-dimensional discrete analogue of Rogers' result classifying all diffusions Y such that $2\overline{Y} - Y$ is a diffusion (see [Rog81]). We are unable to solve the problem in full generality. However, there is a restriction of it which is fairly easy to deal with. First, define the function F_η by

$$\mathbb{P}[(\eta_1, \ldots, \eta_n) = w] = F_\eta(\mathcal{RSK}(w)). \tag{7.35}$$

Then, the same line of reasoning as in the proof of Theorem 7.3.1 shows that $\widetilde{X}$ is a Markov chain if and only if for all $\lambda \nearrow \lambda'$ the value

$$\frac{\sum_{\mathrm{sh} R' = \lambda'} F_\eta(R', S')}{\sum_{\mathrm{sh} R = \lambda} F_\eta(R, S)}$$

only depends on λ, λ' and not on the standard tableaux S, S' such that $\mathrm{sh}S = \lambda, \mathrm{sh}S' = \lambda'$.

Proposition 7.4.4. *If $F_\eta(R,S) = F_\eta(R)$ then $\widetilde{X}$ is a Markov chain. If X is also a Markov chain, then η is exchangeable.*

Proof. The first part is trivial from the previous discussion. We denote by $P(\alpha,\beta)$ the transition probabilities of the chain X, by $R(w)$ the R-tableau obtained by applying RSK to the word w and by (w,l) the word $(w_1,\ldots,w_n,l)$ if $w = (w_1,\ldots,w_n)$ and $l \in [k]$. Then, use the Markov property of X to get that

$$\begin{aligned} F_\eta(R(w,l)) &= \mathbb{P}[(\eta_1,\ldots,\eta_{n+1}) = (w,l)] \\ &= \mathbb{P}[(\eta_1,\ldots,\eta_n) = w]\, P(\mathrm{type}(w), \mathrm{type}(w) + e_l) \\ &= F_\eta(R(w))\, P\left(\mathrm{type}(w), \mathrm{type}(w) + e_l\right). \end{aligned}$$

Recalling that $R(w,l) = R(w) \leftarrow l$ (tableau obtained by row-insertion of l in $R(w)$), we have

$$\frac{F_\eta(R(w) \leftarrow l)}{F_\eta(R(w))} = P\left(\mathrm{type}(w), \mathrm{type}(w) + e_l\right),$$

proving that $\frac{F_\eta(R(w)\leftarrow l)}{F_\eta(R(w))} = \frac{F_\eta(R(w')\leftarrow l)}{F_\eta(R(w'))}$ if $\mathrm{type}(w') = \mathrm{type}(w)$. We can easily iterate this property for succesive insertions of letters $l_1,\ldots,l_j$:

$$\frac{F_\eta(R(w) \leftarrow l_1,\ldots,l_j)}{F_\eta(R(w))} = \frac{F_\eta(R(w') \leftarrow l_1,\ldots,l_j)}{F_\eta(R(w'))}.$$

Knowing that $\mathrm{type}(R(w)) = \mathrm{type}(w)$ and that RSK is onto, we can say that if the tableaux R, R' have the same type, then

$$\frac{F_\eta(R \leftarrow l_1,\ldots,l_j)}{F_\eta(R)} = \frac{F_\eta(R' \leftarrow l_1,\ldots,l_j)}{F_\eta(R')}. \tag{7.36}$$

Now, we need the following combinatorial

Lemma 7.4.1. *If the tableaux R, R' have the same type, there exist letters $l_1,\ldots,l_j$ such that $R \leftarrow l_1,\ldots,l_j = R' \leftarrow l_1,\ldots,l_j$.*

Proof. We proceed by induction on the cardinality of the alphabet. If $k=1$, $\mathrm{type}(R) = \mathrm{type}(R')$ implies $R = R'$. Suppose $k \geq 2$, $\alpha = \mathrm{type}(R) = \mathrm{type}(R')$ and define i (resp. i') to be the number of letters different from 1 in the 1st line of R (resp. R'). When we insert $m := \max(i,i')$ letters 1 in both tableaux R and R', we obtain two tableaux $\overline{R}$ and $\overline{R'}$ with a first line filled with $\alpha_1 + m$ letters 1. The other lines of $\overline{R}$ and $\overline{R'}$ form two tableaux $\widetilde{R}$ and $\widetilde{R}'$, not containing 1 and of the same type. By induction, there exist letters $l'_1,\ldots,l'_m$ in $\{2,3,\ldots,k\}$ such that $\widetilde{R} \leftarrow l'_1,\ldots,l'_m = \widetilde{R}' \leftarrow l'_1,\ldots,l'_m$. Therefore, inserting letters $l'_1, 1, l'_2, 1, \ldots, l'_m, 1$ makes the tableaux $\overline{R}$ and $\overline{R'}$ equal, which proves our claim. ∎

Then, if type(R) = type(R'), find letters $l_1, \ldots, l_j$ as in Lemma 7.4.1 and use equation (7.36) to get $F_\eta(R) = F_\eta(R')$. This shows that $\mathbb{P}[(\eta_1, \ldots, \eta_n) = w]$ just depends on the type of w, which concludes our proof that η is exchangeable. □

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Troisième partie

Matrix-valued diffusion processes

Chapitre 8

Some properties of the Wishart processes and a matrix extension of the Hartman-Watson law

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(Dedicated to Gérard Letac on the occasion of his retirement,
and to Marie-France Bru who started the whole thing...)

Abstract : The aim of this paper is to discuss for Wishart processes some properties which are analogues of the corresponding well-known ones for Bessel processes. In fact, we mainly concentrate on the local absolute continuity relationship between the laws of Wishart processes with different dimensions, a property which, in the case of Bessel processes, has proven to play a rather important role in a number of applications.

Key words : Bessel processes, Wishart processes, Time inversion, Hartman-Watson distributions.
Mathematics Subject Classification (2000) : 60J60 - 60J65 - 15A52

8.1 Introduction and main results

(1.0) To begin with, we introduce some notations concerning sets of matrices :

- $M_{n,m}(\mathbb{R}), M_{n,m}(\mathbb{C})$: the set of $n \times m$ real and complex matrices
- $\mathcal{S}_m(\mathbb{R}), \mathcal{S}_m(\mathbb{C})$: the set of $m \times m$ real and complex symmetric (not self-adjoint) matrices
- $\mathcal{S}_m^+$: the set of $m \times m$ real non-negative definite matrices
- $\widetilde{\mathcal{S}}_m^+$: the set of $m \times m$ real strictly positive definite matrices
- For $A \in M_{n,m}(\mathbb{R})$, A' denotes its transpose. Note that $\widehat{A} \stackrel{\text{def}}{=} A'A \in \mathcal{S}_m^+$.

(1.1) The present paper constitutes a modest contribution to the studies of matrix valued diffusions which are being undertaken in recent years, due to the growing interest in random matrices; see O'Connell [O'C03] for some recent survey. More precisely, we engage here in finding some analogues for Wishart processes of certain important properties for squared Bessel processes, which we now recall (for some similar efforts concerning the Bessel processes, see [Yor01], pp.64–67 and [GY93]).

(1.a) Definition of BESQ processes For $x \geqq 0$ and $\delta \geqq 0$, the stochastic differential equation

$$dX_t = 2\sqrt{X_t}\, dB_t + \delta\, dt, \quad X_0 = x, \tag{8.1}$$

with the constraint $X_t \geqq 0$ admits one and only one solution, i.e., (8.1) enjoys pathwise uniqueness. The process is called a squared Bessel process, denoted as BESQ(δ), and its distribution on the canonical space $C(\mathbb{R}_+, \mathbb{R}_+)$ is denoted by Q_x^δ, where, abusing the notation, we shall still denote the process of coordinates by X_t, $t \geqq 0$, and its filtration by $\mathcal{X}_t = \sigma\{X_s, s \leqq t\}$.

The family $\{Q_x^\delta\}_{\delta \geqq 0, x \geqq 0}$ enjoys a number of remarkable properties, among which

(1.b) Additivity property of BESQ laws We have

$$Q_x^\delta * Q_{x'}^{\delta'} = Q_{x+x'}^{\delta+\delta'} \tag{8.2}$$

for every $\delta, \delta', x, x' \geqq 0$. This property was found by Shiga-Watanabe [SW73] and considered by Pitman-Yor [PY82] who established a Lévy-Khintchine type representation of (each of) the infinitely divisible Q_x^δ's.

(1.c) Local absolute continuity property Writing $\delta = 2(1+\nu)$, with $\nu \geqq -1$, and $Q_x^\delta = Q_x^{(\nu)}$, there is the relationship : for $\nu \geqq 0$,

$$Q_x^{(\nu)}|_{\mathcal{X}_t} = \left(\frac{X_t}{x}\right)^{\nu/2} \exp\left(-\frac{\nu^2}{2}\int_0^t \frac{ds}{X_s}\right) \cdot Q_x^{(0)}|_{\mathcal{X}_t}, \tag{8.3}$$

from which we can deduce that the $Q_x^{(0)}$-conditional law of $\int_0^t (X_s)^{-1} ds$ given $X_t = y$ is the Hartman-Watson distribution $\eta_r(du)$, $r > 0$, $u > 0$. It is characterized by

$$\int_0^\infty \exp\left(-\frac{\nu^2 u}{2}\right) \eta_r(du) = \frac{I_\nu(r)}{I_0(r)},$$

where I_ν denotes the usual modified Bessel function ; precisely, there is the following consequence of (8.3) : for $\nu \geqq 0$,

$$Q_x^{(0)}\left[\exp\left(-\frac{\nu^2}{2}\int_0^t \frac{ds}{X_s}\right)|X_t = y\right] = \frac{I_\nu(r)}{I_0(r)}, \tag{8.4}$$

where $r = \sqrt{xy}/t$, and more generally,

$$Q_x^{(\nu)}\left[\exp\left(-\frac{\mu^2}{2}\int_0^t \frac{ds}{X_s}\right)|X_t = y\right] = \frac{I_{\sqrt{\nu^2+\mu^2}}(r)}{I_\nu(r)}.$$

The relation (8.3) was obtained and exploited by Yor [Yor80] to yield, in particular, the distribution at time t of a continuous determination θ_t of the angular argument of planar Brownian motion, thus recovering previous calculations by Spitzer [Spi58], from which one may derive Spitzer's celebrated limit law for θ_t :

$$\frac{2\theta_t}{\ln(t)} \overset{\text{(law)}}{\longrightarrow} C_1 \quad \text{as} \;\; t \to \infty, \tag{8.5}$$

where C_1 denotes the standard Cauchy variable, with parameter 1. It is also known that

$$\frac{4}{(\ln(t))^2}\int_0^t \frac{ds}{X_s} \overset{\text{(law)}}{\longrightarrow} T_{(1/2)} \quad \text{as} \quad t \to \infty, \tag{8.6}$$

where $T_{(1/2)}$ denotes the standard stable (1/2) variable. We recall that

$$E[\exp(i\lambda C_1)] = E[\exp(-\frac{\lambda^2}{2}T_{(1/2)})] = \exp(-|\lambda|), \; \lambda \in \mathbb{R}.$$

The absolute continuity property (8.3) has been of some use in a number of problems, see, e.g., Kendall [Ken91] for the computation of a shape distribution for triangles, Geman-Yor [GY93] for the pricing of Asian options, Hirsch-Song [HS99] in connection with the flows of Bessel processes, and more recently by Werner [Wer04] who deduces the computation of Brownian intersection exponents also from the relationship (8.3).

(1.d) Time inversion Let X_t be a Q_x^δ distributed process and define $i(X)_t = t^2 X(1/t)$, then $i(X)$ is a generalized squared Bessel process with drift $\sqrt{x}$, starting from 0. (See [Wat75] and [PY81] for the definitions of generalized Bessel processes). As an application, Pitman and Yor [PY80] give a "forward" skew product representation for the d-dimensional Brownian motion with drift.

(1.e) Intertwining property If $Q_t^\delta(x, dy)$ denotes the semigroup of the BESQ(δ) process, there is the intertwining relation

$$Q_t^{\delta+\delta'}\Lambda_{\delta,\delta'} = \Lambda_{\delta,\delta'}Q_t^\delta, \tag{8.7}$$

where $\Lambda_{\delta,\delta'}$ denotes the multiplication operator associated with $\beta_{\delta/2,\delta'/2}$, a beta variable with parameter $(\delta/2, \delta'/2)$, i.e.,

$$\Lambda_{\delta,\delta'} f(x) = E[f(x\beta_{\delta/2,\delta'/2})],$$

for every Borel function $f : \mathbb{R}_+ \to \mathbb{R}_+$. The relation (8.7) may be proven purely in an analytical manner, but it may also be shown in a more probabilistic way, with the help of time inversion, using a realization of $X^{\delta+\delta'}$ as the sum $X^{\delta} + X^{\delta'}$ of two independent BESQ processes (see [CPY98] for details).

(1.2) With the help of the above presentation of the BESQ processes, it is not difficult to discuss and summarize the main results obtained so far by M.F. Bru ([Bru89a, Bru91]) concerning the family of Wishart processes, which take values in $\mathcal{S}_m^+$ for some $m \in \mathbb{N}$, to be fixed throughout the sequel.

For values of δ to be discussed later, WIS(δ, m, x) shall denote such a Wishart process with "dimension" δ, starting at x, to be defined as the solution of the following stochastic differential equation :

$$dX_t = \sqrt{X_t}\, dB_t + dB_t'\sqrt{X_t} + \delta I_m\, dt, \quad X_0 = x, \tag{8.8}$$

where $\{B_t, t \geqq 0\}$ is an $m \times m$ Brownian matrix whose components are independent one-dimensional Brownian motions, and I_m is the identity matrix in $M_{m,m}(\mathbb{R})$. We denote the distribution of WIS(δ, m, x) on $C(\mathbb{R}_+, \mathcal{S}_m^+)$ by $\mathbf{Q}_x^{\delta}$.

Assume that $x \in \mathcal{S}_m^+$ and that x has distinct eigenvalues, which we denote by $\lambda_1(0) > \cdots > \lambda_m(0) \geqq 0$. Then, M.F. Bru [Bru91] has shown the following

Theorem 8.1.1. (i) *If $\delta \in (m-1, m+1)$, then* (8.8) *has a unique solution in $\mathcal{S}_m^+$ in the sense of probability law.*
(ii) *If $\delta \geqq m+1$, then* (8.8) *has a unique strong solution in $\tilde{\mathcal{S}}_m^+$.*
(iii) *The eigenvalue process $\{\lambda_i(t), t \geqq 0, 1 \leq i \leq m\}$ never collides, that is, almost surely,*

$$\lambda_1(t) > \cdots > \lambda_m(t) \geqq 0, \qquad \forall t > 0.$$

Moreover, if $\delta \geqq m+1$, then $\lambda_m(t) > 0$ for all $t > 0$ almost surely and the eigenvalues satisfy the stochastic differential equation

$$\begin{aligned} d\lambda_i(t) &= 2\sqrt{\lambda_i(t)}\, d\beta_i(t) + \Big\{\delta + \sum_{k \neq i} \frac{\lambda_i(t) + \lambda_k(t)}{\lambda_i(t) - \lambda_k(t)}\Big\}\, dt, \quad i = 1, ..., m, \\ &= 2\sqrt{\lambda_i(t)}\, d\beta_i(t) + \Big\{\delta - m + 1 + 2\sum_{k \neq i} \frac{\lambda_i(t)}{\lambda_i(t) - \lambda_k(t)}\Big\}\, dt, \end{aligned} \tag{8.9}$$

where $\beta_1(t), ..., \beta_m(t)$ are independent Brownian motions.
(iv) *If $\delta \geqq m+1$, then*

$$d(\det(X_t)) = 2\det(X_t)\sqrt{\mathrm{tr}(X_t^{-1})}\, d\beta(t) + (\delta - m + 1)\det(X_t)\mathrm{tr}(X_t^{-1})\, dt \tag{8.10}$$

and

$$d(\log(\det(X_t))) = 2\sqrt{\mathrm{tr}(X_t^{-1})}\, d\beta(t) + (\delta - m - 1)\mathrm{tr}(X_t^{-1})\, dt, \tag{8.11}$$

where $\beta = \{\beta(t), t \geqq 0\}$ *is a Brownian motion.*
(v) *For any* $\Theta \in \mathcal{S}_m^+$,

$$\begin{aligned} &\mathbf{Q}_x^\delta[\exp(-\mathrm{tr}(\Theta X_t))] \\ &\quad = (\det(I + 2t\Theta))^{-\delta/2}\exp(-\mathrm{tr}(x(I+2t\Theta)^{-1}\Theta)] \\ &\quad = \exp(-\mathrm{tr}(x/2t))(\det(I+2t\Theta))^{-\delta/2}\exp\big(\frac{1}{2t}\mathrm{tr}(x(I+2t\Theta)^{-1})\big). \end{aligned} \tag{8.12}$$

For the sake of clarity, we postpone the discussion of further properties of Wishart processes as presented in M.F. Bru [Bru91] to Section 8.2.
(1.3) We now present some of our main results and, in particular, the extension for Wishart processes of the absolute continuity property (8.3).

Theorem 8.1.2. *With the above notation, we have for* $\nu \geqq 0$ *:*

$$\mathbf{Q}_x^{m+1+2\nu}|_{\mathcal{F}_t} = \left(\frac{\det(X_t)}{\det(x)}\right)^{\nu/2}\exp\left(-\frac{\nu^2}{2}\int_0^t \mathrm{tr}(X_s^{-1})\, ds\right)\cdot \mathbf{Q}_x^{m+1}|_{\mathcal{F}_t}. \tag{8.13}$$

Just as in the case of squared Bessel processes, the semigroup of $\mathrm{WIS}(\delta, m, x)$ is explicitly known, and we deduce from Theorem 8.1.2 our main result in this paper :

Corollary 8.1.3. *Let* $\nu \geqq 0$. *Then we have*

$$\begin{aligned} \mathbf{Q}_x^{m+1}\left[\exp\left(-\frac{\nu^2}{2}\int_0^t \mathrm{tr}(X_s^{-1})\, ds\right)|X_t = y\right] &= \left(\frac{\det(x)}{\det(y)}\right)^{\nu/2}\frac{\mathbf{q}_t^{(\nu)}(x,y)}{\mathbf{q}_t^{(0)}(x,y)} \\ &= \frac{\mathbf{\Gamma}_m((m+1)/2)}{\mathbf{\Gamma}_m((m+1)/2+\nu)}(\det(z))^{\nu/2}\frac{{}_0\mathbf{F}_1((m+1)/2+\nu; z)}{{}_0\mathbf{F}_1((m+1)/2; z)} \\ &= \frac{\widetilde{\mathbf{I}}_\nu(z)}{\widetilde{\mathbf{I}}_0(z)} \end{aligned} \tag{8.14}$$

where $z = xy/4t^2$, $\mathbf{q}_t^{(\nu)}$ *denotes the transition probability of the Wishart process of dimension* $\delta = m+1+2\nu$, $\mathbf{\Gamma}_m$ *is the multivariate gamma function,* ${}_0\mathbf{F}_1$ *is a hypergeometric function (see the appendix for the definition of* $\mathbf{\Gamma}_m$ *and* ${}_0\mathbf{F}_1$ *) and* $\widetilde{\mathbf{I}}_\nu(z)$ *is the function defined by*

$$\widetilde{\mathbf{I}}_\nu(z) = \frac{(\det(z))^{\nu/2}}{\mathbf{\Gamma}_m((m+1)/2+\nu)}\, {}_0\mathbf{F}_1((m+1)/2+\nu; z). \tag{8.15}$$

Note that in the case $m=1$, $\widetilde{\mathbf{I}}_\nu(z)$ is related to the usual modified Bessel function $I_\nu(z)$ (see [Leb72]) by $\widetilde{\mathbf{I}}_\nu(z) = I_\nu(2z^{1/2})$. Clearly, formula (8.14) appears as a generalization of the result (8.4) for $m=1$.

Notation : In general, quantities related to Wishart processes will appear in boldface.

Proofs and extensions of (8.13), with two general dimensions instead of $m+1$ and $m+1+2\nu$, are given in Section 8.2.

As in the case of the Bessel processes, we obtain the absolute continuity relationship for the negative indexes in the following way.

Theorem 8.1.4. *Assume* $0<\nu<1$ *and let* T_0 *be the first hitting time of* 0 *for* $\{\det(X_t)\}$. *Then we have*

$$\begin{aligned} \mathbf{Q}_x^{m+1-2\nu}|_{\mathcal{F}_t \cap \{t<T_0\}} &= \left(\frac{\det(X_t)}{\det(x)}\right)^{-\nu/2} \exp\left(-\frac{\nu^2}{2}\int_0^t \mathrm{tr}(X_s^{-1})\,ds\right)\cdot \mathbf{Q}_x^{m+1}|_{\mathcal{F}_t} \\ &= \left(\frac{\det(X_t)}{\det(x)}\right)^{-\nu}\cdot \mathbf{Q}_x^{m+1+2\nu}|_{\mathcal{F}_t}. \end{aligned} \tag{8.16}$$

From formula (8.16) we may deduce the law of T_0 for $\mathrm{WIS}(m+1-2\nu,m,x)$, which will also be given in Section 8.2. In particular, we obtain :

Corollary 8.1.5. *For* $0<\nu<1$, *we have*

$$\mathbf{Q}_x^{(-\nu)}(T_0>t\,|\,X_t=y) = \left(\frac{\widetilde{\mathbf{I}}_\nu}{\widetilde{\mathbf{I}}_{-\nu}}\right)\left(\frac{xy}{4t^2}\right). \tag{8.17}$$

(1.4) In this paper, we also obtain some extension of the time inversion results for Bessel processes (see **(1.d)**). For this, we need to introduce Wishart processes with drift. For $\delta=n$ an integer, we define a Wishart process with drift $\widehat{\Theta}\equiv\Theta'\Theta$ as the process

$$X_t^\Theta = (B_t+\Theta t)'(B_t+\Theta t) \equiv \widehat{B_t+\Theta t},$$

where $\{B_s, s\geqq 0\}$ is an $n\times m$ Brownian matrix starting from 0 and $\Theta=(\Theta_{ij})\in M_{n,m}(\mathbb{R})$. Its law turns out to only depend on $\widehat{\Theta}=\Theta'\Theta$. In Section 8.3, we extend the definition of these processes to a non-integer dimension δ and we show that these processes are time-inversed Wishart processes.

8.2 Some properties of Wishart processes and proofs of theorems

8.2.1 First properties of Wishart processes

(2.a) Wishart processes of integral dimension In the case $\delta=n$ is an integer, $\mathrm{WIS}(n,m,x)$ is the law of the process $\{X_s=B_s'B_s\equiv\widehat{B}_s, s\geqq 0\}$, where $\{B_s\}$ is an

$n \times m$ Brownian matrix starting from B_0 with $\widehat{B}_0 = B_0' B_0 = x$.
(2.b) Transition function Let $\delta > m-1$. Formula (8.12) shows that the distribution of X_t for fixed t is the non-central Wishart distribution $W_m(\delta, tI_m, t^{-1}x_1)$ (Muirhead's notation), see Theorem 10.3.3 in Muirhead [Mui82]. The transition probability density $\mathbf{q}_\delta(t, x, dy)$ with respect to the Lebesgue measure $dy = \prod_{i \leqq j} dy_{ij}$ of the Wishart process $\{X_t\}$ is thus given by

$$\begin{aligned} &\mathbf{q}_\delta(t,x,y) \\ &= \frac{1}{(2t)^{\delta m/2}\mathbf{\Gamma}_m(\delta/2)} \exp\big(-\frac{1}{2t}\mathrm{tr}(x+y)\big)(\det(y))^{(\delta-m-1)/2}\, {}_0\mathbf{F}_1\big(\frac{\delta}{2}; \frac{xy}{4t^2}\big) \qquad (8.18) \\ &= \frac{1}{(2t)^{m(m+1)/2}} \exp\big(-\frac{1}{2t}\mathrm{tr}(x+y)\big) \left(\frac{\det(y)}{\det(x)}\right)^{(\delta-m-1)/4} \widetilde{\mathbf{I}}_{(\delta-m-1)/2}\big(\frac{xy}{4t^2}\big), \end{aligned}$$

where $\mathbf{\Gamma}_m$ is the multivariate gamma function, ${}_0\mathbf{F}_1$ is a hypergeometric function (see their definitions in the appendix) and $\widetilde{\mathbf{I}}_\nu(z)$ is the function defined by (8.15). The transition probability density $\mathbf{q}_\delta(t, x, y)$ may be continuously extended in x belonging to $\mathcal{S}_m^+$, and we can consider the Wishart processes starting from degenerate matrices. Indeed, the Wishart processes starting from 0 will play some role in the following. Note that

$$\mathbf{q}_\delta(t,0,y) = \frac{1}{(2t)^{\delta m/2}\mathbf{\Gamma}_m(\delta/2)} \exp\big(-\frac{1}{2t}\mathrm{tr}(y)\big)(\det(y))^{(\delta-m-1)/2}.$$

(2.c) Additivity property We have the following property (see [Bru91]) : If $\{X_t\}$ and $\{Y_t\}$ are two independent Wishart processes WIS(δ, m, x) and WIS(δ', m, y), then $\{X_t + Y_t\}$ is a Wishart process WIS$(\delta+\delta', m, x+y)$. Nevertheless, the laws $\mathbf{Q}_x^\delta$ of WIS(δ, m, x) are not infinitely divisible since the parameter δ cannot take all the positive values, in fact, δ needs to belong to the so-called Gindikin's ensemble $\Lambda_m = \{1, 2, ..., m-1\} \bigcup (m-1, \infty)$ (see Lévy [Lév48] for the Wishart distribution).
(2.d) The eigenvalue process The drift in the stochastic differential equation (8.9) giving the eigenvalues of the Wishart process is a repelling force between these eigenvalues (which may be thought as positions of particles) which prohibits collisions. We now discuss some other models of non colliding processes. In [KO01], König and O'Connell consider the eigenvalues of the Laguerre process (defined as in **(2.a)** replacing the Brownian motion B by a complex Brownian motion and the transpose by the adjoint for $n \geqq m$). Then, the eigenvalue process satisfies the same equation as (8.9) except that the drift is multiplied by "2". It is shown that this process evolves like m independent squared Bessel processes conditioned never to collide.

Gillet [Gil03] considers a stochastic differential equation for an m-dimensional process, called a watermelon, whose paths don't intersect. It turns out that this process corresponds to the square roots of the eigenvalues of a Laguerre process and then can be interpreted as the process obtained from m independent three dimensional Bessel processes conditioned to stay in the Weyl chamber $W = \{(x_1, x_2, \ldots, x_m); x_1 > x_2 > \ldots > x_m\}$

We also refer to Cépa-Lépingle [CL01] and Grabiner [Gra99] for other closely related studies about non-colliding particles.

We now study the filtration of the processes which appear in the density (8.13).

Proposition 8.2.1. (i) *Let* $\{\mathcal{D}_t, t \geqq 0\}$ *be the filtration generated by the process* $\{D_t = \det(X_t)\}$. *Then* $\{\mathcal{D}_t\}$ *is equal to the filtration generated by the eigenvalues* $\{\lambda_i(t), i = 1, \ldots, m, t \geqq 0\}$ *of the process* $\{X_t\}$. *Therefore, the density in* (8.13) *is* $\mathcal{D}_t$ *measurable.*
(ii) *Let* $\Lambda^{\delta}_{\bar{\lambda}}$ *the probability law of the eigenvalues* $(\lambda_i(t); i = 1, \ldots, m)$ *of a* $\mathrm{WIS}(\delta, m, x)$ *with* $\bar{\lambda}$ *the vector of the eigenvalues of* x; *i.e., the solution of* (8.9) *starting from* $\bar{\lambda}$. *Then, the absolute continuity relation* (8.13) *reads*

$$\Lambda^{m+1+2\nu}_{\bar{\lambda}}|_{\mathcal{D}_t} = \left(\frac{\prod_{i=1}^m \lambda_i(t)}{\prod_{i=1}^m \lambda_i(0)}\right)^{\nu/2} \exp\left(-\frac{\nu^2}{2}\int_0^t (\sum_{i=1}^m \frac{1}{\lambda_i(s)})\, ds\right) \cdot \Lambda^{m+1}_{\bar{\lambda}}|_{\mathcal{D}_t}.$$

Proof. (i) Denote by $L_t = \ln(D_t) = \sum_{i=1}^m \ln(\lambda_i(t))$. L_t is $\mathcal{D}_t$ measurable. According to equation (8.9), we have

$$\ln(\lambda_i(t)) = \frac{2}{\sqrt{\lambda_i(t)}} d\beta_i(t) + K_i(\lambda(t))dt$$

for a function K_i on $\mathbb{R}^m$ and

$$\langle L, L\rangle_t = 4\int_0^t \sum_{i=1}^m (\frac{1}{\lambda_i(s)})\, ds = 4\int_0^t \mathrm{tr}(X_s^{-1})\, ds,$$

which shows that $\mathrm{tr}(X_t^{-1}) = d\langle L, L\rangle_t/dt$ is $\mathcal{D}_t$ measurable.

Now, let us define $L_p(t) = \mathrm{tr}(X_t^{-p})$, $p \in \mathbb{N}$ with $L_0(t) \equiv L(t)$. It is easy to verify that

$$\frac{d}{dt}\langle L_p, L_q\rangle_t = L_{p+q+1}(t)$$

and therefore, it follows that all the processes $L_p(t) = \sum_{i=0}^n (\lambda_i(t))^{-p}$ are $\mathcal{D}_t$ measurable. Now, from the knowledge of all the processes L_p, $p \in \mathbb{N}$, we can recover the m-dimensional process $\{\lambda_i(t), i = 1, \ldots, m, t \geqq 0\}$.
(ii) We just write the density in terms of the eigenvalues. □

8.2.2 Girsanov formula

Here, after writing the Girsanov formula in our context, we prove Theorem 8.1.2, i.e., the absolute continuity relationship between the laws of Wishart processes of different dimensions. We also show that we may obtain, by using the Girsanov formula, a process which may be called a squared Ornstein-Uhlenbeck type Wishart process.

Let $\mathbf{Q}_x^\delta, x \in \widetilde{\mathcal{S}}_m^+, \delta > m-1$, be the probability law of WIS(δ, m, x) process $\{X_t, t \geqq 0\}$, which is considered as the unique solution of

$$dX_t = \sqrt{X_t}\, dB_t + dB_t'\sqrt{X_t} + \delta I_m dt, \quad X_0 = x, \tag{8.19}$$

where $\{B_t\}$ is an $m \times m$ Brownian matrix under $\mathbf{Q}_x^\delta$. We consider a predictable process $H = \{H_s\}$, valued in $\mathcal{S}_m$, such that

$$\mathcal{E}_t^H = \exp\left(\int_0^t \mathrm{tr}(H_s dB_s) - \frac{1}{2}\int_0^t \mathrm{tr}(H_s^2)\, ds\right)$$

is a martingale with respect to $\mathbf{Q}_x^\delta$ and denote by $\mathbf{Q}_x^{\delta,H}$ the probability measure such that

$$\mathbf{Q}_x^{\delta,H}|_{\mathcal{F}_t} = \mathcal{E}_t^H \cdot \mathbf{Q}_x^\delta|_{\mathcal{F}_t}, \tag{8.20}$$

where $\{\mathcal{F}_t\}$ is the natural filtration of $\{X_t\}$. Then the process $\{\beta_t\}$ given by

$$\beta_t = B_t - \int_0^t H_s\, ds$$

is a Brownian matrix under $\mathbf{Q}_x^{\delta,H}$ and $\{X_t\}$ is a solution of

$$dX_t = \sqrt{X_t}\, d\beta_t + d\beta_t'\sqrt{X_t} + (\sqrt{X_t}H_t + H_t\sqrt{X_t} + \delta I_m)\, dt. \tag{8.21}$$

We consider two special cases : $H_t = \nu X_t^{-1/2}$, $\nu > 0$, and $H_t = \lambda\sqrt{X_t}$, $\lambda \in \mathbb{R}$.

Remark 8.2.1. *Here is a slight generalization of* (8.20) *: let $\{H_s\}$ be a predictable process with values in $M_{n,m}(\mathbb{R})$ and $\{B_s\}$ be an $n \times m$ Brownian matrix under $\mathbf{P}$. Then, if $\mathbf{P}^H$ is given by*

$$d\mathbf{P}^H|_{\mathcal{F}_t} = \exp\left(\int_0^t \mathrm{tr}(H_s'\, dB_s) - \frac{1}{2}\int_0^t \mathrm{tr}(\widehat{H}_s) ds\right) \cdot d\mathbf{P}|_{\mathcal{F}_t},$$

$\beta_t = B_t - \int_0^t H_s ds$ is an $n \times m$ Brownian matrix under $\mathbf{P}^H$.

Case 1 Let $H_t = \nu X_t^{-1/2}$. Then the equation (8.21) becomes

$$dX_t = \sqrt{X_t}\, d\beta_t + d\beta_t'\sqrt{X_t} + (\delta + 2\nu)I_m dt,$$

which is the stochastic differential equation for a WIS$(\delta+2\nu, m, x)$ process. That is, we have obtained

$$\mathbf{Q}_x^{\delta+2\nu}|_{\mathcal{F}_t} = \exp\left(\nu\int_0^t \mathrm{tr}(X_s^{-1/2}\, dB_s) - \frac{\nu^2}{2}\int_0^t \mathrm{tr}(X_s^{-1}) ds\right) \cdot \mathbf{Q}_x^\delta|_{\mathcal{F}_t}. \tag{8.22}$$

We can write the stochastic integral on the right hand side in a simpler way when $\delta = m+1$ and thus obtain Theorem 8.1.2, as we now show.

Proof of Theorem 8.1.2. Developing the determinant of $y \in \widetilde{\mathcal{S}}_m^+$ in terms of its cofactors, we obtain $\nabla_y(\det(y)) = \det(y) y^{-1}$ and, hence,

$$\nabla_y(\log(\det(y))) = y^{-1}. \tag{8.23}$$

We know, from (8.11), that $\{\log(\det(X_t))\}$ is a local martingale when $\delta = m+1$. Moreover, by (8.23), we obtain from Itô's formula

$$\begin{aligned}\log(\det(X_t)) &= \log(\det(x)) + \int_0^t \mathrm{tr}(X_s^{-1}(\sqrt{X_s}dB_s + dB_s'\sqrt{X_s})') \\ &= \log(\det(x)) + 2\int_0^t \mathrm{tr}(X_s^{-1/2}dB_s).\end{aligned}$$

Hence, by (8.22), we obtain

$$\mathbf{Q}_x^{m+1+2\nu}|_{\mathcal{F}_t} = \left(\frac{\det(X_t)}{\det(x)}\right)^{\nu/2} \exp\left(-\frac{\nu^2}{2}\int_0^t \mathrm{tr}(X_s^{-1})\, ds\right) \cdot \mathbf{Q}_x^{m+1}|_{\mathcal{F}_t}. \qquad \square$$

Remark 8.2.2. *According to Theorem 8.1.2, we have the following absolute continuity relationship, for* $\delta = m+1+2\lambda$ *and* $\delta' = m+1+2\nu$, $\lambda, \nu \geqq 0$,

$$\mathbf{Q}_x^{\delta'}|_{\mathcal{F}_t} = \left(\frac{\det(X_t)}{\det(x)}\right)^{(\nu-\lambda)/2} \exp\left(-\frac{\nu^2-\lambda^2}{2}\int_0^t \mathrm{tr}(X_s^{-1})\, ds\right) \cdot \mathbf{Q}_x^{\delta}|_{\mathcal{F}_t}, \tag{8.24}$$

from which we deduce for $\alpha \in \mathbb{R}$

$$\mathbf{Q}_x^{\delta}\left[\left(\frac{\det(X_t)}{\det(x)}\right)^{\alpha} \exp\left(-\frac{\nu^2-\lambda^2}{2}\int_0^t \mathrm{tr}(X_s^{-1})\, ds\right)\right] = \mathbf{Q}_x^{\delta'}\left[\left(\frac{\det(X_t)}{\det(x)}\right)^{\alpha-(\nu-\lambda)/2}\right].$$

The moments of $\det(X_t)$ *are given by the following formula (see [Mui82] p. 447) :*

$$\mathbf{Q}_x^{\delta}[(\det(X_t))^s] = (2t)^{ms}\frac{\mathbf{\Gamma}_m(s+\delta/2)}{\mathbf{\Gamma}_m(\delta/2)}\ {}_1\mathbf{F}_1\big(-s; \frac{\delta}{2}; -\frac{x}{2t}\big).$$

For $x = 0$, *we have*

$$\mathbf{Q}_0^{\delta}[(\det(X_t))^s] = (2t)^{ms}\frac{\mathbf{\Gamma}_m(s+\delta/2)}{\mathbf{\Gamma}_m(\delta/2)} = (2t)^{ms}\frac{\prod_{i=1}^m \Gamma(s+\delta/2-(i-1)/2)}{\prod_{i=1}^m \Gamma(\delta/2-(i-1)/2)}$$

for $s > 0$, *which is the Mellin transform of the distribution of* $\det(X_t)$ *under* $\mathbf{Q}_0^{\delta}$. *Hence, letting* $Y_1, ..., Y_m$ *be independent gamma variables whose densities are given by*

$$\frac{1}{\Gamma(\delta/2-(i-1)/2)} e^{-\xi}\xi^{\delta/2-(i-1)/2-1}, \ \xi > 0, \ i = 1, ..., m,$$

we see that the distribution of $\det(X_t)$ *under* $\mathbf{Q}_0^{\delta}$ *coincides with that of* $(2t)^m Y_1 \cdots Y_m$. *This result is a consequence of Bartlett's decomposition (cf. [Mui82, Theorem 3.2.14]).*

Case 2 Let $H_t = \lambda\sqrt{X_t}, \lambda \in \mathbb{R}$. Then (8.21) becomes

$$dX_t = \sqrt{X_t}\, d\beta_t + d\beta_t'\sqrt{X_t} + (2\lambda X_t + \delta I_m)\, dt.$$

By (8.19), we obtain

$$d(\mathrm{tr}(X_t)) = 2\mathrm{tr}(\sqrt{X_t}\, dB_t) + m\delta\, dt$$

and

$$\int_0^t \mathrm{tr}(\sqrt{X_s}\, dB_s) = \frac{1}{2}(\mathrm{tr}(X_t) - \mathrm{tr}(x) - m\delta t).$$

Hence, from (8.20), we have obtained that the probability measure ${}^{\lambda}\mathbf{Q}_x^{\delta}$ given by

$${}^{\lambda}\mathbf{Q}_x^{\delta}|_{\mathcal{F}_t} = \exp\left(\frac{\lambda}{2}(\mathrm{tr}(X_t) - \mathrm{tr}(x) - m\delta t) - \frac{\lambda^2}{2}\int_0^t \mathrm{tr}(X_s)\, ds\right) \cdot \mathbf{Q}_x^{\delta}|_{\mathcal{F}_t} \tag{8.25}$$

is the probability law of the process given by

$$dX_t = (\sqrt{X_t}\, d\beta_t + d\beta_t'\sqrt{X_t}) + (2\lambda X_t + \delta I_m)dt, \quad X_0 = x, \tag{8.26}$$

for a Brownian matrix $\{\beta_t\}$ (under ${}^{\lambda}\mathbf{Q}_x^{\delta}$). See M.F. Bru [Bru91] for a study of squared Ornstein Uhlenbeck processes and related computations of Laplace transforms.

8.2.3 Generalized Hartman-Watson laws

We concentrate on the case $\delta \geqq m+1$ for a while and write $\delta = m+1+2\nu$. We denote by $\mathbf{q}_t^{(\nu)}(x,y)$ the transition probability density with respect to the Lebesgue measure of the generalized Wishart process (a solution to (8.8)) $\{X_t^{(\nu)}\}$ given by (8.18). Then, we have

$$\begin{aligned}\frac{\mathbf{q}_t^{(\nu)}(x,y)}{\mathbf{q}_t^{(0)}(x,y)} &= \frac{(2t)^{m(m+1)/2}\mathbf{\Gamma}_m((m+1)/2)}{(2t)^{m(m+1+2\nu)/2}\mathbf{\Gamma}_m((m+1)/2+\nu)}(\det(y))^{\nu} \\ &\quad\times \frac{{}_0\mathbf{F}_1((m+1)/2+\nu; xy/4t^2)}{{}_0\mathbf{F}_1((m+1)/2; xy/4t^2)} \\ &= \frac{\mathbf{\Gamma}_m((m+1)/2)}{\mathbf{\Gamma}_m((m+1)/2+\nu)}(\det\frac{y}{2t})^{\nu}\ \frac{{}_0\mathbf{F}_1((m+1)/2+\nu; xy/4t^2)}{{}_0\mathbf{F}_1((m+1)/2; xy/4t^2)}.\end{aligned}$$

Denoting the law of $\{X_t^{(\nu)}\}$ by $\mathbf{Q}_x^{(\nu)}$, we showed in the previous subsection

$$\frac{d\mathbf{Q}_x^{(\nu)}}{d\mathbf{Q}_x^{(0)}}|_{\mathcal{F}_t} = \left(\frac{\det(X_t)}{\det(x)}\right)^{\nu/2} \exp\left(-\frac{\nu^2}{2}\int_0^t \mathrm{tr}(X_u^{-1})\, du\right),$$

which yields

$$\frac{\mathbf{q}_t^{(\nu)}(x,y)}{\mathbf{q}_t^{(0)}(x,y)} = \left(\frac{\det(y)}{\det(x)}\right)^{\nu/2} \mathbf{Q}_x^{(0)}[\exp\left(-\frac{\nu^2}{2}\int_0^t \operatorname{tr}(X_u^{-1})\,du\right)|X_t = y]. \tag{8.27}$$

Therefore we obtain

$$\begin{aligned}&\mathbf{Q}_x^{(0)}\left[\exp\left(-\frac{\nu^2}{2}\int_0^t \operatorname{tr}(X_u^{-1})\,du\right)|X_t = y\right] \\ &\qquad = \frac{\mathbf{\Gamma}_m((m+1)/2)}{\mathbf{\Gamma}_m((m+1)/2+\nu)}(\det(z))^{\nu/2}\frac{{}_0\mathbf{F}_1((m+1)/2+\nu;z)}{{}_0\mathbf{F}_1((m+1)/2;z)}\end{aligned} \tag{8.28}$$

with $z = xy/4t^2$, proving Corollary 8.1.3.

Using the function $\widetilde{\mathbf{I}}_\nu$ defined by (8.15), we may also write

$$\mathbf{Q}_x^{(0)}\left[\exp\left(-\frac{\nu^2}{2}\int_0^t \operatorname{tr}(X_u^{-1})\,du\right)|X_t = y\right] = \frac{\widetilde{\mathbf{I}}_\nu(z)}{\widetilde{\mathbf{I}}_0(z)}, \tag{8.29}$$

which is precisely (8.4) when $m = 1$.

We can extend (8.29) as follows :

Proposition 8.2.2. *Let $\lambda \geqq 0$, $\nu \geqq 0$,*

$$\begin{aligned}&\mathbf{Q}_x^{(0)}\left[\exp\left(-\frac{\lambda^2}{2}\int_0^t \operatorname{tr}(X_u)\,du - \frac{\nu^2}{2}\int_0^t \operatorname{tr}(X_u^{-1})\,du\right)|X_t = y\right] \\ &= \left(\frac{\lambda t}{\sinh(\lambda t)}\right)^{m(m+1)/2} \exp(-a_\lambda(t)\operatorname{tr}(x+y))\frac{\widetilde{\mathbf{I}}_\nu(\lambda^2 xy/4\sinh^2(\lambda t))}{\widetilde{\mathbf{I}}_0(xy/4t^2)},\end{aligned} \tag{8.30}$$

where $a_\lambda(t) = (2t)^{-1}(\lambda t\coth(\lambda t) - 1)$.

Remark 8.2.3. *(i) The computation in the case $\nu = 0$ was done by M.F. Bru in [Bru91].*
(ii) In the case $m = 1$, formula (8.30) was obtained in [PY82] and yields to the joint characteristic function of the stochastic area and winding number of planar Brownian motion $\{Z_u, u \leqq t\}$.

Proof. From the absolute continuity relationships (8.13) and (8.25), we obtain

$$\begin{aligned}\frac{d^\lambda \mathbf{Q}_x^{(\nu)}}{d\mathbf{Q}_x^{(0)}}|_{\mathcal{F}_t} &= \left(\frac{\det(X_t)}{\det(x)}\right)^{\nu/2} \exp\left(\frac{\lambda}{2}(\operatorname{tr}(X_t) - \operatorname{tr}(x) - m\delta t)\right) \\ &\qquad \times \exp\left(-\frac{\lambda^2}{2}\int_0^t \operatorname{tr}(X_u)\,du - \frac{\nu^2}{2}\int_0^t \operatorname{tr}(X_u^{-1})\,du\right),\end{aligned}$$

from which we deduce

$$\mathbf{Q}_x^{(0)}\left[\exp\left(-\frac{\lambda^2}{2}\int_0^t \mathrm{tr}(X_u)\,du - \frac{\nu^2}{2}\int_0^t \mathrm{tr}(X_u^{-1})\,du\right)|X_t = y\right]$$
$$= \frac{{}^{\lambda}\mathbf{q}_t^{(\nu)}(x,y)}{\mathbf{q}_t^{(0)}(x,y)}\left(\frac{\det(x)}{\det(y)}\right)^{\nu/2}\exp\left(-\frac{\lambda}{2}(\mathrm{tr}(y)-\mathrm{tr}(x)-m\delta t)\right),$$

where ${}^{\lambda}\mathbf{q}^{(\nu)}$ is the transition density of the squared Ornstein Uhlenbeck process ${}^{\lambda}X$, the solution of (8.26). Since ${}^{\lambda}X_t = e^{2\lambda t}X((1-e^{-2\lambda t})/2\lambda)$ for some Wishart process X, we have

$${}^{\lambda}\mathbf{q}^{(\nu)}(t,x,y) = e^{-\lambda m(m+1)t}\mathbf{q}^{(\nu)}(\frac{1-e^{-2\lambda t}}{2\lambda}, x, ye^{-2\lambda t}).$$

Straightforward computations give (8.30). □

8.2.4 The case of negative indexes

We first give a proof of Theorem 8.1.4 and then discuss about the law of T_0, the first hitting time of 0 by $\{\det(X_t)\}$.

Proof of Theorem 8.1.4. We consider the local martingale $\{M_t\}$ under $\mathbf{Q}_x^{(0)}$ defined by

$$M_t = \left(\frac{\det(X_t)}{\det(x)}\right)^{-\nu/2}\exp\left(-\frac{\nu^2}{2}\int_0^t \mathrm{tr}(X_s^{-1})\,ds\right).$$

Note that, for $\eta > 0$, $\{M_{t\wedge T_\eta}\}$ is a bounded martingale, where $T_\eta = \inf\{t; \det(X_t) \leqq \eta\}$. Then, applying the Girsanov theorem, we find

$$\mathbf{Q}_x^{(-\nu)}|_{\mathbb{F}_{t\wedge T_\eta}} = M_{t\wedge T_\eta}\cdot \mathbf{Q}_x^{(0)}|_{\mathbb{F}_{t\wedge T_\eta}}.$$

Hence, letting η tend to 0, we obtain the result, since $T_0 = \infty$ a.s. on the right hand side. □

Proof of Corollary 8.1.5. From the second equality in (8.16), we obtain

$$\mathbf{Q}_x^{(-\nu)}(T_0 > t\,|\,X_t = y) = \left(\frac{\det(x)}{\det(y)}\right)^{\nu}\frac{\mathbf{q}_t^{(\nu)}(x,y)}{\mathbf{q}_t^{(-\nu)}(x,y)}.$$

Now, using the expression of the semigroup $\mathbf{q}_t^{(\nu)}(x,y)$ given in (8.18), we obtain (8.17). □

We next give the tail of the law of T_0 under $\mathbf{Q}_x^{(-\nu)}$.

Proposition 8.2.3. *For any $t > 0$, we have*

$$\mathbf{Q}_x^{(-\nu)}(T_0 > t) = \frac{\mathbf{\Gamma}_m((m+1)/2)}{\mathbf{\Gamma}_m(\delta/2)}\big(\det(\frac{x}{2t})\big)^{\nu}e^{-\mathrm{tr}(x/2t)}\;{}_1\mathbf{F}_1\big(\frac{m+1}{2};\frac{\delta}{2};\frac{x}{2t}\big) \tag{8.31}$$
$$= \frac{\mathbf{\Gamma}_m((m+1)/2)}{\mathbf{\Gamma}_m(\delta/2)}\big(\det(\frac{x}{2t})\big)^{\nu}\;{}_1\mathbf{F}_1\big(\nu;\frac{\delta}{2};-\frac{x}{2t}\big), \tag{8.32}$$

where $\delta = m+1+2\nu$.

Proof. By Theorem 8.1.4, we have

$$\mathbf{Q}_x^{(-\nu)}(T_0 > t) = \mathbf{Q}_x^{(\nu)}\left[\left(\frac{\det(x)}{\det(X_t)}\right)^{\nu}\right] \tag{8.33}$$

and compute the right hand side by using the explicit expression (8.18) for the semi-group of $\{X_t\}$.

We have by (8.18)

$$\mathbf{Q}_x^{(\nu)}\left[\left(\frac{\det(x)}{\det(X_t)}\right)^{\nu}\right] = \frac{\exp(-\mathrm{tr}(x)/2t)(\det(x))^{\nu}}{(2t)^{m\delta/2}\mathbf{\Gamma}_m(\delta/2)} \int_{\mathcal{S}_m^+} e^{-\mathrm{tr}(y)/2t}\ {}_0\mathbf{F}_1\big(\frac{\delta}{2}; \frac{xy}{4t^2}\big)\ dy.$$

Noting that ${}_0\mathbf{F}_1(\delta/2; xy/4t^2) = {}_0\mathbf{F}_1(\delta/2; \sqrt{x}y\sqrt{x}/4t^2)$ from definition, we change the variables by $z = \sqrt{x}y\sqrt{x}/4t^2$ to obtain

$$\begin{aligned}
&\mathbf{Q}_x^{(\nu)}\left[\left(\frac{\det(x)}{\det(X_t)}\right)^{\nu}\right] \\
&= \frac{\exp(-\mathrm{tr}(x)/2t)(\det(x))^{\nu-(m+1)/2}}{(2t)^{m(\delta/2-m-1)}\mathbf{\Gamma}_m(\delta/2)} \int_{\mathcal{S}_m^+} e^{-2t\mathrm{tr}(x^{-1}z)}\ {}_0\mathbf{F}_1\big(\frac{\delta}{2}; z\big)\ dz.
\end{aligned} \tag{8.34}$$

For the formula for the Jacobian, see Theorem 2.1.6, p.58, in [Mui82].

Then, using the fact that the Laplace transform of a ${}_p\mathbf{F}_q$ function is a ${}_{p+1}\mathbf{F}_q$ function (cf. Theorem 7.3.4, p.260, in [Mui82]), we get (8.31) and then, using the Kummer relation (Theorem 7.4.3, p.265, in [Mui82]), (8.32). □

Remark 8.2.4. *When $m = 1$, we can explicitly compute the right hand side of* (8.33) *and show that T_0 is distributed as $x/2\gamma_\nu$, where γ_ν is a gamma variable with parameter ν. It may be also obtained by using the integral relation*

$$\frac{1}{X_t^\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty u^{\nu-1} e^{-uX_t}\ du,$$

and then the explicit expression of $Q_x^{(\nu)}[e^{-uX_t}]$. A third method consists in using the time reversal between BES(ν) *and* BES($-\nu$) *; see paper #1 in [Yor01] for details.*

Remark 8.2.5. *As the knowledge of the law of T_0 under $Q_x^{(-\nu)}$ has played an important rôle in several questions for $m = 1$ (in the pricing of Asian options in particular, see, e.g [GY93]), it seems worth looking for some better expression than* (8.31) *or* (8.32)*. First, let us define $S_0 = (2T_0)^{-1}$, and note that, from* (8.32)*, we have*

$$\mathbf{Q}_x^{(-\nu)}(S_0 \leqq u) = \frac{\mathbf{\Gamma}_m((m+1)/2)}{\mathbf{\Gamma}_m(\delta/2)}\big(\det(x)\big)^{\nu} u^{m\nu}\ {}_1\mathbf{F}_1\big(\nu; \frac{\delta}{2}; -ux\big). \tag{8.35}$$

Note in particular that the right-hand side of (8.35) *is a distribution function in u.*

From (8.34), we also have the following expression

$$\mathbf{Q}_x^{(-\nu)}\left[S_0 \leqq u\right] = \frac{(\det(x))^{\nu-(m+1)/2}}{\mathbf{\Gamma}_m(\delta/2)} \exp(-u\mathrm{tr}(x))u^{m(\delta/2-m-1)} \times \int_{\mathcal{S}_m^+} e^{-\mathrm{tr}(x^{-1}z)/u}\ {}_0\mathbf{F}_1\big(\frac{\delta}{2}; z\big)\ dz,$$

from which we obtain the following Laplace transform

$$\mathbf{Q}_x^{(-\nu)}\left[\exp(-\lambda S_0)\right] = \lambda \int_0^\infty e^{-\lambda u}\, \mathbf{Q}_x^{(-\nu)}\left[S_0 \leqq u\right] du$$
$$= \frac{(\det(x))^{\nu-(m+1)/2}}{\mathbf{\Gamma}_m(\delta/2)}\, 2\lambda\, (\lambda + \mathrm{tr}(x))^{-\alpha/2} \times \int_{\mathcal{S}_m^+} K_\alpha(2\sqrt{(\lambda+\mathrm{tr}(x))\mathrm{tr}(x^{-1}z)})\, (\mathrm{tr}(x^{-1}z))^{\alpha/2}\ {}_0\mathbf{F}_1\big(\frac{\delta}{2}; z\big)\ dz,$$

where $\alpha = m(\delta/2-m-1)+1$, K_α is the usual modified Bessel (Macdonald) function and we have used the integral representation for K_α given in formula (5.10.25) in [Leb72] .

In the case where $m = 1$, we obtain

$$\mathbf{Q}_x^{(-\nu)}\left[\exp(-\lambda S_0)\right] = \frac{\lambda}{x} \frac{1}{(1+\lambda/x)^{\nu/2}} \int_0^\infty t\, K_\nu(t\,\sqrt{1+\lambda/x})\, I_\nu(t)\ dt$$

by using the fact that $\widetilde{\mathbf{I}}_\nu(x) = I_\nu(2z^{1/2})$. Now, we recall the formula (cf. formula (5.15.6) in [Leb72])

$$\int_0^\infty t\, K_\nu(at)\, I_\nu(t)\ dt = \frac{1}{a^\nu(a^2-1)}, \qquad a \geqq 1,$$

from which we deduce

$$\mathbf{Q}_x^{(-\nu)}\left[\exp(-\lambda S_0)\right] = \frac{1}{(1+\lambda/x)^\nu}.$$

Hence, we again recover the well-known fact that xS_0 obeys the Gamma(ν) distribution.

Now we go back to Theorem 8.1.4. We may replace t by any stopping time T in (8.16). In particular, we may consider

$$T_r = \inf\{t; \det(X_t) = r\} \qquad \text{for} \quad 0 < r < \det(x).$$

We have $T_r < T_0$ a.s., and (8.16) implies

$$\mathbf{Q}_x^{(-\nu)}[H_{T_r}] = \left(\frac{r}{\det(x)}\right)^\nu \mathbf{Q}_x^{(\nu)}[H_{T_r}; T_r < \infty]$$

for any non-negative $(\mathcal{F}_t)$-predictable process $\{H_t\}$, and, in particular, we obtain

$$\mathbf{Q}_x^{(-\nu)}(T_r < \infty) = \left(\frac{r}{\det(x)}\right)^{\nu} < 1.$$

This result is in complete agreement with the fact that $\{(\det(x)/\det(X_t))^{\nu}\}$ is a local martingale, which converges almost surely to 0 as $t \to \infty$. Therefore we obtain (see Chapter II, (3.12)Exercise, [RY99]), for a uniform random variable U,

$$\sup_{t \geqq 0} \Big(\frac{\det(x)}{\det(X_t)}\Big)^{\nu} \overset{\text{(law)}}{=} \frac{1}{U} \qquad \text{or} \qquad \inf_{t \geqq 0} \frac{\det(X_t)}{\det(x)} \overset{\text{(law)}}{=} U^{1/\nu}.$$

8.3 Wishart processes with drift

(3.1) In this section, we define Wishart processes with drift and show in particular that they are Markov processes. Recall that, in the one-dimensional case, Bessel processes with drift have been introduced by Watanabe [Wat75] and studied by Pitman-Yor (see [PY81]). They play an essential role in the study of diffusions on $\mathbb{R}_+$ which are globally invariant under time inversion. Let us first consider the case of the integral dimension, $\delta = n \in \mathbb{N}$.

Theorem 8.3.1. *Let $\{B_s, s \geqq 0\}$ be an $n \times m$ Brownian matrix starting from 0 and let $\Theta = (\Theta_{ij}) \in M_{n,m}(\mathbb{R})$. Then, setting $X_t^{\Theta} = (B_t + \Theta t)'(B_t + \Theta t) \equiv (\widehat{B_t + \Theta t})$, we have*

$$E[G(X_t^{\Theta}, t \leqq s)] = E[G(X_t, t \leqq s)\, {}_0\mathbf{F}_1(\frac{n}{2}; \frac{1}{4}\widehat{\Theta} X_s) \exp\big(-\frac{1}{2}\mathrm{tr}(\widehat{\Theta})s\big)] \tag{8.36}$$

for any $s > 0$ and for any non-negative functional G, where $\widehat{\Theta} = \Theta'\Theta$ and $X_t \equiv X_t^0$ is an n-dimensional Wishart process.

Proof. By the usual Cameron-Martin relationship, we have

$$E[G(X_t^{\Theta}, t \leqq s)] = E[G(X_t, t \leqq s) \exp\left(\sum_{i=1}^{n}\sum_{j=1}^{m} \Theta_{ij} B_{ij}(s) - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} (\Theta_{ij})^2 s\right)].$$

Since $\sum_i \sum_j \Theta_{ij} B_{ij}(s) = \mathrm{tr}(\Theta' B_s)$, the rotational invariance of Brownian motions ($OB \overset{\text{(law)}}{=} B$ for any $O \in O(n)$) yields

$$\begin{aligned} E[G(X_t, t \leqq s) \exp\big(\mathrm{tr}(\Theta' B_s)\big)] &= E[G((OB_t)'(OB_t)), t \leqq s) \exp(\mathrm{tr}(\Theta' O B_s))] \\ &= E[G(X_t, t \leqq s) \exp(\mathrm{tr}(B_s \Theta' O))]. \end{aligned}$$

Since the last equality holds for any $O \in O(n)$, the integral representation (8.47) given in the appendix gives

$$\begin{aligned} E[G(X_t, t \leqq s) \exp\big(\mathrm{tr}(\Theta' B_s)\big)] &= E[G(X_t, t \leqq s) \int_{O(n)} \exp(\mathrm{tr}(B_s \Theta' O))\, dO] \\ &= E[G(X_t, t \leqq s)\ {}_0\mathbf{F}_1\big(\frac{n}{2}; \frac{1}{4} B_s \Theta' \Theta B_s'\big)], \end{aligned}$$

where dO is the normalized Haar measure on $O(n)$. The last expression shows that the law of $\{X_t^{\Theta}\}$ depends on Θ only through the product $\widehat{\Theta} = \Theta'\Theta$; hence, we shall also denote X_t^{Θ} by $X_t^{(\widehat{\Theta})}$. Moreover from Lemma 8.5.1 in the Appendix, we see

$$E[G(X_t, t \leqq s) \exp\big(\mathrm{tr}(\Theta' B(s))\big)] = E[G(X_t, t \leqq s)\ {}_0\mathbf{F}_1\big(\frac{n}{2}; \frac{1}{4} \Theta X_s \Theta'\big)]. \tag{8.37}$$

Finally, by using Lemma 8.5.1 again, we obtain the better expression (8.36). □

Proposition 8.3.2. (i) *Keeping the notations in Theorem 8.3.1, the stochastic process* $\{X_t^{\Theta}\}$ *now denoted by* $\{X_t^{(\widehat{\Theta})}\}$ *is a Markov process, which we shall refer to* $\mathrm{WIS}^{(\widehat{\Theta})}(n, m)$, *whose transition probabilities* $\mathbf{q}_n^{(\widehat{\Theta})}(t, x, dy)$ *are given by*

$$\begin{aligned} \mathbf{q}_n^{(\widehat{\Theta})}(t, x, dy) &= \frac{{}_0\mathbf{F}_1(n/2; \widehat{\Theta} y/4)}{{}_0\mathbf{F}_1(n/2; \widehat{\Theta} x/4)} \exp\big(-\frac{1}{2}\mathrm{tr}(\widehat{\Theta})t\big) \mathbf{q}_n^{(0)}(t, x, dy) \\ &= \frac{1}{(2t)^{nm/2} \mathbf{\Gamma}_m(n/2)} \exp\big(-\frac{1}{2t}\mathrm{tr}(x + y)\big)(\det(y))^{(n-m-1)/2} \\ &\quad \times \frac{{}_0\mathbf{F}_1(n/2; xy/4t^2)\ {}_0\mathbf{F}_1(n/2; \widehat{\Theta} y/4)}{{}_0\mathbf{F}_1(n/2; \widehat{\Theta} x/4)} \exp\big(-\frac{1}{2}\mathrm{tr}(\widehat{\Theta})t\big)\ dy. \end{aligned} \tag{8.38}$$

(ii) *The conditional law of* B_s *given* $\{X_t, t \leqq s\}$ *is given by*

$$E[\exp(\mathrm{tr}(\Theta' B_s)) | \{X_t, t \leqq s\}, X_s = y] = {}_0\mathbf{F}_1\big(\frac{n}{2}, \frac{\widehat{\Theta} y}{4}\big).$$

Proof. The first assertion follows from formula (8.36), which describes $\{X_t^{(\widehat{\Theta})}, t \geqq 0\}$ as an h-transform of $\{X_t, t \geqq 0\}$ with

$$h(X_s, s) = {}_0\mathbf{F}_1\big(\frac{n}{2}, \frac{\widehat{\Theta} X_s}{4}\big) \exp\big(-\frac{1}{2}\mathrm{tr}(\widehat{\Theta})s\big).$$

In fact, we have from (8.36), for $u > s$

$$\begin{aligned} &E[G(X_u^{(\widehat{\Theta})})|\{X_t^{(\widehat{\Theta})}, t \leqq s\}] \\ &\qquad = \frac{E[G(X_u)\,{}_0\mathbf{F}_1(n/2; \widehat{\Theta}X_u/4)\exp(-\mathrm{tr}(\widehat{\Theta})u/2)|\{X_t^{(\widehat{\Theta})}, t \leqq s\}]}{{}_0\mathbf{F}_1(n/2; \widehat{\Theta}X_s/4)\exp(-\mathrm{tr}(\widehat{\Theta})s/2)} \\ &\qquad = \frac{\mathbf{Q}_{u-s}^n[G(\cdot)\,{}_0\mathbf{F}_1(n/2; \widehat{\Theta}\cdot/4)]\exp(-\mathrm{tr}(\widehat{\Theta})(u-s)/2)}{{}_0\mathbf{F}_1(n/2; \widehat{\Theta}X_s/4)}, \end{aligned}$$

where $\mathbf{Q}_t^n$, $t \geqq 0$ denotes the semigroup of the original Wishart process.

The second assertion is nothing else but (8.37). $\square$

Remark 8.3.1. *We can also see Propositions 8.3.1 and 8.3.2 as consequences of a result by Rogers and Pitman [RP81]. Indeed, for $y \in \mathcal{S}_m^+$, define*

$$\Sigma(y) = \{\alpha \in M_{n,m}(\mathbb{R}); \widehat{\alpha} \equiv \alpha'\alpha = y\},$$

and let $\Lambda(y, .)$ be the uniform measure on $\Sigma(y)$ given by

$$\Lambda f(y) = \int_{O(n)} f(O\alpha)\, dO,$$

where $\alpha \in \Sigma(y)$ (independent of the choice of α). Then, by the rotational invariance of Brownian motion, the semigroups $\mathbf{P}_t$ of $\{B_t\}$ and $\mathbf{Q}_t$ of $\{X_t = \widehat{B}_t\}$ satisfy

$$\mathbf{Q}_t\Lambda = \Lambda\mathbf{P}_t.$$

Set $f_\Theta(\alpha) = \exp(\mathrm{tr}(\Theta'\alpha))$, then the law of $B_t^\Theta \equiv B_t + \Theta t$, the Brownian matrix with drift Θ, satisfies

$$\mathbf{P}_t^\Theta(\alpha, d\beta) = \exp\big(-\frac{1}{2}\mathrm{tr}(\widehat{\Theta})t\big)\frac{f_\Theta(\beta)}{f_\Theta(\alpha)}\,\mathbf{P}_t(\alpha, d\beta).$$

Setting $g_\Theta = \Lambda f_\Theta$, we have (see [RP81])

$$\mathbf{Q}_t^\Theta(x, dy) = \exp\big(-\frac{1}{2}\mathrm{tr}(\widehat{\Theta})t\big)\frac{g_\Theta(y)}{g_\Theta(x)}\,\mathbf{Q}_t(x, dy)$$

and $\Lambda^\Theta\mathbf{P}_t^\Theta = \mathbf{Q}_t^\Theta\Lambda^\Theta$, where the kernel Λ^Θ is given by

$$\Lambda^\Theta(y, d\alpha) = \frac{f_\Theta(\alpha)}{g_\Theta(y)}\Lambda(y, d\alpha).$$

We are now in a position to define Wishart processes with drift in general dimensions δ.

Definition *Let $\delta > m-1$ and $\Delta \in \widetilde{\mathcal{S}}_m^+$. We define a Wishart process* $\mathrm{WIS}^{(\Delta)}(\delta, m, x)$ *of dimension δ and drift Δ as the $\widetilde{\mathcal{S}}_m^+$-valued Markov process, starting from x, with semigroup given by*

$$\begin{aligned} \mathbf{q}_\delta^{(\Delta)}(t,x,dy) &= \frac{{}_0\mathbf{F}_1(\delta/2;\Delta y/4)}{{}_0\mathbf{F}_1(\delta/2;\Delta x/4)} \exp\big(-\frac{1}{2}\mathrm{tr}(\Delta)t\big)\mathbf{q}_\delta^{(0)}(t,x,dy) \\ &= \frac{1}{(2t)^{\delta m/2}\mathbf{\Gamma}_m(\delta/2)} \exp\big(-\frac{1}{2t}\mathrm{tr}(x+y)\big)(\det(y))^{(\delta-m-1)/2} \\ &\quad \times \frac{{}_0\mathbf{F}_1(\delta/2;xy/4t^2)\;{}_0\mathbf{F}_1(\delta/2;\Delta y/4)}{{}_0\mathbf{F}_1(\delta/2;\Delta x/4)} \exp\big(-\frac{1}{2}\mathrm{tr}(\Delta)t\big)\;dy. \end{aligned} \tag{8.39}$$

However, we need to prove the semigroup property of $\mathbf{q}_\delta^{(\Delta)}$, which is done in the following.

Proposition 8.3.3. (i) *Let X be a Wishart process* $\mathrm{WIS}(\delta,m,a)$, $a \in \mathcal{S}_m^+$, *then the process $i(X)$ obtained by time inversion is a* $\mathrm{WIS}^{(a)}(\delta,m,0)$ *process.*
(ii) *More generally, if X is a* $\mathrm{WIS}^{(\Delta)}(\delta,m,a)$ *process, then $i(X)$ is a* $\mathrm{WIS}^{(a)}(\delta,m,\Delta)$ *process.*

Sketch of Proof. (i) After a straightforward computation, we see that the distribution of $i(X)_t$ is $\mathbf{q}_\delta^{(a)}(t,0,dy)$ given by (8.39). Next, we compute $E[f(i(X)_s)g(i(X)_t)]$ for $s<t$ in terms of the process X and the semigroup $\mathbf{q}_\delta(t,a,dy)$. We then obtain that $i(X)$ is a Markov process with semigroup $\mathbf{q}'_\delta$ (a priori non homogeneous) given by the transition probability density

$$\mathbf{q}'_\delta(s,t;x,y) = \frac{1}{t^{m(m+1)}} \frac{\mathbf{q}_\delta(1/t,a,y/t^2)}{\mathbf{q}_\delta(1/s,a,x/s^2)} \mathbf{q}_\delta\big(\frac{1}{s}-\frac{1}{t},\frac{y}{t^2},\frac{x}{s^2}\big),$$

from which we obtain after some computations that

$$\mathbf{q}'_\delta(s,t;x,y) = \mathbf{q}_\delta^{(a)}(t-s,x,y).$$

The proof of (ii) is similar. □

Remark 8.3.2. *The semigroup property of* $\mathbf{q}^{(\Delta)}$ *entails that*

$$\mathbf{L}^\delta({}_0\mathbf{F}_1(\delta/2;\Delta x/4)) = \frac{1}{2}\mathrm{tr}(\Delta)\;{}_0\mathbf{F}_1(\delta/2;\Delta x/4), \tag{8.40}$$

where $\mathbf{L}^\delta$ denotes the infinitesimal generator of the Wishart process of dimension δ. Note that the differential equations satisfied by ${}_0\mathbf{F}_1$ given in Theorem 7.5.6, [Mui82], in terms of eigenvalues do not directly yield (8.40)*. But one can translate those equations into differential equations with respect to the matrix entries.*

As an application of time inversion, we give an interpretation of the Hartman-Watson distribution in terms of the Wishart processes with drift.

Proposition 8.3.4. *Let $x, y \in \widetilde{\mathcal{S}}_m^+$ and let $\mathbf{Q}_y^{\delta,(x)}$ denote the distribution of the Wishart process* $\mathrm{WIS}^{(x)}(\delta, m, y)$ *of dimension δ and drift x, starting from y. Then,*

$$\mathbf{Q}_y^{m+1,(x)}\left[\exp\left(-\frac{\nu^2}{2}\int_0^\infty \mathrm{tr}(X_s^{-1})\,ds\right)\right] = \frac{\widetilde{\mathbf{I}}_\nu(xy/4)}{\widetilde{\mathbf{I}}_0(xy/4)}, \tag{8.41}$$

where $\widetilde{\mathbf{I}}_\nu$ is defined in (8.15).

Proof. Let f be a bounded function. From time inversion and the Markov property, we have

$$\begin{aligned}
&\mathbf{Q}_x^{m+1}\left[f(X_t)\exp\left(-\frac{\nu^2}{2}\int_0^t \mathrm{tr}(X_u^{-1})\,du\right)\right]\\
&= \mathbf{Q}_0^{m+1,(x)}\left[f(t^2X_{1/t})\exp\left(-\frac{\nu^2}{2}\int_{1/t}^\infty \mathrm{tr}(X_u^{-1})\,du\right)\right]\\
&= \mathbf{Q}_0^{m+1,(x)}\left[f(t^2X_{1/t})\mathbf{Q}_{X_{1/t}}^{m+1,(x)}\left[\exp\left(-\frac{\nu^2}{2}\int_0^\infty \mathrm{tr}(X_u^{-1})\,du\right)\right]\right].
\end{aligned} \tag{8.42}$$

On the other hand, according to (8.29), the first line of the above identities is equal to

$$\mathbf{Q}_x^{m+1}\left[f(X_t)\frac{\widetilde{\mathbf{I}}_\nu(xX_t/4t^2)}{\widetilde{\mathbf{I}}_0(xX_t/4t^2)}\right] = \mathbf{Q}_0^{m+1,(x)}\left[f(t^2X_{1/t})\frac{\widetilde{\mathbf{I}}_\nu(xX_{1/t}/4)}{\widetilde{\mathbf{I}}_0(xX_{1/t}/4)}\right]. \tag{8.43}$$

By comparison of the last two terms in (8.42) and (8.43), we obtain (8.41). □

Remark 8.3.3. *We also note that, by time inversion, the left hand side of* (8.41) *equals*

$$\mathbf{Q}_x^{m+1,(y)}\left[\exp\left(-\frac{\nu^2}{2}\int_0^\infty \mathrm{tr}(X_s^{-1})\,ds\right)\right],$$

from which we deduce the identity

$$\frac{\widetilde{\mathbf{I}}_\nu(xy)}{\widetilde{\mathbf{I}}_0(xy)} = \frac{\widetilde{\mathbf{I}}_\nu(yx)}{\widetilde{\mathbf{I}}_0(yx)}.$$

But, in fact, independently from the preceding probabilistic argument, the equality $\widetilde{\mathbf{I}}_\mu(xy) = \widetilde{\mathbf{I}}_\mu(yx)$ holds as a consequence of the property that $\widetilde{\mathbf{I}}_\mu(z)$ depends only on the eigenvalues of the matrix z (we apply this remark to both $\mu = \nu$ and $\mu = 0$).

Proposition 8.3.4 is a particular relation between the Wishart bridge and the Wishart process with drift. We refer to Theorem 5.8 in [PY81] for other relations in the Bessel case which can be extended in our context.

(3.2) Intertwining property The extension for Wishart processes of the intertwining relation (8.7) is given in the following proposition, which M.F. Bru in [Bru89b] predicted, from the results in [Yor89], that it would hold.

Proposition 8.3.5. *For $\delta, \delta' \geqq m-1$ and every t,*

$$\mathbf{Q}_t^{\delta+\delta'} \Lambda_{\delta,\delta'} = \Lambda_{\delta,\delta'} \mathbf{Q}_t^{\delta}, \tag{8.44}$$

where, letting $\beta_{\delta/2,\delta'/2}$ be a Beta_m *variable with parameter $(\delta/2, \delta'/2)$ as defined in Def.3.3.2 in [Mui82], $\Lambda_{\delta,\delta'}(x,dy)$ denotes the kernel whose action on any bounded Borel function f is given by*

$$\Lambda_{\delta,\delta'} f(x) = E[f(\sqrt{x}\beta_{\frac{\delta}{2},\frac{\delta'}{2}}\sqrt{x})], \quad x \in \widetilde{\mathcal{S}}_m^+.$$

Note that (8.44) may be understood as a Markovian extension of the relation (8.49) given in the Appendix (see [Yor89] in the Bessel case). Indeed, from (8.44), we have

$$\mathbf{Q}_t^{\delta+\delta'} \Lambda_{\delta,\delta'} f(0) = \Lambda_{\delta,\delta'} \mathbf{Q}_t^{\delta} f(0),$$

which is equivalent to

$$E[f(t\sqrt{\gamma_{\delta+\delta'}}\beta_{\delta/2,\delta'/2}\sqrt{\gamma_{\delta+\delta'}})] = E[f(t\gamma_\delta)],$$

where γ_p is a Wishart distribution $W_m(p, I_m)$, β is a Beta_m variable (see **(5.b)**) and, on the left-hand side, the two random variables are independent.

Proof. At least two proofs may be given for this result.

(i) *an analytical proof*, in which we just check that the Laplace transforms of both hand sides of (8.44) are equal. Indeed, take $f_\Theta(x) = \exp(-\mathrm{tr}(\Theta x))$ with $\Theta \in \mathcal{S}_m^+$. We compute $\Lambda_{\delta,\delta'} \mathbf{Q}_t^{\delta} f_\Theta(x)$ using (8.12).

On the other hand, using Theorem 7.4.2 in [Mui82], we have

$$\begin{aligned}
\Lambda_{\delta,\delta'} f_\Theta(x) &= E[\exp(-\mathrm{tr}(\Theta\sqrt{x}\beta_{\delta/2,\delta'/2}\sqrt{x})) \\
&= {}_1\mathbf{F}_1(\delta/2; (\delta+\delta')/2; \sqrt{x}\,\Theta\sqrt{x}) \\
&= {}_1\mathbf{F}_1(\delta/2; (\delta+\delta')/2; \sqrt{\Theta}\,x\sqrt{\Theta}) \\
&= E[\exp(-\mathrm{tr}(\sqrt{\Theta}\beta_{\delta/2,\delta'/2}\sqrt{\Theta}\,x))
\end{aligned}$$

We then use (8.12) again to compute $\mathbf{Q}_t^{\delta+\delta'} \Lambda_{\delta,\delta'} f_\Theta(x)$. The equality $\mathbf{Q}_t^{\delta+\delta'} \Lambda_{\delta,\delta'} f_\Theta(x) = \Lambda_{\delta,\delta'} \mathbf{Q}_t^{\delta} f_\Theta(x)$ follows from a change of variable formula.

(ii) *a probabilistic proof.* The proof of this result follows from the same lines as the proof of the corresponding result (8.7) for the squared Bessel processes given in [CPY98]. The main ingredients are the time inversion invariance of Wishart processes, starting from 0, and the relation (8.49) given in the Appendix. Indeed, let X and X' be two independent Wishart processes with respective dimension δ and δ', starting at 0. Set $Y = X + X'$, $\mathcal{X}_t = \sigma\{X_s, X'_s, s \leqq t\}$ and $\mathcal{Y}_t = \sigma\{Y_s, s \leqq t\}$. Then Y is a Wishart process of dimension $\delta + \delta'$ and we have

$$\begin{aligned} E[F(Y_u, u \leqq t) f(X_t)] &= E[F(u^2 Y_{1/u}, u \leqq t) f(t^2 X_{1/t})] \\ &= E[E[F(u^2 Y_{1/u}, u \leqq t) | Y_{1/t}] f(t^2 X_{1/t})] \\ &= E[E[F(u^2 Y_{1/u}, u \leqq t | Y_{1/t}] \Lambda_{\delta,\delta'} f(t^2 Y_{1/t})] \\ &= E[F(u^2 Y_{1/u}, u \leqq t) \Lambda_{\delta,\delta'} f(t^2 Y_{1/t})] \\ &= E[F(Y_u, u \leqq t) \Lambda_{\delta,\delta'} f(Y_t)], \end{aligned}$$

where we have used the Markov property of $\{t^2 Y_{1/t}\}$ with respect to $\mathcal{X}_{1/t}$ for the second equality and used (8.49) for the third one. We deduce from the above equation

$$E[f(X_t) | \mathcal{Y}_t] = \Lambda_{\delta,\delta'} f(Y_t),$$

which implies the intertwining relation (8.44).

8.4 Some developments ahead

We hope that the present paper is the first of a series of two or three papers devoted to the topics of Wishart processes ; indeed, in the present paper, we concentrated on the extension to Wishart processes of the Hartman-Watson distribution for Bessel processes, but there are many other features of Bessel processes which may also be extended to Wishart processes. What seems to be the most accessible for now are some extensions of Spitzer type limiting results, i.e., (8.5) and (8.6) ; for instance, in [DMDMY], we prove that

$$\left(\frac{2}{m \ln(t)}\right)^2 \int_0^t \operatorname{tr}(X_u^{-1})\, du \xrightarrow[t\to\infty]{\text{(law)}} T_1(\beta), \tag{8.45}$$

where X is our $\mathrm{WIS}(m+1, m, x)$, for $x \in \widetilde{\mathcal{S}}_m^+$, and that, if $\delta > m+1$ and X is a $\mathrm{WIS}(\delta, m, x)$,

$$\frac{1}{m(\ln(t))} \int_0^t \operatorname{tr}(X_u^{-1})\, du \xrightarrow[t\to\infty]{\text{(a.s.)}} \frac{1}{\delta - (m+1)}. \tag{8.46}$$

We also hope that a number of probabilistic results concerning Bessel functions, as discussed in Pitman-Yor [PY81], may be extended to their matrix counterparts.

For the moment, we show, a little informally, how (8.45) may be deduced from the absolute continuity relationship (8.13) in Theorem 1.2 (in the case $m = 1$, this kind of arguments has been developed in Yor [Yor97], with further refinements given in Pap-Yor [PY00], and Bentkus-Pap-Yor [BPY03]). Indeed, with our notation, we have :

$$\mathbf{Q}_x^{(0)}[\exp\left(-\frac{\nu^2}{2}\int_0^t \operatorname{tr}(X_s^{-1})\,ds\right)] = \mathbf{Q}_x^{(\nu)}[\left(\frac{\det(x)}{\det(X_t)}\right)^{\nu/2}].$$

We then replace ν by $\nu/c\ln(t)$, for some constant c, which we shall choose later, to see

$$\begin{aligned}
&\mathbf{Q}_x^{(0)}[\exp\left(-\frac{\nu^2}{2(c\ln(t))^2}\int_0^t \operatorname{tr}(X_s^{-1})\,ds\right)] \\
&= \mathbf{Q}_x^{(\nu/(c\ln(t)))}[\left(\frac{\det(x)}{\det(X_t)}\right)^{\nu/(2c\ln(t))}] \\
&\simeq \mathbf{Q}_x^{(0)}[\exp\left(-\frac{\nu}{2c\ln(t)}\ln(\det(X_t))\right)] \\
&\simeq \mathbf{Q}_{x/t}^{(0)}[\exp\left(-\frac{\nu}{2c\ln(t)}\ln(t^m\det(X_1))\right)] \\
&\longrightarrow \exp(-\nu)
\end{aligned}$$

as $t \to \infty$ for the choice $c = m/2$. A similar argument easily leads to (8.46), while with the weaker convergence in probability result, instead of almost sure convergence, under $\mathbf{Q}_x^\delta$, for $\delta > m+1$.

8.5 Appendix

(5.a) We recall the definition of hypergeometric functions of matrix arguments. We refer to the book of Muirhead, Chapter 7 [Mui82]. For $a_i \in \mathbb{C}, b_j \in \mathbb{C}\setminus\{0, \frac{1}{2}, \frac{2}{2}, ..., \frac{m-1}{2}\}$ and $X \in \mathcal{S}_m(\mathbb{C})$, the hypergeometric function ${}_p\mathbf{F}_q$ is defined by

$${}_p\mathbf{F}_q(a_1,...,a_p;b_1,...,b_q;X) = \sum_{k=0}^{\infty}\sum_{\kappa}\frac{(a_1)_\kappa\cdots(a_p)_\kappa}{(b_1)_\kappa\cdots(b_q)_\kappa}\frac{C_\kappa(X)}{\kappa!},$$

where $\sum_\kappa$ denotes the summation over all partitions $\kappa = (k_1,..,k_m)$, $k_1 \geqq \cdots \geqq k_m \geqq 0$, of k, $\kappa! = k_1!\cdots k_m!$, $k = \sum_{i=1}^m k_i$,

$$(a)_\kappa = \prod_{i=1}^m \left(a - \frac{i-1}{2}\right)_{k_i}, \qquad (a)_k = a(a+1)\cdots(a+k-1), \qquad (a)_0 = 1.$$

$C_\kappa(X)$ is the zonal polynomial corresponding to κ, which is originally defined for $X \in \mathcal{S}_m(\mathbb{R})$ and is a symmetric, homogeneous polynomial of degree k in the eigenvalues of

X. For $X \in \mathcal{S}_m(\mathbb{R})$ and $Y \in \mathcal{S}_m^+$, since the eigenvalues of YX are the same as those of $\sqrt{Y}X\sqrt{Y}$, we define $C_\kappa(YX)$ by

$$C_\kappa(YX) = C_\kappa(\sqrt{Y}X\sqrt{Y}).$$

Hence we can also define ${}_p\mathbf{F}_q(a_1, \cdots; b_1, \cdots; YX)$. For details, see Chapter 7 of Muirhead [Mui82]. Moreover we find in [Mui82] that

$${}_0\mathbf{F}_0(X) = \exp(\operatorname{tr}(X)), \qquad {}_1\mathbf{F}_0(a; X) = \det(I_m - X)^{-a}$$

and also that, for $X \in M_{m,n}(\mathbb{R}), m \leqq n$, and for $H = (H_1 : H_2) \in O(n)$ with $H_1 \in M_{n,m}$,

$$\int_{O(n)} \exp(\operatorname{tr}(XH_1))\, dH = {}_0\mathbf{F}_1\big(\frac{n}{2}; \frac{1}{4}XX'\big), \tag{8.47}$$

where dH is the normalized Haar measure on $O(n)$.

We also recall the definition of the multivariate gamma function $\mathbf{\Gamma}_m(\alpha), \operatorname{Re}(\alpha) > (m-1)/2$:

$$\mathbf{\Gamma}_m(\alpha) = \int_{\tilde{\mathcal{S}}_m^+} \exp(-\operatorname{tr}(A))(\det(A))^{\alpha-(m+1)/2}\, dA.$$

It may be worthwhile noting that the multivariate gamma function $\mathbf{\Gamma}_m(\alpha)$ is represented as a product of the usual gamma function by

$$\mathbf{\Gamma}_m(\alpha) = \pi^{m(m-1)/4} \prod_{i=1}^m \Gamma\big(\alpha - \frac{i-1}{2}\big), \quad \operatorname{Re}(\alpha) > \frac{m-1}{2}.$$

We now give a lemma which plays an important role in Section 8.3.

Lemma 8.5.1. *Let X be an $m \times m$ symmetric matrix and Θ be an $n \times m$ matrix. Then, one has*

$${}_0\mathbf{F}_1(b; \Theta X \Theta') = {}_0\mathbf{F}_1(b; \Theta'\Theta X) \tag{8.48}$$

if $b \notin \mathbb{C} \setminus \{0, \frac{1}{2}, \frac{2}{2}, ..., \frac{m\vee n-1}{2}\}$.

Proof. Note that the argument $\Theta X \Theta'$ on the left-hand side of (8.48) is an $n \times n$ matrix and that $\Theta'\Theta X$ on the right-hand side is an $m \times m$ matrix. Note also that the non-zero eigenvalues of $\Theta X \Theta'$ and $\Theta'\Theta X$ coincide. Then, we obtain the same type of equalities for the zonal polynomials and therefore (8.48).

(5.b) The beta-gamma algebra for matrices Let X and Y be two independent Wishart matrices with respective distributions $W_m(\delta, I_m)$ and $W_m(\delta', I_m)$ (Muirhead's notation, [Mui82] p.85) with $\delta + \delta' > m - 1$. Then, $S = X + Y$ is invertible and the matrice Z defined by $Z = S^{-1/2}XS^{-1/2}$ is a Beta_m distribution with parameter $(\delta/2, \delta'/2)$, see [Mui82, Def. 3.3.2] for the definition of Beta matrices. Moreover, Z and

S are independent, see Olkin and Rubin [OR62, OR64], Casalis and Letac [CL96] for an extension to Wishart distributions on symmetric cones and [BW02]. We thus have the following identity in law :

$$((X_\delta + X_{\delta'})^{-1/2} X_\delta (X_\delta + X_{\delta'})^{-1/2}, X_\delta + X_{\delta'}) \stackrel{\text{(law)}}{=} (X_{\delta,\delta'}, X_{\delta+\delta'}), \tag{8.49}$$

where, on the left-hand side, X_δ and $X_{\delta'}$ are independent and Wishart distributed, and, on the right-hand side, the two variables are independent and $X_{\delta,\delta'}$ is Beta_m distributed.

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Chapitre 9

Matrix Jacobi processes

Abstract : We discuss a matrix-valued generalization of Jacobi processes. These are defined through a stochastic differential equation whose solution we study existence and uniqueness of. The invariant measures of such processes are given as well as absolute continuity relations between different dimensions. In the case of integer dimensions, we interpret those processes as push-forwards of Brownian motion on orthogonal groups.

9.1 Introduction

Suppose $\Theta = (\theta_i)_{1\leq i\leq n}$ is a Brownian motion on the unit sphere $S^{n-1} \subset \mathbb{R}^n$. For $p \leq n$, the invariance of the law of Θ under isometries ensures that $X := (\theta_i)_{1\leq i\leq p} \in \mathbb{R}^p$ and $J = \| X \|^2 = \sum_{i=1}^p \theta_i^2 \in [0,1]$ are Markov processes. J is known as the Jacobi process of dimensions $(p, n-p)$. In fact, such a process can also be defined for non-integer dimensions (p, q).
The aim of this note is to provide a matrix-valued generalization of this process. For integer dimensions, it naturally comes from projection of Brownian motion on some orthogonal group. The process can also be defined for non-integer dimensions through a stochastic differential equation whose solution we study existence and uniqueness of. We also present the matrix extensions of some of the basic properties satisfied by the one-dimensional Jacobi processes. In particular, we establish absolute continuity relations between the laws of processes with different dimensions. The invariant measure of such processes turns out to be a Beta matrix distribution allowing us to recover results by Olshanski [Ol′90] and Collins [Col03] about some push-forward of Haar measure on orthogonal groups. We can also describe the trajectories performed by the eigenvalues and interpret those as some process conditioned to stay in a Weyl chamber.

The upper-left corner of Haar measure on $\mathcal{O}_n(\mathbb{R})$

Before studying the stochastic processes themselves, we would like to present the fixed-time picture, which served as a starting point for this work. Let $M \in \mathcal{M}_{m,n}(\mathbb{R})$ $(m \leq n)$ be a random $m \times n$ matrix whose entries are standard iid Gaussian random variables. We can decompose $M = (M_1, M_2)$ with $M_1 \in \mathcal{M}_{m,p}(\mathbb{R})$ and $M_2 \in \mathcal{M}_{m,q}(\mathbb{R})$ $(p + q = n)$. Then, $W_1 := M_1 M_1^*$ and $W_2 := M_2 M_2^*$ are independent Wishart matrices with parameters p and q such that $MM^* = W_1 + W_2$. The matrix

$$Z := (W_1 + W_2)^{-1/2} W_1 (W_1 + W_2)^{-1/2} \tag{9.1}$$

has a Beta matrix distribution with parameters p, q. Let us now look at the singular values decomposition of $M : M = UDV$ with $U \in \mathcal{O}_m(\mathbb{R})$, $V \in \mathcal{O}_n(\mathbb{R})$, $D = (\Delta, 0) \in \mathcal{M}_{m,n}(\mathbb{R})$ and Δ is diagonal in $\mathcal{M}_{m,m}(\mathbb{R})$ with nonnegative entries. In fact, U and V are not uniquely determined but they can be chosen such that U and V are Haar-distributed on $\mathcal{O}_m(\mathbb{R})$ and $\mathcal{O}_n(\mathbb{R})$ respectively and that U, V, Δ are independent. Then, $MM^* = U\Delta^2 U^*$ and

$$\sqrt{W_1 + W_2} = \sqrt{MM^*} = U\Delta U^*. \tag{9.2}$$

Now, call X the $m \times p$ upper-left corner of V. A simple block-calculation shows that $M_1 = U\Delta X$. Therefore, seeing (9.2),

$$M_1 M_1^* = U\Delta X X^* \Delta U^* = \sqrt{MM^*}(UXX^*U^*)\sqrt{MM^*}. \tag{9.3}$$

It follows from (9.3) that $Z = (UX)(UX)^*$. Now, the law of X is invariant under left multiplication by an element of $\mathcal{O}_m(\mathbb{R})$ (coming from the inclusion $\mathcal{O}_m(\mathbb{R}) \subset \mathcal{O}_n(\mathbb{R})$ and the left-invariance of V under $\mathcal{O}_n(\mathbb{R})$-multiplication). Since U and X are independent, UX has then the same law as X and Z has the same law as XX^*. In conclusion, there are two equivalent ways to construct a Beta-distributed random matrix (with integer parameters), either from two independent Wishart distributions as in the one-dimensional case, or from the upper-left corner X of some Haar-distributed orthogonal matrix. This idea of corner projection (due to Collins [Col03] for the fixed-time situation) can be used at the process level, ie if one starts from Brownian motion on $\mathcal{O}_n(\mathbb{R})$ instead of Haar measure.

Notations

If M and N are semimartingales, we write $M \sim N$ (or $dM \sim dN$) when $M - N$ has finite variation and we use the notation $dM\, dN = d\langle M, N\rangle$.
As for matrices, $\mathcal{M}_{m,p}$ is the set on $m \times p$ real matrices, $\mathcal{S}_m$ the set on $m \times m$ real symmetric matrices, $\mathcal{SO}_n(\mathbb{R})$ the special orthogonal group, $\mathcal{A}_n$ the set of $n \times n$ real skew-symmetric matrices, 0_m and 1_m are the zero and identity matrices in $\mathcal{M}_{m,m}$ and $*$ denotes transposition. We also need

- $\Pi_{m,p} = \{M \in \mathcal{M}_{m,p} \,|\, MM^* \leq 1_m\}$,
- $\widehat{\mathcal{S}}_m = \{x \in \mathcal{S}_m \,|\, 0_m < x < 1_m\}$, $\bar{\mathcal{S}}_m = \{x \in \mathcal{S}_m \,|\, 0_m \leq x \leq 1_m\}$,
- $\widehat{\mathcal{S}}'_m$ (resp. $\bar{\mathcal{S}}'_m$) the set of matrices in $\widehat{\mathcal{S}}_m$ (resp. $\bar{\mathcal{S}}_m$) with distinct eigenvalues.

9.2 The case of integer dimensions

9.2.1 The upper-left corner process

Let Θ be a Brownian motion on $\mathcal{SO}_n(\mathbb{R})$. Θ is characterized by the stochastic differential equation :

$$d\Theta = \Theta \circ dA = \Theta\, dA + \frac{1}{2} d\Theta\, dA, \tag{9.4}$$

where $A = (a_{ij})$ is a Brownian motion on $\mathcal{A}_n$, the Lie algebra of $\mathcal{SO}_n(\mathbb{R})$. This means that the $(a_{ij},\, i < j)$ are independent real standard Brownian motions.

Remark 9.2.1. *It is an easy check that $dA' = \Theta \circ dA \circ \Theta^*$ defines another Brownian motion A' on $\mathcal{A}_n$. Thus, Θ can also be defined by $d\Theta = dA' \circ \Theta$. This corresponds to the fact that, on a compact Lie group, left-invariant and right-invariant Brownian motions coincide and allows one to talk about Brownian motion on $\mathcal{SO}_n(\mathbb{R})$ without further precision.*

For $h \in \mathcal{SO}_n(\mathbb{R})$, call $\pi_{m,p}(h) \in \mathcal{M}_{m,p}$ the upper-left corner of h with m lines and q columns.

Theorem 9.2.1. *If Θ is Brownian motion on $\mathcal{SO}_n(\mathbb{R})$, then $X = \pi_{m,p}(\Theta)$ is a diffusion on $\Pi_{m,p}$ whose infinitesimal generator is $\frac{1}{2}\Delta_{n,m,p}$, where :*

$$\begin{aligned} \Delta_{n,m,p} F \quad = \quad & \sum_{1 \leq i,i' \leq m,\, 1 \leq j,j' \leq p} (\delta_{ii'}\delta_{jj'} - X_{ij'}X_{i'j}) \frac{\partial^2 F}{\partial X_{ij} \partial X_{i'j'}} \\ & - (n-1) \sum_{1 \leq i \leq m,\, 1 \leq j \leq p} X_{ij} \frac{\partial F}{\partial X_{ij}}. \end{aligned}$$

Remark 9.2.2. *When $m = 1$, X is just the projection on the p first coordinates of the first line of Θ, which performs a Brownian motion on the unit sphere $S^{n-1} \subset \mathbb{R}^n$. So, it corresponds to the process described in the introduction. Its generator $\Delta_{n,1,p}$ is the generator $\Delta_{n-1,p}$ considered in [Bak96].*

Since the law of Brownian motion on $\mathcal{SO}_n(\mathbb{R})$ is invariant under transposition (see Remark 9.2.1), the following proposition is obvious :

Proposition 9.2.2. *If X is a diffusion governed by the infinitesimal generator $\frac{1}{2}\Delta_{n,m,p}$, then X^* is a diffusion governed by $\frac{1}{2}\Delta_{n,p,m}$.*

9.2.2 The Jacobi process

Theorem 9.2.3. *Let X be the diffusion governed by $\frac{1}{2}\Delta_{n,m,p}$. Then $J := XX^*$ is a diffusion process on $\bar{\mathcal{S}}_m$. If $p \geq m+1$, $q \geq m+1$ and $0_m < J_0 < 1_m$ then J satifies the following SDE :*

$$dJ = \sqrt{J} dB \sqrt{1_m - J} + \sqrt{1_m - J} dB^* \sqrt{J} + (pI - (p+q)J)\, dt, \tag{9.5}$$

where B is a Brownian motion on $\mathcal{M}_{m,m}$ and $q = n - p$. J will be called a Jacobi process of dimensions (p, q).

To express its infinitesimal generator, we need some notations. If g is a function from $\mathcal{S}_m$ to $\mathbb{R}$, the matrix $Dg = (D_{ij} g)_{1 \leq i,j \leq m} \in \mathcal{S}_m$ is defined by :

$$\begin{cases} D_{ij} g = D_{ji} g = \frac{1}{2} \frac{\partial g}{\partial x_{ij}} & \text{if } i < j \,, \\ D_{ii} g = \frac{\partial g}{\partial x_{ii}} \,. \end{cases}$$

D is just the gradient operator when $\mathcal{S}_m$ is given the Euclidean structure $< x, y >= \operatorname{tr}(xy)$ for $x, y \in \mathcal{S}_m$. The matrix-product rule is used to define compositions of differential operators, for instance $D^2 g = (D^2_{ij} g)_{1 \leq i,j \leq m} \in \mathcal{S}_m$ where $D^2_{ij} g = \sum_k D_{ik} D_{kj} g$. Then, the generator $\mathbf{A}_{p,q}$ of the Jacobi process of dimensions (p, q) is given by :

$$\mathbf{A}_{p,q}\, g(x) = \operatorname{tr}\left(2\left(xD^2 g - x(xD)^* Dg\right) + (p - (p+q)x)\, Dg\right) \,. \tag{9.6}$$

Remark 9.2.3. *The integer m does not appear in the previous generator. It only parametrizes the state space $\bar{\mathcal{S}}_m$. This is due to the special choice of XX^* and not X^*X, which breaks the "symmetry" of the roles played by m and p in $\Delta_{n,m,p}$.*

Remark 9.2.4. *When $m = 1$, J is the usual one-dimensional Jacobi process on $[0, 1]$ described in the introduction.*

Here is a proposition showing some symmetry between the roles of p and q. It can easily be seen either from the geometric interpretation or direcly from equation (9.5) :

Proposition 9.2.4. *If J is a Jacobi process of dimensions (p, q), then $1_m - J$ is a Jacobi process of dimensions (q, p).*

9.3 Study of the SDE for non-integer dimensions

The differential operator $\mathbf{A}_{p,q}$ makes good sense even if p and q are not integers. But, this is not enough to guarantee the existence of the corresponding stochastic process. This is why a careful examination of the SDE is necessary. A related investigation is carried out in [Bru91] for Wishart processes and has been of great inspiration to us.

Several ideas are directly borrowed from [Bru91]. However, the use of local times in the study of weak solutions makes our presentation of the case p or $q \in (m-1, m+1)$ simpler and more self-contained than the corresponding one for $\alpha \in (m-1, m+1)$ in [Bru91].

Theorem 9.3.1. *Suppose $x \in \bar{\mathcal{S}}_m$ and consider the SDE :*

$$dJ = \sqrt{J}\, dB\, \sqrt{1_m - J} + \sqrt{1_m - J}\, dB^* \sqrt{J} + (p1_m - (p+q)J)\, dt, \;\; J_0 = x, \tag{9.7}$$

where $J \in \bar{\mathcal{S}}_m$ and B is a Brownian motion on $\mathcal{M}_{m,m}$.

(i) *If $p \wedge q \geq m+1$ and $x \in \widehat{\mathcal{S}}_m$, (9.7) has a unique strong solution in $\widehat{\mathcal{S}}_m$.*

(ii) *If $p \wedge q > m-1$ and $x \in \bar{\mathcal{S}}'_m$, (9.7) has a unique solution in law in $\bar{\mathcal{S}}_m$.*

(iii) *If initially distinct, the eigenvalues of J remain so forever and can be labeled $\lambda_1 > \cdots > \lambda_m$. They satisfy the following SDE :*

$$d\lambda_i = 2\sqrt{\lambda_i(1-\lambda_i)}\, db_i + \Big\{ (p - (p+q)\lambda_i) + \sum_{j\,(\neq i)} \frac{\lambda_i(1-\lambda_j) + \lambda_j(1-\lambda_i)}{\lambda_i - \lambda_j} \Big\} dt, \tag{9.8}$$

for $1 \leq i \leq m$ and independent real Brownian motions $b_1, \ldots, b_m$.

Remark 9.3.1. *(iii) says that the eigenvalues perform a diffusion process governed by the generator :*

$$\begin{aligned} \boldsymbol{G}_{p,q} &= 2\sum_i \lambda_i(1-\lambda_i)\partial_i^2 + \sum_i (p-(p+q)\lambda_i)\partial_i + \sum_{i\neq j} \frac{\lambda_i + \lambda_j - 2\lambda_i\lambda_j}{\lambda_i - \lambda_j}\partial_i \\ &= 2\sum_i \lambda_i(1-\lambda_i)\partial_i^2 + \sum_i \left(p - (m-1) - (p+q-2(m-1))\lambda_i\right)\partial_i \\ &\quad + \sum_{i\neq j} \frac{2\lambda_i(1-\lambda_i)}{\lambda_i-\lambda_j}\partial_i \end{aligned} \tag{9.9}$$

In the language of Chapter 12, we have $\mathbf{G}_{p,q} = \mathcal{L}^{(1)}$ for $\mathbf{L} = \sum_i (a(\lambda_i)\partial_i^2 + b(\lambda_i)\partial_i)$, $a(\lambda) = 2\lambda(1-\lambda)$, $b(\lambda) = \alpha - (\alpha+\beta)\lambda$, $\alpha = p-(m-1)$ and $\beta = q-(m-1)$.

The rest of this section is devoted to some details about the architecture of the proof of Theorem 9.3.1. The first step is the computation of some relevant stochastic differentials :

Proposition 9.3.2. *The following relations are valid up to time*

$$T = \inf\{t \mid J_t \notin \widehat{\mathcal{S}}_m\} = \inf\{t \mid \det(J_t)\det(1_m - J_t) = 0\} :$$

- $$\begin{aligned} d\big(\det(J)\big) &= 2\det(J)\mathrm{tr}\big[(1_m - J)^{1/2}J^{-1/2}dB\big] \\ &\quad + \det(J)\big[(p-m+1)\mathrm{tr}(J^{-1}) - m(p+q-m+1)\big]dt, \end{aligned}$$
- $d\big(\alpha \log\det(J) + \beta\log\det(1_m - J)\big) = \mathrm{tr}\big(H^{\alpha,\beta}\, dB\big) + V^{\alpha,\beta}\, dt,$

where

$$H^{\alpha,\beta} = 2\big(\alpha(1_m - J)^{1/2}J^{-1/2} - \beta(1_m - J)^{-1/2}J^{1/2}\big)$$

and

$$V^{\alpha,\beta} = \alpha(p-m-1)\,\mathrm{tr}(J^{-1}) + \beta(q-m-1)\,\mathrm{tr}\big((1_m - J)^{-1}\big) - (\alpha+\beta)m(p+q-m-1).$$

Equipped with such relations, we can prove strong existence and uniqueness in the following easy case where the process never hits the boundary :

Proposition 9.3.3. *If $p \wedge q \geq m+1$ and $J_0 = x \in \widehat{\mathcal{S}}_m$ then (9.7) has a unique strong solution in $\widehat{\mathcal{S}}_m$.*

Then, we can establish non-collision of the eigenvalues and describe their trajectories :

Proposition 9.3.4. *If J is a solution of (9.7) and $J_0 = x \in \widehat{\mathcal{S}}'_m$ then $\forall t > 0$, $J_t \in \widehat{\mathcal{S}}'_m$ and the eigenvalues $\lambda_1(t) > \cdots > \lambda_m(t)$ satisfy (9.8).*

If p or $q \in (m-1, m+1)$, the process may hit the boundary of $\bar{\mathcal{S}}_m$ and then might exit $\bar{\mathcal{S}}_m$, which causes trouble for the extraction of square roots in equation (9.7). This is analogous to the one-dimensional Bessel squared process situation in which case the problem is circumvented by writing the equation with some absolute value and then by proving that the process remains nonnegative forever (see Revuz-Yor [RY99]). Similarly, we introduce an auxiliary equation involving positive parts to deal with our case of p or $q \in (m-1, m+1)$. Because of the positive parts, the coefficients in this new equation won't be smooth functions anymore but only 1/2-Hölder. Consequently, some work will have to be done concerning existence and uniqueness of solutions in this multi-dimensional context. For $x \in \mathcal{S}_m$, $x = h\,\mathrm{diag}(\lambda_1, \ldots, \lambda_m)h^*$ with $h \in \mathcal{O}_m(\mathbb{R})$, we define $x^+ = h\,\mathrm{diag}(\lambda_1^+, \ldots, \lambda_m^+)h^*$ where $\lambda_i^+ = \max(\lambda_i, 0)$. Note that $x \mapsto x^+$ is continuous on $\mathcal{S}_m$.

Proposition 9.3.5. *For all $p, q \in \mathbb{R}$, the SDE*

$$dJ = \sqrt{J^+}\,dB\,\sqrt{(1_m - J)^+} + \sqrt{(1_m - J)^+}\,dB^*\,\sqrt{J^+} + (p1_m - (p+q)J)\,dt, \tag{9.10}$$

with $J_0 = x \in \mathcal{S}_m$ has a solution $J_t \in \mathcal{S}_m$ defined for all $t \geq 0$.
If the eigenvalues of $J_0 = x$ are $\lambda_1(0) > \cdots > \lambda_m(0)$, the following SDE is verified up to time $\tau = \inf\{t \mid \exists\, i < j,\ \lambda_i(t) = \lambda_j(t)\}$:

$$\begin{aligned} d\lambda_i &= 2\sqrt{\lambda_i^+(1-\lambda_i)^+}\,db_i + \Big\{ (p - (p+q)\lambda_i) \\ &\qquad + \sum_{j\,(\neq i)} \frac{\lambda_i^+(1-\lambda_j)^+ + \lambda_j^+(1-\lambda_i)^+}{\lambda_i - \lambda_j} \Big\}dt, \end{aligned} \tag{9.11}$$

for $1 \leq i \leq m$ and independent real Brownian motions $b_1, \ldots, b_m$.

Then, we need to show that eigenvalues of J stay in $[0, 1]$ if starting in $[0, 1]$. We can imitate the intuitive argument from [Bru91] to support this claim. Suppose the smallest eigenvalue satisfies $\lambda_m(t) = 0$. For $1 \leq i \leq m-1$, we have $\lambda_i(t) \geq \lambda_m(t) = 0$, thus $\lambda_i^+(t) = \lambda_i(t)$. Seeing the equation governing λ_m, the infinitesimal drift received by λ_m between times t and $t + dt$ becomes $(p - (m-1))dt > 0$, forcing λ_m to stay nonnegative. The same reasoning shows that λ_1 will stay below 1 since $q > m - 1$. Indeed, we can make this rigorous by proving the following

Proposition 9.3.6. *If (9.11) is satisfied with* $1 \geq \lambda_1(0) > \cdots > \lambda_m(0) \geq 0$ *and* $p \wedge q > m - 1$, *then*

(i) *calling* $L_t^a(\xi)$ *the local time spent by a process* ξ *at level* a *before time* t, *we have* $L_t^0(\lambda_m) = L_t^1(\lambda_1) = 0$ *for* $t < \tau$,

(ii) *for all* t, $\mathbb{P}\left[t < \tau, (\lambda_m(t) < 0) \text{ or } (\lambda_1(t) > 1)\right] = 0$,

(iii) $\left\{t < \tau;\, (\lambda_m(t) = 0) \text{ or } (\lambda_1(t) = 1)\right\}$ *has zero Lebesgue measure a.s.*

The previous proposition says that $J = J^+$ and $(1_m - J)^+ = 1_m - J$ up to time τ, which makes it possible to perform the same computations as for Proposition 9.3.4 and to prove the

Proposition 9.3.7. *If* J *is a solution of (9.10), then* $\tau = \infty$ *a.s. Hence, all eigenvalues of* J *are in* $[0, 1]$ *forever and* J *is solution of (9.7).*

This concludes the existence part when $p \wedge q > m - 1$ and $1 \geq \lambda_1(0) > \cdots > \lambda_m(0) \geq 0$.

For uniqueness in law if $p \wedge q > m - 1$, we can appeal to Girsanov relations (see Theorem 9.4.3) to change dimensions and invoke uniqueness (pathwise hence weak) for $p \wedge q \geq m + 1$. This proves uniqueness in law up to time T, since the Girsanov are stated on the sigma-fields $\mathcal{F}_t \cap \{T > t\}$. But we can repeat this argument between T and the next hitting time of the boundary and so on to conlude about uniqueness in law.

Remark 9.3.2. *When* p *or* $q \in (m-1, m+1)$, *we conjecture that existence and uniqueness in law hold even if the eigenvalues are not distinct initially. But the absence of explicit expression for the semi-group makes it difficult for us to carry an approximation argument as in [Bru91].*

Remark 9.3.3. *Even when* $p \wedge q \geq m + 1$, *we don't know how to prove that*

$$\mathbb{P}_x\left(\forall t > 0,\, \forall i \neq j,\, \lambda_i(t) \neq \lambda_j(t)\right) = 1 \quad (\text{resp. } \mathbb{P}_x\left(\forall t > 0,\, 0_m < J_t < 1_m\right) = 1)$$

when the eigenvalues of x *are not necessarily distinct (resp. when* $\lambda_1(0) = 1$ *or* $\lambda_m(0) = 0$*). By the Markov property and the result when the eigenvalues of* x *are distinct (resp.*

when $\lambda_1(0) < 1$ and $\lambda_m(0) > 0$), it would be enough to prove that, for fixed $t > 0$, we have $\mathbb{P}_x(\forall i \neq j,\, \lambda_i(t) \neq \lambda_j(t)) = 1$ (resp. $\mathbb{P}_x(0_m < J_t < 1_m) = 1$). So it would be sufficient to know that the semi-group has a density with respect to its invariant measure (since this one is absolutely continuous with respect to Lebesgue measure on $\bar{\mathcal{S}}_m$, see Section 9.4.1). We believe this property to be true as in the one-dimensional case but we are unable to find a relevant general theorem in the literature.

9.4 Properties of the Jacobi process

9.4.1 Invariant measures

Proposition 9.4.1. *Suppose $n \geq m + p$. Then the generator $\Delta_{n,m,p}$ has reversible probability measure $\nu_{n,m,p}$ defined by*

$$\nu_{n,m,p}(dX) = c_{n,m,p} \det(1_m - XX^*)^{(n-1-p-m)/2} \mathbf{1}_{\Pi_{m,p}}(X)\, dX\,,$$

and associated "carré du champ" given by :

$$\mathbf{\Gamma}(F,G) = \sum_{1\leq i,i'\leq m\,,\,1\leq j,j'\leq p} (\delta_{ii'}\delta_{jj'} - X_{ij'}X_{i'j})\, \frac{\partial F}{\partial X_{ij}} \frac{\partial G}{\partial X_{i'j'}}\,.$$

Thus, for F, G vanishing on the boundary of $\Pi_{m,p}$, the following integration by parts formula holds :

$$\int G\, \Delta_{n,m,p} F\, d\nu_{n,m,p} = \int F\, \Delta_{n,m,p} G\, d\nu_{n,m,p} = -\int \mathbf{\Gamma}(F,G)\, d\nu_{n,m,p}.$$

Remark 9.4.1. *Since Haar measure is the invariant measure of Brownian motion on $\mathcal{SO}_n(\mathbb{R})$, this proposition incidently shows that its push-forward by projection on the upper-left corner is $\nu_{n,m,p}$. This result was first derived in [Ol'90] and [Col03] by direct computations of Jacobians.*

Proposition 9.4.2. *Suppose $p > m - 1$ and $q > m - 1$. Let us define the probability measure $\mu_{p,q}$ on $\mathcal{S}_m$ by :*

$$\mu_{p,q}(dx) = \frac{\Gamma_m((p+q)/2)}{\Gamma_m(p/2)\Gamma_m(q/2)} \det(x)^{(p-m-1)/2} \det(1_m - x)^{(q-m-1)/2} \mathbf{1}_{0_m \leq x \leq 1_m}\, dx\,,$$

where Γ_m is the multi-dimensional Gamma function (see section 8.5 for a definition). Then the generator $\mathbf{A}_{p,q}$ has reversible probability measure $\mu_{p,q}$ and associated "carré du champ" given by

$$\mathbf{\Gamma}(f,g) = 2\,\mathrm{tr}(xDfDg - xDfxDg)\,.$$

Thus, for f, g vanshing on the boundary of $\{0_m \leq x \leq 1_m\}$, the following integration by parts formula holds :

$$\int g\, \mathbf{A}_{p,q} f\, d\mu_{p,q} = \int f\, \mathbf{A}_{p,q} g\, d\mu_{p,q} = -\int \mathbf{\Gamma}(f, g)\, d\mu_{p,q}$$

Remark 9.4.2. $\det(x)^\alpha \det(1_m - x)^\beta$ *is integrable on $\{0_m \leq x \leq 1_m\} \subset \mathcal{S}_m$ if and only if $\alpha > -1$ and $\beta > -1$, which corresponds to the constraint $p > m-1$ and $q > m-1$ in Proposition 9.4.2. In the case of integers p and $q = n - p$, this is equivalent to $p \geq m$ and $n \geq p + m$. This is consistent with the fact that, when $p < m$ for example, $J = XX^*$ is of rank at most p and thus has no density with respect to Lebesgue measure on $\mathcal{S}_m$.*

9.4.2 Girsanov relations

Our goal is to establish absolute continuity relations for Jacobi processes of different dimensions. In the case of Wishart processes, such relations have turned out to be of some interest, in particular to define matrix extensions of the Hartman-Watson distributions (see [DMDMY04], which is Chapter 8 of this thesis). Here, for example, we obtain an expression for the law of the hitting time T of the boundary in terms of negative moments of the fixed-time distribution (see corollary 9.4.6) . Unfortunately, such moments don't seem to be easily computed.

We use a matrix version of Girsanov theorem as stated in [DMDMY04] and which we now recall. We denote by $\mathbb{P}_x^{p,q}$ the law of the Jacobi process of dimensions (p, q) and starting from x. Suppose that B is a $\mathbb{P}_x^{p,q}$-Brownian $m \times m$ matrix and that H is a $\mathcal{S}_m$-valued predictable process such that

$$\mathcal{E}_t = \exp\left(\int_0^t \operatorname{tr}(H_s dB_s) - \frac{1}{2} \int_0^t \operatorname{tr}(H_s^2) ds \right)$$

is a $\mathbb{P}_x^{p,q}$-martingale. Define the new probability measure by

$$\widehat{\mathbb{P}}_x^{p,q}|_{\mathcal{F}_t} = \mathcal{E}_t \,.\, \mathbb{P}_x^{p,q}|_{\mathcal{F}_t}.$$

Then $\widehat{B}_t = B_t - \int_0^t H_s\, ds$ is a $\widehat{\mathbb{P}}_x^{p,q}$-Brownian matrix.

We apply this with $H = H^{\alpha,\beta}$ defined in Proposition 9.3.2. Then, we have

$$dJ = \sqrt{J}\, d\widehat{B}\, \sqrt{1_m - J} + \sqrt{1_m - J}\, d\widehat{B}^* \sqrt{J} + (p' 1_m - (p' + q')J)\, dt,$$

with $p' = p + 4\alpha$ and $q' = q + 4\beta$. Thanks to Proposition 9.3.2, we can compute $\mathcal{E}_t$ more explicitly (see Section 9.5) to get :

Theorem 9.4.3. *If* $T = \inf\{t \mid \det J_t(1_m - J_t) = 0\}$, *we have :*

$$\mathbb{P}_x^{p',q'}|_{\mathcal{F}_t \cap \{T>t\}} = \left(\frac{\det J_t}{\det x}\right)^{\alpha} \left(\frac{\det(1_m - J_t)}{\det(1_m - x)}\right)^{\beta}$$

$$\exp\left(-\int_0^t ds\left(c + u\,\mathrm{tr}\big(J_s^{-1}\big) + v\,\mathrm{tr}\big((1_m - J_s)^{-1}\big)\right)\right) \mathbb{P}_x^{p,q}|_{\mathcal{F}_t \cap \{T>t\}},$$

where $\alpha = (p'-p)/4$, $\beta = (q'-q)/4$, $u = \frac{p'-p}{4}\left(\frac{p'+p}{2} - m - 1\right)$, $v = \frac{q'-q}{4}\left(\frac{q'+q}{2} - m - 1\right)$, $c = m\left(\frac{p'+q'-p-q}{4}\right)\left(m + 1 - \frac{p'+q'+p+q}{2}\right)$.

Corollary 9.4.4. *If* $p + q = 2(m+1)$, *then*

$$\mathbb{P}_x^{q,p}|_{\mathcal{F}_t \cap \{T>t\}} = \left(\frac{\det\big(J_t(1_m - J_t)^{-1}\big)}{\det\big(x(1_m - x)^{-1}\big)}\right)^{(q-p)/4} \mathbb{P}_x^{p,q}|_{\mathcal{F}_t \cap \{T>t\}}.$$

Since $\mathbb{P}_x^{p,q}(T = \infty) = 1$ for $p \wedge q \geq m + 1$ and $0_m < x < 1_m$, we also get :

Corollary 9.4.5. *If* $\mathbb{P}^{(\mu,\nu)}$ *denotes* $\mathbb{P}^{m+1+2\mu, m+1+2\nu}$, *then, for* $0 \leq \mu, \nu < 1$,

$$\mathbb{P}_x^{(-\mu,-\nu)}|_{\mathcal{F}_t \cap \{T>t\}} = \left(\frac{\det J_t}{\det x}\right)^{-\mu} \left(\frac{\det(1_m - J_t)}{\det(1_m - x)}\right)^{-\nu} \mathbb{P}_x^{(\mu,\nu)}|_{\mathcal{F}_t}.$$

Corollary 9.4.6. *For* $0 \leq \mu, \nu < 1$,

$$\mathbb{P}_x^{(-\mu,-\nu)}(T > t) = \mathbb{E}_x^{(\mu,\nu)}\left[\left(\frac{\det J_t}{\det x}\right)^{-\mu} \left(\frac{\det(1_m - J_t)}{\det(1_m - x)}\right)^{-\nu}\right].$$

Remark 9.4.3. *We refer to [Mui82] (see also section 8.5) for a definition of matrix hypergeometric functions and the partial differential equations they satisfy. We notice that, if* $\phi(x) = {}_2\mathbf{F}_1(a, b; c; x)$, *we find that* ϕ *is an eigenfunction for* $\boldsymbol{A}_{p,q}$ *:*

$$\boldsymbol{A}_{p,q}\phi = \boldsymbol{G}_{p,q}\phi = \mu\,\phi,$$

if $p = 2c$, $p + q = 2(a + b) + m + 1$ *and* $\mu = 2mab$. *Therefore,* $\phi(J_t)e^{-\mu t}$ *is a* $\mathbb{P}^{p,q}$ *local martingale. However, unlike in the one-dimensional case, this is not enough to compute the law of* T. *From a different point of view, it would be interesting to relate the following known properties of hypergeometric functions :*

$$\begin{aligned} {}_2\mathbf{F}_1(a, b; c; x) &= \det(1_m - x)^{-b}\ {}_2\mathbf{F}_1(c - a, b; c; -x(1_m - x)^{-1}) \\ &= \det(1_m - x)^{c-a-b}\ {}_2\mathbf{F}_1(c - a, c - b; c; x) \end{aligned}$$

to properties of the Jacobi process (Girsanov relations in particular).

9.4.3 Connection with real Jacobi processes conditioned to stay in a Weyl chamber

If we start from Brownian motion on the group $\mathcal{U}_n(\mathbb{C})$ of complex unitary matrices instead of $\mathcal{SO}_n(\mathbb{R})$, we can define a Hermitian Jacobi process with values in the space of Hermitian $m\times m$ matrices. Its eigenvalues will still perform a diffusion process whose generator we call $\mathbf{H}_{p,q}$ (see Section 9.5). On the other hand, consider the generator $\mathbf{L}_{\alpha,\beta}$ of m real i.i.d. Jacobi processes of dimensions (α,β). Now, define $h(\lambda)=\prod_{i<j}(\lambda_i-\lambda_j)$ for $\lambda\in\mathbb{R}^m$. Then, h is positive on $W=\{\lambda_1>\cdots>\lambda_m\}$ and is an eigenfunction of $\mathbf{L}_{\alpha,\beta}$ (sse Section 9.5). Thus, we can define the Doob-transform of $\mathbf{L}_{\alpha,\beta}$ by h, which gives the new generator : $\widehat{\mathbf{L}}_{\alpha,\beta}f=\mathbf{L}_{\alpha,\beta}f+\Gamma(\log h,f)$ where $\Gamma(f,g)=\mathbf{L}_{\alpha,\beta}(fg)-f\mathbf{L}_{\alpha,\beta}g-g\mathbf{L}_{\alpha,\beta}f$ (see, for example, Part 3, Chap. VIII in [RY99]). $\widehat{\mathbf{L}}_{\alpha,\beta}$ can be thought of as the generator of m real i.i.d. Jacobi processes of dimensions (α,β) conditioned to stay in W forever.

Proposition 9.4.7. *We have $\widehat{\boldsymbol{L}}_{2(p-m+1),2(q-m+1)}=\boldsymbol{H}_{p,q}$. In other words, the eigenvalues of the Hermitian Jacobi process of dimensions (p,q) perform m real iid Jacobi processes of dimensions $(2(p-m+1),2(q-m+1))$ conditioned never to collide (in the sense of Doob).*

Remark 9.4.4. *A very similar discussion appears in [KO01]. It is shown that Bessel squared processes of dimensions $2(p-m+1)$ conditioned never to collide have the same law as the eigenvalues of the Laguerre process of dimension p on the space of Hermitian $m\times m$ matrices.*

9.5 Proofs

Proof of Proposition 9.2.1. Let us adopt the following block notations :

$$\Theta=\begin{pmatrix} X & Y\\ Z & W\end{pmatrix}\quad\text{and}\quad A=\begin{pmatrix}\alpha & \beta\\ \gamma & \varepsilon\end{pmatrix}$$

where $X\in\mathcal{M}_{m,p}$, $Y\in\mathcal{M}_{m,n-m}$, $\alpha\in\mathcal{A}_m$, $\beta=-\gamma*\in\mathcal{M}_{m,n-p}$, $\varepsilon\in\mathcal{A}_{n-m}$. Then,

$$\begin{cases} dX=X\circ d\alpha+Y\circ d\gamma=Xd\alpha+Yd\gamma+\frac{1}{2}(dXd\alpha+dYd\gamma)\\ dY=X\circ d\beta+Y\circ d\varepsilon=Xd\beta+Yd\varepsilon+\frac{1}{2}(dXd\beta+dYd\varepsilon)\end{cases}\tag{9.12}$$

Seeing (9.12) and using the independence between α,β,ε, $d\alpha d\alpha=-(p-1)1_m dt$, $d\beta d\beta^*=(n-p)1_m dt$, we get

$$\begin{cases} dXd\alpha=Xd\alpha d\alpha+Yd\gamma d\alpha=Xd\alpha d\alpha=-(p-1)Xdt,\\ dYd\gamma=Xd\beta d\gamma+Yd\varepsilon d\gamma=Xd\beta d\gamma=-(n-p)Xdt.\end{cases}\tag{9.13}$$

Thus, the finite variation part of dX is $-\frac{n-1}{2}Xdt$. Then, noting that $\alpha_{jj}=0$,

$$dX_{ij}\sim\sum_{k\neq j}X_{ik}d\alpha_{kj}+\sum_{l}Y_{il}d\gamma_{lj}.$$

Then, we use relations $d\alpha_{kj}d\alpha_{k'j'}=(\delta_{kk'}\delta_{jj'}-\delta_{kj'}\delta_{k'j})\,dt$ and $d\gamma_{lj}d\gamma_{l'j'}=\delta_{ll'}\delta_{jj'}\,dt$ to compute

$$\begin{aligned}dX_{ij}dX_{i'j'} &= \left(\sum_{k\neq j}X_{ik}d\alpha_{kj}+\sum_{l}Y_{il}d\gamma_{lj}\right)\left(\sum_{k'\neq j'}X_{i'k'}d\alpha_{k'j'}+\sum_{l'}Y_{i'l'}d\gamma_{l'j'}\right)\\ &= \sum_{k\neq j,\,k'\neq j'}X_{ik}X_{i'k'}d\alpha_{kj}d\alpha_{k'j'}+\sum_{l}Y_{il}Y_{i'l'}d\gamma_{lj}d\gamma_{l'j'}\\ &= (1)\;+\;(2).\end{aligned}$$

Let us consider the terms separately :

$$\begin{aligned}(1) &= \begin{cases}\sum_{k,k'\neq j}X_{ik}X_{i'k'}d\alpha_{kj}d\alpha_{k'j}=\sum_{k\neq j}X_{ik}X_{i'k}dt & \text{if } j=j'\\ -X_{ij'}X_{i'j}dt \quad \text{if } j\neq j'\end{cases}\\ (2) &= \delta_{jj'}\sum_{l}Y_{il}Y_{i'l}dt.\end{aligned}$$

Since $\sum_{k\neq j}X_{ik}X_{i'k}+\sum_{l}Y_{il}Y_{i'l}=\delta_{ii'}-X_{ij}X_{i'j}$, this eventually leads to :

$$dX_{ij}dX_{i'j'}=(\delta_{ii'}\delta_{jj'}-X_{ij'}X_{i'j})dt. \tag{9.14}$$

From this, it is standard to deduce that X is a diffusion process with generator $\frac{1}{2}\Delta_{n,p,r}$. □

Proof of Proposition 9.2.3. The fact that J is a Markov process is a direct consequence of the invariance of the law of X under maps $M\mapsto Mt$ for $t\in SO(p)$. We can also apply $\Delta_{n,m,p}$ to a function $F(M)=g(MM^*)$ and see that the resulting function only depends on MM^*, which gives the expression of $\mathbf{L}_{p,q}g$.
If $p\geq m+1$, $q\geq m+1$ and $0_m<J_0<1_m$, we present a direct approach towards the SDE. We keep notations of the proof of Theorem 9.2.1. If $T=\inf\{t\,|\,J_t\notin\bar{\mathcal{S}}_m\}$, the following computations are valid up to time T :

$$\begin{aligned}dJ &= dX\,X^*+XdX^*+dXdX^* \\ &\sim (Xd\alpha+Yd\gamma)X^*+X(-d\alpha X^*+d\gamma^*Y^*)\\ &\sim Yd\gamma X^*+Xd\gamma^*Y^*\\ &= \sqrt{1_m-J}dB^*\sqrt{J}+\sqrt{J}dB\sqrt{1_m-J},\end{aligned} \tag{9.15}$$

if we define $B_t = \int_0^t J_s^{-1/2} X_s d\gamma_s^* Y_s^* (1_m - J_s)^{-1/2} \in \mathcal{M}_{m,m}$. Then,

$$dB_{ij} = \sum_{k,l,s,t} (J^{-1/2})_{ik} X_{kl} d\gamma_{sl} Y_{ts} ((1_m - J)^{-1/2})_{tj}.$$

Using the symmetry of $J = XX^*$ and $1_m - J = YY^*$, we get

$$\begin{aligned} &\frac{dB_{ij}\, dB_{i'j'}}{dt} \\ &= \sum_{k,k',l,l',s,t} (J^{-1/2})_{ik} X_{kl} Y_{ts} ((1_m - J)^{-1/2})_{tj} \, (J^{-1/2})_{i'k'} X_{k'l} Y_{t's} ((1_m - J)^{-1/2})_{t'j'} \\ &= \sum_{k,k',l,l'} (J^{-1/2})_{ik} (XX^*)_{kk'} (J^{-1/2})_{k'i'} ((1_m - J)^{-1/2})_{jt} (YY^*)_{tt'} ((1_m - J)^{-1/2})_{t'j'} \\ &= (J^{-1/2} J J^{-1/2})_{ii'} ((1_m - J)^{-1/2} (1_m - J)(1_m - J)^{-1/2})_{jj'} = \delta_{ii'} \delta_{jj'} \,. \end{aligned}$$

This proves that B is a Brownian motion on $\mathcal{M}_{m,m}$. From (9.15), we deduce that the finite variation part of dJ is

$$\begin{aligned} & -(n-1)Jdt + (Xd\alpha + Yd\gamma)(-d\alpha X^* + d\gamma^* Y^*) \\ = \;& -(n-1)Jdt - Xd\alpha d\alpha X^* + Yd\gamma d\gamma^* Y^* \\ = \;& -(n-1)Jdt + (p1_m - J)dt \\ = \;& (p1_m - nJ)dt, \end{aligned}$$

which establishes the equation satisfied by J up to time T. But, as will be shown in the proof of Proposition 9.3.3, if J satifies such an equation, then $\mathbb{P}(T = +\infty) = 1$, which finishes the proof. ☐

Proof of Proposition 9.3.2. First, dy differentiation of the determinant, we find

$$\begin{aligned} d(\det J) &= \operatorname{tr}(\tilde{J} dJ) + (1-m)\operatorname{tr}(\tilde{J})dt + m(m-1)\det(J)dt \\ &= 2\det(J)\operatorname{tr}\left[(1_m - J)^{1/2} J^{-1/2} dB\right] \\ &+ \det(J)\left[(p - m + 1)\operatorname{tr}(J^{-1} - m(p + q - m + 1)\right] dt, \end{aligned}$$

where $\tilde{J}$ is the comatrix of J. Thus, $d\langle \det J \rangle = 4(\det J)^2 (\operatorname{tr}(J^{-1}) - m)dt$ and

$$\begin{aligned} d(\log\det J) = 2\operatorname{tr}\left[(1_m - J)^{1/2} J^{-1/2} dB\right] + \\ \left[(p - m - 1)\operatorname{tr}(J^{-1}) - m(p + q - m - 1)\right] dt. \end{aligned}$$

Since $1_m - J$ is a Jacobi process of dimensions (q, p) governed by $-B^*$, we deduce the analogous relation for $d(\log\det(1_m - J))$ and we can conclude the proof. ☐

Proof of Proposition 9.3.3. $x \mapsto \sqrt{x}$ and $x \mapsto \sqrt{1_m - x}$ are analytic on $\widehat{\mathcal{S}}_m$ (see, for example, p. 134 in [RW00]). Thus, (9.7) has a unique strong solution in $\widehat{\mathcal{S}}_m$ up to T. We use a modification of Mc Kean's celebrated argument to show that $T = \infty$ a.s. If $\Gamma = \log(\det(J)\ \det(1_m - J))$, Proposition 9.3.2 gives

$$d\Gamma = \operatorname{tr}(H^{1,1}dB) + V^{1,1}dt.$$

Since the local martingale part is a time-change of Brownian motion β_{C_t} and $V^{1,1} \geq -2m(p+q-m-1)$, we have

$$\Gamma_t - \Gamma_0 + 2m(p+q-m-1)t \geq \beta_{C_t}.$$

On $\{T < \infty\}$, $\lim_{t\to T}\Gamma_t = -\infty$ so that $\lim_{t\to T}\beta_{C_t} = -\infty$. Since Brownian motion never goes to infinity without oscillating, we get $\mathbb{P}(T < \infty) = 0$. □

Proof of Proposition 9.3.4. Let $\tau = \inf\{t \mid \exists\, i < j,\ \lambda_i(t) = \lambda_j(t)\}$. The derivation of the equations satisfied by the eigenvalues up to time τ is classical in similar contexts (see [NRW86] or [Bru91]). This computation is detailed just after this proof. Our task is to show that $\tau = \infty$ a.s. Set $V(\lambda_1, \ldots, \lambda_m) = \sum_{i>j}\log(\lambda_i - \lambda_j)$, and compute $\mathbf{G}_{p,q}V = C_{m,p,q}$ (see Lemma 9.5.1). If $\Omega_t = V\big(\lambda_1(t), \ldots, \lambda_m(t)\big)$, we have

$$\Omega_t = \Omega_0 + t\, C_{m,p,q} + \text{ local martingale},$$

which allows for the same modification of Mc Kean's argument as for the proof of Proposition 9.3.3. □

Equations for the eigenvalues. Let us diagonalise $J = U\Lambda U^*$ with U (resp. Λ) a continuous semimartingale with values in $\mathcal{SO}(m)$ (resp. in the diagonal $m \times m$ matrices). This can be done up to time τ since $J \mapsto (U, \Lambda)$ is smooth as long as the eigenvalues of J are distinct (again, this is standard in such a context, see [NRW86]). Define $dX = dU^* \circ U \in \mathcal{A}_m$ and $dN = U^* \circ dJ \circ U$. Then, $d\Lambda = dX \circ \Lambda - \Lambda \circ dX + dN$, which can be written

$$\begin{cases} d\lambda_i = dN_{ii} \\ 0 = \lambda_j \circ dX_{ij} - \lambda_i \circ dX_{ij} + dN_{ij}. \end{cases} \tag{9.16}$$

Therefore, $dX_{ij} = \frac{1}{\lambda_i - \lambda_j} \circ dN_{ij}$. From (9.7), we can compute :

$$dJ_{st}dJ_{s't'} = J_{ss'}(1_m - J)_{tt'} + J_{st'}(1_m - J)_{ts'} + J_{ts'}(1_m - J)_{st'} + J_{tt'}(1_m - J)_{ss'},$$

and

$$\begin{aligned}
dN_{ik}dN_{k'j} &= \left(\sum_{s,t} U_{si}dJ_{st}U_{tk}\right)\left(\sum_{s',t'} U_{s'k'}dJ_{s't'}U_{t'j}\right) \\
&= \sum_{s,t,s',t'} U_{si}U_{tk}U_{s'k'}U_{t'j}\Big\{J_{ss'}(1_m-J)_{tt'} + J_{st'}(1_m-J)_{ts'} \\
&\quad + J_{ts'}(1_m-J)_{st'} + J_{tt'}(1_m-J)_{ss'}\Big\}\,dt \\
&= \Big\{(U^*JU)_{ik'}(U^*(1_m-J)U)_{kj} + (U^*JU)_{ij}(U^*(1_m-J)U)_{kk'} \\
&\quad +(U^*JU)_{kk'}(U^*(1_m-J)U)_{ij} + (U^*JU)_{kj}(U^*(1_m-J)U)_{ik'}\Big\}\,dt \\
&= \Big\{\Lambda_{ik'}(1_m-\Lambda)_{kj} + \Lambda_{ij}(1_m-\Lambda)_{kk'} \\
&\qquad +\Lambda_{kk'}(1_m-\Lambda)_{ij} + \Lambda_{kj}(1_m-\Lambda)_{ik'}\Big\}\,dt.
\end{aligned}$$

Let dM (resp. dF) be the local martingale (resp. the finite variation) part of dN. We have

$$\begin{aligned}
dF &= U^*(p1_m-(p+q)J)U + \frac{1}{2}(dU^*dJU + U^*dJdU) \\
&= (p1_m-(p+q)\Lambda) + \frac{1}{2}\left((dU^*U)(U^*dJU) + (U^*dJU)(U^*dU)\right) \\
&= (p1_m-(p+q)\Lambda) + \frac{1}{2}\left(dXdN + (dXdN)^*\right).
\end{aligned}$$

Then,

$$\begin{aligned}
(dXdN)_{ij} &= \sum_{k\,(\neq i)} dX_{ik}dN_{kj} = \sum_{k\,(\neq i)} \frac{1}{\lambda_i-\lambda_k}dN_{ik}dN_{kj} \\
&= \delta_{ij}\sum_{k\,(\neq i)} \frac{\lambda_i(1-\lambda_k)+\lambda_k(1-\lambda_i)}{\lambda_i-\lambda_k}\,dt.
\end{aligned}$$

This shows that $(dXdN)^* = dXdN$ and that

$$dF_{ij} = \delta_{ij}\Big((p-(p+q)\lambda_i) + \sum_{k\,(\neq i)} \frac{\lambda_i(1-\lambda_k)+\lambda_k(1-\lambda_i)}{\lambda_i-\lambda_k}\Big)\,dt.$$

Next,

$$\begin{aligned}
dM_{ii}dM_{jj} &= dN_{ii}dN_{jj} \\
&= \Big(\Lambda_{ij}(1_m-\Lambda)_{ij} + \Lambda_{ij}(1_m-\Lambda)_{ij} \\
&\quad +\Lambda_{ij}(1_m-\Lambda)_{ij} + \Lambda_{ij}(1_m-\Lambda)_{ij}\Big)\,dt \\
&= 4\delta_{ij}\lambda_i(1-\lambda_i)\,dt,
\end{aligned}$$

which proves $dM_{ii} = 2\sqrt{\lambda_i(1-\lambda_i)}\, db_i$ for some independent Brownian motions $b_1, \dots, b_m$. The proof is finished if we look back at the first line of (9.16). □

Lemma 9.5.1. *If* $V(\lambda) = \sum_{i<j} \log(\lambda_i - \lambda_j)$*, we have*

$$\boldsymbol{G}_{p,q} V = C_{m,p,q},$$

where $C_{m,p,q} = m(m-1)\left(\frac{m+1}{3} - \frac{p+q}{2}\right)$.

Proof of Lemma 9.5.1. With the notations of Chapter 12, we recall that $\mathbf{G}_{p,q} = \mathcal{L}^{(1)}$ for $\mathbf{L} = \sum_i \left(a(\lambda_i)\partial_i^2 + b(\lambda_i)\partial_i\right)$, $a(\lambda) = 2\lambda(1-\lambda)$, $b(\lambda) = \alpha - (\alpha+\beta)\lambda$, $\alpha = p-(m-1)$ and $\beta = q-(m-1)$. But, $\mathcal{L}^{(1)}V = \mathbf{L}h/h = -m(m-1)\left(2(m-2)/3 + (\alpha+\beta)/2\right)$ has been computed in the example 12.3.1 of Chapter 12. □

Proof of Proposition 9.3.5. Mappings $j \mapsto \sqrt{j^+}$ and $j \mapsto \sqrt{(1_m - j)^+}$ are continuous on $\mathcal{S}_m$ so the solution is defined up to its explosion time e (see [IW89] for example). If $\parallel \dot{} \parallel$ is a multiplicative norm, we have :

$$(\,2 \parallel j^+ \parallel \parallel (1_m - j)^+ \parallel)^2 + \parallel p(1_m - j)^+ - qj^+ \parallel^2 \leq K(1 + \parallel j \parallel^2),$$

which proves that $e = \infty$ a.s. (see [IW89]). Equations for the eigenvalues are proved exactly in the same way as in the previous computations. □

Proof of Proposition 9.3.6. Let us prove the statements for λ_m, things being entirely similar for λ_1. Although we won't constantly recall it, all that follows is valid up to time τ. Calling $L_t^a(\lambda_m)$ the local time of λ_m at time t and level a and using the occupation times formula :

$$\begin{aligned}
\int_0^1 (4a(1-a))^{-1}\, L_t^a(\lambda_m)\, da &= \int_0^t \mathbf{1}_{(1>\lambda_m(s)>0)} \frac{d\langle \lambda_m \rangle_s}{4\lambda_m(s)(1-\lambda_m(s))} \\
&= \int_0^t \mathbf{1}_{(1>\lambda_m(s)>0)}\, ds \\
&\leq t.
\end{aligned}$$

This proves that $L_t^0(\lambda_m) = 0$, otherwise the previous integral would diverge.
Now, call

$$b(s) = p - (p+q)\lambda_m(s) + \sum_{i=1}^{m-1} \frac{\lambda_m^+(s)\,(1-\lambda_i(s))^+ + \lambda_i^+(s)\,(1-\lambda_m(s))^+}{\lambda_m(s) - \lambda_i(s)} \tag{9.17}$$

and use $L_t^0(\lambda_m) = 0$ as well as Tanaka's formula to get

$$\mathbb{E}\left[(-\lambda_m(t))^+\right] = -\mathbb{E}\left[\int_0^t \mathbf{1}_{(\lambda_m(s)<0)}\, b(s)\, ds\right]. \tag{9.18}$$

We remark that, on $\{\lambda_m(s) < 0\}$, we have $\lambda_m(s)^+ = 0$, $\frac{\lambda_i^+(s)\,(1-\lambda_m(s))^+}{\lambda_m(s)-\lambda_i(s)} \geq -1$ for $1 \leq i \leq m-1$ and then $b(s) \geq p-(m-1)$. Plugging this in (9.18) yields

$$\mathbb{E}\left[(-\lambda_m(t))^+\right] \leq -(p-(m-1)) \int_0^t \mathbb{P}(\lambda_m(s) < 0)\, ds,$$

which shows $\mathbb{E}\left[(-\lambda_m(t))^+\right] = 0$, proving claim (ii) for λ_m.
Seeing $\lambda_m(t) \geq 0$ a.s., we have $L_t^{0-}(\lambda_m) = 0$ and therefore (see Theorem (1.7), Chap. IV in [RY99] for example)

$$L_t^0(\lambda_m) = 2\int_0^t \mathbf{1}_{(\lambda_m(s)=0)}\, b(s)\, ds,$$

with $b(s)$ as in (9.17). Now, on $\{\lambda_m(s) = 0\}$, we have $b(s) = p-(m-1)$ proving that

$$L_t^0(\lambda_m) = 2(p-(m-1)) \int_0^t \mathbf{1}_{(\lambda_m(s)=0)}\, ds.$$

Since $L_t^0(\lambda_m) = 0$ and $p-(m-1) > 0$, claim (iii) is proved. □

Proof of Proposition 9.3.7. Again, we consider the process $\Omega_t = V\big(\lambda(t)\big)$ for $t < \tau$. Its infinitesimal drift is given by $\mathbf{G}_{p,q}^+ V(\lambda(t))$, where

$$\begin{aligned}\mathbf{G}_{p,q}^+ = 2\sum_i \lambda_i^+(1-\lambda_i)^+ \frac{\partial^2}{\partial\lambda_i^2} + \sum_i (p-(p+q)\lambda_i)\frac{\partial}{\partial\lambda_i} \\ + \sum_{i\neq j} \frac{\lambda_i^+(1-\lambda_j)^+ + \lambda_j^+(1-\lambda_i)^+}{\lambda_i-\lambda_j}\frac{\partial}{\partial\lambda_i}.\end{aligned}$$

By Proposition 9.3.6, all the eigenvalues are in $[0,1]$ up to time τ and thus $\mathbf{G}_{p,q}^+ V(\lambda(t)) = \mathbf{G}_{p,q} V(\lambda(t))$ for $t < \tau$. This allows for the same argument as in Proposition 9.3.4 to prove $\tau = \infty$. □

Proof of Proposition 9.4.1. Let us use the following notations : $\alpha = (n-1-m-p)/2$, $A_{i,i',j,j'} = \delta_{ii'}\delta_{jj'} - X_{ij'}X_{i'j}$, $\partial_{ij} = \frac{\partial}{\partial X_{ij}}$, $\rho(X) = \det(1_m - XX^*)^\alpha$ and $L = \sum_{i,i',j,j'} A_{i,i',j,j'}\,\partial_{ij}\partial_{i'j'}$. Then, integration by parts yields :

$$\begin{aligned}\int (LF)\, G\, \rho\, dX &= -\sum_{i,i',j,j'} \int \partial_{ij}F\ \partial_{i'j'}\left(G\, A_{i,i',j,j'}\rho\right) dX \\ &= -\sum_{i,i',j,j'} \int A_{i,i',j,j'}\ \partial_{ij}F\ \partial_{i'j'}G\ \rho\, dX \\ &\quad -\sum_{i,i',j,j'} \int G\ \partial_{ij}F\ \partial_{i'j'}(A_{i,i',j,j'}\,\rho)\, dX \qquad (9.19)\end{aligned}$$

Let us recall that if $\phi(X) = \det(X)$, then $d\phi_X(H) = \det(X)\mathrm{tr}(X^{-1}H)$. Thus $d\rho_X(H) = -\alpha\,\rho(X)\mathrm{tr}\,((1_m - XX^*)^{-1}(HX^* + XH^*))$, which implies that :

$$\partial_{i'j'}\rho = -2\alpha\,\rho(1_m - XX^*)^{-1}X)_{i'j'}.$$

It is also easy to check that :

$$\partial_{i'j'}\,A_{i,i',j,j'} = -(\delta_{ii'} + \delta_{jj'})X_{ij}.$$

Consequently, writing $Y = (1_m - XX^*)^{-1}$, the second term of the right-hand side in (9.19) equals :

$$\begin{aligned}
&\sum_{i,i',j,j'} \int G\frac{\partial F}{\partial X_{ij}}\,(-2\alpha(\delta_{ii'}\delta_{jj'} - X_{ij'}X_{i'j})(YX)_{i'j'} - (\delta_{ii'} + \delta_{jj'})X_{ij})\,\rho\,dX \\
= \;& \sum_{ij}\int G\frac{\partial F}{\partial X_{ij}}\,(-2\alpha\,(YX)_{ij} + 2\alpha\,(XX^*YX)_{ij} - (m+p)X_{ij})\,\rho\,dX \\
= \;& (-2\alpha - m - p)\sum_{ij}\int G\frac{\partial F}{\partial X_{ij}}\,X_{ij}\,\rho\,dX \\
= \;& (n-1)\int G\sum_{ij}\frac{\partial F}{\partial X_{ij}}\,X_{ij}\,\rho\,dX\;, \qquad (9.20)
\end{aligned}$$

where we used $-YX + XX^*YX = -X$. Now, (9.19) together with (9.20) gives the result. □

Proof of Proposition 9.4.2. Computations are similar to those of the previous proof. We write $\psi(x) = (\det x)^\alpha\,(\det(1_m - x))^\beta$ and we use $D_{ij}\psi = (\alpha x^{-1}_{ij} - \beta(1_m - x)^{-1}_{ij})\,\psi$ (y^{-1}_{ij} denotes the term (i,j) of the matrix y^{-1}). Suppose f, g have compact support and compute

$$\int f\mathbf{L}_{p,q}\psi\,dx = 2(A - B) + \int f\mathrm{tr}\,((p - (p+q)x)Dg)\,\psi\,dx,$$

where $A = \int f\mathrm{tr}(xD^2g)\psi\,dx$ and $B = \int f\mathrm{tr}(x(xD)^*Dg)\psi\,dx$. We use integration by parts to get :

$$\begin{aligned}
A &= \sum_{i,j,k}\int f x_{ij} D_{jk} D_{ki} g\,\psi\,dx = -\sum_{i,j,k}\int D_{ki} g D_{jk}(f x_{ij}\psi)\,dx \\
&= -\sum_{i,j,k}\left(\int D_{ki} g D_{jk}(f) x_{ij}\psi)\,dx + \int f D_{ki} g (D_{jk}(x_{ij})\psi + x_{ij} D_{jk}\psi)\,dx\right) \\
&= -\left\{\int \mathrm{tr}(xDfDg)\psi\,dx + \sum_{i,j,k}\int f D_{ki} g (\frac{\mathbf{1}_{k=i\neq j}}{2} + \mathbf{1}_{k=i=j})\psi\,dx \right. \\
&\qquad \left. + \sum_{i,j,k}\int f D_{ki} g x_{ij} (\alpha x^{-1}_{jk} - \beta(1_m - x)^{-1}_{jk})\psi\,dx \right\} \\
&= -\left\{\int \mathrm{tr}(xDfDg)\psi\,dx + (\frac{m-1}{2} + 1)\sum_{i}\int D_{ii} g f\psi\,dx \right. \\
&\qquad \left. + \sum_{i,j,k}\int f D_{ki} g x_{ij} (\alpha x^{-1}_{jk} - \beta(1_m - x)^{-1}_{jk})\psi\,dx \right\} \\
&= -\int \mathrm{tr}(xDfDg)\psi\,dx - \frac{m+1}{2}\int f\,\mathrm{tr}(Dg)\psi\,dx \\
&\qquad - \int f\,\mathrm{tr}\left(\left[(\alpha+\beta)1_m - \beta(1_m - x)^{-1}\right] Dg\right)\psi\,dx.
\end{aligned}$$

Similarly,

$$\begin{aligned}
B &= \sum_{i,j,k,l}\int f x_{il} x_{jk} D_{ik} D_{jl} g\,\psi\,dx = -\sum_{i,j,k,l}\int D_{jl} g D_{ik}(f x_{il} x_{jk}\,\psi)\,dx \\
&= -\sum_{i,j,k,l}\left\{\int x_{il} x_{jk} D_{ik} f D_{jl} g\,\psi\,dx + \int f D_{jl} g (\frac{\mathbf{1}_{k=l\neq i}}{2} + \mathbf{1}_{k=i=l}) x_{jk}\,\psi\,dx \right. \\
&\qquad + \int f D_{jl} g (\frac{\mathbf{1}_{i=j\neq k}}{2} + \mathbf{1}_{i=j=k}) x_{il}\,\psi\,dx \\
&\qquad \left. + \int f D_{jl} g x_{il} x_{jk} (\alpha x^{-1} - \beta(1_m - x)^{-1})_{ik}\,\psi\,dx \right\} \\
&= -\mathrm{tr}(xDfxDg) - (m+1)\int f\,\mathrm{tr}(xDg)\psi\,dx - \\
&\qquad \int f\,\mathrm{tr}\left(\left[(\alpha+\beta)x - \beta(1_m - x)^{-1}x\right] Dg\right)\psi\,dx.
\end{aligned}$$

Therefore,

$$2(A-B) = \int \left\{-\Gamma(f,g) + f\,\mathrm{tr}\left((a - bx)Dg\right)\right\}\psi\,dx, \tag{9.21}$$

where $a = m + 1 + 2\alpha$ and $b = 2(m+1) + 2(\alpha+\beta)$. Now, (9.21) shows that $\psi\, dx$ is reversible if and only if $a = p$ and $b = p + q$, which corresponds to $\alpha = \frac{p-m-1}{2}$ and $\beta = \frac{q-m-1}{2}$. □

Proof of Proposition 9.4.3. Thanks to Proposition 9.3.2, if

$$\phi(x) = (\det x)^{\alpha}(\det(1_m - x))^{\beta}, H_s = H_s^{\alpha,\beta} \text{ and } V_s = V_s^{\alpha,\beta},$$

we have

$$\frac{\phi(J_t)}{\phi(J_0)} = \exp\left(\int_0^t \operatorname{tr}(H_s dB_s) + \int_0^t V_s\, ds\right),$$

from which we deduce that

$$\mathcal{E}_t = \frac{\phi(J_t)}{\phi(J_0)} \exp\left(\int_0^t \left(\frac{1}{2}\operatorname{tr}(H_s^2) + V_s\right) ds\right).$$

Now,

$$\frac{1}{2}\operatorname{tr}(H_s^2) + V_s = \left(\alpha(p-m-1) + 2\alpha^2\right) \operatorname{tr}(J_s^{-1}) +$$
$$\left(\beta(q-m-1) + 2\beta^2\right) \operatorname{tr}((1_m - J_s)^{-1}) - (\alpha+\beta)m(p+q+2(\alpha+\beta)-m-1),$$

which finishes the proof. □

Proof of Proposition 9.4.7. Computations similar to the real case ones show that $\mathbf{H}_{p,q}$ differs from $\mathbf{G}_{p,q}$ by a factor 2 on the drift :

$$\begin{aligned}\mathbf{H}_{p,q} &= 2\sum_i \lambda_i(1-\lambda_i)\partial_i^2 + 2\sum_i (p - (p+q)\lambda_i)\partial_i + 2\sum_{i\neq j} \frac{\lambda_i + \lambda_j - 2\lambda_i\lambda_j}{\lambda_i - \lambda_j}\partial_i \\ &= 2\sum_i \lambda_i(1-\lambda_i)\partial_i^2 + \sum_i (\alpha - (\alpha+\beta)\lambda_i)\partial_i + \sum_{i\neq j} \frac{4\lambda_i(1-\lambda_i)}{\lambda_i - \lambda_j}\partial_i,\end{aligned}$$

with $\alpha = 2(p-(m-1))$ and $\beta = 2(q-(m-1))$. On the other hand, the example 12.3.1 of Chapter 12 asserts that : $\mathbf{L}_{\alpha,\beta} = -m(m-1)\left(\frac{2(m-2)}{3} + \frac{\alpha+\beta}{2}\right) h$. Moreover, it is easy to see that the "carré du champ" of $\mathbf{L}_{\alpha,\beta}$ is :

$$\Gamma(\log h, f) = \sum_{i\neq j} \frac{4\lambda_i(1-\lambda_i)}{\lambda_i - \lambda_j}\partial_i f.$$

This shows equality between $\widehat{\mathbf{L}}_{\alpha,\beta} f = \mathbf{L}_{\alpha,\beta} f + \Gamma(\log h, f)$ and $\mathbf{H}_{p,q}$ when $\alpha = 2(p - (m-1))$ and $\beta = 2(q-(m-1))$. □

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Quatrième partie

Brownian motion and reflection groups

Chapitre 10

Exit problems associated with finite reflection groups

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Abstract : We obtain a formula for the distribution of the first exit time of Brownian motion from a fundamental region associated with a finite reflection group. In the type A case it is closely related to a formula of de Bruijn and the exit probability is expressed as a Pfaffian. Our formula yields a generalisation of de Bruijn's. We derive large and small time asymptotics, and formulas for expected first exit times. The results extend to other Markov processes. By considering discrete random walks in the type A case we recover known formulas for the number of standard Young tableaux with bounded height.

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10.1 Introduction

The reflection principle is a protean concept which has given rise to many investigations in probability and combinatorics. Its most famous embodiment may be the ballot problem of counting the number of walks with unit steps staying above the origin. In the context of a one-dimensional Brownian motion $(B_t, t \geq 0)$ with transition density $p_t(x, y)$, the reflection principle gives a simple expression for the transition density $p^*_t(x, y)$ of the Brownian motion started in $(0, \infty)$ and killed when it first hits zero :

$$p^*_t(x, y) = p_t(x, y) - p_t(x, -y). \tag{10.1}$$

The exit probability is recovered by integrating over $y > 0$. If $\mathbb{P}_x$ denotes the law of B started at $x > 0$ and T is the first exit time from $(0, \infty)$, then

$$\mathbb{P}_x(T > t) = \mathbb{P}_x(B_t > 0) - \mathbb{P}_x(B_t < 0). \tag{10.2}$$

The formula (10.1) extends to the much more general setting of Brownian motion in a fundamental region associated with a finite reflection group. For example, if B is a Brownian motion in $\mathbb{R}^n$ with transition density $p_t(x, y)$ and $C = \{x \in \mathbb{R}^n : \ x_1 > x_2 > \cdots > x_n\}$ then the transition density of the Brownian motion, killed when it first exits C, is given by

$$p^*_t(x, y) = \sum_{\pi \in \mathfrak{S}_n} \varepsilon(\pi) p_t(x, \pi y), \tag{10.3}$$

where $\pi y = (y_{\pi(1)}, \ldots, y_{\pi(n)})$ and $\varepsilon(\pi)$ denotes the sign of π. Equivalently,

$$p^*_t(x, y) = (2\pi t)^{-n/2} \det[\exp(-(x_i - y_j)^2/2t)]^n_{i,j=1}. \tag{10.4}$$

This is referred to as the 'type A' case and the associated reflection group is isomorphic to $\mathfrak{S}_n$. The formula (10.4) is a special case of a more general formula due to Karlin and McGregor [KM59] ; it can be verified in this setting by noting that the right-hand-side satisfies the heat equation with appropriate boundary conditions. Integrating (10.3) over $y \in C$ yields a formula for the exit probability

$$\mathbb{P}_x(T > t) = \sum_{\pi \in \mathfrak{S}_n} \varepsilon(\pi) \mathbb{P}_x(B_t \in \pi C). \tag{10.5}$$

This formula involves some complicated multi-dimensional integrals, but thanks to an integration formula of de Bruijn [dB55], it can be re-expressed as a Pfaffian which only involves one-dimensional integrals. More precisely, if we set $\gamma(a) = \sqrt{\frac{2}{\pi}} \int_0^a e^{-y^2/2} dy$ and $p_{ij} = \gamma\left(\frac{x_i - x_j}{\sqrt{2t}}\right)$ then, for $x \in C$,

$$\mathbb{P}_x(T > t) = \begin{cases} \operatorname{Pf}\,(p_{ij})_{i,j \in [n]} & \text{if } n \text{ is even,} \\ \sum_{l=1}^n (-1)^{l+1} \operatorname{Pf}\,(p_{ij})_{i,j \in [n] \setminus \{l\}} & \text{if } n \text{ is odd.} \end{cases} \tag{10.6}$$

(Observe that $p_{ji} = -p_{ij}$ since γ is an odd function and see appendix for a definition of the Pfaffian.) For example, when $n = 3$, we recover the simple formula

$$\mathbb{P}_x(T > t) = p_{12} + p_{23} - p_{13}, \tag{10.7}$$

which was obtained in [OU92] by direct reflection arguments.

The formula (10.3) extends naturally to Brownian motion in a fundamental region C associated with any finite reflection group (for discrete versions see Gessel and Zeilberger [GZ92] and Biane [Bia92]). As above, this can be integrated to give a formula for the exit probability involving multi-dimensional integrals. The main point of this paper is that there is an analogue of the simplified formula (10.6) in the general case which can be obtained directly. This leads to a generalisation of de Bruijn's formula and can be used to obtain asymptotic results as well as formulae for expected exit times. Our approach is not restricted to Brownian motion. For example, if we consider discrete random walks in the type A case we recover results of Gordon [Gor83] and Gessel [Ges90] on the number of standard Young tableaux with bounded height.

The outline of the paper is as follows. In the next section we introduce the reflection group setting and state the main results. These results involve a condition which we refer to as 'consistency'. This is discussed in detail for the various types of reflection groups in section 3. In section 4 we apply our results to give formulae for the exit probability of Brownian motion from a fundamental domain and use these formulae to obtain small and large time asymptotic expansions and to compute expected exit times. In section 5, we present a generalisation of de Bruijn's formula and in section 6 we describe some related combinatorics. The final section is devoted to proofs.

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10.2 The main result

10.2.1 The reflection group setting

For background on root systems and finite reflection groups see, for example, [Hum90]. Let V be a real Euclidean space endowed with a positive symmetric bilinear form (λ, μ). Let Φ be a (reduced) root system in V with associated reflection group W. Let Δ be a simple system in Φ with corresponding positive system Π and fundamental chamber $C = \{\lambda \in V : \forall \alpha \in \Delta, (\alpha, \lambda) > 0\}$. Denote the reflections in W by s_α ($\alpha \in \Pi$).

Definition 10.2.1. A subset of V is said to be **orthogonal** if its distinct elements are pairwise orthogonal. If $E \subset V$, we will denote by $\mathcal{O}(E)$ the set of all orthogonal subsets of E.

Definition 10.2.2 (Consistency).
- We will say that $I \subset \Pi$ satisfies hypothesis (C1) if there exists $J \in \mathcal{O}(\Delta \cap I)$ such that if $w \in W$ with $J \subset wI \subset \Pi$ then $wI = I$.
- We will say that $I \subset \Pi$ satisfies hypothesis (C2) if the restriction of the determinant to the subgroup $U = \{w \in W : \ wI = I\}$ is trivial, ie $\forall w \in U$, $\varepsilon(w) = \det w = 1$.
- I will be called **consistent** if it satisfies (C1) and (C2).

Suppose $I \subset \Pi$ is consistent. Set $W^I = \{w \in W : \ wI \subset \Pi\}$ and $\mathcal{I} = \{wI : \ w \in W^I\}$. The hypothesis (C2) makes it possible to attribute a sign to every element of $\mathcal{I}$ by setting $\varepsilon_A := \varepsilon(w)$ for $A \in \mathcal{I}$, where w is any element of W^I with $wI = A$.

For example, $I = \Delta$ is consistent with $W^I = U = \{\text{id}\}$ and $\mathcal{I} = \{\Delta\}$.

Section 10.3 will be devoted to a study of the consistency condition in the different types of root systems. Most root systems will turn out to possess a non-trivial (and useful) consistent subset $I \subset \Pi$.

10.2.2 The exit problem

Let $I \subset \Pi$ be consistent, and define ε_A for $A \in \mathcal{I}$ as above. Let $X = (X_t, t \geq 0)$ be a standard Brownian motion in V and write $\mathbb{P}_x$ for the law of X started at $x \in C$. For $\alpha \in \Pi$, set $T_\alpha = \inf\{t \geq 0 : \ (\alpha, X_t) = 0\}$. For $A \subset \Pi$ write $T_A = \min_{\alpha \in A} T_\alpha$, and set $T = T_\Delta = \inf\{t \geq 0 : \ X_t \notin C\}$.

Denote by $p_t(x,y)$ (respectively $p_t^*(x,y)$) the transition density of X (respectively that of X started in C and killed at time T). The analogue of the formula (10.3) in this setting is

$$p_t^*(x,y) = \sum_{w \in W} \varepsilon(w) p_t(x, wy), \tag{10.8}$$

which can be integrated to obtain

$$\mathbb{P}_x(T > t) = \sum_{w \in W} \varepsilon(w) \mathbb{P}_x(X_t \in wC). \tag{10.9}$$

A discrete version of this formula was obtained by Gessel and Zeilberger [GZ92] and Biane [Bia92] ; it is readily verified in the continuous setting by observing that the expression given satisfies the heat equation with appropriate boundary conditions. As remarked in the introduction, this formula typically involves complicated multi-dimensional integrals. Our main result is the following alternative.

Proposition 10.2.3.

$$\mathbb{P}_x(T > t) = \sum_{A \in \mathcal{I}} \varepsilon_A \, \mathbb{P}_x(T_A > t). \tag{10.10}$$

In fact, we will prove Proposition 10.2.3 in the following, slightly more general, context. Let $X = (X_t, t \geq 0)$ be a Markov process with W-invariant state space $E \subset V$, infinitesimal generator $\mathcal{L}$, and write $\mathbb{P}_x$ for the law of the process started at x. Assume that the law of X is W-invariant, that is,

$$\mathbb{P}_x \circ (wX)^{-1} = \mathbb{P}_{wx} \circ X^{-1},$$

and that X is sufficiently regular so that :

(i) $u_I(x,t) = \mathbb{P}_x(T_I > t)$ satisfies the boundary-value problem :

$$\frac{\partial u_I}{\partial t} = \mathcal{L} u_I \qquad \begin{cases} u_I(x,0) = 1 & \text{if } x \in O = \{\lambda \in V :\ \forall \alpha \in I,\ (\alpha, \lambda) > 0\}, \\ u_I(x,t) = 0 & \text{if } x \in \partial O. \end{cases} \tag{10.11}$$

(ii) $u(x,t) = \mathbb{P}_x(T > t)$ is the *unique* solution to

$$\frac{\partial u}{\partial t} = \mathcal{L} u \qquad \begin{cases} u(x,0) = 1 & \text{if } x \in C, \\ u(x,t) = 0 & \text{if } x \in \partial C. \end{cases} \tag{10.12}$$

These hypotheses are satisfied if X is a standard Brownian motion in V or, in the crystallographic case, a continuous-time W-invariant simple random walk.

Note that for $I = \Delta$, the sum in (10.10) has only one term and the formula is a tautology. However, as we shall see, in general we can find more interesting and useful choices of I.

Remark 10.2.1. *There are explicit formulae of a different nature for the distribution of the exit time from a general convex cone $C \subset \mathbb{R}^k$. Typically, these are expressed as infinite series whose terms involve eigenfunctions of the Laplace-Beltrami operator on $C \cap S^{k-1}$ with Dirichlet boundary conditions. See, for example, [DeB87], [DeB01] and references therein.*

10.2.3 The orthogonal case

If I is orthogonal, the summation in (10.10) is over orthogonal subsets of Π, and Proposition 10.2.3 is therefore most effective when X has independent components in orthogonal directions. In this case, (10.10) becomes :

$$\mathbb{P}_x(T > t) = \sum_{A \in \mathcal{I}} \varepsilon_A \prod_{\alpha \in A} \mathbb{P}_x(T_\alpha > t), \tag{10.13}$$

where $\mathbb{P}_x(T_\alpha > t) = \mathbb{P}_x\left((X_t, \alpha) > 0\right) - \mathbb{P}_x\left((X_t, \alpha) < 0\right)$. For example, if X is Brownian motion, we have

$$\mathbb{P}_x(T > t) = \sum_{A \in \mathcal{I}} \varepsilon_A \prod_{\alpha \in A} \gamma\left(\widehat{\alpha}(x)/\sqrt{t}\right), \tag{10.14}$$

where $\widehat{\alpha}(x) = (\alpha, x)/|\alpha|$ and $\gamma(a) = \sqrt{\frac{2}{\pi}} \int_0^a e^{-y^2/2} dy$.

Consider the polynomial $Q \in \mathbb{Z}[X_\alpha,\, \alpha \in \Pi]$ defined by

$$Q = \sum_{A \in \mathcal{I}} \varepsilon_A \prod_{\alpha \in A} X_\alpha. \tag{10.15}$$

Then $\mathbb{P}_x(T > t)$ is equal to the polynomial Q evaluated at the variables $\mathbb{P}_x(T_\alpha > t)$, $\alpha \in \Pi$. Note that the polynomial Q is homogeneous of degree $|I|$. A useful property of Q is the following, which we record here for later reference.

Proposition 10.2.4. *For $x \in V$, set $P(x) = Q\left((\alpha, x),\, \alpha \in \Pi\right)$. If $I \neq \Pi$, then $P = 0$.*

10.2.4 A dual formula

In the orthogonal case, there is an analogue of the formula (10.13) for the complementary probability $\mathbb{P}_x(T \leq t)$. This will prove to be useful when analyzing the small time behaviour (see Section 10.4.6).

For $\alpha \in \Delta$, $B \in \mathcal{O}(\Pi)$, define $\alpha.B \in \mathcal{O}(\Pi)$ by :

$$\alpha.B = \begin{cases} B & \text{if } \alpha \in B; \\ \{\alpha\} \cup B & \text{if } \alpha \in B^\perp; \\ s_\alpha B & \text{otherwise.} \end{cases}$$

We can then define the "length" $l(B)$ for $B \in \mathcal{O}(\Pi)$ by :

$$l(B) = \inf\{l \in \mathbb{N} :\ \exists\, \alpha_1, \alpha_2, \ldots, \alpha_l \in \Delta,\ B = \alpha_l \ldots \alpha_2.\alpha_1.\emptyset\}. \tag{10.16}$$

Proposition 10.2.5. *For all $B \in \mathcal{O}(\Pi)$, $l(B) < \infty$. In other words, any $B \in \mathcal{O}(\Pi)$ can be obtained from the empty set by successive applications of the simple roots.*

Proposition 10.2.6. *Suppose I is consistent and orthogonal. Then,*

$$\mathbb{P}_x(T \leq t) = \sum_{B \in \mathcal{O}(\Pi) \setminus \{\emptyset\}} (-1)^{l(B)-1}\, \mathbb{P}_x[\forall \beta \in B,\ T_\beta \leq t]. \tag{10.17}$$

If we introduce the polynomial $R \in \mathbb{Z}[X_\alpha,\, \alpha \in \Pi]$,

$$R = \sum_{B \in \mathcal{O}(\Pi) \setminus \{\emptyset\}} (-1)^{l(B)-1} \prod_{\alpha \in B} X_\alpha, \tag{10.18}$$

then (10.17) is essentially equivalent to the following relation between Q and R :

$$1 - Q\left(1 - X_\alpha,\, \alpha \in \Pi\right) = R\left(X_\alpha,\, \alpha \in \Pi\right).$$

Note that R is not homogeneous.

10.2.5 The semi-orthogonal case

Definition 10.2.7. We say $E \subset V$ is **semi-orthogonal** if it can be partitioned into blocks (ρ_i) such that $\rho_i \perp \rho_j$ for $i \neq j$ and each ρ_i is either a singlet or a pair of vectors whose mutual angle is $3\pi/4$. The set of the blocks ρ_i will be denoted by E^*.

Remark 10.2.2. *A prototypical pair of vectors in a semi-orthogonal subset is* $\{e_1 - e_2, e_2\}$*, where* (e_1, e_2) *is orthonormal.*

If I is consistent and semi-orthogonal and if X has independent components in orthogonal directions, the formula (10.10) becomes :

$$\mathbb{P}_x(T > t) = \sum_{A \in \mathcal{I}} \varepsilon_A \prod_{\rho \in A^*} \mathbb{P}_x(T_\rho > t). \tag{10.19}$$

Call Π' the set of pairs of positive roots whose mutual angle is $3\pi/4$. The relevant polynomial to consider is $S \in \mathbb{Z}[X_\alpha,\ \alpha \in \Pi;\ X_{\{\alpha,\beta\}},\ \{\alpha, \beta\} \in \Pi']$,

$$S = \sum_{A \in \mathcal{I}} \varepsilon_A \prod_{\rho \in A^*} X_\rho. \tag{10.20}$$

Proposition 10.2.8. *Suppose* $2|I| < |\Pi|$*. For* $x \in V$*, the evaluation of* S *with* $X_\alpha = (\alpha, x),\ \alpha \in \Pi$ *and* $X_{\{\alpha,\beta\}} = (\alpha, x)(\beta, x)(s_\alpha\beta, x)(s_\beta\alpha, x),\ \{\alpha, \beta\} \in \Pi'$*, is equal to zero.*

10.3 Consistency

Lemma 10.3.1. *Suppose there exists* $J \in \mathcal{O}(\Delta)$ *which is uniquely extendable to a maximal orthogonal (resp. semi-orthogonal) subset* $I \subset \Pi$*, maximal meaning that there is no orthogonal (resp. semi-orthogonal) subset strictly larger than* I*. In this case,* I *satisfies condition* (C1).

Proof. If $J \subset wI \subset \Pi$ then wI is a maximal orthogonal (resp. semi-orthogonal) subset of Π and the unique extension property says that $wI = I$.

◊

10.3.1 The dihedral groups

The dihedral group $I_2(m)$ is the group of symmetries of a regular m-sided polygon centered at the origin. It is a reflection group acting on $V = \mathbb{R}^2 \simeq \mathbb{C}$. Define $\beta = i$, $\alpha_l = e^{il\pi/m}(-\beta)$ for $1 \leq l \leq m$ and $\alpha = \alpha_1$. Then we can take $\Pi = \{\alpha_1, \ldots, \alpha_m\}$ and $\Delta = \{\alpha, \beta\}$.

Set $I = \{\alpha\}$ if m is odd and $I = \{\alpha, \alpha' = e^{i\pi/2}\alpha\}$ if $m \equiv 2 \bmod 4$. Then I is orthogonal and consistent. In the first case, $\mathcal{I} = \{\{\alpha_1\}, \ldots, \{\alpha_m\}\}$ with $\varepsilon_{\{\alpha_i\}} =$

$(-1)^{i-1}$. In the second case, $\mathcal{I} = \{\{\alpha_1, \alpha'_1\}, \ldots, \{\alpha_m, \alpha'_m\}\}$ and $\varepsilon_{\{\alpha_i, \alpha'_i\}} = (-1)^{i-1}$. With notations $X_j = X_{\alpha_j}$ and $X'_j = X_{\alpha'_j}$, the polynomial Q can be written

$$Q = \begin{cases} \sum_{j=1}^{m} (-1)^{j-1} X_j & \text{if } m \text{ is odd} \\ \sum_{j=1}^{m} (-1)^{j-1} X_j X'_j & \text{if } m \equiv 2 \mod 4\,. \end{cases} \tag{10.21}$$

10.3.2 The A_{k-1} case

Consider $W = \mathfrak{S}_k$ acting on $\mathbb{R}^k$ by permutation of the canonical basis vectors. Then we can take $V = \mathbb{R}^k$ or $\{x \in \mathbb{R}^k :\ x_1 + \cdots + x_k = 0\}$, $\Pi = \{e_i - e_j,\ 1 \le i < j \le k\}$ and $\Delta = \{e_i - e_{i+1},\ 1 \le i \le k-1\}$.

The choice of I depends on the parity of k. If k is even, we take

$$I = \{e_1 - e_2, e_3 - e_4, \ldots, e_{k-1} - e_k\}.$$

If k is odd, then

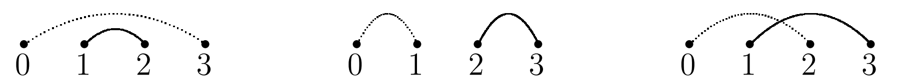
$$I = \{e_1 - e_2, e_3 - e_4, \ldots, e_{k-2} - e_{k-1}\}.$$

Proposition 10.3.1. (i) *I is consistent and orthogonal.*

(ii) *The set $\mathcal{I}$ can be identified with the set $P_2(k)$ of partitions of $[k]$ into $k/2$ pairs if k is even and into $(k-1)/2$ pairs and a singlet if k is odd.*

(iii) *Under this identification, the sign ε is just the parity of the number of crossings (if k is odd, we consider an extra pair made of the singlet and a formal dot 0 strictly at the left of 1 and use this pair to compute the number of crossings).*

The proof of this proposition will be provided in Section 10.7.1. In Figures 1 and 2, we give examples of the identification between $A \in \mathcal{I}$ and $\pi \in P_2(k)$, using the notation $c(\pi)$ for the number of crossings.



$\pi = \{\{1,2\},\{3\}\}$ $\quad$ $\pi = \{\{1\},\{2,3\}\}$ $\quad$ $\pi = \{\{1,3\},\{2\}\}$

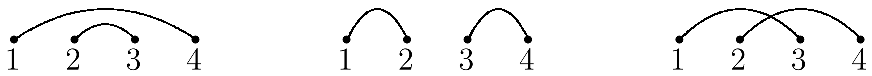
$A = \{e_1 - e_2\}$ $\quad$ $A = \{e_2 - e_3\}$ $\quad$ $A = \{e_1 - e_3\}$

$c(\pi) = 0$ $\quad$ $c(\pi) = 0$ $\quad$ $c(\pi) = 1$

FIG. 10.1 – Pair partitions and their signs for A_2.

Now, recall the polynomial Q defined in (10.15) and write for simplicity $X_{ij} = X_{e_i - e_j}$, $i < j$. Then,

$$Q = \sum_{\pi \in P_2(k)} (-1)^{c(\pi)} \prod_{\{i<j\} \in \pi} X_{ij}, \tag{10.22}$$



$\pi = \{\{1,4\},\{2,3\}\}$	$\pi = \{\{1,2\},\{3,4\}\}$	$\pi = \{\{1,3\},\{2,4\}\}$
$A = \{e_1 - e_4, e_2 - e_3\}$	$A = \{e_1 - e_2, e_3 - e_4\}$	$A = \{e_1 - e_3, e_2 - e_4\}$
$c(\pi) = 0$	$c(\pi) = 0$	$c(\pi) = 1$

FIG. 10.2 – Pair partitions and their signs for A_3.

which can be expressed as a Pfaffian (see Appendix),

$$Q = \begin{cases} \text{Pf } (X_{ij})_{i,j \in [k]} & \text{if } k \text{ is even ,} \\ \sum_{l=1}^{k} (-1)^{l+1} \text{Pf } (X_{ij})_{i,j \in [k] \setminus \{l\}} & \text{if } k \text{ is odd,} \end{cases} \tag{10.23}$$

with the convention that $X_{ij} = -X_{ji}$ for $i > j$.

Remark 10.3.1. *It is interesting to make the combinatorial meaning of Proposition 10.3.1 explicit. Suppose k is even for simplicity. If $\pi = \{\{j_1, j_1'\}, \ldots, \{j_p, j_p'\}\}$ is a pair partition of $[k]$ with $j_i < j_i'$ then we can define $\sigma \in \mathfrak{S}_k$ by $\sigma(2i-1) = j_i, \sigma(2i) = j_i'$. This definition depends on the numbering of the blocks of π, giving rise to $(k/2)!$ such permutations σ. The result is that they all have the same sign which is precisely $(-1)^{c(\pi)}$. If we order the blocks in such a way that $j_1 < j_2 < \cdots < j_p$, then we can be even more precise. Let $i(\sigma)$ denote the number of inversions of σ and $b(\pi)$ the number of bridges of π, that is of pairs $i < l$ with $j_i < j_l < j_l' < j_i'$. Then,*

$$i(\sigma) = c(\pi) + 2b(\pi). \tag{10.24}$$

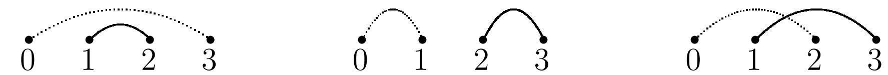
10.3.3 The D_k case

We consider the group W of evenly signed permutations on $\{1, \ldots, k\}$. More precisely, $f : \mathbb{R}^k \to \mathbb{R}^k$ is a sign flip with support $\overline{f}$ if $(fx)_i = -x_i$ when $i \in \overline{f}$ and $(fx)_i = x_i$ when $i \notin \overline{f}$. The elements of W are all $f\,\sigma$ where $\sigma \in \mathfrak{S}_k$ and f is a sign flip whose support has even cardinality. W is a reflection group and we take $V = \mathbb{R}^k$, $\Pi = \{e_i \pm e_j,\ 1 \le i < j \le k\}$ and $\Delta = \{e_1 - e_2, e_2 - e_3, \ldots, e_{k-1} - e_k, e_{k-1} + e_k\}$.

For even k (resp. odd k), we take $I = \{e_1 \pm e_2, e_3 \pm e_4, \ldots, e_{k-1} \pm e_k\}$ (resp. $I = \{e_2 \pm e_3, e_4 \pm e_5, \ldots, e_{k-1} \pm e_k\}$). Proposition (10.3.1) is exactly the same in this case. The identification between $\mathcal{I}$ and $P_2(k)$ is performed as shown in the following examples :

Writing $X_{ij} = X_{e_i - e_j} = -X_{ji}$, $\bar{X}_{ij} = X_{e_i + e_j} = \bar{X}_{ji}$, $i < j$, we have

$$Q = \sum_{\pi \in P_2(k)} (-1)^{c(\pi)} \prod_{\{i<j\} \in \pi} X_{ij}\, \bar{X}_{ij}, \tag{10.25}$$



$\pi = \{\{1,2\},\{3\}\}$	$\pi = \{\{1\},\{2,3\}\}$	$\pi = \{\{1,3\},\{2\}\}$
$A = \{e_1 - e_2, e_1 + e_2\}$	$A = \{e_2 - e_3, e_2 + e_3\}$	$A = \{e_1 - e_3, e_1 + e_3\}$
$c(\pi) = 0$	$c(\pi) = 0$	$c(\pi) = 1$

FIG. 10.3 – Pair partitions and their signs for D_3.

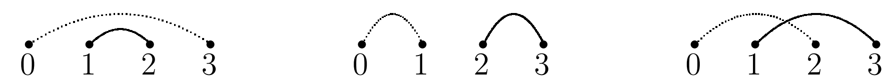
which can be expressed as a Pfaffian,

$$Q = \begin{cases} \mathrm{Pf}\,\left(X_{ij}\,\bar{X}_{ij}\right)_{i,j\in[k]} & \text{if } k \text{ is even },\\ \sum_{l=1}^{k}(-1)^{l+1}\,\mathrm{Pf}\,(X_{ij}\,\bar{X}_{ij})_{i,j\in[k]\setminus\{l\}} & \text{if } k \text{ is odd.}\end{cases} \tag{10.26}$$

10.3.4 The B_k case

W is the group of signed permutations on $\{1,\ldots,k\}$, that is $W = \{f\,\sigma \ : \sigma \in \mathfrak{S}_k,\ f \text{ is a sign flip }\}$. W is a reflection group acting on $V = \mathbb{R}^k$ and the root system is determined by $\Pi = \{e_i - e_j,\ 1 \le i < j \le k\,;\ e_i,\ 1 \le i \le k\}$ and $\Delta = \{e_1 - e_2, e_2 - e_3, \ldots, e_{k-1} - e_k, e_k\}$.

If k is even (resp. odd), set $I = \{e_1 - e_2, e_2, e_3 - e_4, e_4, \ldots, e_{k-1} - e_k, e_k\}$ (resp. $I = \{e_1 - e_2, e_2, e_3 - e_4, e_4, \ldots, e_{k-2} - e_{k-1}, e_{k-1}, e_k\}$). Then I is not orthogonal but only semi-orthogonal. Proposition (10.3.1) is still unchanged. The identification between $\mathcal{I}$ and $P_2(k)$ is performed as shown in Figure 4.



$\pi = \{\{1,2\},\{3\}\}$	$\pi = \{\{1\},\{2,3\}\}$	$\pi = \{\{1,3\},\{2\}\}$
$A = \{e_1 - e_2, e_2, e_3\}$	$A = \{e_1, e_2 - e_3, e_3\}$	$A = \{e_1 - e_3, e_3, e_2\}$
$c(\pi) = 0$	$c(\pi) = 0$	$c(\pi) = 1$

FIG. 10.4 – Pair partitions and their signs for B_3.

Writing $X_{ij} = X_{e_i - e_j} = -X_{ji},\ \widehat{X}_{ij} = \widehat{X}_{ji} = X_{e_j} = X_j,\ i < j$, we have

$$Q = \sum_{\pi \in P_2(k)} (-1)^{c(\pi)} X_{s(\pi)} \prod_{\{i<j\}\in\pi} X_{ij}\,\widehat{X}_{ij}, \tag{10.27}$$

where $s(\pi)$ is the singlet of π, the term $X_{s(\pi)}$ being absent if k is even. (10.27) can be expressed as a Pfaffian,

$$Q = \begin{cases} \mathrm{Pf}\left(X_{ij}\,\widehat{X}_{ij}\right)_{i,j\in[k]} & \text{if } k \text{ is even },\\ \sum_{l=1}^{k}(-1)^{l+1}\,X_l\;\mathrm{Pf}\,(X_{ij}\,\widehat{X}_{ij})_{i,j\in[k]\setminus\{l\}} & \text{if } k \text{ is odd.}\end{cases} \tag{10.28}$$

10.3.5 H_3 and H_4

The underlying Euclidean space is $\mathbb{R}^4$ identified with the quaternions $\mathbb{H}$ with standard basis $(1, i, j, k)$. We adopt the notations of [Hum90] which we refer to for the definitions of the root systems in these cases. For H_3, we take $I = \{a-i/2+bj, 1/2+bi-aj, b+ai+j/2\}$ and for H_4, we take $I = \{-a+i/2+bj, -1/2-ai+bk, b+aj+k/2, bi-j/2+ak\}$. Then I is orthogonal and consistent in both cases.

10.3.6 F_4

The underlying Euclidean space is $\mathbb{R}^4$ with standard basis (e_1, e_2, e_3, e_4) and again we refer to [Hum90] for definition of the root system. We choose $I = \{e_2 - e_3, e_3, e_1 - e_4, e_4\}$. I is semi-orthogonal and consistent.

10.4 Applications to Brownian motion

10.4.1 Brownian motion in a wedge and the dihedral groups

In this case T is the exit time of a planar Brownian motion from a wedge of angle π/m :

$$C = \{re^{i\theta} :\ r \geq 0,\ 0 < \theta < \pi/m\} \subset \mathbb{C} \simeq \mathbb{R}^2.$$

Recall that $\alpha_l = e^{i\pi(l/m-1/2)}$ and $\alpha'_l = e^{i\pi/2}\alpha_l$.

Formula (10.10) reads

$$\mathbb{P}_x(T > t) = \begin{cases} \sum_{i=1}^{m}(-1)^{i-1}\mathbb{P}_x(T_{\alpha_i} > t) & \text{if } m \text{ is odd },\\ \sum_{i=1}^{m}(-1)^{i-1}\mathbb{P}_x(T_{\{\alpha_i,\alpha'_i\}} > t) & \text{if } m \equiv 2 \ (\mathrm{mod}\ 4).\end{cases} \tag{10.29}$$

In the case of Brownian motion, the formulae (10.29) can be rewritten respectively as

$$\mathbb{P}_x(T > t) = \sum_{i=1}^{m}(-1)^{i-1}\gamma\left(\frac{-r\sin(\theta - i\pi/m)}{\sqrt{t}}\right), \tag{10.30}$$

and

$$\mathbb{P}_x(T > t) = \sum_{i=1}^{m}(-1)^{i-1}\gamma\left(\frac{-r\sin(\theta - i\pi/m)}{\sqrt{t}}\right)\gamma\left(\frac{r\cos(\theta - i\pi/m)}{\sqrt{t}}\right), \tag{10.31}$$

where $x = re^{i\theta} \in C$.

These should be compared with the formulae presented in [CD03]. If we take the previous formulae with $r = 1$, integrate over $\theta \in (0, \pi/m)$ and compute a Laplace transform, we recover results of [CD03] for m not a multiple of 4. However, the results in [CD03] are valid for all m. The case $m \equiv 0 \pmod 4$ will be discussed in Section 10.4.5.

General results on exit times from wedges can be found in the paper of Spitzer [Spi58]. We recover Spitzer's results for the special angles discussed above.

10.4.2 The A_{k-1} case and non-colliding probability

The fundamental chamber is $C = \{x \in V : \ x_1 > x_2 > \cdots > x_k\}$ where $V = \mathbb{R}^k$ or $\{x \in \mathbb{R}^k : \ x_1 + \cdots + x_k = 0\}$. Thus T is the first 'collision time' between any two coordinates of X.

Using notation $\{i < j\} \in \pi$ to mean that $\{i, j\} \in \pi$ and $i < j$, formula (10.14) reads :

$$\mathbb{P}_x(T > t) = \sum_{\pi \in P_2(k)} (-1)^{c(\pi)} \prod_{\{i<j\} \in \pi} p_{ij}, \tag{10.32}$$

where $p_{ij} = \mathbb{P}_x(T_{e_i - e_j} > t) = \gamma\left(\frac{x_i - x_j}{\sqrt{2t}}\right)$.

When $k = 3$, the formula is

$$\mathbb{P}_x(T > t) = p_{12} + p_{23} - p_{13}. \tag{10.33}$$

When $k = 4$, the formula becomes

$$\mathbb{P}_x(T > t) = p_{12}p_{34} + p_{14}p_{23} - p_{13}p_{24}. \tag{10.34}$$

For odd k, we can isolate the singlet in (10.32) to deduce the following relation between the problem with k particles and the problem with $k - 1$ particles :

$$\mathbb{P}_x(T > t) = \sum_{l=1}^{k} (-1)^{l-1} \mathbb{P}_x(T_l > t), \tag{10.35}$$

where $T_l = \inf\{t : \ \exists\, i \neq j \in [k] \backslash \{l\},\, X_i(t) = X_j(t)\}$ so that $\mathbb{P}_x(\widehat{T_l} > t)$ only depends on $(x_i)_{i \in [k] \backslash \{l\}}$.

Recalling definition and expression of the Pfaffian given in the Appendix, formula (10.32) reads :

$$\mathbb{P}_x(T > t) = \begin{cases} \operatorname{Pf}\,(p_{ij})_{i,j \in [k]} & \text{if } k \text{ is even ,} \\ \sum_{l=1}^{k} (-1)^{l+1} \operatorname{Pf}\,(p_{ij})_{i,j \in [k] \backslash \{l\}} & \text{if } k \text{ is odd,} \end{cases} \tag{10.36}$$

with the convention that $p_{ji} = -p_{ij}$ for $i \leq j$. The merit of these formulae is to replace the alternating sums by closed-form expressions which are easier to compute (Pfaffians are just square roots of determinants).

10.4.3 The D_k case

The chamber is $C = \{x \in \mathbb{R}^k : \; x_1 > \cdots > x_{k-1} > |x_k|\}$. The formula (10.14) becomes :

$$\mathbb{P}_x(T > t) = \sum_{\pi \in P_2(k)} (-1)^{c(\pi)} \prod_{\{i<j\} \in \pi} \gamma\left(\frac{x_i - x_j}{\sqrt{2t}}\right) \gamma\left(\frac{x_i + x_j}{\sqrt{2t}}\right). \tag{10.37}$$

We can write it down in terms of Pfaffians :

$$\mathbb{P}_x(T > t) = \begin{cases} \text{Pf } (p_{ij}\,\bar{p}_{ij})_{i,j \in [k]} & \text{if } k \text{ is even },\\ \sum_{l=1}^{k} (-1)^{l+1} \text{Pf } (p_{ij}\,\bar{p}_{ij})_{i,j \in [k]\setminus\{l\}} & \text{if } k \text{ is odd,} \end{cases} \tag{10.38}$$

where $\bar{p}_{ij} = \bar{p}_{ji} = \mathbb{P}_x(T_{e_i+e_j} > t) = \gamma\left(\frac{x_i+x_j}{\sqrt{2t}}\right)$.

10.4.4 The B_k case

The chamber is $C = \{x \in \mathbb{R}^k : \; x_1 > \cdots > x_{k-1} > x_k > 0\}$. The formula for B_k is :

$$\mathbb{P}_x(T > t) = \sum_{\pi \in P_2(k)} (-1)^{c(\pi)}\, \mathbb{P}_x(T_{e_{s(\pi)}} > t) \prod_{\{i<j\} \in \pi} \mathbb{P}_x(T_{e_i - e_j, e_j} > t),$$

where $s(\pi)$ is the singlet of π, the term $\mathbb{P}_x(T_{e_{s(\pi)}} > t)$ being absent when k is even.

The result is a polynomial involving the probabilities of exiting orthants associated to $\{e_i - e_j, e_j\}$. Those are wedges of angle $\pi/4$.

Those formulae can be rewritten in terms of Pfaffians :

$$\mathbb{P}_x(T > t) = \begin{cases} \text{Pf } (\widehat{p}_{ij})_{i,j \in [k]} & \text{if } k \text{ is even },\\ \sum_{l=1}^{k} (-1)^{l+1}\, p_l \,\text{Pf } (\widehat{p}_{ij})_{i,j \in [k]\setminus\{l\}} & \text{if } k \text{ is odd,} \end{cases} \tag{10.39}$$

where $p_l = \mathbb{P}_x(T_{e_l} > t) = \gamma(x_l/\sqrt{t})$ and $\widehat{p}_{ij} = -\widehat{p}_{ji} = \mathbb{P}_x(T_{e_i - e_j, e_j} > t)$ for $i < j$.

10.4.5 Wedges of angle $\pi/4n$

First, let us pay attention to the case of an angle $\pi/4$ where $C = \{x_1 > x_2 > 0\}$. Consider the group $W = I_2(4)$ of isometries of the square : $W = \{\text{id}, r, r^2, r^3, s_1, s_2, s_3, s_4\}$, r being the rotation of angle $\pi/2$ and s_1 (resp. s_2, s_3, s_4) the symmetry with respect to the line $y = -x$ (resp. $y = 0$, $y = x$, $x = 0$). By translation and W-invariance, formula (10.9) can be rewritten

$$\mathbb{P}_x(T > t) = \sum_{w \in W} \varepsilon(w) \mathbb{P}(X_t \in -w(x) + C). \tag{10.40}$$

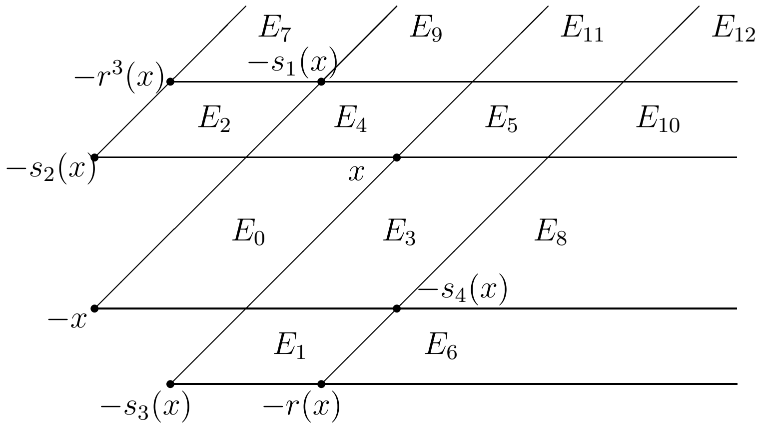


FIG. 10.5 – The regions E_i

Now, define regions E_i, $1 \le i \le 12$ as in Figure 5 and $P_i = \mathbb{P}(X_t \in E_i)$.

Then, we can decompose the terms in (10.40) into $\mathbb{P}(X_t \in -x + C) = P_0 + P_3 + P_4 + P_5 + P_8 + P_9 + P_{10} + P_{11} + P_{12}$, $\mathbb{P}(X_t \in -r(x) + C) = P_6 + P_8 + P_{10} + P_{12}$, $\mathbb{P}(X_t \in -r^2(x)+C) = P_5+P_{10}+P_{11}+P_{12}$, $\mathbb{P}(X_t \in -r^3(x)+C) = P_7+P_9+P_{11}+P_{12}$, $\mathbb{P}(X_t \in -s_1(x)+C) = P_9+P_{10}+P_{11}+P_{12}$, $\mathbb{P}(X_t \in -s_2(x)+C) = P_2+P_4+P_5+P_7+P_9+P_{10}+P_{11}+P_{12}$, $\mathbb{P}(X_t \in -s_3(x)+C) = P_1+P_3+P_5+P_6+P_8+P_{10}+P_{11}+P_{12}$, $\mathbb{P}(X_t \in -s_4(x)+C) = P_8+P_{10}+P_{12}$. In the end, terms cancel in an abundant way to result in

$$\mathbb{P}_x(T > t) = P_0 - P_1 - P_2 = P_0 - 2P_1, \tag{10.41}$$

since E_1 and E_2 are symmetric with respect to the origin. The interest of (10.41) is two-fold : first, the number of terms is significantly lower than in (10.40) and second, the integrals involved are over bounded regions. Indeed, write $x_i' = x_i/\sqrt{t}$ for simplicity, then

$$P_0 = \frac{1}{\pi}\int_0^{x_1'-x_2'} dy \int_{-x_2'}^{x_2'} dz \; e^{-\left((y+z)^2+z^2\right)/2},$$

$$P_1 = \frac{1}{2\pi}\int_{x_1'-x_2'}^{x_1'+x_2'} dy \int_{-x_1'}^{-x_2'} dz \; e^{-\left((y+z)^2+z^2\right)/2},$$

so that

$$\begin{aligned}\mathbb{P}_x(T > t) &= \frac{1}{\pi}\int_0^{x_1'-x_2'} dy \int_{-x_2'}^{x_2'} dz \; - \int_{x_1'-x_2'}^{x_1'+x_2'} dy \int_{-x_1'}^{-x_2'} dz \; e^{-\left((y+z)^2+y^2\right)/2} \\ &:= H(x_1', x_2'). \end{aligned} \tag{10.42}$$

The integral formula defines H on $\mathbb{R}^2$ and implies that $H(y_1, -y_2) = H(y_1, y_2)$ and

$H(y_2, y_1) = -H(y_1, y_2)$. Thus, expanding G in power series, we have

$$H(y_1, y_2) = \sum_{p,q \geq 0} a_{pq} \, (y_1 \, y_2)^{2p+1} \, (y_1^{2q} - y_2^{2q}). \tag{10.43}$$

This type of result generalizes to angles $\pi/4n$ with the same kind of reasoning. Write $C_0 = \{re^{i\theta};\ r \geq 0,\ 0 \leq \theta \leq \pi/4n\}$ the fundamental chamber, $(C_i)_{1 \leq i \leq 8n-1}$ the other chambers in counterclockwise order of appearance and x_i the W-image of $x \in C_0$ which lies in C_i. Call $\mathcal{P}(x, y)$ the parallelogram which has x and y as two opposite vertices and whose sides are parallel to the walls of the chamber C_0. Define $P_i = P_i(x, t) := \mathbb{P}_0(X_t \in \mathcal{P}(x_i, x_{4n-i}))$. Then, the formula is

$$\mathbb{P}_x(T > t) = P_0 + 2 \sum_{i=1}^{2n-1} (-1)^i P_i.$$

10.4.6 Asymptotic expansions

We will now consider the asymptotic behaviour of $\mathbb{P}_x(T > t)$ in two different regimes where t is either large or small compared with the initial distance between x and the boundary ∂C.

Long time behaviour

We will suppose that $|\Pi|$ has the same parity as $|I|$, which is the case in all of our examples.

Proposition 10.4.1. *If I is consistent and orthogonal (or semi-orthogonal), the following expansion holds :*

$$\mathbb{P}_x(T > t) = h(x) \sum_{q \geq 0} E_q(x) \, t^{-(q+n/2)}, \tag{10.44}$$

where $n = |\Pi|$, $h(x) = \prod_{\alpha \in \Pi} (x, \alpha)$, $E_q(x)$ is a W-invariant polynomial of degree $2q$ and the series is convergent for all $x \in C$, $t > 0$. In particular, there exists a constant κ such that :

$$\mathbb{P}_x(T > t) \sim \frac{\kappa \, h(x)}{t^{n/2}} \ \text{ as } t \to \infty. \tag{10.45}$$

Remark 10.4.1. *The leading term in (10.45) was obtained in Grabiner [Gra99] for the classical root systems of types A, B and D. More general results on expected exit times from cones and their moments can be found in [Bur55], [DZ94] and [DeB87].*

Remark 10.4.2. *The polynomials E_q satisfy $\Delta_h E_{q+1} = -(q+n/2)\,E_q$, where*

$$\Delta_h f = \frac{1}{2}\Delta f + (\nabla \log h)\cdot \nabla f.$$

This follows from the fact that $u(x,t) = \mathbb{P}_x(T > t)$ satisfies the heat equation and $\Delta h = 0$.

For $I_2(m)$ with $m \neq 0 \mod 4$, one has $|\Pi| = m$ so that

$$\mathbb{P}_x(T > t) \sim \kappa\,\frac{h(x)}{t^{m/2}}.$$

Let us indicate how to compute the constant κ for odd $m = 2m'+1$. We have $\kappa\, h(x) = a_{m'} \sum_1^m (-1)^{j-1}(x,\alpha_j)^m$. Suppose $x = e^{i\theta}$ then $(x,\alpha_j) = \sin(j\pi/m - \theta)$. Let $s = \pi/m - \theta$ go to 0 so that $h(x) \sim a\, s$ with $a = \prod_{j=1}^{m-1}\lambda_j = \prod_{j=1}^{m'}\lambda_j^2$, $\lambda_j = \sin(j\pi/m)$. The numbers $0, \lambda_j, -\lambda_j$, $1 \leq j \leq m'$ are the roots of the Tchebycheff polynomial T_m such that $T_m(\cos\theta) = \cos m\theta$. The leading coefficient of T_m being 2^{m-1}, we have

$$a = (-1)^{m'} 2^{1-m} T_m'(0) = m 2^{1-m}.$$

On the other hand, $\sum_1^m (-1)^{j-1}(x,\alpha_j)^m \sim b\, s$ with

$$b = m \sum_1^{m-1} (-1)^j \sin^{m-1}(j\pi/m)\cos(j\pi/m).$$

Thus, $\kappa = a_{m'} b/a$.

For A_{k-1},

$$\mathbb{P}_x(T > t) \sim \kappa\,\frac{\prod_{1\leq i<j\leq k}(x_i - x_j)}{t^{k(k-1)/4}}.$$

For D_k,

$$\mathbb{P}_x(T > t) \sim \kappa\,\frac{\prod_{1\leq i<j\leq k}(x_i^2 - x_j^2)}{t^{k(k-1)/2}}.$$

The constant κ is related to the Selberg-type integral

$$\Omega = \int_C e^{-|z|^2/2}\, h(z)\, dz.$$

In the example of A_{k-1}, (10.3) yields

$$\begin{aligned}\mathbb{P}_x(T > t) &= \int_C \det(p_t(x_i, y_j))_{1\leq i,j\leq k}\, dy \\ &= \frac{e^{-|x|^2/2t}}{(2\pi)^{k/2}} \int_C e^{-|z|^2/2} \det(e^{x_i z_j/\sqrt{t}})_{1\leq i,j\leq k}\, dz.\end{aligned}$$

Taking $x = \delta = (k-1, k-2, \dots, 0)$,

$$\det(e^{x_i z_j/\sqrt{t}})_{1\le i,j\le k} = h(z/\sqrt{t}) = h(z)/t^{n/2}$$

so that

$$\mathbb{P}_x(T > t) \sim \frac{t^{-n/2}}{(2\pi)^{k/2}}\Omega.$$

On the other hand, $h(\delta) = \prod_{j=1}^{k-1} j!$ from which it follows that $\Omega = (2\pi)^{-k/2}\prod_{j=1}^{k-1} j!\,\kappa$.

Similar formulae can be obtained in other cases relating the constant κ to the corresponding Selberg-type integral. General formulae for Selberg integrals associated with reflection groups are given by the Macdonald-Mehta conjectures (now proved) and can be found for example in [Meh91].

Small time behaviour

Now the quantity of interest is $\mathbb{P}_x(T \le t)$ which goes to 0 as $t \to 0$. Such asymptotics for general domains are the subject of a vast literature in the setting of large deviations (see [FW84]). The interest of our direct approach is to provide very precise results when the domain is the chamber of some finite reflection group. Let us introduce notation $\theta(u) = 1 - \gamma(u) \sim \sqrt{\frac{2}{\pi}}u^{-1}e^{-u^2/2}$ as $u \to \infty$.

We will suppose that I is orthogonal and consistent, which only rules out B_k and F_4. Then,

$$\mathbb{P}_x(T \le t) = \sum_{B\in\mathcal{O}(\Pi)\setminus\{\emptyset\}} (-1)^{l(B)-1}\prod_{\alpha\in B}\theta(\widehat{\alpha}(x)/\sqrt{t}). \tag{10.46}$$

The case of $I_2(m)$ with odd $m = 2m'+1$ is particularly simple since $\mathcal{O}(\Pi) = \big\{\{\alpha_i\},\, 1 \le i \le m\big\}$ and $\mathbb{P}_x(T \le t) = \sum_i (-1)^{i-1}\theta\left((x,\alpha_i)/\sqrt{t}\right)$. Suppose $(x,\alpha_1) \le (x,\alpha_m)$. Then we have $(x,\alpha_1) \le (x,\alpha_m) < (x,\alpha_2) \le (x,\alpha_{m-1}) < (x,\alpha_3) \le (x,\alpha_{m-2}) < \cdots < (x,\alpha_{m'+1})$, which gives the hierarchy of terms in the asymptotic behaviour of $\mathbb{P}_x(T \le t)$ and enables the expansion with any given precision. For example, if x is fixed, $t \to \infty$ and noting $a_i = (x,\alpha_i)/\sqrt{t}$,

$$\mathbb{P}_x(T \le t) = \theta(a_1) + \theta(a_m) - \theta(a_2) - \theta(a_{m-1}) + o\left(\theta\left(\min(a_2, a_{m-1})\right)\right).$$

Similarly, one could deal with $I_2(m)$ for $m \equiv 2 \mod 4$.

Let us consider the general case and suppose there is only one root length, which is the case in all the examples where I is orthogonal. Although formula (10.46) allows to deal with general starting point, we will suppose for simplicity that x is at equal distance from all the walls of C, ie $\forall\alpha \in \Delta$, $d(x,\alpha^\perp) = (\alpha,x)/|\alpha| = c$. Then, $\widehat{\alpha}(x) = c\,\mathrm{ht}(\alpha)$

where $\mathrm{ht}(\alpha) > 0$ is the sum of the coordinates of $\alpha \in \Pi$ in the basis Δ. Noting $a = c/\sqrt{t}$, the following equivalent hold :

$$q_B := \prod_{\alpha \in B} \theta(\widehat{\alpha}(x)/\sqrt{t}) \sim c(B)\, a^{-|B|}\, e^{-n(B)a^2/2} \ \text{ as } a \to \infty,$$

where $n(B) = \sum_{\alpha \in B} \mathrm{ht}(\alpha)^2$ and $c(B) = \left(\frac{2}{\pi}\right)^{|B|/2} \left(\prod_{\alpha \in B} \mathrm{ht}(\alpha)\right)^{-1}$. Thus,

$$q_{B'} = o(q_B) \iff (n(B') > n(B) \text{ or } (n(B') = n(B) \text{ and } |B'| > |B|))\,.$$

Proposition 10.4.2. *The expansion up to some "order" n is given by*

$$\mathbb{P}_x(T \le t) = \sum_{n(B) \le n} (-1)^{l(B)-1} \prod_{\alpha \in B} \theta(ht(\alpha)a) + o\left(a^{-r}\, e^{-na^2/2}\right) \ \ \text{as } a \to \infty, \qquad (10.47)$$

where $r = \max\{|B| : \ n(B) = n\}$.

For instance, let us look at formula (10.47) for "small orders" $n = 1, 2, 3, 4$ in the crystallographic cases. Here they correspond to A_{k-1} and D_k. In this case, $\mathrm{ht}(\alpha) \in \mathbb{N}^*$ for all $\alpha \in \Pi$ so that $\mathrm{ht}(\alpha) = 1 \Leftrightarrow \alpha \in \Delta$. Recalling $n(B) = \sum_{\alpha \in B} \mathrm{ht}(\alpha)^2$, we see that

$$n(B) = i \Leftrightarrow (B \in \mathcal{O}(\Delta),\, |B| = i) \ \text{ for } i = 1, 2, 3.$$

Then

$$n(B) = 4 \Leftrightarrow (B \in \mathcal{O}(\Delta),\, |B| = 4) \ \text{ or } \ (B = \{\alpha\},\, \mathrm{ht}(\alpha) = 2)\,.$$

The result can thus be written :

$$\mathbb{P}_x(T \le t) = \lambda_1\theta(a) - \lambda_2\theta(a)^2 + \lambda_3\theta(a)^3 - \lambda_4'\theta(2a) - \lambda_4\theta(a)^4 + o\left(a^{-4}e^{-4a^2/2}\right), \ (10.48)$$

where $\lambda_i = |\{A \in \mathcal{O}(\Delta) : \ |A| = i\}|$ for $1 \le i \le 4$ and $\lambda_4' = \sum_{\mathrm{ht}(\alpha)=2} (-1)^{l(\{\alpha\})}$.

The case of A_{k-1} can be interpreted as k Brownian particles with equal distance c between consecutive neighbours. The constants in (10.48) can be explicitly computed : $\lambda_1 = k-1$,$\lambda_2 = (k-2)(k-3)/2$, $\lambda_3 = (k-5)(k-4)(k-3)/6$, $\lambda_4 = (k-7)(k-6)(k-5)(k-4)/24$ and $\lambda_4' = k-2$. This extends a previous result of [OU92] where the expansion was given up to "order" 2.

In the case of D_k, $\lambda_1 = k$, $\lambda_2 = (k-1)(k-2)/2$, $\lambda_3 = (k-4)(k-3)(k-2)/6 + 1$, $\lambda_4 = (k-6)(k-5)(k-4)(k-3)/24 + k - 5$ and $\lambda_4' = k - 1$.

10.4.7 Expected exit times

The dihedral case

A well-known result of Spitzer [Spi58] is that $\mathbb{E}_x(T^r) < \infty$ if and only if $m > r$, independently of x. Note that this also follows from the results of Section 10.4.6. In particular, $\mathbb{E}_x(T)$ is always finite. Let us concentrate on the case of odd m. It is impossible to get expectations directly by integrating (10.29) since the T_{α_i} are not integrable. The strategy is to use formula (10.29) to compute Laplace transforms :

$$\mathbb{E}_x(e^{-\lambda T}) = \sum_{i=1}^{m}(-1)^{i-1}\mathbb{E}_x(e^{-\lambda T_{\alpha_i}}) = \sum_{i=1}^{m}(-1)^{i-1}\exp(-(x,\alpha_i)\sqrt{2\lambda}),\ \lambda > 0. \tag{10.49}$$

By differentiation,

$$\mathbb{E}_x(Te^{-\lambda T}) = \frac{1}{\sqrt{2\lambda}}\left(\sum_{i=1}^{m}(-1)^{i-1}(x,\alpha_i)\exp(-(x,\alpha_i)\sqrt{2\lambda})\right).$$

Since $\sum_{i=1}^{m}(-1)^{i-1}(x,\alpha_i) = 0$ (cf Proposition 10.2.4), we can let $\lambda \to 0$ and get

$$\mathbb{E}_x(T) = \sum_{i=1}^{m}(-1)^{i}(x,\alpha_i)^2. \tag{10.50}$$

Remark 10.4.3. *In fact, identifying coefficients in the asymptotic expansion of (10.49) with respect to $\lambda \to 0$, we can express the moments :*

$$\mathbb{E}_x(T^r) = \frac{2^r\, r!}{(2r)!}\sum_{i=1}^{m}(-1)^{i+r-1}(x,\alpha_i)^{2r},\ 0 \le r \le m-1. \tag{10.51}$$

Writing $\cos^2\theta = \frac{1+\mathcal{R}(e^{i2\theta})}{2}$, the sum in (10.50) is computable using elementary geometric series : $\mathbb{E}_x(T) = \frac{r^2}{2}\left(\frac{\cos(2\theta-\pi/m)}{\cos(\pi/m)} - 1\right)$ for $x = re^{i\theta}$ with $r \ge 0$, $0 \le \theta \le \pi/m$. This formula makes good sense even if the angle α of the cone is not a fraction of π. It satisfies the Poisson equation $\Delta f = -2$ with the correct boundary conditions so that the exit time from the cone $C_\alpha = \{x = re^{i\theta} :\ r > 0,\ 0 < \theta < \alpha\}$ is :

$$\mathbb{E}_x(T) = \frac{r^2}{2}\left(\frac{\cos(2\theta-\alpha)}{\cos\alpha} - 1\right),\ x = re^{i\theta} \in C_\alpha. \tag{10.52}$$

Remark 10.4.4. *The formula (10.52) can be deduced from Spitzer's results. See, for example, Bramson and Griffeath [BG91].*

The A_{k-1} case

Since $\mathbb{P}_x(T > t) \sim C\, t^{-k(k-1)/4}$ as $t \to \infty$ (see Section 10.4.6), $\mathbb{E}_x(T^r) < \infty$ if and only if $r < k(k-1)/4$. In particular, $\mathbb{E}_x(T) = \infty$ for $k = 2$ (which is well-known) and $\mathbb{E}_x(T) < \infty$ for $k \geq 3$. Since A_2 is isomorphic to $I_2(3)$, it follows from the previous section that $\mathbb{E}_x(T) = (x_1 - x_2)(x_2 - x_3)$, which can be checked directly with the Poisson equation. Since $\gamma(a/\sqrt{t}) \sim C/\sqrt{t}$, formula (10.32) can only be integrated term by term if $k \geq 6$.

Let us first deal with the case $k = 4$ for which we use a kind of Laplace transform trick again and set :

$$E(\lambda) := \int_0^\infty e^{-\lambda t}\mathbb{P}_x(T > t)\, dt \ , \ \lambda > 0, \tag{10.53}$$

where $\mathbb{P}_x(T > t) = \gamma(x_{12}/\sqrt{t})\gamma(x_{34}/\sqrt{t}) - \gamma(x_{13}/\sqrt{t})\gamma(x_{24}/\sqrt{t}) + \gamma(x_{14}/\sqrt{t})\gamma(x_{23}/\sqrt{t})$ and $x_{ij} = x_i - x_j$. We write $\gamma(a/\sqrt{t}) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^a e^{-z^2/2t} \frac{dz}{\sqrt{t}}$ so that

$$\begin{aligned} J_\lambda(x_{ij}, x_{lm}) &:= \int_0^\infty e^{-\lambda t}\gamma(x_{ij}/\sqrt{t})\gamma(x_{lm}/\sqrt{t})\, dt \\ &= \frac{2}{\pi}\int_0^{x_{ij}}\!\!\int_0^{x_{lm}} \left(\int_0^\infty e^{-\lambda t - (z^2+w^2)/2t}\frac{dt}{t}\right) dzdw \end{aligned} \tag{10.54}$$

The integral between parenthesis can be written explicitly in terms of the Bessel function K_0 of the second kind (see, for example, [AS64, (29.3.119)])

$$J_\lambda(x_{ij}, x_{lm}) = \frac{4}{\pi}\int_0^{x_{ij}}\!\!\int_0^{x_{lm}} K_0\left(\sqrt{2\lambda(z^2+w^2)}\right) dzdw. \tag{10.55}$$

We want to let $\lambda \to 0$ and use $K_0(x) = -\log(x/2) - c + \varepsilon(x)$, where c is the Euler constant and $\varepsilon(x) = o(x)$ as $x \to 0$ (see the Bessel function formula 9.6.13). Thus

$$J_\lambda(x_{ij}, x_{lm}) = \frac{4}{\pi}\left(x_{ij}x_{lm}c(\lambda) + \frac{1}{2}I(x_{ij}, x_{lm}) + r(\lambda)\right), \tag{10.56}$$

where $c(\lambda) = -\log(\sqrt{\lambda/2}) - c$, $I(a,b) := \frac{-1}{2}\int_0^a\!\int_0^b \log(z^2+w^2)\, dzdw$ and $r(\lambda) \to 0$ as $\lambda \to 0$. Since $x_{12}x_{34} - x_{13}x_{24} + x_{23}x_{14} = 0$ (see Proposition 10.2.4), we obtain

$$E(\lambda) = \frac{2}{\pi}\left(I(x_{12}, x_{34}) - I(x_{13}, x_{24}) + I(x_{14}, x_{23})\right) + r'(\lambda),$$

where $r'(\lambda) \to 0$ as $\lambda \to 0$. Letting $\lambda \to 0$ yields

$$\mathbb{E}_x(T) = F_2(x_1 - x_2, x_3 - x_4) - F_2(x_1 - x_3, x_2 - x_4) + F_2(x_1 - x_4, x_2 - x_3), \tag{10.57}$$

where we can explicitly compute

$$F_2(a,b) = \frac{2}{\pi}I(a,b) = \frac{2}{\pi}\left(3ab + (a^2 - b^2)\arctan(a/b) - \pi a^2/2 - ab\log(a^2+b^2)\right). \tag{10.58}$$

Remark 10.4.5. *We have $(\partial_a^2+\partial_b^2)F_2=-2$ from which we can check the Poisson equation for (10.57). The correct boundary conditions follow from $F_2(a,b)=F_2(b,a)$ and $F_2(0,b)=0$.*

The case $k=5$ is deduced from the previous one thanks to (10.35) :

$$\mathbb{E}_x(T)=\sum_{l=1}^{5}(-1)^{l-1}\mathbb{E}_x(T_l), \tag{10.59}$$

where $\mathbb{E}_x(T_l)$ is the expression (10.57) applied to $(x_i)_{1\le i\neq l\le 5}$.

Let us now consider $k\ge 6$, note $p=\lfloor k/2\rfloor\ge 3$, use $\gamma(a/\sqrt{t})=\left(\frac{2}{\pi}\right)^{1/2}\int_0^a e^{-z^2/2t}\frac{dz}{\sqrt{t}}$ and integrate (10.32) term by term :

$$\mathbb{E}_x(T)=\sum_{\pi\in P_2(k)}(-1)^{c(\pi)}\left(\frac{2}{\pi}\right)^{p/2}\int_\pi\left(\int_0^\infty e^{-\sum z_{ij}^2/2t}\frac{dt}{t^{p/2}}\right)\prod_{\{i<j\}\in\pi}dz_{ij}, \tag{10.60}$$

where $\int_\pi$ denotes integration over $\prod_{\{i<j\}\in\pi}[0,x_{ij}]$. Now, we use

$$\int_0^\infty e^{-k/2t}\frac{dt}{t^q}=\frac{2^{q-1}\Gamma(q)}{k^{q-1}(q-1)},\ k>0,$$

and set

$$F_p(y_1,\ldots,y_p)=\frac{2^{p+1}\Gamma(p/2)}{\pi^{p/2}(p-2)}\int_0^{y_1}\cdots\int_0^{y_p}\frac{dz_1\ldots dz_p}{(z_1^2+\cdots+z_p^2)^{p/2-1}}, \tag{10.61}$$

to result in

$$\mathbb{E}_x(T)=\sum_{\pi\in P_2(k)}(-1)^{c(\pi)}F_p(x_\pi), \tag{10.62}$$

where $x_\pi=(x_i-x_j)_{\{i<j\}\in\pi}\in\mathbb{R}_+^p$. The previous result holds for $p=\lfloor k/2\rfloor\ge 2$.

Remark 10.4.6. *The formula (10.62) cannot be expressed as a Pfaffian since F_p does not have a product form.*

The D_k case

Since $\mathbb{P}_x(T>t)\sim C\,t^{-k(k-1)/2}$ as $t\to\infty$ (see Section 10.4.6), $\mathbb{E}_x(T^r)<\infty$ if and only if $r<k(k-1)/2$. In particular, $\mathbb{E}_x(T)=\infty$ for $k=2$ and $\mathbb{E}_x(T)<\infty$ for $k\ge 3$. The method of computation is the same as for A_{k-1}, with the same obstacle for D_3 as for A_3. The general result for D_k is

$$\mathbb{E}_x(T)=\sum_{\pi\in P_2(k)}(-1)^{c(\pi)}F_{2p}(x'_\pi), \tag{10.63}$$

where $p=\lfloor k/2\rfloor\ge 2$, F_{2p} is defined in (10.58) an (10.61) and $x'_\pi=(x_i-x_j,x_i+x_j)_{\{i<j\}\in\pi}\in\mathbb{R}_+^{2p}$.

10.5 A generalisation of de Bruijn's formula

Suppose I is consistent. For $A \in \mathcal{I}$, denote dy W_A the group generated by reflections s_α, $\alpha \in A$. Denote by C_A the chamber associated to A, $C_A = \{x \in V : \ \forall \alpha \in A,\ (x,\alpha) > 0\}$. We will assume that C_A is a fundamental region for the reflection group W_A, which is certainly the case if I is orthogonal or semi-orthogonal.

Proposition 10.5.1. *If $f : V \to \mathbb{R}$ is integrable, then*

$$\int_C \sum_{w\in W} \varepsilon(w) f(wy)\,dy = \sum_{A\in\mathcal{I}} \varepsilon_A \sum_{w\in W_A} \varepsilon(w) \int_{C_A} f(wy)\,dy. \tag{10.64}$$

If I is orthogonal and $A = \{\alpha_1, \dots, \alpha_l\} \in \mathcal{I}$, then $W_A \simeq \langle s_{\alpha_1}\rangle \times \cdots \times \langle s_{\alpha_l}\rangle$.

If I is semi-orthogonal and $A = \rho_1 \cup \ldots \cup \rho_l \in \mathcal{I}$, then $W_A \simeq \langle \rho_1\rangle \times \cdots \times \langle \rho_l\rangle$, where each factor is either $\mathbb{Z}/2$ or $I_2(4)$ according whether ρ is a singlet or a pair.

10.5.1 The dihedral case

Recall that C is the wedge of angle π/m, $\alpha_j = e^{i(j\pi/m-\pi/2)}$, $\alpha'_j = e^{i(j\pi/m)}$, $E_j = \{x : (x,\alpha_j) > 0\}$, $E'_j = \{x : \ (x,\alpha'_j) > 0\}$ for $1 \le j \le m$.

Proposition 10.5.2. *If $f : \mathbb{R}^2 \to \mathbb{R}$ is integrable, then*

$$\int_C \sum_{w\in W} \varepsilon(w) f(wy)\,dy$$
$$= \begin{cases} \sum_{j=1}^m (-1)^{j-1} \int_{E_j} \left(f(y) - f(s_j y)\right)\,dy & \text{if } m \text{ is odd} \\ \sum_{j=1}^m (-1)^{j-1} \int_{E_j\cap E'_j} \left(f(y) - f(s_j y) - f(-s_j y) + f(-y)\right)\,dy & \text{if } m \equiv 2 \mod 4 \end{cases}.$$

10.5.2 Type A

Proposition 10.5.3. *If $f(y) = f_1(y_1)\ldots f_k(y_k)$ for integrable functions $f_i : \mathbb{R} \to \mathbb{R}$, then*

$$\int_C \det\left(f_i(y_j)\right)_{i,j\in[k]}\,dy_1 \ldots dy_k = \mathrm{Pf}\,\left(I(f_i,f_j)\right)_{i,j\in[k]},\ \text{if } k \text{ is even and,} \tag{10.65}$$

$$\int_C \det\left(f_i(y_j)\right)_{i,j\in[k]}\,dy_1 \ldots dy_k = \sum_{l=1}^k (-1)^{l+1} \int_{\mathbb{R}} f_l\ \mathrm{Pf}\,\left(I(f_i,f_j)\right)_{i,j\in[k]\setminus\{l\}},\ \text{if } k \text{ is odd,} \tag{10.66}$$

where I is the skew-symmetric bilinear form

$$I(f,g) = \int_{y>z} \left(f(y)g(z) - f(z)g(y)\right)\,dydz = \int sgn(y-z) f(y) g(z)\,dydz. \tag{10.67}$$

Remark 10.5.1. *This formula was first obtained by de Bruijn [dB55] using completely different methods. For a recent discussion with interesting connections to shuffle algebras, see [LT02].*

10.5.3 Type D

Proposition 10.5.4. *If $f(y) = f_1(y_1)\dots f_k(y_k)$ for integrable functions $f_i : \mathbb{R} \to \mathbb{R}$, then*

$$\int_C \det\left(f_i(y_j)\right)_{i,j\in[k]} dy_1 \dots dy_k = 0, \tag{10.68}$$

if the functions f_i are odd;

$$2^{(k-2)/2} \int_C \det\left(f_i(y_j)\right)_{i,j\in[k]} dy_1 \dots dy_k = \text{Pf}\ \left(K(f_i, f_j)\right)_{i,j\in[k]}, \tag{10.69}$$

if the functions f_i are even and k is even;

$$2^{(k-1)/2} \int_C \det\left(f_i(y_j)\right)_{i,j\in[k]} dy_1 \dots dy_k = \sum_{l=1}^{k} (-1)^{l+1} \int_{\mathbb{R}} f_l \ \text{Pf}\ \left(K(f_i, f_j)\right)_{i,j\in[k]\setminus\{l\}}, \tag{10.70}$$

if the functions f_i are even and k is odd; here, K is the skew-symmetric bilinear form

$$K(f,g) = \frac{1}{2} \int_{y>|z|} \left(f(y)g(z) - f(z)g(y)\right) dy\, dz. \tag{10.71}$$

Remark 10.5.2. *It is also possible to translate (10.64) into concrete terms for type B but the formula obtained is just a special case of Proposition 10.5.3.*

10.6 Random walks and related combinatorics

Let I_ν denote the Bessel function of index ν and recall that

$$I_\nu(x) = \sum_{l\geq 0} \frac{1}{l!\,\Gamma(\nu + l + 1)} (x/2)^{2l+\nu}.$$

Proposition 10.6.1. *Let $X_1, \dots, X_k$ be independent Poisson processes with unit rate and write $X = (X_1, \dots, X_k)$. Denote by T the first exit time of X from $C = \{x \in \mathbb{N}^k : x_1 > \dots > x_k\}$. Then,*

$$\mathbb{P}_x(T > t) = \sum_{\pi\in P_2(k)} (-1)^{c(\pi)} \prod_{\{i<j\}\in\pi} q_{x_i - x_j}(t), \tag{10.72}$$

where $q_x(t) = \sum_{l=-x+1}^{x} p_l(t)$ and $p_y(t) = e^{-2t} I_{|y|}(2t)$.

Proposition 10.6.1 allows us to enumerate a certain class of Young tableaux. Let $\mathcal{T}_k(n)$ denote the set of standard Young tableaux with n boxes and height at most k and write $\tau_k(n)$ for the cardinality of $\mathcal{T}_k(n)$. If f^λ is the number of standard Young tableaux of shape λ, then $\tau_k(n) = \sum_{\lambda\vdash n,\, \lambda_1\leq k} f^\lambda$.

Proposition 10.6.2. *The generating function of* $(\tau_k(n), n \geq 0)$ *is given by*

$$y_k(t) := \sum_{n\geq 0} \frac{\tau_k(n)}{n!} t^n = \begin{cases} H_k\left(\gamma(t)\right) & \text{if } k \text{ is even,} \\ e^t\, H_k\left(\gamma(t)\right) & \text{if } k \text{ is odd,} \end{cases} \tag{10.73}$$

where $\gamma(t) = (\gamma_i(t))_{1\leq i\leq k-1}$, $\gamma_i(t) = \sum_{l=-i+1}^{i} I_{|i|}(t) = I_0(t) + 2\sum_{l=1}^{i-1} I_l(t) + I_i(t)$ *and* H_k *is the polynomial*

$$H_k(Y_1, \ldots, Y_{k-1}) = \sum_{\pi\in P_2(k)} (-1)^{c(\pi)} \prod_{\{i<j\}\in\pi} Y_{j-i} = \mathrm{Pf}\,\left(Y_{j-i}\right),$$

with the convention $Y_l = -Y_{-l}$.

Proposition 10.6.3.

$$\tau_k(n) = \sum_{\lambda\vdash n, \lambda_1\leq k} f_\lambda = \begin{cases} \sum_{\pi\in P_2(k)} (-1)^{c(\pi)} p_\pi(n) & \text{if } k \text{ is even;} \\ \sum_{\pi\in P_2(k)} (-1)^{c(\pi)} p'_\pi(n) & \text{if } k \text{ is odd,} \end{cases} \tag{10.74}$$

where, for $\pi = \{\{i_1, i_2\}, \ldots, \{i_{k-1}, i_k\}\}$, $p_\pi(n)$ *(respectively* $p'_\pi(n)$*) is the number of lattice paths of length* n *(respectively at most* n*) in* $\mathbb{N}^{k/2}$ *started at* $x_\pi := (|i_1 - i_2|, \ldots, |i_{k-1} - i_k|)$.

The formula (10.73) is related to a formula obtained by Gordon [Gor83] concerning plane partitions. Following [Ste90] we can transform this Pfaffian expression into a determinantal expression to recover the following well-known identity of Gessel [Ges90] :

$$y_k(t) = \begin{cases} \det[I_{i-j}(2x) + I_{i+j-1}(2x)]_{1\leq i,j\leq k/2} & \text{if } k \text{ is even;} \\ e^t\, \det[I_{i-j}(2x) - I_{i+j}(2x)]_{1\leq i,j\leq (k-1)/2} & \text{if } k \text{ is odd.} \end{cases}$$

Similar formulas exist for $\sum_{\lambda\vdash n,\, \lambda_1\leq k} s_\lambda$ (see, for example, [Ste90]) ; these can also be obtained by our approach if one considers a slightly different class of random walks.

10.7 Proofs

10.7.1 The main result

Recall that, if $\alpha \in \Delta$, then $s_\alpha(\Pi\backslash\{\alpha\}) = \Pi\backslash\{\alpha\}$. (See, for example, [Hum90, Proposition 1.4].) We begin with the following key lemma.

Lemma 10.7.1. *If $K \subset I$ and $\alpha \in \Delta \cap K^\perp$, then $s_\alpha L = L$, where*

$$L = \{w \in W^I : \ K \subset wI, \ \alpha \notin wI\}.$$

Proof. Suppose $w \in L$. α being the only positive root made negative by s_α, ($wI \subset \Pi$ and $\alpha \notin wI$) implies that $s_\alpha wI \subset \Pi$. If $\alpha \in s_\alpha wI$ then $s_\alpha(\alpha) = -\alpha \in wI$, which is absurd since $wI \subset \Pi$. Hence $\alpha \notin s_\alpha wI$. Seeing that $K \subset wI$, we get $K = s_\alpha K \subset s_\alpha wI$. This proves that $s_\alpha w \in L$. Thus $s_\alpha L \subset L$. Applying s_α to the previous inclusion, we get $L \subset s_\alpha L$, whence the equality. ♢

Lemma 10.7.2. *If $g : W^I \to \mathbb{R}$ and $\alpha \in \Delta$ are such that $g(w) = 0$ whenever $\alpha \in wI$, and $g(w) = g(s_\alpha w)$ whenever $\alpha \notin wI$, then $\sum_{w \in W^I} \varepsilon(w) g(w) = 0$.*

Proof. Applying lemma 10.7.1 with $K = \phi$,

$$\sum_{w \in W^I} \varepsilon(w) g(w) = \sum_{w \in W^I, \alpha \notin wI} \varepsilon(w) g(w) = \sum_{w \in W^I, \alpha \notin wI} \varepsilon(s_\alpha w) g(s_\alpha w)$$
$$= - \sum_{w \in W^I, \alpha \notin wI} \varepsilon(w) g(w),$$

which must therefore be zero. ♢

Lemma 10.7.3. *If condition (C1) is satisfied, then*

$$\sum_{w \in W^I} \varepsilon(w) = \sum_{w \in U} \varepsilon(w).$$

Proof. By lemma 10.7.1, for any $K \subset I$ and $\alpha \in \Delta \cap K^\perp$,

$$\sum_{w \in W^I, K \subset wI, \alpha \notin wI} \varepsilon(w) = \sum_{w \in W^I, K \subset wI, \alpha \notin wI} \varepsilon(s_\alpha w) = - \sum_{w \in W^I, K \subset wI, \alpha \notin wI} \varepsilon(w)$$

so that $\sum_{w \in W^I, K \subset wI, \alpha \notin wI} \varepsilon(w) = 0$ and

$$\sum_{w \in W^I, K \subset wI} \varepsilon(w) = \sum_{w \in W^I, K \cup \{\alpha\} \subset wI} \varepsilon(w). \tag{10.75}$$

If $J = \{\alpha_1, \ldots, \alpha_q\}$, we apply (10.75) successively to $(K, \alpha) = (\emptyset, \alpha_1)$, $(K, \alpha) = (\{\alpha_1\}, \alpha_2)$, etc, until exhaustion of J. We then obtain :

$$\sum_{w \in W^I} \varepsilon(w) = \sum_{w \in W^I, J \subset wI} \varepsilon(w).$$

Now, hypothesis (C1) makes sure that the last sum runs over those w such that $wI = I$, that is $w \in U$. Thus, the result is proved. ♢

Proposition 10.7.1. *Suppose I is consistent. Then :*
(i) $\sum_{A\in\mathcal{I}} \varepsilon_A = 1$.
(ii) *If $f : \mathcal{I} \to \mathbb{R}$ and $\alpha \in \Delta$ are such that $f(A) = 0$ whenever $\alpha \in A$, and $f(A) = f(s_\alpha A)$ whenever $\alpha \notin A$, then $\sum_{A\in\mathcal{I}} \varepsilon_A f(A) = 0$.*

Proof. If $f : \mathcal{I} \to \mathbb{R}$ then

$$\sum_{w\in W^I} \varepsilon(w) f(wI) = \sum_{A\in\mathcal{I}} \sum_{wI=A} \varepsilon(w) f(wI) = |U| \sum_{A\in\mathcal{I}} \varepsilon_A f(A)$$

since $|\{w :\ wI = A\}| = |U|$ for every $A \in \mathcal{I}$. Hence (ii) follows from lemma 10.7.2 and (i) follows from lemma 10.7.3 and the previous computation with $f = 1$. ♢

Proof of proposition 10.2.3. We check the boundary conditions for $u(x,t) = \sum_{A\in\mathcal{I}} \varepsilon_A \mathbb{P}_x(T_A > t)$.
If $x \in C$, the regularity hypothesis (10.11) guarantees that $\mathbb{P}_x(T_A > 0) = 1$ for all A, which implies $u(x,0) = 1$ thanks to (i) of Proposition 10.7.1.
Suppose $x \in \partial C$, choose $\alpha \in \Delta$ such that $(\alpha, x) = 0$ and set $f(A) = \mathbb{P}_x(T_A > t)$. If $\alpha \in A$ then $f(A) = 0$. If $\alpha \notin A$ then use invariance of the law of X under s_α and the fact that $s_\alpha x = x$ to get $f(s_\alpha A) = f(A)$. $u(x,t) = 0$ follows from (ii) of Proposition 10.7.1. ♢

10.7.2 Bijection and cancellation lemmas

Lemma 10.7.4. *Suppose $\alpha \in \Delta$.*
(i) *$\theta : A \to s_\alpha A$ is a permutation of $\{A \in \mathcal{I} :\ \alpha \notin A\}$.*
(ii) *Suppose I is orthogonal. If $B \in \mathcal{O}(\Pi)$ and $\alpha \notin B^\perp$ then $\Gamma : A \to s_\alpha A$ is a bijection from $F = \{A \in \mathcal{I} :\ B \subset s_\alpha A\}$ to $G = \{A \in \mathcal{I} :\ B \subset A\}$.*

Proof. (i) follows from lemma 10.7.1 with $K = \emptyset$.
For (ii), suppose $A \in F$ and $s_\alpha A \not\subset \Pi$. Then $\alpha \in A$, so that $s_\alpha A = \{-\alpha\} \cup (A\backslash\{\alpha\})$ and, since $-\alpha \notin B$, $B \subset A\backslash\{\alpha\}$. This implies $\alpha \in B^\perp$, which is absurd. Thus, Γ is well-defined. In the same way, $A \to s_\alpha A$ is well-defined from G to F, which proves (ii). ♢

Lemma 10.7.5. *Suppose I is orthogonal. If a function $G(x, A)$, defined for $x \in V$ and $A \subset \Phi$, satisfies :*
(i) *$G(s_\alpha x, A) = G(x, s_\alpha A)$ for $\alpha \in \Phi$,*
(ii) *$G(x, \{-\alpha\} \cup (A\backslash\{\alpha\}) = -G(x, A)$ for $\alpha \in A$,*
then $F(x) = \sum_{A\in\mathcal{I}} \varepsilon_A G(x, A)$ satisfies $F(wx) = \varepsilon(w)F(x)$ for $w \in W$. In particular, if F is a polynomial in x then $h(x) = \prod_{\alpha\in\Pi}(x, \alpha)$ divides $F(x)$. If I is only semi-orthogonal, the same result holds if one adds to (ii) *the following property : $G(x, s_\alpha\{\alpha, \beta\} \cup (A\backslash\{\alpha, \beta\}) = -G(x, A)$ for $\{\alpha, \beta\} \in A^*$.*

Proof. The proof is done for the orthogonal case but is the same in the semi-orthogonal one, modulo obvious modification. Use successively hypotheses (i), (ii), $\varepsilon_{s_\alpha A} = -\varepsilon_A$ and the bijection given by (i) of lemma 10.7.4 to compute

$$\begin{aligned} F(s_\alpha x) &= \sum_{A\in\mathcal{I}} \varepsilon_A\, G(s_\alpha x, A) = \sum_{A\in\mathcal{I}} \varepsilon_A\, G(x, s_\alpha A) \\ &= \sum_{A\in\mathcal{I},\, \alpha\in A} \varepsilon_A\, G(x, \{-\alpha\}\cup(A\backslash\{\alpha\})) - \sum_{A\in\mathcal{I},\, \alpha\notin A} \varepsilon_{s_\alpha A}\, G(x, s_\alpha A) \\ &= -\sum_{A\in\mathcal{I},\, \alpha\in A} \varepsilon_A\, G(x, A) - \sum_{A\in\mathcal{I},\, \alpha\notin A} \varepsilon_A\, G(x, A) \\ &= -F(x) \end{aligned}$$

$\diamondsuit$

Proof of proposition 10.2.4. Define $G(x, A) = \prod_{\alpha\in A}(x, \alpha)$ which satisfies hypotheses of lemma 10.7.5. Then $F(x) = P(x)$ is a polynomial divisible by $h(x) = \prod_{\alpha\in\Pi}(x, \alpha)$. Since $\deg P = |I| < |\Pi| = \deg h$, we have $P = 0$. $\diamondsuit$

Proof of proposition 10.2.8. Define

$$G(x, A) = \prod_{\{\alpha,\beta\}\in A^*} (\alpha, x)(\beta, x)(s_\alpha\beta, x)(s_\beta\alpha, x) \prod_{\gamma\in A}(x, \gamma)$$

which satisfies hypotheses of lemma 10.7.5. Then $F(x) = P(x)$ is a polynomial divisible by $h(x) = \prod_{\alpha\in\Pi}(x, \alpha)$. Since $\deg P \le 2|I| < |\Pi| = \deg h$, we have $P = 0$. $\diamondsuit$

10.7.3 The dual formula

Proof of proposition 10.2.5. Let $B = \{\beta_1, \ldots, \beta_k\} \in \mathcal{O}(\Pi)$. It suffices to show that $B = \alpha_j.\cdots.\alpha_1.B'$ for some sequence $\alpha_1, \ldots, \alpha_j \in \Delta$ and $B' \in \mathcal{O}(\Pi)$ with $|B'| < |B|$. If B contains a simple root α we are done because then $B = \alpha.(B\backslash\{\alpha\})$. Assume that $B\cap\Delta = \phi$. We can write $\beta_1 = s_{\alpha_j}\cdots s_{\alpha_2}\alpha_1$ for some sequence $\alpha_1, \ldots, \alpha_j \in \Delta$ such that α_{i+1} is neither equal or orthogonal to $s_{\alpha_i}\cdots s_{\alpha_2}\alpha_1$ for all $i = 1, \ldots, j-1$. Now set $B' = (s_{\alpha_2}\cdots s_{\alpha_j}B)\backslash\{\alpha_1\}$. Then $B' \in \mathcal{O}(\Pi)$ and $B = \alpha_j.\cdots.\alpha_1.B'$, as required. To see this, set $B_1 = \alpha_1.B' = s_{\alpha_2}\cdots s_{\alpha_j}B$ and $B_{i+1} = s_{\alpha_{i+1}}B_i$ for $i = 1, \ldots, j-1$. Note that $\alpha_1 \in B_1$ and $s_{\alpha_i}\cdots s_{\alpha_2}\alpha_1 \in B_i$ for $i = 2, \ldots, j-1$. Since B_i is orthogonal, and α_{i+1} is neither equal to, or orthogonal to, $s_{\alpha_i}\cdots s_{\alpha_2}\alpha_1$, it follows that $\alpha_{i+1} \notin B_i$ and hence $s_{\alpha_{i+1}}B_i = \alpha_{i+1}.B_i$ for each $i = 1, \ldots, j-1$ and, moreover, that B' contains only positive roots. $\diamondsuit$

Proof of proposition 10.2.6. Since $\sum_{A\in\mathcal{I}}\varepsilon_A = 1$, we can write

$$\mathbb{P}_x[T \le t] = \sum_{A\in\mathcal{I}} \varepsilon_A \mathbb{P}_x[T_A \le t].$$

By the inclusion-exclusion principle,

$$\mathbb{P}_x[T_A \le t] = \sum_{\emptyset \ne B \subset A} (-1)^{|B|-1} \mathbb{P}_x[\forall \beta \in B,\, T_\beta \le t].$$

Thus

$$\mathbb{P}_x[T \le t] = \sum_{B \in \mathcal{O}(\Pi) \setminus \{\emptyset\}} \nu_B \mathbb{P}_x[\forall \beta \in B,\, T_\beta \le t],$$

where $\nu_B = (-1)^{|B|-1} \sum_{A \in \mathcal{I},\, B \subset A} \varepsilon_A$. We will prove that $\nu_B = (-1)^{l(B)-1}$ by induction on $l(B)$.
The result for $l(B) = 0$ is just (i) of proposition 10.7.1.
Suppose that $l(B) = l \ge 1$, write $B = \alpha_l \ldots \alpha_2.\alpha_1.\emptyset$. Set $\alpha = \alpha_l$ and $B' = \alpha_{l-1} \ldots \alpha_1.\emptyset$ so that $B = \alpha.B'$. We have $l(B') = l - 1$ and $\alpha \notin B'$ (the contrary would contradict $l(B) = l$).
The first case is $\alpha \in B'^{\perp}$. Then $|B'| = l - 1$ and $B = B' \cup \{\alpha\}$. Thus,

$$\begin{aligned} \nu_B &= (-1)^{|B|-1} \sum_{A \in \mathcal{I},\, \alpha \in A,\, B' \subset A} \varepsilon_A \\ &= (-1)^{|B|-1} \Big(\sum_{A \in \mathcal{I},\, B' \subset A} \varepsilon_A - \sum_{A \in \mathcal{I}} \varepsilon_A f(A) \Big), \end{aligned}$$

where $f(A) = \mathbf{1}_{\alpha \notin A,\, B' \subset A}$. If $\alpha \in A$, then $f(A) = 0$. If $\alpha \notin A$, then $f(A) = \mathbf{1}_{B' \subset A}$. Using $\alpha \notin s_\alpha A$ and $s_\alpha B' = B'$, we obtain $f(s_\alpha A) = \mathbf{1}_{B' \subset s_\alpha A} = \mathbf{1}_{s_\alpha B' \subset A} = f(A)$. Again, (ii) of proposition 10.7.1 applies and $\nu_B = (-1)^{|B|-1} \sum_{A \in \mathcal{I},\, B' \subset A} \varepsilon_A = -\nu_{B'}$, which concludes this case.
The second case is $\alpha \notin B'^{\perp}$. Then $|B'| = l$, $B = s_{\alpha_l} B'$ and

$$\nu_B = (-1)^{|B|-1} \sum_{A \in \mathcal{I},\, s_\alpha B' \subset A} \varepsilon_A = (-1)^{|B|} \sum_{A \in \mathcal{I},\, B' \subset s_\alpha A} \varepsilon_{s_\alpha A}. \tag{10.76}$$

Thanks to (ii) of lemma 10.7.4, $A \mapsto s_\alpha A$ is a bijection from $\{A \in \mathcal{I} :\ B' \subset s_\alpha A\}$ to $\{A' \in \mathcal{I} :\ B' \subset A'\}$, so that (10.76) transforms into $\nu_B = (-1)^{|B|} \sum_{A' \in \mathcal{I},\, B' \subset A'} \varepsilon_{A'} = -\nu_{B'}$ and we are done. $\diamondsuit$

10.7.4 Consistency

The dihedral groups

For m odd, we take $J = \{\alpha\}$. The angle between two roots being a multiple of π/m, no two roots are orthogonal. Therefore, the unique extension property of Lemma 10.3.1 is trivially verified with $I = J$. Then, $U = \{\mathrm{id}\}$, which guarantees condition (C2). Thus, I is consistent and $\mathcal{I} = \{\{\alpha_1\}, \ldots, \{\alpha_m\}\}$ with $\varepsilon_{\{\alpha_i\}} = (-1)^{i-1}$.

For $m = 2m'$ with m' odd, we again choose $J = \{\alpha\}$. Now, the unique extension property of Lemma 10.3.1 is verified with $J = \{\alpha\}$, $I = \{\alpha, \alpha' = e^{i\pi/2}\alpha\}$. Suppose w leaves I invariant. Since I is a basis of $\mathbb{R}^2$, if w fixes I pointwise, then $w = \mathrm{id}$. Otherwise, w permutes α and α'. Then, w has to be the reflection with respect to the bisecting line of α and α'. Since $\pi/4$ is not a multiple of π/m, the latter reflection is not in W. Hence, $U = \{\mathrm{id}\}$ and (C2) is satisfied. Writing $\alpha_i' = e^{i\pi/2}(\alpha_i)$, I is consistent again with $\mathcal{I} = \{\{\alpha_1, \alpha_1'\}, \ldots, \{\alpha_m, \alpha_m'\}\}$ and $\varepsilon_{\{\alpha_i, \alpha_i'\}} = (-1)^{i-1}$.

A_{k-1}

We define $p = \lfloor k/2 \rfloor$. The choice of $J = I$ makes Lemma 10.3.1 trivially verified. The following lemma is obvious :

Lemma 10.7.6. *We can characterize the sets W^I and U :*

$$\begin{aligned} W^I &= \{\sigma \in \S_k :\ \forall i \in [p],\ \sigma(2i-1) < \sigma(2i)\}, \\ U &= \{\sigma \in \S_k :\ \sigma \textit{ permutes the consecutive pairs } (1,2), (3,4), \ldots, (2p-1, 2p)\} \\ &= \{\sigma \in \S_k :\ \exists \tau \in \S_p,\ \forall i \in [p],\ \sigma(2i-1) = 2\tau(i) - 1,\ \sigma(2i) = 2\tau(i)\}. \end{aligned}$$

Proof of proposition 10.3.1. We will use the sign function : $s(x) = 1$ if $x \geq 0$ and $s(x) = -1$ if $x < 0$. Let us first suppose that k is even. The sign of $\sigma \in \mathfrak{S}_k$ can be expressed as

$$\varepsilon(\sigma) = \prod_{j<i} s\left(\sigma(i) - \sigma(j)\right) = P_{00}\, P_{11}\, P_{01}\, P_{10}\, Q, \tag{10.77}$$

where

$$P_{ab} = \prod_{1 \leq m < l \leq p} s\left(\sigma(2m-a) - \sigma(2l-b)\right) \quad \text{for } a, b \in \{0,1\}$$

and

$$Q = \prod_{1 \leq l \leq p} s\left(\sigma(2l) - \sigma(2l-1)\right).$$

If $\sigma \in U$, using notation of lemma 10.7.6,

$$P_{00} P_{11} = \prod_{1 \leq m < l \leq p} s\left(\tau(l) - \tau(m)\right)^2 = 1,$$

$$P_{01} P_{10} = \prod_{1 \leq m < l \leq p} s\left(4(\tau(l) - \tau(m))^2 - 1\right) = 1$$

and $Q = 1$. So (i) is proved.

For (ii), we note that

$$\sigma I = \left\{e_{\sigma(1)} - e_{\sigma(2)}, e_{\sigma(3)} - e_{\sigma(4)}, \ldots, e_{\sigma(k-1)} - e_{\sigma(k)}\right\}$$

and we define the mapping θ from $\mathcal{I}$ to $P_2(k)$ by

$$\theta(\sigma I) = \big\{\{\sigma(1), \sigma(2)\}, \{\sigma(3), \sigma(4)\}, \ldots, \{\sigma(k-1), \sigma(k)\}\big\}.$$

This is well-defined since, if $\sigma I = \sigma' I$, then $\sigma = \sigma'\tau$ with $\tau \in U$, so

$$\{\sigma(2i-1), \sigma(2i)\} = \{\sigma'(\tau(2i-1)), \sigma'(\tau(2i))\}.$$

Since τ permutes consecutive pairs $(2i-1, 2i)$, $\theta(\sigma I)$ and $\theta(\sigma' I)$ are the same partition. θ is obviously injective and onto.

For (iii), notice that if $\pi = \{\{j_1, j_1'\}, \ldots, \{j_p, j_p'\}\}$ with $j_i < j_i'$ then

$$(-1)^{c(\pi)} = \prod_{1 \le m < l \le p} s\left((j_l - j_m)(j_l - j_m')(j_l' - j_m)(j_l' - j_m')\right). \tag{10.78}$$

Construct σ by $\sigma(2i-1) = j_i, \sigma(2i) = j_i'$. Then $\theta(\sigma I) = B$ and comparing (10.78) with (10.77) yields $(-1)^{c(\pi)} = \varepsilon(\sigma)$.

If k is odd and $\sigma \in U$, then $\sigma(k) = k$ so that formula (10.77) is still true and the same proof holds for (i). θ is given by

$$\theta(\sigma I) = \big\{\{\sigma(1), \sigma(2)\}, \{\sigma(3), \sigma(4)\}, \ldots, \{\sigma(k-2), \sigma(k-1)\}, \{\sigma(k)\}\big\}.$$

As for (iii), suppose $\sigma \in W^I$, $\theta(\sigma I) = \pi$ and define π' to be the partition π deprived of its singlet $\{\sigma(k)\}$. Then formula (10.77) must be modified into

$$\varepsilon(\sigma) = P_{00}\, P_{11}\, P_{01}\, P_{10}\, Q \prod_{j=1}^{k-1} (\sigma(k) - \sigma(j)).$$

The first part equals $(-1)^{c(\pi')}$. The second is $(-1)^{\sigma(k)+1}$, which exactly corresponds with the number of crossings obtained by adding the pair $\{0, \sigma(k)\}$ with 0 at the left of everything. ♢

Proof of remark 10.3.1. We prove the formula $i(\sigma) = c(\pi) + 2b(\pi)$. Recall that

$$\pi = \big\{\{j_1, j_1'\}, \{j_2, j_2'\}, \ldots, \{j_p, j_p'\}\big\} \in P_2(k), \qquad j_i < j_i', \quad j_1 < j_2 < \cdots < j_p.$$

Construct σ by $\sigma(2i-1) = j_i, \sigma(2i) = j_i'$. We define the set of crossings of π,

$$\mathrm{Cr} = \{(i,l) \in [p]^2 : \ i < l,\ j_i < j_l < j_i' < j_l'\},$$

and the set of bridges of π,

$$\mathrm{Br} = \{(i,l) \in [p]^2 : \ i < l,\ j_i < j_l < j_l' < j_i'\}.$$

If $(i,l) \in \mathrm{Cr}$, then $\sigma(2l-1) < \sigma(2i)$ and $2i < 2l-1$. Hence we set $\phi(i,l) = \{(2i, 2l-1)\} \subset \mathrm{Inv}(\sigma)$.
If $(i,l) \in \mathrm{Br}$, then $\sigma(2l-1) < \sigma(2l) < \sigma(2i)$ and $2i < 2l-1$. Hence we set $\psi(i,l) = \{(2i, 2l-1), (2i, 2l)\} \subset \mathrm{Inv}(\sigma)$.
We claim that :

$$\Big(\bigcup_{(i,l)\in \mathrm{Cr}} \phi(i,l) \Big) \bigcup \Big(\bigcup_{(i,l)\in \mathrm{Br}} \psi(i,l) \Big) = \mathrm{Inv}(\sigma)\,. \tag{10.79}$$

Since $\mathrm{Cr} \cap \mathrm{Br} = \emptyset$, the union in (10.79) is disjoint and (10.24) follows. For (10.79), we have already proved the inclusion of the left-hand side in $\mathrm{Inv}(\sigma)$.
Conversely, take $(a,b) \in \mathrm{Inv}(\sigma)$. Suppose a is odd, $a = 2i-1$, then $\sigma(a) = j_i$ and for $l > i$, we have $j'_l > j_l > j'_i > j_i$, i.e. $\sigma(2l) > \sigma(2l-1) > \sigma(2i) > \sigma(2i-1)$. This means that $(a,b) \notin \mathrm{Inv}(\sigma)$. Hence a is even, $a = 2i$.
If b is even, $b = 2l$, then $i < l$. Hence $\sigma(2i-1) = j_i < \sigma(2l-1) = j_l < \sigma(2l) = j'_l < \sigma(2i) = j'_i$, the first two inequalities being trivial and the last one due to $(a,b) \in \mathrm{Inv}(\sigma)$. Thus $(a,b) \in \psi(i,l)$ and $(i,l) \in \mathrm{Br}$.
If b is odd, $b = 2l-1$, $2i < 2l-1$ and $\sigma(2i-1) = j_i < \sigma(2l-1) = j_l < \sigma(2i) = j'_i$. Two possibilities arise : either $j_l < j'_l < j'_i$, then $(a,b) \in \psi(i,l)$ and $(i,l) \in \mathrm{Br}$, or $j'_i < j'_l$, then $(a,b) \in \phi(i,l)$ and $(i,l) \in \mathrm{Cr}$.
Anyhow, $(a,b) \in \big(\bigcup_{(i,l)\in\mathrm{Cr}} \phi(i,l)\big) \bigcup \big(\bigcup_{(i,l)\in\mathrm{Br}} \psi(i,l)\big)$, which ends the proof. ♢

D_k

We define $p = \lfloor k/2 \rfloor$. By choosing $J = \{e_1 - e_2, e_3 - e_4, \ldots, e_{k-1} - e_k, e_{k-1} + e_k\}$ if k is even and $J = \{e_2 - e_3, e_4 - e_5, \ldots, e_{k-1} - e_k, e_{k-1} + e_k\}$ otherwise, the unique extension property of Lemma 10.3.1 is easily checked.

Lemma 10.7.7. *We can characterize the sets W^I and U :*

$$\begin{aligned}
W^I &= \{f\sigma \in W :\ \forall i \in [p],\, \sigma(2i-1) \notin \overline{f},\, \sigma(2i-1) < \sigma(2i)\} \text{ if } k \text{ is even,}\\
W^I &= \{f\sigma \in W :\ \forall i \in [p],\, \sigma(2i) \notin \overline{f},\, \sigma(2i) < \sigma(2i+1)\} \text{ if } k \text{ is odd,}\\
U &= \{f\sigma \in W :\ \sigma \text{ permutes the consecutive pairs } (1,2),(3,4),\ldots,(k-1,k);\\
&\qquad \forall i \in [p],\, \sigma(2i-1) \notin \overline{f}\} \text{ if } k \text{ is even,}\\
U &= \{f\sigma \in W :\ \sigma \text{ permutes the consecutive pairs } (2,3),(4,5),\ldots,(k-1,k);\\
&\qquad \forall i \in [p],\, \sigma(2i) \notin \overline{f}\} \text{ if } k \text{ is odd.}
\end{aligned}$$

Proof. Let us do proofs for even k. Suppose $f\sigma \in W^I$ then $f\sigma(e_{2i-1} \pm e_{2i}) = f(e_{\sigma(2i-1)}) \pm f(e_{\sigma(2i)}) \in \Pi$, which implies $\sigma(2i-1) \notin \overline{f}$ and then $\sigma(2i) > \sigma(2i-1)$. This proves the first equality. The second one is then obvious. ♢

Proof of proposition 10.3.1 for D_k. The identification between $\mathcal{I}$ and $P_2(k)$ is done via θ which sends $f\sigma I = \{e_{\sigma(2i-1)} \pm e_{\sigma(2i)},\, i \in [p]\}$ to :

$$\begin{cases} \big\{\{\sigma(2i-1), \sigma(2i)\},\, i \in [p]\big\} & \text{if } k \text{ is even,} \\ \big\{\{\sigma(1)\}; \{\sigma(2i-1), \sigma(2i)\},\, i \in [p]\big\} & \text{if } k \text{ is odd.} \end{cases}$$

Since $\varepsilon(f) = 1$ for $f\sigma \in W$, (C2) and $\varepsilon_A = (-1)^{c(\theta(A))}$ immediately come from the analogous facts for A_{k-1}. ♢

B_k

We define $p = \lfloor k/2 \rfloor$. Our choice is $J = \{e_1 - e_2, e_3 - e_4, \ldots, e_{k-1} - e_k\}$ if k is even and $J = \{e_1 - e_2, e_3 - e_4, \ldots, e_{k-2} - e_{k-1}, e_k\}$ if k is odd. Then Lemma 10.3.1 is easily verified.

Lemma 10.7.8. *We can characterize the sets W^I and U :*

$$W^I = \{\sigma \in \mathfrak{S}_k : \ \forall i \in [p],\, \sigma(2i-1) < \sigma(2i)\},$$
$$U = \{\sigma \in \mathfrak{S}_k : \ \sigma \textit{ permutes the consecutive pairs } (1,2), (3,4), \ldots, (2p-1, 2p)\}.$$

Proof. Suppose $f\sigma \in W^I$ then (1) : $f\sigma(e_{2i-1} - e_{2i}) = f(e_{\sigma(2i-1)}) - f(e_{\sigma(2i)}) \in \Pi$ and (2) $f\sigma(e_{2i}) = f(e_{\sigma(2i)}) \in \Pi$. (1) implies $\sigma(2i-1) \notin \overline{f}$ and then (2) yields $\sigma(2i) \notin \overline{f}$. So $f = \mathrm{id}$ and consequently $\sigma(2i) > \sigma(2i-1)$. This proves the first equality. The second and third ones are then obvious. ♢

Proof of proposition 10.3.1 for B_k. The proof of (C2) is the same as for A_{k-1}. The identification between $\mathcal{I}$ and $P_2(k)$ is done via θ. If k is even and $A = \sigma I = \{e_{\sigma(2i-1)} - e_{\sigma(2i)}, e_{\sigma(2i)},\, i \in [p]\}$ then

$$\theta(A) = \big\{\{\sigma(2i-1), \sigma(2i)\},\, i \in [p]\big\}.$$

If k is odd and $A = \sigma I = \{e_{\sigma(2i-1)} - e_{\sigma(2i)}, e_{\sigma(2i)},\, i \in [p];\, e_{\sigma(k)}\}$, then

$$\theta(A) = \big\{\{\sigma(2i-1), \sigma(2i)\},\, i \in [p];\, \{\sigma(k)\}\big\}.$$

Then, $\varepsilon_A = (-1)^{c(\theta(A))}$ directly comes from the A_{k-1} case. ♢

H_3 **and** H_4

Let us first deal with H_3, take $J = \{\alpha = a - i/2 + bj, \beta = 1/2 + bi - aj\}$, define $\gamma = b + ai + j/2$. Then, $I = \{\alpha, \beta, \gamma\}$. Lemma 10.3.1 is trivially verified since the linear span of Φ is three-dimensional. Now, suppose $\varepsilon(w) = -1$ and $wI = I$. w acts as an odd permutation of $\{\alpha, \beta, \gamma\}$ so, for example, w is the transposition (α, β). Thus, $w(\alpha + \beta) = \alpha + \beta$, $w\gamma = \gamma$ and $w(\alpha - \beta) = -\alpha + \beta$, which means that $w = s_{\alpha-\beta}$. This is absurd since $\mathbb{R}(\alpha-\beta)$ contains no root of H_3. The same being true for $\mathbb{R}(\alpha-\gamma)$ and $\mathbb{R}(\beta-\gamma)$, the proof is done.

For H_4, take $J=\{\alpha=-a+i/2+bj, \beta=-1/2-ai+bk\}$, define $\gamma=b+aj+k/2$, $\delta=bi-j/2+ak$. Then, $I=\{\alpha,\beta,\gamma,\delta\}$ satisfies lemma 10.3.1 since $\mathbb{R}^4\equiv\mathbb{H}$ is four-dimensional. Suppose $\varepsilon(w)=-1$ and $wI=I$. w acts as an odd permutation of $\{\alpha,\beta,\gamma,\delta\}$, so is either a transposition or a 4-cycle. A transposition is ruled out by the same analysis as H_3, since the differences between elements of I are not multiples of roots. If w is the 4-cycle $(\alpha\beta\gamma\delta)$ then consider the root $\lambda=(1+i+j+k)/2$. Since $(\lambda,\alpha)=(\gamma,i)=0$, we have $(w\lambda,i)=(\lambda,\gamma)(\delta,i)+(\lambda,\delta)(\alpha,i)=(2a+2b+1)/8\notin\{(\phi,i);\ \phi\in\Phi\}$. Hence, $w\lambda\notin\Phi$, which proves that $w\notin W$.

F_4

Take $J=\{e_2-e_3,e_4\}$ and $I=\{e_2-e_3,e_3,e_1-e_4,e_4\}$. I is semi-orthogonal and lemma 10.3.1 is an easy check. Suppose $wI=I$. Since w sends long roots to long roots and short roots to short roots, $w\in\S_{\{e_3,e_4\}}\times\S_{\{e_2-e_3,e_1-e_4\}}$. If $we_3=e_3$ and $we_4=e_4$ then $w(e_1-e_4)=we_1-e_4\in\{e_2-e_3,e_1-e_4\}$, which forces $we_1=e_1$ and similarly $we_2=e_2$. So $w=\mathrm{id}$ and $\varepsilon(w)=1$. If w transposes e_3 and e_4, we see in the same way that it also transposes e_1 and e_2, so that $\varepsilon(w)=1$.

10.7.5 Asymptotic expansions

Proof of proposition 10.4.1. Let us deal with the orthogonal case first. We start with the formula $\mathbb{P}_x(T>t)=\sum_{A\in\mathcal{I}}\varepsilon_A\prod_{\alpha\in A}\gamma\left(\widehat{\alpha}(x)/\sqrt{t}\right)$. Expanding γ in power series, noting $l=|I|$ and $a_p=\sqrt{\frac{2}{\pi}}\left(\frac{-1}{2}\right)^p\frac{1}{p!(2p+1)}$, we have

$$\begin{aligned}\mathbb{P}_x(T>t) &= \sum_{A\in\mathcal{I}}\varepsilon_A\prod_{\alpha\in A}\sum_{p\geq 0}a_p\left(\frac{\widehat{\alpha}(x)}{\sqrt{t}}\right)^{2p+1}\\ &= \sum_{A=\{\alpha_1,\ldots,\alpha_l\}\in\mathcal{I}}\varepsilon_A\sum_{p\in\mathbb{N}^l}\prod_{j=1}^{l}a_{p_j}\left(\frac{\widehat{\alpha_j}(x))}{\sqrt{t}}\right)^{2p_j+1}\\ &= \sum_{r\geq 0}F_r(x)\left(\frac{1}{\sqrt{t}}\right)^{2r+l},\end{aligned}$$

where $F_r(x)=\sum_{A\in\mathcal{I}}\varepsilon_A\,G_r(x,A)$ and, if $A=\{\alpha_1,\ldots,\alpha_l\}$,

$$G_r(x,A)=\sum_{p\in\mathbb{N}^l,\sum_i p_i=r}\prod_{j=1}^{l}a_{p_j}\,\widehat{\alpha_j}(x)^{2p_j+1}.$$

Note that, unlike the term $\prod_{j=1}^{l} \widehat{\alpha_j}(x)^{2p_j+1}$, the previous sum only depends on A and not on the enumeration $\alpha_1, \ldots, \alpha_l$ of its elements so the notation $G_r(x, A)$ is legitimate. Now, lemma 10.7.5 applies to $F_r(x)$ which is a polynomial of degree $2r + l$ so that $F_r(x) = 0$ if $2r+l < n$ and $F_{n+p}(x) = h(x)\, E'_p(x)$ if $p \geq 0$ where $E'_p(x)$ is a W-invariant polynomial of degree p. Since F_r has degree $2r+l$ and $n \equiv l \mod 2$, $E'_p = 0$ if p is odd and we set $E_q = E'_{2q}$ to conclude the proof.

In the semi-orthogonal case, we will call m (resp. l) the number of pairs (resp. singlets) of I^*, so that $|I| = 2m+l$. Whenever we write a pair (α, β), it will be normalized so that it is isometric to $(e_1 - e_2, e_2)$, so that $\mathbb{P}_x(T_{\alpha,\beta} > t) = H\left((x, \alpha+\beta)/\sqrt{t}, (x, \beta)/\sqrt{t}\right)$ with H defined in (10.42). So, writing $(x, \alpha+\beta) = x_{\alpha\beta}$, $(x, \beta) = x_\beta$ and $(x, \gamma) = x_\gamma$,

$$\begin{aligned}
\mathbb{P}_x(T > t) &= \sum_{A \in \mathcal{I}} \varepsilon_A \prod_{(\alpha,\beta)\in A^*} \sum_{p,q \geq 0} a_{pq} \left(\frac{x_{\alpha\beta} x_\beta}{t}\right)^{2p+1} \left(\left(\frac{x_{\alpha\beta}}{\sqrt{t}}\right)^{2q} - \left(\frac{x_\beta}{\sqrt{t}}\right)^{2q}\right) \\
&\qquad \prod_{\{\gamma\}\in A^*} \sum_{s \geq 0} a_s \left(\frac{x_\gamma}{\sqrt{t}}\right)^{2s+1} \\
&= \sum_{r \geq 0} F_r(x) \left(\frac{1}{\sqrt{t}}\right)^r,
\end{aligned}$$

where $F_r(x) = \sum_{A \in \mathcal{I}} \varepsilon_A G_r(x, A)$ and, for $A^* = \{(\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m), \{\gamma_1\}, \ldots, \{\gamma_l\}\}$,

$$G_r(x, A) = \sum_{p,\,q,\,s} \prod_{j=1}^{m} a_{p_j q_j} (x_{\alpha_j\beta_j}\, x_{\beta_j})^{2p_j+1} \left(x_{\alpha_j\beta_j}^{2q_j} - x_{\beta_j}^{2q_j}\right) \prod_{i=1}^{l} a_{s_i} x_{\gamma_i}^{2s_i+1},$$

where the sum runs over those $p, q \in \mathbb{N}^m$, $s \in \mathbb{N}^l$ such that $4|p|+2|q|+2|s|+2m+l = r$. Again, the previous expression does not depend on the enumeration of pairs (α_j, β_j) or of singlets $\{\gamma_i\}$. Now,

$$(x_{\alpha_j\beta_j}\, x_{\beta_j})^{2p_j+1} (x_{\alpha_j\beta_j}^{2q_j} - x_{\beta_j}^{2q_j})$$

is sign changed when x is replaced by $s_{\alpha_j} x$ or $s_{\beta_j} x$, so that lemma 10.7.5 applies to $F_r(x)$ which is a polynomial of degree r, null when $r < n$ and divisible by h otherwise. Write $F_{p+n}(x) = h(x) E'_p(x)$ for $p \geq 0$. We remark that $F_r = 0$ if $r \not\equiv l \pmod 2$. Since $n \equiv l \pmod 2$, $E'_p(x) = 0$ if p is odd. Hence the conclusion with $E_q = E'_{2q}$. ♢

Remark 10.7.1. *Actually, (10.44) can be directly obtained by expanding each term in (10.9) as a power series in t. As in the previous proof, the factor $h(x)$ appears thanks to the skew-symmetry coming from the alternating sum over W. This approach gives an alternative expression for the polynomials E_q.*

10.7.6 de Bruijn formulae

Proof of proposition 10.5.1. Without loss of generality, we can assume that f is continuous and compactly supported, since these functions are dense in $L_1(V)$. Integration of formula (10.8) yields

$$\mathbb{P}_x(T > t) = \int_C \sum_{w \in W} \varepsilon(w) p_t(x, wy)\, dy.$$

For $A \in \mathcal{I}$, we have similarly

$$\mathbb{P}_x(T_A > t) = \int_{C_A} \sum_{w \in W_A} \varepsilon(w) p_t(x, wy)\, dy.$$

Now, using formula (10.10), we get

$$\int_C \sum_{w \in W} \varepsilon(w) p_t(x, wy)\, dy = \sum_{A \in \mathcal{I}} \varepsilon_A \sum_{w \in W_A} \varepsilon(w) \int_{C_A} p_t(x, wy)\, dy.$$

Integrating this with respect to $f(x)\, dx$ and using Fubini's theorem,

$$\int_C \sum_{w \in W} \varepsilon(w) P_t f(wy)\, dy = \sum_{A \in \mathcal{I}} \varepsilon_A \sum_{w \in W_A} \varepsilon(w) \int_{C_A} P_t f(wy)\, dy,$$

where $P_t f(z) = \int f(x) p_t(x, z)\, dx$ is the Brownian semi-group. Now let $t \to 0$ in the above formula and use the fact that $P_t f$ converges in $L_1(V)$ to f to get the result. $\diamondsuit$

type A

Proof of proposition 10.5.3. Let us suppose k is even. $\mathcal{I}$ is identified with $P_2(k)$. If $A \in \mathcal{I}$ corresponds with $\pi \in P_2(k)$, then

$$\eta \in \{\pm 1\}^\pi \mapsto w_\eta = \prod_{\{i<j\} \in \pi} \tau_{ij}^{\eta'_{ij}} \in W_A$$

is an isomorphism, where τ_{ij} is the transposition of i and j and $\eta'_{ij} = (1 - \eta_{ij})/2$. Then, C_A corresponds with

$$C_\pi = \cap_{\{i<j\} \in \pi} \{y : \; y_i > y_j\}.$$

Since $f(y) = \prod_{\{i<j\} \in \pi} f_i(y_i) f_j(y_j)$, we have

$$f(w_\eta y) = \prod_{\{i<j\} \in \pi,\, \eta_{ij} = 1} f_i(y_i) f_j(y_j) \prod_{\{i<j\} \in \pi,\, \eta_{ij} = -1} f_i(y_j) f_j(y_i).$$

Thus, the right-hand side of (10.64) reads

$$\sum_{\eta\in\{\pm1\}^\pi}\prod_{\{i<j\}\in\pi}\eta_{ij}\int_{C_\pi}\prod_{\{i<j\}\in\pi,\,\eta_{ij}=1} f_i(y_i)f_j(y_j)\prod_{\{i<j\}\in\pi,\,\eta_{ij}=-1} f_i(y_j)f_j(y_i)\,dy$$

$$=\sum_{\eta\in\{\pm1\}^\pi}\prod_{\{i<j\}\in\pi,\,\eta_{ij}=1}\int_{y>z} f_i(y)f_j(z)\,dydz\prod_{\{i<j\}\in\pi,\,\eta_{ij}=-1}-\int_{y>z} f_i(z)f_j(y)\,dydz$$

$$=\prod_{\{i<j\}\in\pi}\left(\int_{y>z} f_i(y)f_j(z)\,dydz-\int_{y>z} f_i(z)f_j(y)\,dydz\right)=\prod_{\{i<j\}\in\pi} I(f_i,f_j).$$

On the other hand hand, the left-hand side of (10.64) is

$$\int_C\sum_{\sigma\in\mathfrak{S}_k}\varepsilon(\sigma)\prod_{i=1}^k f_i(y_{\sigma(i)})=\int_C \det\left(f_i(y_j)\right)_{i,j\in[k]}\,dy,$$

which concludes the proof. The case of odd k is treated similarly. ♢

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Proof of proposition 10.5.4. A first remark is that

$$\int_C \det\left(f_i(y_j)\right)_{i,j\in[k]}\,dy=0$$

for odd functions $f_i:\mathbb{R}\to\mathbb{R}$, $1\le i\le k$. This can be seen by the change of variable $y_k\to -y_k$ which leaves C invariant and changes the sign of the determinant in the previous integral. So we deal with even functions f_i and suppose k is even. $\mathcal{I}$ is identified with $P_2(k)$. If $A\in\mathcal{I}$ corresponds with $\pi\in P_2(k)$, then

$$(\eta,\bar\eta)\in\{\pm1\}^\pi\times\{\pm1\}^\pi\mapsto w_\eta=\prod_{\{i<j\}\in\pi}\tau_{ij}^{\eta'_{ij}}\left(-\tau_{ij}\right)^{\bar\eta'_{ij}}\in W_A$$

is an isomorphism, where τ_{ij} is the transposition of i and j and $\eta'_{ij}=(1-\eta_{ij})/2$, $\bar{\eta_{ij}}'=(1-\bar{\eta_{ij}})/2$. Then, C_A corresponds with

$$C_\pi=\cap_{\{i<j\}\in\pi}\{y:\ y_i>|y_j|\}.$$

Writing $\prod'$ for $\prod_{\{i<j\}\in\pi}$,

$$\begin{aligned}f(w_\eta y)\ &=\ \prod_{\eta_{ij}=1,\,\bar\eta_{ij}=1}' f_i(y_i)f_j(y_j)\prod_{\eta_{ij}=-1\,\bar\eta_{ij}=-1}' f_i(-y_i)f_j(-y_j)\\ &\quad\prod_{\eta_{ij}=1,\,\bar\eta_{ij}=-1}' f_i(-y_j)f_j(-y_i)\prod_{\eta_{ij}=-1,\,\bar\eta_{ij}=1}' f_i(y_j)f_j(y_i).\end{aligned}$$

Since the f_i are even, $\sum_{w\in W_A}\varepsilon(w)\int_{C_A} f(wy)\,dy$ can be expressed

$$\sum_{\eta,\bar\eta\in\{\pm1\}^\pi}\prod{}'\eta_{ij}\,\bar\eta_{ij}\int_{C_\pi}\prod_{\eta_{ij}=\bar\eta_{ij}}{}' f_i(y_i)f_j(y_j)\prod_{\eta_{ij}=-\bar\eta_{ij}}{}' f_i(y_j)f_j(y_i)\,dy$$

$$= 2^{|\pi|}\sum_{\widehat\eta\in\{\pm1\}^\pi}\prod_{\widehat\eta_{ij}=1}{}'\int_{y>|z|} f_i(y)f_j(z)\,dydz\prod_{\widehat\eta_{ij}=-1}{}' -\int_{y>z} f_i(z)f_j(y)\,dydz$$
$$= 2^{|\pi|}\prod_{\{i<j\}\in\pi} K(f_i,f_j).$$

Hence, the right-hand side of (10.64) is

$$2^{k/2}\sum_{\pi\in P_2(k)}(-1)^{c(\pi)}\prod_{\{i<j\}\in\pi}K(f_i,f_j).$$

On the other hand hand, the integrand of the left-hand side of (10.64) is

$$\sum_{\sigma\in\mathfrak{S}_k}\varepsilon(\sigma)\sum_{\eta\in\{-1,1\}^k}\frac{1+\prod_i\eta_i}{2}\prod_i f_i(\eta_i y_{\sigma(i)})$$
$$=\frac{1}{2}\left\{\det\left(f_i(y_j)+f_i(-y_j)\right)_{i,j\in[k]}+\det\left(f_i(y_j)-f_i(-y_j)\right)_{i,j\in[k]}\right\}$$
$$=2^{k-1}\det\left(f_i(y_j)\right)_{i,j\in[k]},$$

which concludes the proof. The case of odd k is treated similarly. $\diamondsuit$

10.7.7 Random walks and related combinatorics

Proof of proposition 10.6.1. Formula (10.13) reads

$$\mathbb{P}_x(T>t)=\sum_{\pi\in P_2(k)}(-1)^{c(\pi)}\prod_{\{i<j\}\in\pi}\mathbb{P}_x(X_i-X_j>0 \text{ on } [0,t]).$$

Under $\mathbb{P}_x$, X_i-X_j is the simple symmetric continuous-time random walk of rate 2 started at x_i-x_j. Now recall that, if Z is a simple symmetric continuous-time random walk of rate 2 started at 0, the fixed-time marginals can be computed :

$$\mathbb{P}(Z(t)=x)=p_x(t)\ x\in\mathbb{Z}, \tag{10.80}$$

and there is an analogue of the classical reflection principle :

$$\mathbb{P}(Z \text{ does not reach } x \text{ before time } t)=q_x(t)=\sum_{l=-x+1}^{x}p_l(t),\ x\in\mathbb{N}, \tag{10.81}$$

which concludes the proof. ♢

Proof of proposition 10.6.2. In (10.72), let us specialize the starting point x to be $\delta = (k-1, k-2, \dots, 1, 0)$. If k is even, each $\pi \in P_2(k)$ has $k/2$ blocks, so that

$$\mathbb{P}_\delta(T > t) = e^{-kt} \sum_{\pi \in P_2(k)} (-1)^{c(\pi)} \prod_{\{i<j\}\in\pi} \gamma_{j-i}(t) = e^{-kt} H_k(\gamma(t)) \,. \tag{10.82}$$

If k is odd, each $\pi \in P_2(k)$ has $(k-1)/2$ blocks, so that

$$\mathbb{P}_\delta(T > t) = e^{-(k-1)t} H_k(\gamma(t)) \,. \tag{10.83}$$

Now, introduce the "weak" chamber $\Omega = \{x : \ x_1 \geq \cdots \geq x_k\}$ and the associated exit time for X, T_Ω. By translation, we have $\mathbb{P}_\delta(T > t) = \mathbb{P}(T_\Omega > t)$, where $\mathbb{P}$ governs the process X starting from 0. If we denote by N_t the number of jumps of X before time t, then

$$\mathbb{P}(T_\Omega > t) = \sum_{n \geq 0} \mathbb{P}(T_\Omega > t \,|\, N_t = n)\, \mathbb{P}(N_t = n).$$

Now, $(N_t, t \geq 0)$ is a Poisson process of parameter k, so $\mathbb{P}(N_t = n) = e^{-kt}(kt)^n/n!$, and $\mathbb{P}(T_\Omega > t \,|\, N_t = n) = \mathbb{P}(\xi(1), \dots, \xi(n) \in \Omega)$, where $(\xi(i), i \in \mathbb{N})$ is a random walk in $\mathbb{N}^k$, starting at 0, with increments uniformly distributed on the canonical basis vectors $\{e_1, \dots, e_k\}$. Hence,

$$\mathbb{P}(T_\Omega > t \,|\, N_t = n) = \frac{1}{k^n} \ |\, \{ \text{ paths } 0 \nearrow \lambda^1 \nearrow \cdots \nearrow \lambda^n \} \,|,$$

where $\lambda^1, \dots, \lambda^n$ are partitions with at most k parts and $\mu \nearrow \lambda$ means that λ is obtained from μ by adding a box. Seeing the bijection between such sequences of partitions and elements of $\mathcal{T}_k(n)$, we eventually have

$$\mathbb{P}(T_\Omega > t) = e^{-kt} y_k(t), \tag{10.84}$$

which concludes the proof. ♢

Proof of proposition 10.6.3. Follows from proposition 10.6.2 by de-Poissonization. ♢

10.8 Appendix

10.8.1 A direct proof for A_3

For curiosity, we include a direct proof of formula (10.34) for $k = 4$, involving only probabilistic arguments about path reflections. This proof applies to any strong Markov process X with continuous trajectories and invariant in law under $\mathfrak{S}_k$. Let us first recall how this is done for $k = 3$, in which case the argument comes from [OU92]. We have $X = (X_1, X_2, X_3) \in \mathbb{R}^3$ and we will refer to X_i as the particle i. Define :

$$T_{ij} = \inf\{t : X_i(t) = X_j(t)\},\ E_{ij} = \{T_{ij} \leq t\},\ q_{ij} = \mathbb{P}(E_{ij}),\ T = \inf_{ij} T_{ij}.$$

Our goal is to compute $\mathbb{P}(T \leq t)$. Since $E_{13} \subset E_{12} \cup E_{23}$, it follows that $\mathbb{P}(T \leq t) = \mathbb{P}(E_{12} \cup E_{23}) = q_{12} + q_{23} - \mathbb{P}(E_{12}\,,\, E_{23})$. Now, we split according to the first collision and then use the strong Markov property and the invariance in law under any permutation of the particles to switch two particles after their first collision :

$$\begin{aligned}
\mathbb{P}(E_{12}\,,\, E_{23}) &= \mathbb{P}(E_{12}, E_{23}, T = T_{12}) + \mathbb{P}(E_{12}, E_{23}, T = T_{23}) \\
&= \mathbb{P}(E_{12}, E_{13}, T = T_{12}) + \mathbb{P}(E_{13}, E_{23}, T = T_{23}) \\
&= \mathbb{P}(E_{13}, T = T_{12}) + \mathbb{P}(E_{13}, T = T_{23}) = \mathbb{P}(E_{13}).
\end{aligned}$$

Hence, $\mathbb{P}(T \leq t) = q_{12} + q_{23} - q_{13}$. This is consistent with (10.17). Then, we easily get $\mathbb{P}(T > t) = p_{12} + p_{23} - p_{13}$.

For $k = 4$, let us keep the same notations and the same reasoning :

$$\begin{aligned}
\mathbb{P}(T \leq t) &= \mathbb{P}(E_{12} \cup E_{23} \cup E_{34}) \\
&= q_{12} + q_{23} + q_{34} - \mathbb{P}(E_{12}, E_{23}) - \mathbb{P}(E_{23}, E_{34}) - \mathbb{P}(E_{12}, E_{34}) \\
&\quad + \mathbb{P}(E_{12}, E_{23}, E_{34}) \\
&= q_{12} + q_{23} + q_{34} - q_{13} - q_{24} - q_{12}q_{34} + \mathbb{P}(E_{12}, E_{23}, E_{34}),
\end{aligned}$$

where we used independence and the previous result for three particles. Now, denoting $E' = E_{12} \cap E_{23} \cap E_{34}$, $\mathbb{P}(E')$ can be split into $\mathbb{P}(E', T = T_{12}) + \mathbb{P}(E', T = T_{23}) + \mathbb{P}(E', T = T_{34})$. Then, switch particles 1 and 2 after T_{12} to get

$$\begin{aligned}
\mathbb{P}(E', T = T_{12}) &= \mathbb{P}(E_{12}, E_{13}, E_{34}, T = T_{12}) \\
&= \mathbb{P}(E_{13}, E_{34}, T = T_{12}) \\
&= \mathbb{P}(E_{14}, T = T_{12}),
\end{aligned}$$

where we used the result for $k = 3$ to obtain the last line. Similarly,

$$\mathbb{P}(E', T = T_{34}) = \mathbb{P}(E_{14}, T = T_{34}).$$

Now, the term $\mathbb{P}(E', T = T_{23})$ is dealt with by switching particles 2 and 3 after T_{23},

$$\begin{aligned}
\mathbb{P}(E', T = T_{23}) &= \mathbb{P}(E_{13}, E_{23}, E_{24}, T = T_{23}) \\
&= \mathbb{P}(E_{13}, E_{24}, T = T_{23}) \\
&= \mathbb{P}(E_{13}, E_{24}) - \mathbb{P}(E_{13}, E_{24}, T = T_{12}) - \mathbb{P}(E_{13}, E_{24}, T = T_{34}).
\end{aligned}$$

Again, use independence and exchange particles 1, 2 after T_{12} as well as particles 3, 4 after T_{34},

$$\begin{aligned}\mathbb{P}(E', T_{23}) &= q_{13}q_{24} - \mathbb{P}(E_{23}, E_{14}, T = T_{12}) - \mathbb{P}(E_{14}, E_{23}, T = T_{34}) \\ &= q_{13}q_{24} - \mathbb{P}(E_{23}, E_{14}, T \neq T_{23}) \\ &= q_{13}q_{24} - \mathbb{P}(E_{23}, E_{14}) + \mathbb{P}(E_{23}, E_{14}, T = T_{23}) \\ &= q_{13}q_{24} - q_{23}q_{14} + \mathbb{P}(E_{14}, T = T_{23}).\end{aligned}$$

Gathering terms,

$$\mathbb{P}(E_{12}, E_{23}, E_{34}) = q_{14} + q_{13}q_{24} - q_{23}q_{14},$$

which, in turn, yields

$$\mathbb{P}(T \leq t) = q_{12} + q_{23} + q_{34} - q_{13} - q_{24} + q_{14} - q_{12}q_{34} + q_{13}q_{24} - q_{14}q_{23}.$$

This is consistent with (10.17) and we easily get

$$\mathbb{P}(T > t) = p_{12}p_{34} + p_{14}p_{23} - p_{13}p_{24}.$$

10.8.2 The Pfaffian

If $\operatorname{car}\mathbb{K} \neq 2$, any skew-symmetric matrix $A \in \mathcal{M}_n(\mathbb{K})$ can be written $A = PDP^\top$ with $P \in GL(n, \mathbb{K})$, $D = \operatorname{diag}(B_1, \ldots, B_q)$ and $B_l = 0 \in \mathbb{K}$ or $B_l = J = (j - i)_{1 \leq i,j \leq 2} \in \mathcal{M}_2(\mathbb{K})$. Hence, if n is odd, $\det A = 0$. If n is even, one can use the previous decomposition to prove

Proposition 10.8.1. *There exists a unique polynomial* $\operatorname{Pf} \in \mathbb{Z}[X_{ij}, 1 \leq i < j \leq n]$ *such that if* $A = (a_{ij})$ *is a skew-symmetric matrix of size* n, $\det A = \operatorname{Pf}(A)^2$ *and* $\operatorname{Pf}(\operatorname{diag}(J, \ldots, J)) = 1$.

The Pfaffian has an explicit expansion in terms of the matrix coefficients :

Proposition 10.8.2.

$$\operatorname{Pf}(A) = \sum_{\pi \in P_2(n)} (-1)^{c(\pi)} \prod_{\{i<j\} \in \pi} a_{ij} = \frac{1}{2^n (n/2)!} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) \prod_{i=1}^{n-1} a_{\sigma(i)\sigma(i+1)}.$$

For more on Pfaffians and their properties, see [GW98, Ste90].

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Chapitre 11

Exit problems associated with affine reflection groups

Abstract : Let T be the exit time of a Markov process X from the alcove of an affine Weyl group. If the law of X is invariant under W, we give a general formula for the distribution of T. Applications are explicitly given in the different type cases. In the type $\widetilde{A}_{k-1}$, a striking feature is that the validity of our approach is limited to even k, the case of odd k (corresponding to the equilateral triangle for $k = 3$) remaining most mysterious to us.

11.1 Introduction

The distribution of the exit time of Brownian motion from the semi-infinite interval $(0, \infty) \subset \mathbb{R}$ is well-known and obtained by application of the reflection principle. More generally, we can replace $(0, \infty) \subset \mathbb{R}$ by some convex cone $\mathcal{C} \subset \mathbb{R}^n$ called a chamber and which is the fundamental region of a finite reflection group. Results in this direction are obtained in [DO04], which is Chapter 10 of this thesis.

Now, in the one-dimensional case, there is another type of interval for which the exit time distribution is also given by a reflection principle, namely the bounded interval $(0, 1)$. Indeed, if B denotes one-dimensional Brownian motion and T its exit time from $(0, 1)$, then

$$\phi(x, t) := \mathbb{P}_x(T > t) = \sum_{n \in \mathbb{Z}} \Big(\mathbb{P}_x \left(B_t \in 2n + (0, 1) \right) - \mathbb{P}_x \left(B_t \in 2n - (0, 1) \right) \Big). \quad (11.1)$$

A relevant multi-dimensional generalization of this result consists in replacing $(0, 1)$ by a bounded domain $\mathcal{A}$ called an alcove and which is the fundamental region of an affine (infinite) Weyl group. The aim of this note is to present the formulae for the distribution of T in this case as well as a very convenient language that allows to transfer the proofs directly from [DO04] (Chapter 10 of this thesis) into our context.

11.2 The geometric setting

11.2.1 Affine Weyl groups and alcoves

Let V be a real Euclidean space endowed with a positive symmetric bilinear form (x, y). Let Φ be an irreducible crystallographic root system in V with associated reflection group W. Let Δ be a simple system in Φ with corresponding positive system Φ^+ and fundamental chamber $\mathcal{C} = \{x \in V :\ \forall\alpha \in \Delta,\, (\alpha, x) > 0\}$. Denote the reflections in W by s_α ($\alpha \in \Phi^+$). We will call $\Phi^\vee$ the set of coroots $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ for $\alpha \in \Phi$. Then, $L := \mathbb{Z}-$span of $\Phi^\vee$ is a W-stable lattice called the coroot lattice. The affine Weyl group W_a asociated with Φ is the group generated by all affine reflections with respect to the hyperplanes $H_{\alpha,n} = \{x \in V :\ (x, \alpha) = n\}$, $\alpha \in \Phi$, $n \in \mathbb{Z}$. Alternatively, the elements of W_a are all $\tau(l)w$, where $w \in W$ and $\tau(l)$ is the translation by $l \in L$. Such a decomposition is unique and gives a semi-direct product. For $w_a = \tau(l)w \in W_a$, we define $\varepsilon(w_a) := \varepsilon(w) = \det(w)$. The fundamental alcove is $\mathcal{A} = \{x \in V :\ \forall\alpha \in \Phi^+,\, 0 < (x, \alpha) < 1\} = \{x \in V :\ \forall\alpha \in \Delta,\, 0 < (x, \alpha);\, (x, \widetilde{\alpha}) < 1\}$, where $\widetilde{\alpha}$ is the highest positive root.

11.2.2 Affine root systems

We refer to [Kan01] for this formalism although we use slightly modified notations for the sake of consistency.

Definition 11.2.1. *The affine root system is $\Phi_a := \Phi \times \mathbb{Z}$. If $\lambda = (\alpha, n) \in \Phi_a$, we define $\lambda(x) = \lambda.x := (\alpha, x) - n$, $H_\lambda := \{x \in V :\ \lambda.x = 0\} = H_{\alpha,n}$ and s_λ, the reflection with respect to H_λ.*
We also define the action of $w_a = \tau(l)w \in W_a$ on an affine root $\lambda = (\alpha, n) \in \Phi_a$ by $w_a(\lambda) = (w\alpha, n + (w\alpha, l)) \in \Phi_a$.

In this way, we have $w_a H_\lambda = H_{w_a(\lambda)}$ and $s_\lambda(x) = x - \lambda(x)\alpha^\vee$ if $\lambda = (\alpha, n)$. The isometry property of elements of W extends to $w_a(\lambda).x = \lambda.w_a^{-1}(x)$ for $w_a \in W_a$, $\lambda \in \Phi_a$, $x \in V$. If $\lambda = (\alpha, m)$, $\mu = (\beta, n) \in \Phi_a$, we write $\lambda \perp \mu$ to mean that $(\alpha, \beta) = 0$ and, for $X \subset \Phi_a$, we define

$$\mathcal{O}(X) := \{Y \subset X :\ \forall\lambda \neq \mu \in Y,\, \lambda \perp \mu\}.$$

Moreover, the usual properties of a reflection are preserved : $s_\lambda(\lambda) = -\lambda$ and $s_\lambda(\mu) = \mu$ if $\lambda \perp \mu$.

Definition 11.2.2. *The affine simple system is $\Delta_a := \{(\alpha, 0),\, \alpha \in \Delta;\ (-\widetilde{\alpha}, -1)\}$ and the corresponding positive system is $\Phi_a^+ := \{(\alpha, n) :\ (n = 0$ and $\alpha \in \Phi^+)$ or $n \leq -1\}$.*

This definition is taylor-made so that

$$\mathcal{A} = \{x \in V :\ \forall\lambda \in \Phi_a^+,\, \lambda(x) > 0\} = \{x \in V :\ \forall\lambda \in \Delta_a,\, \lambda(x) > 0\}.$$

11.3 The main result

11.3.1 Consistency

The material here is almost the same as that of Section 2.1 in [DO04] (10.2.1 in Chapter 10 of this thesis). For $I_a \subset \Phi_a^+$, we define $W_a^{I_a} := \{w_a \in W_a : \ w_a I_a \subset \Phi_a^+\}$ and $\mathcal{I}_a := \{A = w_a I_a : \ w_a \in W_a^{I_a}\}$.

Definition 11.3.1 (Consistency).
- We will say that I_a satisfies hypothesis (C1) if there exists $J_a \in \mathcal{O}(\Delta_a \cap I_a)$ such that if $J_a \subset A \in \mathcal{I}_a$ then $A = I_a$.
- We will say that I_a satisfies hypothesis (C2) if the restriction of the determinant to the subgroup $U_a = \{w_a \in W_a : \ w_a I_a = I_a\}$ is trivial, ie $\forall w_a \in U_a$, $\varepsilon(w_a) = 1$.
- We will say that I_a satisfies hypothesis (C3) if $\mathcal{I}_a$ is finite.
- I will be called **consistent** if it satisfies (C1), (C2) and (C3).

Condition (C2) makes it possible to attribute a sign to every element of $\mathcal{I}_a$ by $\varepsilon_A := \varepsilon(w_a)$ for $A \in \mathcal{I}_a$, where w_a is any element of W_a^I with $w_a I_a = A$.

11.3.2 The exit problem

Let $X = (X_t, t \geq 0)$ be a Markov process with W_a-invariant state space $E \subset V$, infinitesimal generator $\mathcal{L}$, and write $\mathbb{P}_x$ for the law of the process started at x. For $\lambda \in \Phi_a^+$, call T_λ the hiting time of H_λ, $T_\lambda = \inf\{t \geq 0 : \ \lambda(X_t) = 0\}$. For $A \subset \Phi_a^+$ write $T_A = \min_{\lambda \in A} T_\lambda$, and set $T = T_{\Delta_a}$. Note that T is the exit time from $\mathcal{A}$. Assume that the law of X is W_a-invariant—that is,

$$\mathbb{P}_x \circ (w_a X)^{-1} = \mathbb{P}_{w_a x} \circ X^{-1}.$$

We also suppose (as in 10.2.1) that X is sufficiently regular so that :

(i) $u_{I_a}(x,t) = \mathbb{P}_x(T_{I_a} > t)$ satisfies the boundary-value problem :

$$\frac{\partial u_{I_a}}{\partial t} = \mathcal{L} u_{I_a} \qquad \begin{cases} u_{I_a}(x,0) = 1 & \text{if } x \in O_a := \{y \in V : \ \forall \lambda \in I_a, \, \lambda(y) > 0\}, \\ u_{I_a}(x,t) = 0 & \text{if } x \in \partial O_a. \end{cases} \tag{11.2}$$

(ii) $u(x,t) = \mathbb{P}_x(T > t)$ is the *unique* solution to

$$\frac{\partial u}{\partial t} = \mathcal{L} u \qquad \begin{cases} u(x,0) = 1 & \text{if } x \in \mathcal{A}, \\ u(x,t) = 0 & \text{if } x \in \partial \mathcal{A}. \end{cases} \tag{11.3}$$

Then, we have

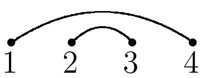
Proposition 11.3.2. *If I_a is consistent, then the exit probability can be expressed as*

$$\mathbb{P}_x(T > t) = \sum_{A \in \mathcal{I}_a} \varepsilon_A \, \mathbb{P}_x(T_A > t). \tag{11.4}$$

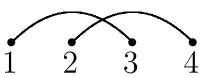
11.4 The different types

11.4.1 The $\widetilde{A}_{k-1}$ case

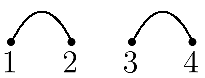
In this case, W is $\mathfrak{S}_k$ acting on $\mathbb{R}^k$ by permutation of the canonical basis vectors, $V = \mathbb{R}^k$ or $\{x \in \mathbb{R}^k : \sum_i x_i = 0\}$, $\Delta = \{e_i - e_{i+1}, 1 \leq i \leq k-1\}$, $\widetilde{\alpha} = e_1 - e_k$, $\mathcal{A} = \{x \in V : 1 + x_k > x_1 > \cdots > x_k\}$, $\alpha^\vee = \alpha$ for $\alpha \in \Phi$ and $L = \{d \in \mathbb{Z}^k : \sum_{i=1}^k d_i = 0\}$. For even $k = 2p$, we take $I_a = \{(e_{2i-1} - e_{2i}, 0), (-e_{2i-1} + e_{2i}, -1) ; 1 \leq i \leq p\}$. Then I_a is consistent and $\mathcal{I}_a$ can be identified with the set $P_2(k)$ of partitions of $[k]$ as shown in the following example for $k = 4$. Under this identification, the sign ε_A is just the parity of the number $c(\pi)$ of crossings.



$\pi = \{\{1,2\},\{3,4\}\}$
$A = \{(e_1 - e_4, 0), (e_2 - e_3, 0),$
$(-e_1 + e_4, -1), (-e_2 + e_3, -1)\}$
$c(\pi) = 0$



$\pi = \{\{1,2\},\{3,4\}\}$
$A = \{(e_1 - e_3, 0), (e_2 - e_4, 0),$
$(-e_1 + e_3, -1), (-e_2 + e_4, -1)\}$
$c(\pi) = 1$



$\pi = \{\{1,2\},\{3,4\}\}$
$A = \{(e_1 - e_2, 0), (e_3 - e_4, 0),$
$(-e_1 + e_2, -1), (-e_3 + e_4, -1)\}$
$c(\pi) = 0$

FIG. 11.1 – Pair partitions and their signs for $\widetilde{A}_3$.

Hence, the formula can be written as

$$\mathbb{P}_x(T > t) = \sum_{\pi \in P_2(k)} (-1)^{c(\pi)} \prod_{\{i<j\} \in \pi} \widetilde{p}_{ij} = \mathrm{Pf}\ (\widetilde{p}_{ij})_{i,j \in [k]} \tag{11.5}$$

where $\widetilde{p}_{ij} = \mathbb{P}_x(T_{(e_i-e_j,0),(-e_i+e_j,-1)} > t) = \mathbb{P}_x(\forall s \le t,\ 0 < X_s^i - X_s^j < 1) = \phi(x_i - x_j, 2t)$.

For odd k, our results don't appear to apply. Indeed, the same choice of I_a is no longer consistent (see Remark 11.6.1), ie the sign ε_A is not well-defined. The difference between even and odd k can be seen directly at the level of pair partitions : if you interchange 1 and k in the blocks of $\pi \in P_2(k)$ (which corresponds to the reflection with respect to $\{x_1 - x_k = 1\}$, which is the affine hyperplane of the alcove), this will change the sign of π if k is even while it won't affect the sign if k is odd. The case of the equilateral triangle is $\widetilde{A}_2$, so it unfortunately corresponds to $k = 3$ and we have no formula for this case! This seems rather mysterious to us, especially since an explicit formula for the expected exit time $\mathbb{E}_x(T)$ is known in this case and related to a well-known ruin problem for three gamblers (see [AFR] for example).

11.4.2 The $\widetilde{C}_k$ case

In this case, W is the group of signed permutations acting on $V = \mathbb{R}^k$, $\Delta = \{2e_k, e_i - e_{i+1}, 1 \le i \le k-1\}$, $\widetilde{\alpha} = 2e_1$, $\mathcal{A} = \{x \in \mathbb{R}^k :\ 1/2 > x_1 > \cdots > x_k > 0\}$ and $L = \mathbb{Z}^k$.
For even $k = 2p$, we take

$$I_a = \{(e_{2i-1} - e_{2i}, 0),\ (2e_{2i}, 0),\ (-2e_{2i-1}, -1);\ 1 \le i \le p\}.$$

For odd $k = 2p+1$,

$$I_a = \{(e_{2i-1} - e_{2i}, 0),\ (2e_{2i}, 0),\ (2e_k, 0),\ (-2e_{2i-1}, -1),\ (-2e_k, -1);\ 1 \le i \le p\}.$$

Again, $\mathcal{I}_a$ can be identified with $P_2(k)$ and the formula is :

$$\mathbb{P}_x(T > t) = \sum_{\pi \in P_2(k)} (-1)^{c(\pi)}\, \check{p}_{s(\pi)} \prod_{\{i<j\} \in \pi} \check{p}_{ij} \tag{11.6}$$

where

$$\check{p}_{ij} = \mathbb{P}_x(T_{(e_i-e_j,0),(-e_i+e_j,-1),(e_j,0)} > t) = \mathbb{P}_x(\forall s \le t,\ 1/2 > X_s^i > X_s^j > 0),$$
$$\check{p}_i = \mathbb{P}_x(T_{(2e_i,0),(-2e_i,-1)} > t) = \mathbb{P}_x(\forall s \le t,\ 1/2 > X_s^i > 0),$$

and $s(\pi)$ is the singlet of π, the term $\check{p}_{s(\pi)}$ being absent for even k.

Everything can be rewritten in terms of Pfaffian :

$$\mathbb{P}_x(T > t) = \begin{cases} \mathrm{Pf}\ (\check{p}_{ij})_{i,j \in [k]} & \text{if } k \text{ is even,} \\ \sum_{l=1}^k (-1)^{l-1}\, \check{p}_l\, \mathrm{Pf}\ (\check{p}_{ij})_{i,j \in [k] \setminus \{l\}} & \text{if } k \text{ is odd.} \end{cases} \tag{11.7}$$

Remark 11.4.1. *This formula can be obtained directly by applying the exit time formula for the chamber of type C_k (the same as B_k) to the Brownian motion killed when reaching $1/2$. But it was natural to include it in our framework.*

11.4.3 The $\widetilde{B}_k$ case

W is the group of signed permutations acting on $V = \mathbb{R}^k$, $\Delta = \{e_k, e_i - e_{i+1},\ 1 \leq i \leq k-1\}$, $\widetilde{\alpha} = e_1 + e_2$, $\mathcal{A} = \{x \in \mathbb{R}^k :\ x_1 > \cdots > x_k > 0,\ x_1 + x_2 < 1\}$ and $L = \{d \in \mathbb{Z}^k :\ \sum_i d_i \text{ is even }\}$.
For even $k = 2p$, we take

$$I_a = \{(e_{2i-1} - e_{2i}, 0),\ (e_{2i}, 0),\ (-e_{2i-1} - e_{2i}, -1);\ 1 \leq i \leq p\}.$$

For odd $k = 2p+1$,

$$I_a = \{(e_{2i-1} - e_{2i}, 0),\ (e_{2i}, 0),\ (-e_{2i-1} - e_{2i}, -1),\ (e_k, 0),\ (-e_k, -1);\ 1 \leq i \leq p\}.$$

In this case, the formula is :

$$\mathbb{P}_x(T > t) = \sum_{\pi \in P_2(k)} (-1)^{c(\pi)} \bar{p}_{s(\pi)} \prod_{\{i<j\} \in \pi} \bar{p}_{ij} \tag{11.8}$$

where

$$\begin{aligned} \bar{p}_{ij} &= \mathbb{P}_x(T_{(e_i - e_j, 0), (-e_i - e_j, -1), (e_j, 0)} > t) = \mathbb{P}_x(\forall s \leq t,\ 1 - X_s^j > X_s^i > X_s^j > 0), \\ \bar{p}_i &= \mathbb{P}_x(T_{(e_i, 0), (-e_i, -1)} > t) = \mathbb{P}_x(\forall s \leq t,\ 1 > X_s^i > 0) \end{aligned}$$

and $s(\pi)$ denotes the singlet of π, the term $\bar{p}_{s(\pi)}$ being absent for even k.

Everything can be rewritten in terms of Pfaffian :

$$\mathbb{P}_x(T > t) = \begin{cases} \text{Pf } (\bar{p}_{ij})_{i,j \in [k]} & \text{if } k \text{ is even,} \\ \sum_{l=1}^k (-1)^{l-1} \bar{p}_l \text{ Pf } (\bar{p}_{ij})_{i,j \in [k] \setminus \{l\}} & \text{if } k \text{ is odd.} \end{cases} \tag{11.9}$$

11.4.4 The $\widetilde{D}_k$ case

W is the group of evenly signed permutations acting on $V = \mathbb{R}^k$, $\Delta = \{e_i - e_{i+1}, e_{k-1} + e_k,\ 1 \leq i \leq k-1\}$, $\widetilde{\alpha} = e_1 + e_2$, $\mathcal{A} = \{x \in \mathbb{R}^k :\ x_1 > \cdots > x_{k-1} > |x_k|,\ x_1 + x_2 < 1\}$ and $L = \{d \in \mathbb{Z}^k :\ \sum_i d_i \text{ is even }\}$.
For even $k = 2p$, we take

$$I_a = \{(e_{2i-1} - e_{2i}, 0),\ (-e_{2i-1} + e_{2i}, -1),\ (e_{2i-1} + e_{2i}, 0),\ (-e_{2i-1} - e_{2i}, -1);\ 1 \leq i \leq p\}.$$

For odd $k = 2p+1$,

$$I_a = \{(e_{2i} - e_{2i+1}, 0),\ (-e_{2i} + e_{2i+1}, -1),\ (e_{2i} + e_{2i+1}, 0),\ (-e_{2i} - e_{2i+1}, -1);\ 1 \leq i \leq p\}.$$

The formula then becomes :

$$\mathbb{P}_x(T > t) = \sum_{\pi \in P_2(k)} (-1)^{c(\pi)} \prod_{\{i<j\} \in \pi} \breve{p}_{ij} \tag{11.10}$$

where

$$\begin{aligned} \breve{p}_{ij} &= \mathbb{P}_x(T_{(e_i-e_j,0),(-e_i+e_j,-1),(e_i+e_j,0),(-e_i-e_j,-1)} > t) = \acute{p}_{ij}\,\grave{p}_{ij}, \\ \acute{p}_{ij} &= \mathbb{P}_x(\forall s \le t,\, 1 > X_s^i - X_s^j > 0) = \phi(x_i - x_j, 2t), \\ \grave{p}_{ij} &= \mathbb{P}_x(\forall s \le t,\, 1 > X_s^i + X_s^j > 0) = \phi(x_i + x_j, 2t). \end{aligned}$$

Everything can be rewritten in terms of Pfaffian :

$$\mathbb{P}_x(T > t) = \begin{cases} \text{Pf } (\breve{p}_{ij})_{i,j\in[k]} & \text{if } k \text{ is even,} \\ \sum_{l=1}^{k} (-1)^{l-1} \text{Pf } (\breve{p}_{ij})_{i,j\in[k]\setminus\{l\}} & \text{if } k \text{ is odd.} \end{cases} \tag{11.11}$$

11.4.5 The $\widetilde{G}_2$ case

Here, $V = \{x \in \mathbb{R}^3, \sum_i x_i = 0\}$, $\Phi^+ = \{e_3 - e_1, e_3 - e_2, e_1 - e_2, -2e_1 + e_2 + e_3, -2e_2 + e_1 + e_3, 2e_3 - e_1 - e_2\}$, $\widetilde{\alpha} = 2e_3 - e_1 - e_2$, $\Delta = \{e_1 - e_2, -2e_1 + e_2 + e_3\}$ and $L = \{d \in V : \forall i,\, 3d_i \in \mathbb{Z}\}$.
We take $I_a = \{(e_1 - e_2, 0), (-e_1 + e_2, -1), (2e_3 - e_1 - e_2, 0), (-2e_3 + e_1 + e_2, -1)\}$, which is consistent and we can describe $\mathcal{I}_a = \{I_a, A_1, A_2\}$ with $A_1 = \{(e_3 - e_1, 0), (-e_3 + e_1, -1), (-2e_2 + e_1 + e_3, 0), (2e_2 - e_1 - e_3, -1)\}$, $\varepsilon_{A_1} = -1$, $A_2 = \{(e_3 - e_2, 0), (-e_3 + e_2, -1), (-2e_1 + e_2 + e_3, 0), (2e_1 - e_2 - e_3, -1)\}$, $\varepsilon_{A_2} = 1$.

In this case, the chamber $\mathcal{A}$ is a triangle ABC with angles $(\pi/2, \pi/3, \pi/6)$ as represented in Figure 11.2. If T_R denotes the exit time from the region R of the plane and $\mathbb{P}(R) = \mathbb{P}_x(T_R > t)$, then the formula in this case is

$$\mathbb{P}(ABC) = \mathbb{P}(ADEF) - \mathbb{P}(FJCG) + \mathbb{P}(FHCI), \tag{11.12}$$

where $ADEF$, $FJCG$, $FHCI$ are rectangles, as shown in Figure 11.2.

11.4.6 The $\widetilde{F}_4$ case

Recall that $V = \mathbb{R}^4$, $\Phi^+ = \{e_i \pm e_j,\, 1 \le i < j \le 4;\, e_i,\, 1 \le i \le 4;\, (e_1 \pm e_2 \pm e_3 \pm e_4)/2\}$, $\Delta = \{e_2 - e_3, e_3 - e_4, e_4, (e_1 - e_2 - e_3 - e_4)/2\}$, $\widetilde{\alpha} = e_1 + e_2$ and $L = \{d \in \mathbb{Z}^4 : \sum_i d_i \text{ is even }\}$.
$I_a := \{(e_2 - e_3, 0), (-e_2 + e_3, -1), (e_1 - e_4, 0), (-e_1 + e_4, -1), (e_3, 0), (e_4, 0)\}$ turns out to be consistent.

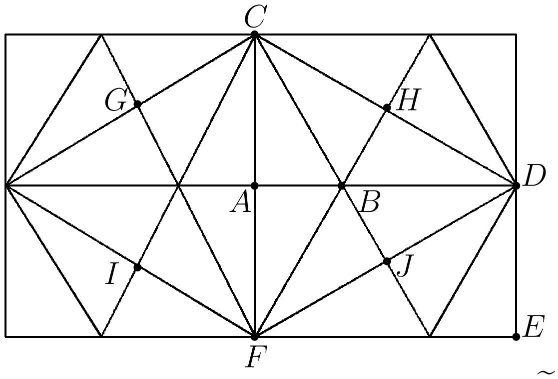


FIG. 11.2 – Tiling associated with $\widetilde{G}_2$

11.5 Expansion and expectation for the exit time in the $\widetilde{A}_{k-1}$ case

We will use the following expansion for the exit time $T_{(0,1)}$ of one-dimensional Brownian motion from $(0,1)$:

$$\phi(x,t) = \mathbb{P}_x(T_{(0,1)} > t) = \sum_{l\in\mathbb{O}} c_l e^{-\lambda_l t} \sin(\pi l x),$$

where $\mathbb{O} = 2\mathbb{N}+1$ ($\mathbb{O}$ is for "odd"), $c_l = 4/(l\pi)$, $\lambda_l = (l\pi)^2/2$ (see [Bas95]). Inserting this in (11.5) and noting $m = k/2 \in \mathbb{N}$, $x_{ij} = x_i - x_j$,

$$\begin{aligned}\mathbb{P}_x(T>t) &= \sum_{\pi=\{\{i_s<j_s\},\,1\le s\le m\}} (-1)^{c(\pi)} \prod_{s=1}^m \left(\sum_{l\in\mathbb{O}} c_l e^{-2\lambda_l t} \sin(\pi l x_{i_s j_s})\right)\\ &= \sum_{\pi=\{\{i_s<j_s\},\,1\le s\le m\}} (-1)^{c(\pi)} \sum_{l\in\mathbb{O}^m} e^{-\pi^2(l_1^2+\cdots+l_m^2)t} \prod_{s=1}^m c_{l_s} \sin(\pi l_s x_{i_s j_s}).\end{aligned}$$

Now, for $\pi = \{\{i_s < j_s\},\ 1 \le s \le m\}$, define

$$G_r(x,\pi) = \sum_{l\in\mathbb{O}^m,\,N(l)=r} \prod_{s=1}^m c_{l_s} \sin(\pi l_s x_{i_s j_s}),$$

where $N(l) = l_1^2 + \cdots + l_m^2$ and

$$F_r(x) = \sum_{\pi\in P_2(k)} (-1)^{c(\pi)} G_r(x,\pi).$$

Remark that the sum defining $G_r(x, \pi)$ runs over a $\mathfrak{S}_m$-invariant set of indices and, thus, does not depend on the enumeration of the blocks of π but only of π itself. With those definitions, we can write

$$\mathbb{P}_x(T > t) = \sum_r e^{-\pi^2 rt} F_r(x). \tag{11.13}$$

As for expectations, we have

$$\begin{aligned}
\mathbb{E}_x(T) &= \int_0^\infty \mathbb{P}_x(T > t)\, dt \\
&= \sum_r \frac{1}{r\pi^2} F_r(x) \\
&= \frac{2^{2m}}{\pi^{m+2}} \sum_{\pi \in P_2(k)} (-1)^{c(\pi)} \sum_{l \in \mathbb{O}^m} \frac{\prod_{s=1}^m \sin(\pi l_s x_{i_s j_s})}{l_1 l_2 \dots l_m (l_1^2 + \cdots + l_m^2)}.
\end{aligned} \tag{11.14}$$

When $k = 2$, $m = k/2 = 1$, the previous formula becomes

$$\mathbb{E}_x(T) = \sum_{n \in \mathbb{N}} \frac{4}{\pi^3} \frac{\sin(\pi(2n+1)x_{12})}{(2n+1)^3} = \frac{1}{2} x_{12}(1 - x_{12}), \qquad 0 < x_{12} = x_1 - x_2 < 1,$$

which is a well-known formula in Fourier series.

Let us come back to formula (11.13). F_r appears as an eigenfunction for half the Dirichlet Laplacian on the alcove and the associated eigenvalue is $-\pi^2 r$. But a general formula for eigenfunctions and eigenvalues is known for crystallographic groups (see [Bér80]) and expressed as an alternating sum over the group elements. Our result yields an alternative formula, expressing eigenfunctions as an alternating sum over $\mathcal{I}_a \simeq P_2(k)$. In particular, the smallest eigenvalue is known, from general theory (see [Bér80]), to be $-2\pi^2|\rho|^2$, where ρ is half the sum of the positive roots,

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = \frac{1}{2} \left(-(2m-1), -(2m-3), \dots, -1, 1, \dots, 2m-3, 2m-1 \right),$$

in the $\widetilde{A}_{k-1}$ case, with $k = 2m$. Thus, the smallest eigenvalue is $-\pi^2 r_0$ with $r_0 = k(k-1)(k+1)/6$. We indicate a simple and direct lemma to recover this result :

Lemma 11.5.1. *The function*

$$H(x) := \prod_{i<j} \sin(\pi(x_i - x_j)) \tag{11.15}$$

is an eigenfunction for $\Delta/2$ with Dirichlet boundary conditions on the alcove $\mathcal{A}$ with associated eigenvalue $-\pi^2 r_0$ where $r_0 = k(k-1)(k+1)/6$. Since H is positive on $\mathcal{A}$, $-\pi^2 r_0$ is the smallest eigenvalue.

Consequently, the first terms in (11.13) cancel and (11.13) can be written :

$$\mathbb{P}_x(T > t) = \sum_{r=r_0}^{\infty} e^{-\pi^2 r t} F_r(x). \tag{11.16}$$

Remark 11.5.1. *It can be noticed that all F_r are divisible by H in the ring of trigonometric polynomials. Indeed, we have the following*

Lemma 11.5.2. *If $F(X) = F(X_1, \dots, X_m)$ is a polynomial in the $(\sin X_i, \cos X_i)_{1\leq i\leq m}$ such that*

$$\forall i \neq j,\ (\sin(X_i - X_j) = 0 \Longrightarrow F(X) = 0)$$

then there exists a polynomial $G(X)$ in the $(\sin X_i, \cos X_i)_{1\leq i\leq m}$ such that

$$F(X) = \prod_{i<j} \sin(X_i - X_j)\ G(X).$$

We can appeal to Lemma 10.7.5 in the case of A_{k-1} to see that $F_r(x_\sigma) = \varepsilon(\sigma) F_r(x)$, so that $F_r(x) = 0$ if $x_i = x_j$ for $i \neq j$. It is also clear that $F_r(x+y) = (-1)^{\sum y_i} F_r(x)$ if $y \in \mathbb{Z}^k$. Those two properties just express the fact that eigenfunctions are alternating under the action of the affine Weyl group, which is well-known. Thus, Lemma 11.5.2 applies with $X_i = \pi x_i$.

11.6 Proofs

11.6.1 The main result

All the formalism of affine root systems has been set for the proofs to be the same as those in [DO04] (Chapter 10 of this thesis). Therefore, we only state the lemmas (without proofs) to show how they have to be modified in this context.

Lemma 11.6.1. *If $K_a \subset I_a$ and $\lambda \in \Delta_a \cap K_a^{\perp}$, then $s_\lambda \mathcal{L} = \mathcal{L}$, where*

$$\mathcal{L} = \{A \in \mathcal{I}_a :\ K_a \subset A,\ \lambda \notin A\}.$$

Lemma 11.6.2. *Suppose (C3) is satisfied and $f : \mathcal{I}_a \to \mathbb{R}$, $\lambda \in \Delta_a$ are such that $f(A) = 0$ whenever $\lambda \in A$, and $f(A) = f(s_\lambda A)$ whenever $\lambda \notin A$, then $\sum_{A\in\mathcal{I}_a} \varepsilon_A f(A) = 0$.*

Lemma 11.6.3. *If conditions (C1) and (C3) are satisfied, we have : $\sum_{A\in\mathcal{I}_a} \varepsilon_A = 1$.*

11.6.2 The $\widetilde{A}_{k-1}$ case

Let us first determine $\mathcal{I}_a$. If $w_a = \tau(d)\sigma \in W_a^{I_a}$, then

$$w_a\{(e_{2i-1} - e_{2i}, 0), (-e_{2i-1} + e_{2i}, -1)\} =$$

$$\{(e_{\sigma(2i-1)} - e_{\sigma(2i)}, n), (-e_{\sigma(2i-1)} + e_{\sigma(2i)}, -1 - n)\},$$

where $n = d_{\sigma(2i-1)} - d_{\sigma(2i)}$. Thus, $n \leq 0$ and $-1 - n \leq 0$, ie $n \in \{0, -1\}$. If $n = 0$, $d_{\sigma(2i-1)} = d_{\sigma(2i)}$ and $\sigma(2i-1) < \sigma(2i)$. If $n = -1$, $d_{\sigma(2i-1)} = d_{\sigma(2i)} - 1$ and $\sigma(2i-1) > \sigma(2i)$. In any case,

$$w_a\{(e_{2i-1} - e_{2i}, 0), (-e_{2i-1} + e_{2i}, -1)\} =$$

$$\{(e_{\min(\sigma(2i-1),\sigma(2i))} - e_{\max(\sigma(2i-1),\sigma(2i))}, 0), (-e_{\min(\sigma(2i-1),\sigma(2i))} + e_{\max(\sigma(2i-1),\sigma(2i))}, -1)\}.$$

Thus, the identification between $\pi = \{\{i_l < j_l\}, 1 \leq l \leq p\} \in P_2(k)$ and $A = \{(e_{i_l} - e_{j_l}, 0), (-e_{i_l} + e_{j_l}, -1)\,;\, 1 \leq l \leq p\} \in \mathcal{I}_a$. Then, we take $J_a = \{(e_{2i-1} - e_{2i}, 0)\,;\, 1 \leq i \leq p\} \in \mathcal{O}(\Delta_a)$. From the previous description of $\mathcal{I}_a$, (C1) and (C3) are obvious. Now, it is clear that

$$U_a = \{\tau(d)\sigma :\ \sigma \text{ permutes sets } \{1,2\}, \{3,4\}, \ldots, \{k-1,k\} \text{ and } \forall i \leq p,$$

$$(d_{\sigma(2i-1)} = d_{\sigma(2i)},\, \sigma(2i-1) < \sigma(2i)) \text{ or } (d_{\sigma(2i-1)} = d_{\sigma(2i)} - 1,\, \sigma(2i-1) > \sigma(2i))\}.$$

Thus, if $\tau(d)\sigma \in U_a$, we can write $\sigma = \sigma_1\sigma_2$, where σ_2 permutes pairs $(1,2), \ldots, (k-1,k)$ and σ_1 is the product of the transpositions $(\sigma(2i-1), \sigma(2i))$ for which $d_{\sigma(2i-1)} = d_{\sigma(2i)} - 1$. Then, $\varepsilon(\sigma_2) = 1$ from [DO04] (Chapter 10 of this thesis) so that $\varepsilon(\sigma) = \varepsilon(\sigma_1) = (-1)^m$, where $m = |\{i :\ d_{\sigma(2i-1)} = d_{\sigma(2i)} - 1\}|$. But, since $d \in L$,

$$0 = \sum_j d_j = \sum_{i=1}^{p} \left(d_{\sigma(2i-1)} + d_{\sigma(2i)}\right) \tag{11.17}$$

$$= 2 \sum_{i,\, d_{\sigma(2i-1)} = d_{\sigma(2i)}} d_{\sigma(2i)} + 2 \sum_{i,\, d_{\sigma(2i-1)} = d_{\sigma(2i)} - 1} d_{\sigma(2i)} - m, \tag{11.18}$$

which proves that m is even. Hence $\varepsilon(\sigma_1)) = 1$. The fact that $\varepsilon_A = (-1)^{c(\pi)}$ comes from the analogous fact in [DO04] (Chapter 10 of this thesis).

Remark 11.6.1. *In the case of odd $k = 2p + 1$, the same discussion carries over by adding singlets to the pair partitions and with $\sigma(k) = k$ if $\tau(d)\sigma \in U_a$. But equality (11.17) is no longer valid, which explains why the sign is not well-defined for such k..*

11.6.3 The $\widetilde{B}_k$ case

Let us first suppose k is even, $k = 2p$. Suppose $d \in L$, f is a sign change with support $\bar{f}$ and $\sigma \in \mathfrak{S}_k$ such that $w_a = \tau(d)f\sigma \in W_a^{I_a}$. Then,

$$w_a\big\{ (e_{2i-1} - e_{2i}, 0),\ (e_{2i}, 0),\ (-e_{2i-1} - e_{2i}, -1) \big\} = \big\{ \big(f(e_{\sigma(2i-1)}) - f(e_{\sigma(2i)}), m - n\big)\,, \big(f(e_{\sigma(2i)}), n\big)\,,\ \big(-f(e_{\sigma(2i-1)}) - f(e_{\sigma(2i)}), -1 - m - n\big) \big\} := S,$$

with $m = f(\sigma(2i-1))d_{\sigma(2i-1)}$ and $n = f(\sigma(2i))d_{\sigma(2i)}$. Thus, $m - n \le 0$, $n \le 0$, $-1 - m - n \le 0$, which forces $m = n = 0$ or $m = -1, n = 0$. If $m = n = 0$, then $f(e_{\sigma(2i-1)}) - f(e_{\sigma(2i)}) \in \Phi^+$, $f(e_{\sigma(2i)}) \in \Phi^+$, which implies $\sigma(2i-1), \sigma(2i) \notin \overline{f}$ and $\sigma(2i-1) < \sigma(2i)$. If $m = -1, n = 0$, then $-f(e_{\sigma(2i-1)}) - f(e_{\sigma(2i)}) \in \Phi^+$, $f(e_{\sigma(2i)}) \in \Phi^+$, which implies $\sigma(2i-1) \in \overline{f}$, $\sigma(2i) \notin \overline{f}$ and $\sigma(2i-1) < \sigma(2i)$. In any case,

$$S = \big\{ (e_{\sigma(2i-1)} - e_{\sigma(2i)}, 0),\ (e_{\sigma(2i)}, 0),\ (-e_{\sigma(2i-1)} - e_{\sigma(2i)}, -1) \big\}$$

and

$$\begin{aligned} W_a^{I_a} = \Big\{ & \tau(d)f\sigma \in W_a:\ \forall i,\ \big(d_{\sigma(2i-1)} = d_{\sigma(2i)} = 0,\ \sigma(2i-1), \sigma(2i) \notin \overline{f}, \\ & \sigma(2i-1) < \sigma(2i)\big) \text{ or } \big(d_{\sigma(2i-1)} = 1,\ d_{\sigma(2i)} = 0,\ \sigma(2i-1) \in \overline{f}, \\ & \sigma(2i) \notin \overline{f},\ \sigma(2i-1) < \sigma(2i)\big) \Big\}. \end{aligned}$$

Then, $\mathcal{I}_a$ clearly identifies with $P_2(k)$ through the correspondance between $\pi = \{\{i_l < j_l\}, 1 \le l \le p\} \in P_2(k)$ and $A = \{(e_{i_l} - e_{j_l}, 0),\ (e_{j_l}, 0),\ (-e_{i_l} - e_{j_l}, -1)\,;\, 1 \le l \le p\}$. So, (C1) and (C3) are obvious by taking $J_a = \{(e_{2i-1} - e_{2i}, 0),\ (-e_1 - e_2, -1)\}$. Now,

$$U_a = \{\tau(d)f\sigma \in W_a^{I_a} : \sigma \text{ permutes pairs } (1,2), \ldots, (2p-1, 2p)\},$$

so that, if $\tau(d)f\sigma \in U_a$, $\varepsilon(\tau(d)f\sigma) = \varepsilon(f)\varepsilon(\sigma) = (-1)^{|\overline{f}|}$. But, $|\overline{f}| = \sum_i d_{\sigma(2i-1)} = \sum_j d_j$ is even, which proves (C2).

For odd $k = 2p+1$, $\mathcal{I}_a$ identifies with $P_2(k)$ through the correspondance between $\pi = \{\{i_l < j_l\}, 1 \le l \le p;\ \{s\}\} \in P_2(k)$ and $A = \{(e_{i_l} - e_{j_l}, 0), (e_{j_l}, 0), (-e_{i_l} - e_{j_l}, -1)\,, 1 \le l \le p;\ (e_s, 0), (-e_s, -1)\}$. Elements $\tau(d)f\sigma \in U_a$ are described in the same way with the extra condition that $\sigma(k) = k$ and $d_k = 0, k \notin \overline{f}$ or $d_k = 1, k \in \overline{f}$. So the proof of (C2) carries over.

11.6.4 The $\widetilde{D}_k$ case

Let us first suppose k is even, $k = 2p$. Suppose $d \in L$, f is an even sign change and $\sigma \in \mathfrak{S}_k$ such that $w_a = \tau(d)f\sigma \in W_a^{I_a}$. Then,

$$\begin{aligned} & w_a \big\{ (e_{2i-1} - e_{2i}, 0),\ (-e_{2i-1} + e_{2i}, -1),\ (e_{2i-1} + e_{2i}, 0)\ (-e_{2i-1} - e_{2i}, -1) \big\} \\ &= \Big\{ \big(f(e_{\sigma(2i-1)}) - f(e_{\sigma(2i)}), m - n\big)\,, \big(-f(e_{\sigma(2i-1)}) + f(e_{\sigma(2i)}), -1 - (m - n)\big)\,, \\ & \big(f(e_{\sigma(2i-1)}) + f(e_{\sigma(2i)}), m + n\big)\,, \big(-f(e_{\sigma(2i-1)}) - f(e_{\sigma(2i)}), -1 - (m + n)\big) \Big\} := S, \end{aligned}$$

with $m = f(\sigma(2i-1))d_{\sigma(2i-1)}$ and $n = f(\sigma(2i))d_{\sigma(2i)}$. Thus, $m-n \leq 0$, $-1-(m-n) \leq 0$, $m+n \leq 0$, $-1-(m+n) \leq 0$, which forces $m = n = 0$ or $m = -1, n = 0$. If $m = n = 0$, then $f(e_{\sigma(2i-1)}) \pm f(e_{\sigma(2i)}) \in \Phi^+$, which implies $\sigma(2i-1) \notin \overline{f}$ and $\sigma(2i-1) < \sigma(2i)$. If $m = -1, n = 0$, then $-f(e_{\sigma(2i-1)}) \pm f(e_{\sigma(2i)}) \in \Phi^+$, which implies $\sigma(2i-1) \in \overline{f}$ and $\sigma(2i-1) < \sigma(2i)$. In any case, we have

$$S = \Big\{ (e_{\sigma(2i-1)} - e_{\sigma(2i)}, 0), (-e_{\sigma(2i-1)} + e_{\sigma(2i)}, -1)\,, \\ (e_{\sigma(2i-1)}) + e_{\sigma(2i)}, 0)\,(e_{\sigma(2i-1)}) + e_{\sigma(2i)}, 0) \Big\}\,,$$

and

$$W_a^{I_a} = \Big\{ \tau(d)f\sigma \in W_a : \forall i, \big(d_{\sigma(2i-1)} = d_{\sigma(2i)} = 0,\ \sigma(2i-1) \notin \overline{f}, \\ \sigma(2i-1) < \sigma(2i)\big) \text{ or } \big(d_{\sigma(2i-1)} = 1,\ d_{\sigma(2i)} = 0,\ \sigma(2i-1) \in \overline{f}, \\ (2i-1) < \sigma(2i)\,\big) \Big\}.$$

The correspondance between $\pi = \{\{i_l < j_l\}, 1 \leq l \leq p\} \in P_2(k)$ and $A = \{(e_{i_l} - e_{j_l}, 0), (-e_{i_l} + e_{j_l}, -1), (e_{i_l} + e_{j_l}, 0)(-e_{i_l} - e_{j_l}, -1)\,;\, 1 \leq l \leq p\}$ identifies $\mathcal{I}_a$ with $P_2(k)$. (C1) and (C3) are obvious with $J_a = \{(e_{2i-1} - e_{2i}, 0), 1 \leq i \leq p;\, (e_{k-1} + e_k, 0)\}$. Moreover,

$$U_a = \{\tau(d)f\sigma \in W_a^{I_a} : \sigma \text{ permutes pairs } (1,2), \ldots, (2p-1, 2p)\},$$

which makes (C2) easy since $\varepsilon(f) = 1$ for $\tau(d)f\sigma \in W_a$.
The case of odd k is an obvious modification.

11.6.5 The $\widetilde{G}_2$ case

Call $\alpha_1 = e_1 - e_2$, $\alpha_2 = 2e_3 - e_1 - e_2 = \widetilde{\alpha}$ and take $J_a = \{(\alpha_1, 0), (-\alpha_2, -1)\}$. We remark that I_a can be written

$$\{(\alpha_1, 0), (-\alpha_1, -1), (\alpha_2, 0), (-\alpha_2, -1)\} \text{ with } \alpha_1 \text{ short}, \alpha_2 \text{ long}, \alpha_1 \perp \alpha_2. \tag{11.19}$$

If $w_a = \tau(d)w \in W_a^{I_a}$ then $(w\alpha_i, d) \in \mathbb{Z}$, $(w\alpha_i, d) \leq 0$ and $-1 - (w\alpha_i, d) \leq 0$, which imposes $(w\alpha_i, d) \in \{0, -1\}$ for $i = 1, 2$. Thus, $A = w_a I_a$ can be also be written as in (11.19) for some α_1', α_2'. This guarantees condition (C3) and if $J_a \subset A$ then obviously $\alpha_1 = \alpha_1'$, $\alpha_2 = \alpha_2'$ so that $A = I_a$, which proves condition (C1). Writing I_a as in (11.19) allows us to see that if $w_a = \tau(d)w \in W_a$, then $w_a I_a = \{(w\alpha_1, m_1), (-w\alpha_1, -1-m_1), (w\alpha_2, m_2), (-w\alpha_2, -1-m_2)\}$ where $m_i = (w\alpha_i, d) \in \mathbb{Z}$. Since W sends long (short) roots to long (short) roots, $w_a \in U_a$ implies $w\alpha_i \in \{\pm\alpha_i\}$ for $i = 1, 2$. If $w\alpha_i = \alpha_i$ for $i = 1, 2$ (respectively $w\alpha_i = -\alpha_i$ for $i = 1, 2$), then $w = \text{id}$ (respectively $w = -\text{id}$) and $\varepsilon(w) = 1$ (recall that $\dim V = 2$). If $w\alpha_1 = \alpha_1$ and

$w\alpha_2 = -\alpha_2$ then $(\alpha_1, d) = 0$ and $(\alpha_2, d) = 1$. This implies $d = (-1/6, -1/6, 1/3) \notin L$, which is absurd. The same absurdity occurs if $w\alpha_1 = -\alpha_1$ and $w\alpha_2 = \alpha_2$.

For the determination of $\mathcal{I}_a$, it is easy to see that all sets of the form (11.19) are I_a, A_1, A_2. The sign of the transformation sending (α_1, α_2) to $(e_3 - e_1, -2e_2 + e_1 + e_3)$ is 1 so that $\varepsilon_{A_1} = -1$ and A_2 is obtained from A_1 by transposing e_1 and e_2, which finishes the proof.

11.6.6 The $\widetilde{F}_4$ case

Call $\alpha_1 = e_2 - e_3$, $\alpha'_1 = e_3$, $\alpha_2 = e_1 - e_4$, $\alpha'_2 = e_4$. Then I_a can be written

$$\{(\alpha_1, 0), (-\alpha_1, -1), (\alpha'_1, 0), (\alpha_2, 0), (-\alpha_2, -1), (\alpha'_2, 0)\}, \tag{11.20}$$

with α_1, α_2 long, α'_1, α'_2 short, $\{\alpha_1, \alpha'_1\} \perp \{\alpha_2, \alpha'_2\}$ and $(\alpha_i, \alpha'_i) = -1$. The same kind of reasoning as in the $\widetilde{G}_2$ case shows conditions (C1) and (C3), with $J_a = \{\alpha_1, \alpha'_2\}$. Let us prove (C2). If $w_a = \tau(d)w \in U_a$, then $w_a I_a =$

$$\{(w\alpha_1, m_1), (-w\alpha_1, -1 - m_1), (w\alpha'_1, m'_1), (w\alpha_2, m_2), (-w\alpha_2, -1 - m_2), (w\alpha'_2, m'_2)\},$$

with $m_i = (w\alpha_i, d)$, $m'_i = (w\alpha'_i, d)$. Since w sends long (short) roots to long (short) roots, necessarily $w\{\alpha'_1, \alpha'_2\} = \{\alpha'_1, \alpha'_2\}$ and $m'_1 = m'_2 = 0$.
Suppose $w\alpha'_i = \alpha'_i$, $i = 1, 2$. Since $(w\alpha_2, \alpha'_1) = (\alpha_2, \alpha'_1) = 0 \neq -1$, we have $w\alpha_1 \in \{\alpha_1, -\alpha_1\}$ and $w\alpha_2 \in \{\alpha_2, -\alpha_2\}$. If $w\alpha_1 = -\alpha_1$, $w\alpha_2 = \alpha_2$ then $m_1 = 1$, $m_2 = 0 = m'_1 = m'_2$, which leads to $d = (0, 1, 0, 0) \notin L$, absurd! If $w\alpha_1 = \alpha_1$, $w\alpha_2 = -\alpha_2$, a similar reasoning leads to the absurdity $d = (1, 0, 0, 0) \notin L$. Hence, $w\alpha_1 = \alpha_1$, $w\alpha_2 = \alpha_2$ or $w\alpha_1 = -\alpha_1$, $w\alpha_2 = -\alpha_2$. Then, using the basis $(\alpha_1, \alpha'_1, \alpha_2, \alpha'_2)$, $\varepsilon(w) = 1$ is an obvious check.
Suppose now $w\alpha'_1 = \alpha'_2$, $w\alpha'_2 = \alpha'_1$. Similar arguments show that $w\alpha_2 \in \{\alpha_1, -\alpha_1\}$ and $w\alpha_1 \in \{\alpha_2, -\alpha_2\}$. If $w\alpha_1 = \alpha_2$, $w\alpha_2 = \alpha_1$ or $w\alpha_1 = -\alpha_2$, $w\alpha_2 = -\alpha_1$ then $\varepsilon(w) = 1$. Suppose $w\alpha_1 = \alpha_2$, $w\alpha_2 = -\alpha_1$, then $m_1 = 0$, $m_2 = -1$, which, as before, leads to $d = (0, 1, 0, 0) \notin L$. If $w\alpha_1 = -\alpha_2$, $w\alpha_2 = \alpha_1$, then $m_1 = -1$, $m_2 = 0$, which also gives $d = (1, 0, 0, 0) \notin L$.

Proof of Lemma 11.5.1. Set $x_{ij} = x_i - x_j$ and $h(x) = \prod_{1 \le i < j \le k} \sin x_{ij}$. Computation of the logarithmic derivative gives

$$\partial_i h = h \sum_{j\,(\neq i)} \frac{\cos x_{ij}}{\sin x_{ij}},$$

which yields

$$\begin{aligned}\partial_i^2 h &= h\left\{\sum_{j,l\,(\neq i)} \frac{\cos x_{ij}\cos x_{il}}{\sin x_{ij}\sin x_{il}} + \sum_{j\,(\neq i)}(-1-\frac{\cos^2 x_{ij}}{\sin^2 x_{ij}})\right\} \\ &= h\left\{\sum_{j\neq l\,(\neq i)} \frac{\cos x_{ij}\cos x_{il}}{\sin x_{ij}\sin x_{il}} - (k-1)\right\},\end{aligned}$$

so that $\Delta h = h(S(x) - k(k-1))$ with

$$S(x) = \sum{}' \frac{\cos x_{ij}\cos x_{il}}{\sin x_{ij}\sin x_{il}},$$

where $\sum'$ runs over i, j, l pairwise distinct. By circular permutation, we get

$$\begin{aligned}3S(x) &= \sum{}' \frac{\cos x_{ij}\cos x_{il}}{\sin x_{ij}\sin x_{il}} + \frac{\cos x_{jl}\cos x_{ji}}{\sin x_{jl}\sin x_{ji}} + \frac{\cos x_{li}\cos x_{lj}}{\sin x_{li}\sin x_{lj}} \\ &= \sum{}' \frac{\cos x_{ij}\cos x_{il}\sin x_{jl} - \cos x_{jl}\cos x_{ij}\sin x_{il} + \cos x_{il}\cos x_{jl}}{\sin x_{ij}\sin x_{il}\sin x_{jl}}.\end{aligned}$$

But magical trigonometry shows that each term in the previous sum equals -1, so that $S(x) = -k(k-1)(k-2)/3$, which concludes the proof. □

Proof of Lemma 11.5.2. Let $P \in \mathbb{C}[S_i, C_i\,;\, 1 \leq i \leq m]$ be such that

$$F(X) = P(\sin X_i, \cos X_i).$$

P cancels whenever $S_iC_j - S_jC_i,\ i < j$, cancels. Since the $S_iC_j - S_jC_i,\ i < j$, are irreducible and relatively prime, their product divides P (we can invoke Hilbert's zeroes theorem). □

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Cinquième partie

Appendix

Chapitre 12

About generators and the Vandermonde function

The Vandermonde determinant plays a recurrent part in most of the investigations revolving around random matrices and non-colliding processes. In particular, it appeared at various places in this thesis. Here are recorded some of its well-known (and maybe not so well-known) remarkable properties related with diffusion generators.

For $x \in \mathbb{R}^m$, we set

$$h(x) = \prod_{1 \le i < j \le m} (x_i - x_j) \quad \text{and} \quad V(x) = \log |h(x)| = \sum_{i>j} \log |x_i - x_j|.$$

12.1 Properties of the Vandermonde function

Here are useful equalities

Proposition 12.1.1.

$$\Delta h = \sum_{i=1}^m \partial_i h = \sum_{i=1}^m x_i \partial_i^2 h = 0, \tag{12.1}$$

$$\sum_{i=1}^m x_i \, \partial_i h(x) = \frac{m(m-1)}{2} \, h(x), \tag{12.2}$$

$$\sum_{i=1}^m x_i^2 \, \partial_i^2 h(x) = \frac{m(m-1)(m-2)}{3} \, h(x). \tag{12.3}$$

Remark 12.1.1. *In fact,*

$$\sum_{i=1}^m x_i^p \, \partial_i^p h(x) = \frac{m(m-1)\dots(m-p)}{p+1} \, h(x). \tag{12.4}$$

This formula generalizes (12.2) and (12.3). Although it does not appear to be of any probabilistic interest for $p > 2$, it is a good-looking formula that we found worthy of being recorded here.

Proof. First, notice that, for integers p, q, $\mathbf{L} := \sum_{i=1}^m x_i^q \, \partial_i^p$ commutes with the action of $\mathfrak{S}_m$ by permutation of variables. Thus, $\mathbf{L}h$ is a skew-symmetric polynomial and, thus, divisible by h. If $q < p$, $\deg \mathbf{L}h < \deg h$, hence $\mathbf{L}h = 0$, which proves (12.1). If $q = p$, then $\deg \mathbf{L}h = \deg h$ and there exists a constant C such that $\mathbf{L}h = C\,h$. Let us see that $C = m(m-1)\dots(m-p)/(p+1)$, which will prove (12.4). Writing $x_{ij} := x_i - x_j$, remark that $\log h = \sum_{i<j} \log x_{ij}$, so that

$$\partial_i h = h \sum_{j\,(\neq i)} \frac{1}{x_{ij}}\,,$$

from which we easily prove by recursion that

$$\partial_i^p h = h \sum \frac{1}{x_{ij_1} x_{ij_2} \dots x_{ij_p}},$$

where the previous sum runs over $j_1, \dots, j_p$ all different from i and pairwise distinct. Thus, $\mathbf{L}h = h\,C(x)$ where

$$C(x) := \sum{}' \frac{x_{j_0}^p}{x_{j_0 j_1} x_{j_0 j_2} \dots x_{j_0 j_p}},$$

where $\sum'$ runs over $j_0, \dots, j_p$ pairwise distinct. But we have seen that $C(x) = C$ does not depend on x. So, we can take $x^{(\varepsilon)} = (\varepsilon^i)_{1 \le i \le k}$ and let ε go to 0 to evaluate $C = \lim C(x^{(\varepsilon)})$. Since $\varepsilon^{j_0} - \varepsilon^{j_l} \sim \pm \varepsilon^{\min(j_0, j_l)}$, we have

$$\frac{\varepsilon^{j_0 p}}{(\varepsilon^{j_0} - \varepsilon^{j_1}) \dots (\varepsilon^{j_0} - \varepsilon^{j_p})} \sim \varepsilon^{j_0 p - \sum_{l=1}^p \min(j_0, j_l)} = \begin{cases} 1 \text{ if } j_0 = \min(j_0, j_1, \dots, j_p), \\ o(1) \text{ otherwise.} \end{cases}$$

Thus,

$$\begin{aligned} C &= \sum{}' \mathbf{1}_{j_0 = \min(j_0, j_1, \dots, j_p)} \\ &= \frac{1}{p+1} \sum{}' \sum_{l=0}^{p} \mathbf{1}_{j_l = \min(j_0, j_1, \dots, j_p)} \\ &= \frac{1}{p+1} \sum{}' 1 = \frac{m(m-1)\dots(m-p)}{p+1}. \end{aligned}$$

□

12.2 h-transforms

Suppose $\mathbf{L}$ is a diffusion generator acting on functions on a subset $I_m \subset \mathbb{R}^m$, with reversible measure $d\mu_m$ and "carré du champ" $\mathbf{\Gamma}$. Then, if k is a positive function on I_m, the measure $k\,d\mu_m$ is reversible for the generator

$$\mathcal{L} := \mathbf{L} + \mathbf{\Gamma}(\log k, .).$$

In particular, if h is the Vandermonde function, the measure $|h|^\beta d\mu_m$ is reversible for

$$\mathcal{L}^{(\beta)} := \mathbf{L} + \beta\,\mathbf{\Gamma}(V, .).$$

Now, suppose L is the generator $a(x)\partial^2 + b(x)\partial$ acting on functions on some subset $I \subset \mathbb{R}$ with reversible measure μ. If $L_i = a(x_i)\partial_i^2 + b(x_i)\partial_i$ is the generator L acting on the ith coordinate, then $\mu_m := \mu^{\otimes m}$ is reversible for

$$\mathbf{L} = \sum_{i=1}^m L_i,$$

and $|h(x)|^\beta \mu^{\otimes m}(dx)$ is reversible for

$$\mathcal{L}^{(\beta)} := \mathbf{L} + \beta \sum_{i=1}^m a(x_i)\partial_i V\,\partial_i.$$

12.3 Orthogonal polynomials and eigenfunctions for $\beta = 2$

If $(f_n)_{n\in\mathbb{N}}$ is an orthonogonal basis for $\mathbb{L}^2(I,\mu)$, then it is shown in [BGU04] that one can construct an orthonogonal basis for $\mathbb{L}^2(I^m, h(x)^2\mu^{\otimes m}(dx))$ in the following way : if

$$\mathcal{P}_m = \{\lambda \in \mathbb{N}^m \mid \lambda_1 \geq \cdots \geq \lambda_m\}$$

and

$$\mathbf{F}_\lambda(x) = \frac{\det\left(f_{\lambda_j+k-j}(x_i)\right)_{1\leq i,j\leq m}}{h(x)},\ \lambda \in \mathcal{P}_m,$$

then $(\mathbf{F}_\lambda)_{\lambda\in\mathcal{P}_m}$ is an orthonogonal basis for $\mathbb{L}^2(I^m, h(x)^2\mu^{\otimes m}(dx))$. We show that, in some cases, if the $(f_n)_{n\in\mathbb{N}}$ are eigenfunctions for L, then the $(\mathbf{F}_\lambda)_{\lambda\in\mathcal{P}_m}$ are eigenfunctions for $\mathcal{L}^{(2)}$.

Proposition 12.3.1. *Suppose that $Lf_n = c_n f_n$ for functions f_n on $\mathbb{R}$ and constants c_n $(n \in \mathbb{N})$. Then,*

$$\mathcal{L}^{(2)}\mathbf{F}_\lambda = \left[\sum_j c_{\lambda_j+k-j} - \frac{\mathbf{L}h}{h}\right]\mathbf{F}_\lambda. \tag{12.5}$$

In particular, if L sends any polynomial to a polynomial of lower degree (ie a, resp. b, is a polynomial of degree at most 2, resp. 1), then $\mathbf{L}h/h$ is a constant and $\mathbf{F}_\lambda$ is an eigenfunction for $\mathcal{L}^{(2)}$.

Remark 12.3.1. *Those determinantal formulae are specific to this $\beta = 2$ case.*

Proof. Let us adopt the following notations : $l_i = (f_{\lambda_j+k-j}(x_i))_{1\le j\le m}$ and $A = (l_1, \dots, l_m)^\top$. Then, $\mathbf{F}_\lambda = \det A/h$ and

$$\partial_i \mathbf{F}_\lambda = \frac{\partial_i \det A}{h} - \partial_i V \mathbf{F}_\lambda,$$

$$\partial_i^2 \mathbf{F}_\lambda = \frac{\partial_i^2 \det A}{h} - 2\frac{\partial_i V \partial_i (\det A)}{h} - \partial_i^2 V \mathbf{F}_\lambda + (\partial_i V)^2 \mathbf{F}_\lambda,$$

so that

$$L_i \mathbf{F}_\lambda = \frac{\det B_i}{h} - (L_i V)\mathbf{F}_\lambda - 2\frac{a(x_i)\partial_i V \partial_i(\det A)}{h} + a(x_i)(\partial_i V)^2 \mathbf{F}_\lambda,$$

where $B_i = (l_1, \dots, l_{i-1}, \tilde{l}_i, l_{i+1}, \dots, l_m)^\top$ and

$$\tilde{l}_i = (L_i f_{\lambda_j+k-j}(x_i))_{1\le j\le m} = (c_{\lambda_j+k-j}\, f_{\lambda_j+k-j}(x_i))_{1\le j\le m}.$$

Hence,

$$L_i \mathbf{F}_\lambda + 2a(x_i)\partial_i V \partial_i \mathbf{F}_\lambda = \frac{\det B_i}{h} - (L_i V)\mathbf{F}_\lambda - a(x_i)(\partial_i V)^2 \mathbf{F}_\lambda.$$

Now, we claim that $\sum_i \det B_i = (\sum_j c_{\lambda_j+k-j}) \det A$. This comes from the following general fact that, if $c, V_1, \dots, V_m \in \mathbb{R}^m$ and $\tilde{V}_i = (c_j V_{ij})_{1\le j\le m}$, then

$$\sum_{i=1}^m \det(V_1, \dots, \tilde{V}_i, \dots, V_m) = \left(\sum_j c_j\right) \det(V_1, \dots, V_m). \tag{12.6}$$

Indeed, the left-hand side is multilinear and skew-symmetric in $V_1, \dots, V_m$, so it is a constant multiple of the determinant. The constant is found by choosing V_i to be the ith canonical basis vector of $\mathbb{R}^m$. In this case, $\tilde{V}_i = c_i V_i$, which finishes the proof of (12.6). Thus, we get

$$\mathcal{L}^{(2)} \mathbf{F}_\lambda = \left[\sum_j c_{\lambda_j+k-j} - \mathcal{L}^{(1)} V\right] \mathbf{F}_\lambda.$$

It remains to show that $\mathcal{L}^{(1)} V = \frac{\mathbf{L}h}{h}$. Recall that $\partial_i V = \partial_i h/h$ and $\partial_i^2 V = \partial_i^2 h/h - (\partial_i V)^2$. Therefore,

$$\mathcal{L}^{(1)} V = \sum_i \left(a(x_i)\frac{\partial_i^2 h}{h} - a(x_i)(\partial_i V)^2 + b(x_i)\frac{\partial_i h}{h} + a(x_i)(\partial_i V)^2 \right) = \frac{\mathbf{L}h}{h}.$$

As to the last assertion of the proposition, it simply comes from the fact that, since $\mathbf{L}$ commutes with the action of $\mathfrak{S}_m$ by permutation of the coordinates, $\mathbf{L}h$ is a skew-symmetric polynomial, hence a multiple of h. Degree considerations show $\mathbf{L}h/h$ is constant. □

We now review the three fundamental examples of diffusions stabilizing polynomials (see [Maz97]).

Example 12.3.1. ***The Jacobi case***
Here, $I = (0,1)$, $L = 2x(1-x)\partial^2 + (\alpha - (\alpha+\beta)x)\,\partial$, $\mu(dx) = x^{\alpha-1}(1-x)^{\beta-1}\,dx$ *and* $\mathbf{L}h = C_{k,\alpha,\beta}h$ *with* $C_{k,\alpha,\beta} = -m(m-1)\left(\frac{2(m-2)}{3} + \frac{\alpha+\beta}{2}\right)$.

Example 12.3.2. ***The Laguerre case***
Here, $I = (0,\infty)$, $L = 2x\partial^2 + (\alpha - x)\partial$, $\mu(dx) = x^{\alpha-1}e^{-x}\,dx$ *and* $\mathbf{L}h = -\frac{m(m-1)}{2}h$.

Example 12.3.3. ***The Hermite case***
Here, $I = \mathbb{R}$, $L = \partial^2 - x\partial$, $\mu(dx) = e^{-x^2/2}\,dx$ *and* $\mathbf{L}h = -\frac{m(m-1)}{2}h$.

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Chapitre 13

L'algorithme RSK

Nous présentons une brève description de l'algorithme RSK dans sa version donnée par Knuth ([Knu70]) pour les permutations généralisées. Une excellente référence est [Ful97]. Nous restons proches des notations de [Joh00]. En particulier, pour $N \in \mathbb{N}$, nous notons $[N] = \{1, \dots, N\}$.

13.1 Permutations généralisées et matrices entières

Definition 13.1.1. *Pour $M, N, l \in \mathbb{N}$, nous définissons $\mathfrak{S}^l_{M,N}$ l'ensemble des permutations généralisées de longueur l, avec antécédents dans $[M]$ et images dans $[N]$, c'est-à-dire l'ensemble des tableaux à deux lignes*

$$\sigma = \begin{pmatrix} i_1 & \dots & i_l \\ j_1 & \dots & j_l \end{pmatrix},$$

où $i_1, \dots, i_l \in [M]$, $j_1, \dots, j_l \in [N]$ et les colonnes $\binom{i_r}{j_r}$, $1 \leq r \leq l$ sont rangées par ordre lexicographique.

Lorsque $l = M$ et $i_r = r$, on dit que σ est un mot à l lettres prises dans l'alphabet $[N]$. Lorsque $l = M = N$, $i_r = r$ et $\{j_1, \dots, j_l\} = [N]$, σ est une permutation usuelle de $\mathfrak{S}_N$.

Definition 13.1.2. *Si $\sigma \in \mathfrak{S}^l_{M,N}$, on appelle $L(\sigma)$ la longueur de la plus longue sous-suite croissante de σ, c'est-à-dire le plus grand entier p tel qu'il existe $r_1 < \dots < r_p$ avec $j_{r_1} \leq \dots \leq j_{r_p}$.*

Nous allons voir que ces objets peuvent être décrits de manière équivalente par des matrices à coefficients entiers.

Definition 13.1.3. *$\mathcal{M}^l_{M,N}$ est l'ensemble des matrices $M \times N$ dont les coefficients sont dans $\mathbb{N}$ et ont une somme égale à l.*

Si $\sigma \in \mathfrak{S}^l_{M,N}$, on peut définir la matrice $f(\sigma) = A = (a_{ij})$ par

$$a_{ij} = \text{ nombre d'occurences de } \binom{M-i}{j} \text{ dans } \sigma.$$

f est une bijection de $\mathfrak{S}^l_{M,N}$ vers $\mathcal{M}^l_{M,N}$. Si l'on introduit l'ensemble Π_{MN} des chemins qui font des pas $(0,1)$ ou $(1,0)$ dans le rectangle $[M] \times [N]$ et

$$G(A) = \max\{ \sum_{(i,j)\in\pi} a_{ij} \; ; \; \pi \in \Pi_{MN}\}$$

alors on a :

$$L(\sigma) = G(f(\sigma)).$$

Exemple 13.1.1. *Si $M = 4$, $N = 3$, $l = 8$ et*

$$\sigma = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 4 \\ \underline{1} & \underline{2} & 3 & \underline{2} & \underline{2} & 1 & 3 & 1 & \underline{2} & \underline{3} \end{pmatrix},$$

on a $L(\sigma) = 6$, une sous-suite croissante de longueur 6 étant soulignée. D'autre part,

$$A = f(\sigma) = \begin{pmatrix} 1 & 1 & \underline{1} \\ 1 & \underline{0} & \underline{1} \\ 0 & \underline{2} & 0 \\ \underline{1} & \underline{1} & 1 \end{pmatrix},$$

et $G(A) = 6$ (on a souligné le chemin qui correspond à la sous-suite croissante de σ et qui réalise le maximum).

13.2 Partitions, tableaux et fonctions de Schur

Definition 13.2.1. *On appelle partition $\lambda = (\lambda_i)$ toute suite décroissante d'entiers nuls à partir d'un certain rang. On écrira $\lambda \vdash l$ lorsque $|\lambda| = \sum_i \lambda_i = l$. A toute partition λ, on associe un diagramme de forme λ : c'est le tableau justifié à gauche dont la i-ème ligne en partant du haut possède λ_i cases.*

Definition 13.2.2. *Si λ est une partition, un tableau semi-standard P à coefficients dans $[M]$ et de forme λ est un remplissage du diagramme de λ par des entiers de $[M]$ qui soit croissant, au sens large, de gauche à droite le long des lignes et de haut en bas le long des colonnes. On note $\mathcal{T}_M$ l'ensemble de tels tableaux.*

Exemple 13.2.1. *Voici un exemple de tableau semi-standard de forme* $(7, 4, 1) \vdash 12$ *et à coefficients dans* $[4]$ *:*

$$P = \begin{matrix} 1 & 1 & 1 & 2 & 2 & 3 & 4 \\ 2 & 2 & 3 & 3 \\ 4 \end{matrix}$$

Definition 13.2.3. *Si* P *est un tableau semi-standard à coefficients dans* $[M]$*, son « type » est le vecteur* $\alpha \in \mathbb{N}^M$ *où* α_i *est le nombre d'occurrences de* i *dans* P*.*

Exemple 13.2.2. *Le type du tableau* P *de l'exemple précédent est* $(3, 4, 3, 2)$*.*

Definition 13.2.4. *On dira qu'un tableau semi-standard à coefficients dans* $[M]$ *et de forme* λ *est « standard » s'il possède* M *cases, ie si* $\lambda \vdash M$*. On note* $\mathcal{S}_M$ *l'ensemble de tels tableaux.*

Exemple 13.2.3. *Voici un exemple de tableau standard de* $\mathcal{S}_8$*, de forme* $(4, 2, 2) \vdash 8$ *:*

$$\begin{matrix} 1 & 3 & 5 & 7 \\ 2 & 4 \\ 6 & 8 \end{matrix}$$

On notera f^λ le nombre de tableaux standards de forme $\lambda \vdash M$. C'est aussi la dimension de la représentation irréductible de $\mathfrak{S}_M$ indexée par λ. Deux formules existent pour f^λ. D'abord, la formule du crochet :

$$f^\lambda = \prod_c h_c,$$

où le produit est pris sur l'ensemble des cases c du diagramme de λ et h_c est la longueur du crochet associé à c, c'est-à-dire le nombre de cases à droites de c dans sa ligne ou en dessous de c dans sa colonne (c incluse). Cette formule peut aussi être écrite :

$$f^\lambda = M! \frac{\prod_{i<j}(l_i - l_j)}{\prod_i l_i},$$

où $l_i = \lambda_i + k - i$ si $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k \geq \lambda_{k+1} = 0)$.

Definition 13.2.5. *Si* λ *est la partition* $\lambda_1 \geq \cdots \geq \lambda_k \geq \lambda_{k+1} = 0$ *et* $x_1, \ldots, x_k$ *des variables, on définit la fonction de Schur associée à* λ *:*

$$s_\lambda(x) = \frac{\det \left(x_i^{\lambda_j + k - j} \right)_{1 \leq i,j \leq k}}{\prod_{i<j}(x_i - x_j)}.$$

s_λ est un polynôme symétrique en les $x_1, \dots, x_k$ qui admet la définition combinatoire équivalente :

$$s_\lambda(x) = \sum_P x^{\mathrm{type}(P)},$$

où la somme est prise sur les tableaux semi-standards P de forme λ et à coefficients dans $[k]$ et $x^\alpha = \prod_i x_i^{\alpha_i}$ pour $\alpha \in \mathbb{N}^k$.

Lorsquee λ parcourt l'ensemble des partitions de M avec au plus k composantes non-nulles, les s_λ forment une base de l'espace des polynômes symétriques homogènes de degré M, en k variables. Ils jouent un rôle fondamental dans les représentations de $\mathfrak{S}_M$ et de $GL(M, \mathbb{C})$.

Definition 13.2.6. *Si $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k \geq \lambda_{k+1} = 0)$ et $\alpha \in \mathbb{N}^k$, on désigne par $K_{\lambda\alpha}$ le nombre (dit de Kostka) de tableaux semi-standards de forme λ et de type α. On a ainsi :*

$$s_\lambda(x) = \sum_\alpha K_{\lambda\alpha} x^\alpha.$$

13.3 Correspondance RSK

Nous définissons maintenant comment insérer une paire $\binom{i}{j}$ dans un couple (P, Q) de tableaux semi-standards de même forme pour obtenir un autre couple (P', Q') de tableaux de même forme. Si j est plus grand (au sens large) que tous les nombres de la $1^{\text{ère}}$ ligne de P alors P' est obtenu en rajoutant une case étiquetée j à l'extrêmité droite de la $1^{\text{ère}}$ ligne de P. Sinon, on remplace par j le 1^{er} nombre, disons k, en partant de la gauche qui soit $> j$. Ensuite, on répète l'opération pour insérer k dans le tableau P privé de sa $1^{\text{ère}}$ ligne, jusqu'à arriver au nouveau tableau P'. Enfin, on obtiendra Q' en ajoutant une case à Q pour qu'il soit de même forme que P' et on remplira cette case de l'entier i.

La correspondance RSK associe à $\sigma \in \mathfrak{S}^l_{MN}$ la paire de tableaux obtenus par insertions successives des colonnes de σ (prises de gauche à droite) à partir du couple de tableaux vides. Cette corespondance est une bijection de $\mathfrak{S}^l_{MN}$ vers les couples $(P, Q) \in \mathcal{T}_N \times \mathcal{T}_M$ de même forme $\lambda \vdash l$. Si l'on restreint l'espace de départ aux mots à $M = l$ lettres (resp. aux permutations usuelles de $\mathfrak{S}_N$, $M = N = l$), alors $(P, Q) \in \mathcal{T}_N \times \mathcal{S}_M$ (resp. $(P, Q) \in \mathcal{S}_M \times \mathcal{S}_M$) et l'on conserve le caractère bijectif.

La correspondance RSK rend facile la lecture de la longueur de la plus longue sous-suite croissante. Précisément, si σ a pour image (P, Q), de forme λ, alors $L(\sigma)$ est égale à la taille de la $1^{\text{ère}}$ ligne de P et Q, $L(\sigma) = \lambda_1$. En fait, on peut étendre cette remarquable propriété de la façon suivante : la somme des longueurs des p premières lignes de λ est égale à la plus grande somme des longueurs de p sous-suites croissantes de σ à supports disjoints. On peut encore traduire ceci du côté des matrices entières : si

$$G_k(A) = \max \sum_{(i,j)\in\pi_1\cup\ldots\cup\pi_p} a_{ij},$$

où le max est pris sur l'ensemble des chemins à supports disjoints $\pi_1, \ldots, \pi_p \in \Pi_{MN}$, alors

$$G_k(f(\sigma)) = \lambda_1 + \cdots + \lambda_p.$$

C'est cette propriété que nous utilisons au chapitre 5.

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Résumé

Cette thèse se divise en trois grandes parties dont les préoccupations sont assez distinctes mais qui gravitent toutes autour de la théorie des matrices aléatoires. Une première partie examine certains des liens qui existent entre les valeurs propres de matrices aléatoires gausssiennes, les processus sans collision et la correspondance de Robinson-Schensted-Knuth. Une deuxième partie est consacrée à des extensions aux matrices symétriques de diffusions classiques en dimension un, les carrés de Bessel et les processus de Jacobi. Dans une troisième partie, nous étudions la distribution du temps de sortie du mouvement brownien de certaines régions de l'espace euclidien qui sont des domaines fondamentaux associés à des groupes de réflexions, finis ou affines.

Abstract

The following thesis falls into three parts. Although they are all closely related to random matrix theory, each of these possesses its own particular concern. The first part deals with some of the existing links between eigenvalues of Gausssian random matrices, non-colliding processes and the Robinson-Schensted-Knuth correspondence. The second part tackles the subject of extensions to symmetric matrices of some classical one-dimensional diffusion processes, namely the Bessel squared processes and the Jacobi processes. Then, the third part hinges round the exit time of Brownian motion from regions which are the fundamental domains associated with finite or affine reflection groups in Euclidean space.

Discipline : Mathématiques, Probabilités.
Mots Clés : Matrices aléatoires - Conditionnement de Doob - Correspondance RSK - Processus de Wishart - Processus de Jacobi - Groupe de réflexions - Chambre de Weyl - Alcôve - Système de racines.

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