Heat flows, geometric and functional inequalities

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Abstract. Heat flow and semigroup interpolations have developed over the years as a major tool for proving geometric and functional inequalities. Main illustrations presented here range over logarithmic Sobolev inequalities, heat kernel bounds, isoperimetric-type comparison theorems, Brascamp-Lieb inequalities and noise stability. Transportation cost inequalities from optimal mass transport are also part of the picture as consequences of new Harnack-type inequalities. The geometric analysis involves Ricci curvature lower bounds via, as a cornerstone, equivalent gradient bounds on the diffusion semigroups. Most of the results presented here are joint with D. Bakry.

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1. Introduction

The last decades have seen important developments of heat flow methods towards a variety of geometric and functional inequalities. Heat flow or semigroup interpolation is a classical analytic tool, going back at least as far as the so-called Duhamel formula, which has been widely used in a number of settings. The modern era, starting in the eighties, emphasized dynamical proofs of Euclidean and Riemannian functional and geometric inequalities under curvature bounds, as put forward in the early contribution [7] by D. Bakry and M. Émery (see also [6]) in the context of hypercontractivity and logarithmic Sobolev inequalities for diffusion operators. The picture encircles today inequalities relevant to heat kernel and gradient bounds, geometric comparison theorems, Sobolev embeddings, convergence to equilibrium, optimal transport, isoperimetry and measure concentration (as illustrated e.g. in [9]). This text surveys some of these achievements with a particular focus on Sobolev-type, isoperimetric and multilinear inequalities, and noise stability.

Section 2 is a first illustration of the power of heat flow monotonicity towards logarithmic Sobolev inequalities, including in the same picture the classical parabolic Li-Yau inequality. Section 3 presents more refined isoperimetric-type inequalities, leading to comparison of the isoperimetric profile of (infinite-dimensional) curved models with the Gaussian profile. Harnack inequalities drawn from heat flow provide links with optimal mass transport and transportation cost inequalities illustrated in Section 4. The classical Brascamp-Lieb inequalities for multilinear integrals of products of functions form another important family of functional and geometric inequalities. While classically analyzed as isoperimetric inequalities by rear-

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rangement methods, recent developments using semigroup interpolation shed new light on their structure and extremizers. The last Section 6 presents some recent progress connecting even further Brascamp-Lieb and isoperimetric inequalities via (Gaussian) noise stability.

One natural framework of investigation is Euclidean space $\mathbb{R}^n$ or a (weighted) Riemannian manifold in which case (Ricci) curvature lower-bounds enter into the picture. Based upon the early achievement [7] (see [6]), the more general setting of Markov Triples (E, μ, Γ) allows us to develop semigroup interpolation in a wide context, concentrating on the basic algebraic Γ-calculus underlying many of the heat flow arguments. The iterated carré du champ operator Γ_2 provides here the natural functional interpretation of the geometric Bochner formula and of curvature-dimension conditions. The recent book [9] gives an overview of semigroup methods in the context of Markov Triples and their applications to functional and geometric inequalities. Most of the results emphasized here are developed in this monograph [9] written jointly with D. Bakry and I. Gentil, to which we refer for further motivation and illustrations.

2. Logarithmic Sobolev and parabolic Li-Yau inequalities

The celebrated logarithmic Sobolev inequality of L. Gross [47], comparing entropy and Fisher information, is one prototypical example of functional inequality which may be investigated by heat flow methods. Let $d\gamma(x) = (2\pi)^{-n/2} e^{-|x|^2/2} dx$ be the standard Gaussian measure on the Borel sets of $\mathbb{R}^n$.

Theorem 2.1 (Gross' logarithmic Sobolev inequality). *For any smooth positive function $f : \mathbb{R}^n \to \mathbb{R}$ such that $\int_{\mathbb{R}^n} f d\gamma = 1$,*

$$\int_{\mathbb{R}^n} f \log f d\gamma \;\leq\; \frac{1}{2} \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f}\, d\gamma.$$

Logarithmic Sobolev inequalities are infinite-dimensional counterparts of the classical Sobolev inequalities, and characterize smoothing properties in the form of hypercontractivity. They prove central in a variety of contexts, including entropic convergence to equilibrium of solutions of evolutionary partial differential equations and of Markov chains and models from statistical mechanics, infinite-dimensional Gaussian analysis and measure concentration (cf. e.g. [6, 9, 52, 76] and the references therein).

While there are numerous different proofs of Gross' logarithmic Sobolev inequality, the perhaps simplest one, put forward by D. Bakry and M. Émery [7] in the mid-eighties, uses semigroup interpolation. Indeed, consider the basic (convolution) heat semigroup $(P_t)_{t\geq 0}$ on $\mathbb{R}^n$ given on suitable functions $f : \mathbb{R}^n \to \mathbb{R}$ by

$$P_t f(x) \;=\; \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} f(y)\, e^{-|x-y|^2/4t} dy, \quad t > 0,\; x \in \mathbb{R}^n.$$

Given the initial condition f, $u = u(t,x) = P_t f(x)$ solves the heat equation $\partial_t u = \Delta u$ with thus $u(0,x) = f(x)$.

Towards the logarithmic Sobolev inequality of Theorem 2.1, consider the entropy of a positive smooth function f on $\mathbb{R}^n$ along the semigroup $(P_t)_{t\geq 0}$ given by, at any $t > 0$ and any point x (omitted below),

$$P_t(f \log f) - P_t f \log P_t f.$$

The heat flow interpolation then amounts to

$$P_t(f\log f) - P_tf\log P_tf \;=\; \int_0^t \frac{d}{ds}\,P_s(P_{t-s}f\log P_{t-s}f)ds.$$

By the heat equation and the chain rule formula, both in time and space, for $s < t$,

$$\frac{d}{ds}P_s(P_{t-s}f\log P_{t-s}f) \;=\; P_s\bigg(\frac{|\nabla P_{t-s}f|^2}{P_{t-s}f}\bigg) \;=\; \phi(s).$$

As gradient and semigroup commute $\nabla P_uf = P_u(\nabla f)$, for every $u > 0$, by the Jensen and Cauchy-Schwarz inequalities (along the Markov operator P_u),

$$|\nabla P_uf|^2 \;\leq\; \big[P_u\big(|\nabla f|\big)\big]^2 \leq P_u\bigg(\frac{|\nabla f|^2}{f}\bigg)P_uf. \tag{2.1}$$

With $u = t - s$, it follows that

$$\phi(s) \;\leq\; P_s\,P_{t-s}\bigg(\frac{|\nabla f|^2}{f}\bigg) \;=\; \phi(t)$$

so that

$$P_t(f\log f) - P_tf\log P_tf \;=\; \int_0^t \phi(s)ds \;\leq\; t\phi(t) \;=\; t\,P_t\bigg(\frac{|\nabla f|^2}{f}\bigg). \tag{2.2}$$

When $t = \frac{1}{2}$, this heat kernel (that is, along the distribution of P_t) inequality is precisely, by homogeneity, the Gross logarithmic Sobolev inequality of Theorem 2.1.

It is a significant observation that the preceding argument may be reversed. Indeed, with $u = s$ and f replaced by $P_{t-s}f$, it holds similarly that $\phi(s) \geq \phi(0)$ so that

$$P_t(f\log f) - P_tf\log P_tf \;\geq\; t\,\phi(0) = t\,\frac{|\nabla P_tf|^2}{P_tf}\,. \tag{2.3}$$

This reverse inequality is a relevant property leading to gradient bounds (see below and [9]).

The preceding analysis actually shows that the map

$$s\in[0,t] \;\mapsto\; \phi(s) \;=\; P_s\bigg(\frac{|\nabla P_{t-s}f|^2}{P_{t-s}f}\bigg) \;=\; P_s\big(P_{t-s}f|\nabla\log P_{t-s}f|^2\big)$$

is increasing. Following [7], an alternative approach to this fact is of course to take derivative (that is, the second derivative of entropy) yielding

$$\phi'(s) = 2\,P_s\big(P_{t-s}f\,\Gamma_2\,(\log P_{t-s}f)\big)$$

where the Γ_2 operator is given, on any smooth function h on $\mathbb{R}^n$, by

$$\Gamma_2(h) \;=\; \frac{1}{2}\,\Delta\big(|\nabla h|^2\big) - \nabla h\cdot\nabla(\Delta h) \;=\; \big|\mathrm{Hess}(h)\big|^2 \;\geq\; 0.$$

Hence $\phi'(s) \geq 0$ and ϕ is increasing.

The same formalism also works in an n-dimensional Riemannian manifold (M, g) along the heat semigroup $(P_t)_{t\geq 0}$ with Laplace-Beltrami operator Δ as infinitesimal generator. In this case, by the classical Bochner identity, the Γ_2 operator takes the form

$$\Gamma_2(h) \,=\, \mathrm{Ric}_g(\nabla h, \nabla h) + \big|\mathrm{Hess}(h)\big|^2$$

where Ric_g denotes the Ricci tensor of the metric g. Whenever (M, g) has non-negative Ricci curvature, we have similarly that $\phi' \geq 0$ yielding the preceding heat kernel inequalities (2.2) and (2.3) in this more general context.

Actually, under $\mathrm{Ric}_g \geq 0$, by the trace inequality,

$$\Gamma_2(h) \,=\, \mathrm{Ric}_g(\nabla h, \nabla h) + \big|\mathrm{Hess}(h)\big|^2 \geq \big|\mathrm{Hess}(h)\big|^2 \geq \frac{1}{n}\,(\Delta h)^2. \tag{2.4}$$

Thus

$$\phi'(s) \,\geq\, \frac{2}{n}\,P_s\big(P_{t-s}f\,[\Delta \log P_{t-s}f]^2\big)$$

retaining dimensional information. A somewhat more involved integration then yields a strengthened dimensional logarithmic Sobolev inequality

$$P_t(f\log f) - P_tf\log P_tf \,\leq\, t\,\Delta P_tf + \frac{n}{2}\,P_tf\log\left(1 - \frac{2t}{n}\,\frac{P_t(f\Delta\log f)}{P_tf}\right)$$

(for f a positive smooth function on M). Of more interest is actually the reverse form, analogue of (2.3),

$$P_t(f\log f) - P_tf\log P_tf \,\geq\, t\,\Delta P_tf - \frac{n}{2}\,P_tf\log\Big(1 + \frac{2t}{n}\,\Delta(\log P_tf)\Big).$$

The latter entails implicitly (and explicitly from the proof) that $1 + \frac{2t}{n}\,\Delta(\log P_tf) > 0$, or equivalently the famous Li-Yau parabolic inequality [55], initially established by the maximum principle and embedded here in a heat flow argument [11].

Theorem 2.2 (Li-Yau parabolic inequality). *For any (smooth) positive function f on a Riemannian manifold (M, g) with non-negative Ricci curvature,*

$$\frac{|\nabla P_tf|^2}{(P_tf)^2} - \frac{\Delta P_tf}{P_tf} \,\leq\, \frac{n}{2t}\,.$$

The Li-Yau parabolic inequality has numerous important applications (cf. [39, 55]), in particular to Harnack inequalities of the type

$$P_tf(x) \,\leq\, P_{t+s}f(y)\Big(\frac{t+s}{t}\Big)^{n/2}e^{d(x,y)^2/4s} \tag{2.5}$$

for $f : M \to \mathbb{R}$ positive and $t, s > 0$, where $d(x, y)$ is the Riemannian distance between $x, y \in M$.

Parallelisms with the Li-Yau gradient estimates and the Perelman F and W entropy functionals (see [65]) are mentioned in the recent contribution [36] of T. Colding where further monotonicity formulas for Ricci curvature with accompanying rigidity theorems are developed (see also [37]).

The preceding heat flow monotonicity principle yielding both logarithmic Sobolev and Li-Yau inequalities may be developed similarly in the extended setting of a weighted n-dimensional Riemannian manifold (M,g) with a weighted measure $d\mu = e^{-V}dx$, where V is a smooth potential on M, invariant and symmetric with respect to the operator

$$\mathrm{L} = \Delta - \nabla V \cdot \nabla$$

for which the Ricci tensor is extended into the so-called Bakry-Émery tensor $\mathrm{Ric}_g + \mathrm{Hess}(V)$. On the basis of Bochner's identity and (2.4), curvature-dimension $CD(K,N)$ conditions

$$\Gamma_2(h) \,=\, \big[\mathrm{Ric}_g + \mathrm{Hess}(V)\big](\nabla h, \nabla h) + \big|\mathrm{Hess}(h)\big|^2 \,\geq\, K|\nabla h|^2 + \frac{1}{N}\,(\mathrm{L}h)^2 \tag{2.6}$$

for every smooth h on (M,g), where $K \in \mathbb{R}$ and $N \geq n$ (not necessarily the topological dimension), encode Ricci curvature lower bounds and dimension. The condition (2.6) is inspired by Lichnerowicz' eigenvalue lower bound [9, 45, 56]. Similar functional and heat kernel inequalities are then achieved under $CD(0,N)$ and also $CD(K,N)$.

The results furthermore extend to the general setting of abstract Markov diffusion operators leading to the concept of Markov Triple [6, 9]. A Markov (diffusion) Triple (E,μ,Γ) consists of a state space E equipped with a diffusion semigroup $(P_t)_{t\geq 0}$ with infinitesimal generator L, carré du champ operator Γ and invariant and reversible σ-finite measure μ. The generator L and the carré du champ operator Γ are intrinsically related by the formula

$$\Gamma(f,g) = \frac{1}{2}\big[\mathrm{L}(fg) - f\,\mathrm{L}g - g\,\mathrm{L}f\big]$$

for functions f,g belonging to a suitable algebra $\mathcal{A}$ of functions (corresponding to smooth functions in a Riemannian setting). The state space E may be endowed with an intrinsic distance d for which Lipschitz functions f are such that $\Gamma(f)$ is bounded (μ-almost everywhere). In the (weighted) Riemannian context, L is the Laplace operator Δ with drift $-\nabla V \cdot \nabla$ with respect to the weighted measure $d\mu = e^{-V}dx$, $\Gamma(f,f) = |\nabla f|^2$ for smooth functions, and d corresponds to the Riemannian metric.

In the Markov Triple setting, the abstract curvature condition

$$CD(K,N),\ K \in \mathbb{R},\ N \geq 1,$$

mimicking (2.6), takes the form

$$\Gamma_2(h) \,\geq\, K\,\Gamma(h) + \frac{1}{N}\,(\mathrm{L}h)^2, \quad h \in \mathcal{A}, \tag{2.7}$$

(with the shorthand notation $\Gamma(h) = \Gamma(h,h)$, $\Gamma_2(h) = \Gamma_2(h,h)$) where the Bakry-Émery Γ_2 operator, going back to [7] (see [6, 9]), is defined from Γ by

$$\Gamma_2(f,g) \,=\, \frac{1}{2}\big[\mathrm{L}\big(\Gamma(f,g)\big) - \Gamma(f,\mathrm{L}g) - \Gamma(g,\mathrm{L}f)\big], \quad (f,g) \in \mathcal{A}\times\mathcal{A}.$$

As a major property emphasized by D. Bakry [5, 6, 9], the curvature condition $CD(K,\infty)$ is translated equivalently into gradient bounds, allowing in particular, along (2.1), for the preceding semigroup interpolation arguments and heat kernel inequalities.

Theorem 2.3 (Gradient bound). *The curvature condition $CD(K,\infty)$ holds true if and only if for any $t \geq 0$ and any $f \in \mathcal{A}$,*

$$\sqrt{\Gamma(P_t f)} \,\leq\, e^{-Kt} P_t\big(\sqrt{\Gamma(f)}\,\big).$$

The curvature-dimension condition $CD(K,N)$ leads on the other hand to dimensional gradient bounds of the type [11, 79]

$$\Gamma(P_t f) \,\leq\, e^{-2Kt} P_t\big(\Gamma(f)\big) - \frac{1 - e^{-2Kt}}{KN}\,(\mathrm{L}P_t f)^2 \tag{2.8}$$

which are central in the comparison with alternative curvature-dimension conditions from optimal transport (see Section 4).

3. Isoperimetric-type inequalities

More refined isoperimetric statements may be achieved by the preceding semigroup interpolation arguments. One prototypical result in this direction is a comparison theorem between the isoperimetric profile of a curved infinite-dimensional diffusion operator (in the preceding sense) and the Gaussian profile.

Denote by $\mathrm{I} : [0,1] \to \mathbb{R}_+$ the Gaussian isoperimetric function defined by $\mathrm{I} = \varphi \circ \Phi^{-1}$ where

$$\Phi(x) \,=\, \int_{-\infty}^{x} e^{-u^2/2} \frac{du}{\sqrt{2\pi}}, \quad x \in \mathbb{R},$$

is the distribution function of a standard normal and $\varphi = \Phi'$ its density. The following theorem ([10]) is presented in the general context of a Markov Triple (E,μ,Γ) (with underlying algebra of smooth functions $\mathcal{A}$), covering in particular the setting of weighted Riemannian manifolds.

Theorem 3.1 (Gaussian isoperimetry for heat kernel measure). *Let (E,μ,Γ) be a Markov Triple satisfying the curvature condition $CD(K,\infty)$ for some $K \in \mathbb{R}$. For every function f in $\mathcal{A}$ with values in $[0,1]$ and every $t \geq 0$,*

$$\mathrm{I}(P_t f) \,\leq\, P_t\Big(\sqrt{\mathrm{I}^2(f) + K(t)\,\Gamma(f)}\,\Big)$$

where $K(t) = \frac{1}{K}\,(1 - e^{-2Kt})$ ($= 2t$ if $K = 0$).

For the example of the standard heat semigroup on $\mathbb{R}^n$ with $t = \frac{1}{2}$, Theorem 3.1 yields that for any smooth function $f : \mathbb{R}^n \to [0,1]$,

$$\mathrm{I}\bigg(\int_{\mathbb{R}^n} f d\gamma\bigg) \,\leq\, \int_{\mathbb{R}^n} \sqrt{\mathrm{I}^2(f) + |\nabla f|^2}\, d\gamma. \tag{3.1}$$

This inequality applied to εf as $\varepsilon \to 0$, together with the asymptotics $\mathrm{I}(v) \sim v\sqrt{2 \log \frac{1}{v}}$ as $v \to 0$, strengthens the logarithmic Sobolev inequality of Theorem 2.1. A smooth approximation f of the characteristic function of a Borel set A in $\mathbb{R}^n$ ensures that

$$\mathrm{I}\big(\gamma(A)\big) \,\leq\, \gamma^+(A) \,=\, \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon}\big[\gamma(A_\varepsilon) - \gamma(A)\big] \tag{3.2}$$

where the right-hand side defines the Minkowski content (surface measure) of A (where, for $\varepsilon > 0$, $A_\varepsilon = \{x \in E\,; d(x,A) \leq \varepsilon\}$). This inequality (3.2) exactly expresses the isoperimetric problem for the Gaussian measure γ on $\mathbb{R}^n$ for which half-spaces

$$H = \{x \in \mathbb{R}^n\,; x \cdot u \leq a\},$$

where u is a unit vector and $a \in \mathbb{R}$, achieve the minimal surface measure at fixed measure. Indeed, if a is chosen such that $\gamma(A) = \Phi(a)$, then $\gamma(A) = \gamma(H)$ and

$$\gamma^+(H) \,=\, \varphi(a) \,=\, \mathrm{I}\big(\Phi(a)\big) \,\leq\, \gamma^+(A).$$

The Gaussian isoperimetric inequality (3.2) goes back to V. Sudakov and B. Tsirel'son [73] and C. Borell [23] relying on the isoperimetric inequality on spheres and a limiting argument. The functional form (3.1) has been put forward by S. Bobkov [21] (see also earlier [42] within Gaussian symmetrization [41]).

The content of Theorem 3.1 is therefore that the isoperimetric profile of the heat kernel measures (of a positively curved diffusion semigroup) is bounded from below, up to a constant, by the isoperimetric profile I of the standard Gaussian measure (in dimension one actually). In particular, if $d\mu = e^{-V}dx$ is a probability measure on $\mathbb{R}^n$ with smooth potential V such that $\mathrm{Hess}(V) \geq K\,\mathrm{Id}$ for some $K > 0$ as symmetric matrices, the curvature condition $CD(K,\infty)$ holds and one may let t tend to ∞ in Theorem 3.1 to see that the isoperimetric profile of μ,

$$\mathrm{I}_\mu(v) \,=\, \inf\big\{\mu^+(A)\,; \mu(A) = v\big\}, \quad v \in (0,1),$$

is bounded from below by $\sqrt{K}\,\mathrm{I}$. In this sense, Theorem 3.1 is the infinite-dimensional analogue of the Lévy-Gromov isoperimetric comparison theorem [59] bounding from below the isoperimetric profile of the (normalized) Riemannian measure of an n-dimensional Riemannian manifold with Ricci curvature bounded from below by $n-1$, by the isoperimetric profile of the standard n-sphere. A heat flow proof of this result is yet to be found. For far-reaching geometric generalizations of the Lévy-Gromov theorem, see [58].

In the same spirit as (2.3), reverse forms of the isoperimetric heat kernel inequalities of Theorem 3.1 are also available. Under the curvature condition $CD(K,\infty)$ for some $K \in \mathbb{R}$, for every function f in $\mathcal{A}$ with values in $[0,1]$ and every $t > 0$,

$$\big[\mathrm{I}(P_t f)\big]^2 - \big[P_t\big(\mathrm{I}(f)\big)\big]^2 \,\geq\, \frac{1}{K}\,(e^{2Kt} - 1)\,\Gamma(P_t f).$$

These (sharp) gradient bounds may then be used to prove new isoperimetric-type Harnack inequalities [8].

Theorem 3.2 (Isoperimetric Harnack inequality). *Let (E,μ,Γ) be a Markov Triple satisfying the curvature condition $CD(K,\infty)$ for some $K \in \mathbb{R}$. For every measurable set A in E, every $t \geq 0$ and every $x, y \in E$ such that $d(x,y) > 0$,*

$$P_t(\mathbb{1}_A)(x) \,\leq\, P_t\big(\mathbb{1}_{A_{d_t}}\big)(y)$$

where $d_t = e^{-Kt}d(x,y)$. In particular, when $K = 0$,

$$P_t(\mathbb{1}_A)(x) \,\leq\, P_t\big(\mathbb{1}_{A_{d(x,y)}}\big)(y).$$

Under the curvature condition $CD(K,\infty)$, it is not possible to expect standard (dimensional) Harnack inequalities of the type (2.5). However, the set inequalities of Theorem 3.2 yield infinite-dimensional analogues first obtained by F.-Y. Wang in [77, 78]. For simplicity, the $CD(0,\infty)$ version states the following.

Theorem 3.3 (Wang's Harnack inequality). *In the preceding context, under the curvature condition $CD(0,\infty)$, for every positive (measurable) function f on E, every $t>0$, every $\alpha>1$, and every $x,y\in E$,*

$$\big(P_tf(x)\big)^\alpha \,\le\, P_t(f^\alpha)(y)\,e^{\alpha d(x,y)^2/4(\alpha-1)t}. \tag{3.3}$$

In the limit as $\alpha\to\infty$, the latter turns into a log-Harnack inequality

$$P_t(\log f)(x) \,\le\, \log P_tf(y)+\frac{d(x,y)^2}{4t} \tag{3.4}$$

for f positive.

4. Transportation cost inequalities

Heat flow methods have developed simultaneously in the context of transportation cost inequalities which are parts of the main recent achievements in optimal transport (cf. [76]). In particular, they may be used to reach the famous HWI inequality of F. Otto and C. Villani [67] connecting (Boltzmann H-) Entropy, Wasserstein distance (W) and Fisher Information (I).

The HWI inequality covers at the same time logarithmic Sobolev and transportation cost inequalities (in the form of the Talagrand quadratic transportation cost inequality [74]). For simplicity, we deal here with a weighted Riemannian manifold (M,g) with weighted probability measure $d\mu=e^{-V}dx$, and restrict ourselves to the non-negative curvature condition $CD(0,\infty)$. The (quadratic) Wasserstein distance $\mathrm{W}_2(\nu,\mu)$ between two probability measures μ and ν on M is defined by

$$\mathrm{W}_2(\nu,\mu) \,=\, \bigg(\int_{M\times M} d(x,y)^2 d\pi(x,y)\bigg)^{1/2}$$

where the infimum is taken over all couplings π with respective marginals ν and μ (cf. [75, 76]).

Theorem 4.1 (Otto-Villani HWI inequality). *Under the curvature condition $CD(0,\infty)$, for any smooth positive function $f: M\to\mathbb{R}$ with $\int_M fd\mu=1$,*

$$\int_M f\log f d\mu \,\le\, \mathrm{W}_2(\nu,\mu)\bigg(\int_M \frac{|\nabla f|^2}{f}\,d\mu\bigg)^{1/2}$$

where $d\nu=fd\mu$.

The starting point towards a semigroup proof (first emphasized in [22]) is the log-Harnack inequality (3.4) which may be translated equivalently as

$$P_t(\log f)\le Q_{2t}(\log P_tf) \tag{4.1}$$

where $(Q_s)_{s>0}$ is the Hopf-Lax infimum-convolution semigroup

$$Q_s\varphi(x) \,=\, \inf_{y\in M}\left[\varphi(y) + \frac{d(x,y)^2}{2s}\right], \quad s>0,\ x\in M.$$

This convolution semigroup is closely related to the Wasserstein distance W_2 via the Kantorovich dual description

$$\frac{1}{2}\,\mathrm{W}_2(\nu,\mu)^2 \,=\, \sup\left[\int_M Q_1\varphi\, d\nu - \int_M \varphi\, d\mu\right] \tag{4.2}$$

where the supremum runs over all bounded continuous functions $\varphi : M \to \mathbb{R}$ (cf. [75, 76]).

Given $f > 0$ a (smooth bounded) probability density with respect to μ and $d\nu = f d\mu$, simple symmetry and scaling properties on the basis of (4.1) and (4.2) yield that

$$\int_M P_t f \log P_t f d\mu \,\leq\, \frac{1}{4t}\,\mathrm{W}_2^2(\nu,\mu). \tag{4.3}$$

The heat flow interpolation scheme illustrated in Section 2 expresses on the other hand that for every $t > 0$,

$$\int_M f\log f d\mu \,\leq\, \int_M P_t f\log P_t f d\mu + t\int_M \frac{|\nabla f|^2}{f}\, d\mu.$$

Together with (4.3), optimization in $t > 0$ yields the HWI inequality. Similar arguments may be developed under $CD(K,\infty)$ for any $K \in \mathbb{R}$ to yield the full formulation of Otto-Villani's HWI inequality (cf. [9, 22]).

The HWI inequality is one important illustration of the description by F. Otto [49, 66] of the heat flow as the gradient flow of entropy, which led to the introduction of curvature lower bounds in metric measure spaces as convexity of entropy along the geodesics of optimal transport by J. Lott, C. Villani [57] and K.-T. Sturm [72] (cf. [76]). Recent major achievements by L. Ambrosio, N. Gigli, G. Savaré [1–3] and M. Erbar, K. Kuwada, K.-T. Sturm [44] establish the equivalence of the curvature and curvature-dimension lower bounds in the sense of the Bakry-Émery Γ_2 operator and of optimal transport in the class of Riemannian Energy (metric) measure spaces with, in particular, the tools of the gradient bounds of Theorem 2.3 and (2.8).

A further by-product of the isoperimetric Harnack Theorem 3.2 in this context is a commutation property between the actions of the heat $(P_t)_{t\geq 0}$ and Hopf-Lax $(Q_s)_{s>0}$ semigroups [8], first emphasized by K. Kuwada [50], at the root of the contraction property in Wasserstein distance along the heat flow [33, 66, 68, 69] (see [75, 76, 78]). The following statement is again restricted, for simplicity, to the non-negative curvature condition.

Theorem 4.2 (Contraction property of the Wasserstein distance). *Under the curvature condition $CD(0,\infty)$, for any $t, s > 0$*

$$P_t(Q_s) \,\leq\, Q_s(P_t).$$

As a consequence,

$$\mathrm{W}_2(\mu_t,\nu_t) \leq \mathrm{W}_2(\mu_0,\nu_0)$$

where $d\mu_t = P_t f d\mu$ and $d\nu_t = P_t g d\mu$, $t \geq 0$, f, g probability densities with respect to μ. Conversely, both properties are equivalent to $CD(0,\infty)$.

5. Brascamp-Lieb inequalities

The Brascamp-Lieb inequalities for multilinear integrals of products of functions in several dimensions were first investigated with rearrangement tools [27, 28]. A later approach, including inverse forms, was developed by F. Barthe via mass transportation [13]. Investigations of E. Carlen, E. Lieb, M. Loss [31] and J. Bennett, A. Carbery, M. Christ, T. Tao [19] promoted heat flow monotonicity as a major tool towards these inequalities and full geometric descriptions of their extremizers.

The basic principle, in a reduced simple instance, is best developed with respect to the so-called Ornstein-Uhlenbeck semigroup $(P_t)_{t\geq 0}$ on $\mathbb{R}^n$ with infinitesimal generator

$$\mathrm{L} = \Delta f - x \cdot \nabla$$

(corresponding therefore to the quadratic potential $V(x) = \frac{1}{2}\,|x|^2$), invariant and symmetric with respect to the standard Gaussian measure γ, and given by the integral representation along suitable functions $f : \mathbb{R}^n \to \mathbb{R}$ by

$$P_t f(x) \,=\, \int_{\mathbb{R}^n} f\big(e^{-t}x + \sqrt{1 - e^{-2t}}\, y\big) d\gamma(y), \quad t \geq 0,\ x \in \mathbb{R}^n. \tag{5.1}$$

Let J be a (smooth) real-valued function on some open rectangle $\mathcal{R}$ of $\mathbb{R}^m$. A composition like $J \circ f$ is implicitly meant for functions f with values in $\mathcal{R}$. Let $f = (f_1, \ldots, f_m)$ be a vector of (smooth) functions on $\mathbb{R}^n$ and consider,

$$\psi(t) \,=\, \int_{\mathbb{R}^n} J \circ P_t f \, d\gamma, \quad t \geq 0$$

(where the Ornstein-Uhlenbeck semigroup $(P_t)_{t\geq 0}$ is extended to functions with values in $\mathbb{R}^m$). By the heat equation $\partial P_t f = \mathrm{L} P_t f$ and integration by parts with respect to the generator L, it holds that

$$\psi'(t) \,=\, -\sum_{k,\ell=1}^{m} \int_{\mathbb{R}^n} \partial_{k\ell} J \circ P_t f \, \nabla P_t f_k \cdot \nabla P_t f_\ell \, d\gamma.$$

Applied to functions $f_k = g_k \circ A_k$, $k = 1, \ldots, m$, on $\mathbb{R}^{rn}$, where $g_k : \mathbb{R}^s \to \mathbb{R}$ and A_k is a (constant) $s \times rn$ matrix such that $A_k\, {}^t\!A_k$ is the identity matrix (of $\mathbb{R}^s$), the argument expresses the following conclusion. For $k, \ell = 1, \ldots, m$, set $M_{k\ell} = A_\ell\, {}^t\!A_k$ (which is an $s \times s$ matrix).

Proposition 5.1. *In the preceding notation, provided the Hessian of J is such that for all vectors v_k in $\mathbb{R}^s$, $k = 1, \ldots, m$,*

$$\sum_{k,\ell=1}^{m} \partial_{k\ell} J \, M_{k\ell} \, v_k \cdot v_\ell \,\leq\, 0, \tag{5.2}$$

then

$$\int_{\mathbb{R}^{rn}} J(g_1 \circ A_1, \ldots, g_m \circ A_m) d\gamma \,\leq\, J\bigg(\int_{\mathbb{R}^{rn}} g_1 \circ A_1 d\gamma, \ldots, \int_{\mathbb{R}^{rn}} g_m \circ A_m d\gamma\bigg).$$

When $s = 1$, the condition (5.2) amounts to the fact that the Hadamard (point-wise) product $\mathrm{Hess}(J) \circ M$ of the Hessian of J and of the matrix $M = (M_{k\ell})_{1 \leq k, \ell \leq n}$ is (semi-) negative definite.

This general proposition encircles a number of illustrations of interest. As a first example, take $s = n$ and $r = m = 2$ and let A_1 and A_2 be the $n \times 2n$ matrices $A_1 = (\mathrm{Id}_n; 0_n)$ and $A_2 = (\rho \, \mathrm{Id}_n; \sqrt{1-\rho^2}\, \mathrm{Id}_n)$ where $\rho \in (0,1)$. In this case, the monotonicity condition (5.2) is expressed by the non-positivity of the matrix

$$\begin{pmatrix} \partial_{11} J & \rho\, \partial_{12} J \\ \rho\, \partial_{12} J & \partial_{22} J \end{pmatrix}. \tag{5.3}$$

For instance, if $J(u,v) = u^\alpha v^\beta$, $(u,v) \in (0,\infty)^2$, the condition is fulfilled with

$$\rho^2 \alpha\beta \, \leq \, (\alpha - 1)(\beta - 1).$$

For this choice of J, Proposition 5.1 indicates that

$$\begin{aligned} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g_1^\alpha(x) g_2^\beta\big(\rho x + \sqrt{1-\rho^2}\, y)\big) d\gamma(x) d\gamma(y) \\ \leq \bigg(\int_{\mathbb{R}^n} g_1 d\gamma \bigg)^\alpha \bigg(\int_{\mathbb{R}^n} g_2 d\gamma \bigg)^\beta. \end{aligned} \tag{5.4}$$

With $\rho = e^{-t}$, by definition of $P_t g_2$ and duality, the preceding amounts to the famous Nelson hypercontractivity property [64] (for the Ornstein-Uhlenbeck semigroup), equivalent to the logarithmic Sobolev inequality of Theorem 2.1 [9, 47].

Theorem 5.2 (Nelson's hypercontractivity). *Whenever $1 < p < q < \infty$ and $e^{2t} \geq \frac{q-1}{p-1}$, for any measurable function $f : \mathbb{R}^n \to \mathbb{R}$,*

$$\|P_t f\|_q \, \leq \, \|f\|_p.$$

This example is actually embedded in the so-called geometric form of the Brascamp-Lieb inequalities emphasized by K. Ball (cf. [12, 15, 19]). For simplicity, consider only the one-dimensional versions $r = s = 1$. Let $A_1, \ldots, A_m$ be unit vectors which decompose the identity in $\mathbb{R}^n$ in the sense that for $0 \leq c_k \leq 1$, $k = 1, \ldots, m$,

$$\sum_{k=1}^m c_k A_k \otimes A_k \, = \, \mathrm{Id}_n. \tag{5.5}$$

Then, for $J(u_1, \ldots, u_m) = u_1^{c_1} \cdots u_m^{c_m}$ on $(0,\infty)^m$ and $f_k(x) = g_k(A_k \cdot x)$, $g_k : \mathbb{R} \to \mathbb{R}$, $k = 1, \ldots, m$, condition (5.2) of Proposition 5.1 amounts to

$$\sum_{k,\ell=1}^m c_k c_\ell A_k \cdot A_\ell \, v_k v_\ell \, \leq \, \sum_{k=1}^m c_k v_k^2 \tag{5.6}$$

for all $v_1, \ldots, v_m \in \mathbb{R}$, which is easily seen to follow from (5.5).

Corollary 5.3 (Geometric Brascamp-Lieb inequality). *Under the decomposition* (5.5)*, for positive measurable functions g_k on $\mathbb{R}$, $k = 1, \ldots, m$,*

$$\int_{\mathbb{R}^n} \prod_{k=1}^m g_k^{c_k}(A_k \cdot x) d\gamma \, \leq \, \prod_{k=1}^m \bigg(\int_{\mathbb{R}} g_k d\gamma \bigg)^{c_k}.$$

These Brascamp-Lieb inequalities are more classically stated with respect to the Lebesgue measure as

$$\int_{\mathbb{R}^n} \prod_{k=1}^m f_k^{c_k}(A_k \cdot x)dx \,\leq\, \prod_{k=1}^m \bigg(\int_{\mathbb{R}} f_k dx \bigg)^{c_k}$$

which is immediately obtained after the change $f_k(x) = g_k(x)e^{-x^2/2}$ (using $\sum_{k=1}^m c_k = n$).

It should be mentioned that inverse Brascamp-Lieb inequalities may also be established along the heat equation as emphasized recently in [34]. C. Borell showed in [25] (see also [14]) that the condition, for non-negative functions f, g, h on $\mathbb{R}^n$ and $\theta \in (0,1)$,

$$h\big(\theta x + (1-\theta)y\big) \,\geq\, f(x)^\theta g(y)^{1-\theta} \tag{5.7}$$

for all $x, y \in \mathbb{R}^n$, is stable under the (standard) heat semigroup $(P_t)_{t\geq 0}$ on $\mathbb{R}^n$ (acting on f, g, h). In the limit as $t \to \infty$, it yields the Prékopa-Leindler theorem indicating that under (5.7),

$$\int_{\mathbb{R}^n} h dx \,\geq\, \bigg(\int_{\mathbb{R}^n} f dx \bigg)^\theta \bigg(\int_{\mathbb{R}^n} g dx \bigg)^{1-\theta}.$$

Specialized to the characteristic functions of sets, this theorem entails the geometric Brunn-Minkowski inequality (in its multiplicative form), and hence the classical isoperimetric inequality in Euclidean space (cf. [46]). C. Borell also provides in [26] the analogous analysis for the Gaussian Brunn-Minkowski and isoperimetric inequalities (as conjectured in [41]).

On the basis of the geometric form Brascamp-Lieb inequalities established by monotonicity along the heat equation, the works [19, 20] of J. Bennett, A. Carbery, M. Christ, T. Tao fully analyze finiteness of constants, structure and existence and uniqueness of centered Gaussian extremals of Euclidean Brascamp-Lieb inequalities (see also [17, 29] for a survey). For applications to the Hausdorff-Young inequality, Euclidean convolution and entropic inequalities, see [18, 34, 38].

One of the motivations of E. Carlen, E. Lieb and M. Loss in [31] was to investigate Brascamp-Lieb and Young inequalities for coordinates on the sphere. Let $\mathbb{S}^{n-1}$ be the standard n-sphere in $\mathbb{R}^n$ and denote by σ the uniform (normalized) measure on it.

Theorem 5.4 (Brascamp-Lieb inequality on the sphere). *Assume that J on $\mathbb{R}^n$, or some open (convex) set in $\mathbb{R}^n$, is separately concave in any two variables. If g_k, $k = 1, \ldots, n$, are, say bounded measurable, functions on $[-1, +1]$, then*

$$\int_{\mathbb{S}^{n-1}} J\big(g_1(x_1), \ldots, g_n(x_n)\big)d\sigma \,\leq\, J\bigg(\int_{\mathbb{S}^{n-1}} g_1(x_1)d\sigma, \ldots, \int_{\mathbb{S}^{n-1}} g_n(x_n)d\sigma \bigg).$$

The proof proceeds as the one of Proposition 5.1 along now the heat flow of the Laplace operator $\Delta = \frac{1}{2}\sum_{k,\ell=1}^n (x_k\partial_\ell - x_\ell\partial_k)^2$ on $\mathbb{S}^{n-1}$. The monotonicity condition on J then takes the form

$$\sum_{k,\ell=1}^n \partial_{k\ell}J\,(\delta_{k\ell} - x_k x_\ell)v_k v_\ell \,\leq\, 0$$

which is easily seen to be satisfied under concavity of J in any two variables. The case considered in [31] corresponds to $J(u_1, \ldots, u_n) = (u_1 \cdots u_n)^{1/2}$ on $\mathbb{R}_+^n$ leading to

$$\int_{\mathbb{S}^{n-1}} g_1(x_1)\cdots g_n(x_n)d\sigma \,\leq\, \bigg(\int_{\mathbb{S}^{n-1}} g_1^2(x_1)d\sigma \bigg)^{1/2} \cdots \bigg(\int_{\mathbb{S}^{n-1}} g_n^2(x_n)d\sigma \bigg)^{1/2}.$$

More general forms under decompositions (5.5) of the identity in Riemannian Lie groups have been studied in [15]. Discrete versions on the symmetric group and multivariate hypergeometric models have been considered analogously [15, 32].

As one further illustration of Proposition 5.1, consider $X = (X_1, \ldots, X_m)$ a centered Gaussian vector on $\mathbb{R}^m$ with covariance matrix $M = A\,{}^tA$ such that $M_{kk} = 1$ for every $k = 1, \ldots, m$. The vector X has the distribution of Ax, $x \in \mathbb{R}^n$, under the standard normal distribution γ on $\mathbb{R}^n$. Applying Proposition 5.1 to the unit vectors ($1 \times n$ matrices) A_k, $k = 1, \ldots, m$, which are the lines of the matrix A and to $f_k(x) = g_k(A_k \cdot x)$, $x \in \mathbb{R}^n$, where $g_k : \mathbb{R} \to \mathbb{R}$, $k = 1, \ldots, m$, with respect to γ, yields that whenever $\mathrm{Hess}(J) \circ M \leq 0$,

$$\mathbb{E}\Big(J\big(g_1(X_1), \ldots, g_m(X_m)\big)\Big) \,\leq\, J\Big(\mathbb{E}\big(g_1(X_1)\big), \ldots, \mathbb{E}\big(g_m(X_m)\big)\Big) \tag{5.8}$$

(under suitable integrability properties on the g_k's). See [34] for the case of Brascamp-Lieb functions J and multidimensional forms.

6. Gaussian noise stability

The study of noise stability (or sensitivity) in Boolean analysis is another field of interest in which links with interpolation along the Ornstein-Uhlenbeck semigroup (for the ideal Gaussian continuous model) were developed. Indeed, as recently demonstrated by E. Mossel and J. Neeman [60, 61], for a suitable choice of the function J, the correlation inequality (5.8) actually entails significant inequalities related to (Gaussian) noise stability and isoperimetry.

Set, for $(u, v) \in [0,1]^2$ and $\rho \in (0,1)$,

$$J_\rho^{\mathrm{B}}(u,v) \,=\, \gamma \otimes \gamma\big((x,y) \in \mathbb{R}^2 \,; x \leq \Phi^{-1}(u), \rho x + \sqrt{1-\rho^2}\, y \leq \Phi^{-1}(v)\big).$$

Equivalently, when $\rho = e^{-t}$, $t > 0$,

$$J_\rho^{\mathrm{B}}(u,v) \,=\, \int_{\mathbb{R}^n} \mathbb{1}_H P_t(\mathbb{1}_K) d\gamma$$

where $(P_t)_{t \geq 0}$ is the Ornstein-Uhlenbeck semigroup (5.1) and H and K are the (parallel) half-spaces

$$H \,=\, \big\{x \in \mathbb{R}^n \,; x_1 \leq \Phi^{-1}(u)\big\}, \quad K \,=\, \big\{x \in \mathbb{R}^n \,; x_1 \leq \Phi^{-1}(v)\big\}.$$

As a main feature, the function J_ρ^{B} is ρ-concave in the sense that the matrix (5.3), which is the Hadamard product of the Hessian of J_ρ^{B} with the covariance matrix of the Gaussian vector $(x, \rho x + \sqrt{1-\rho^2}\, y)$, is non-positive definite. Proposition 5.1 applied to this function J_ρ^{B} as towards hypercontractivity, or equivalently the multidimensional analogue of (5.8), therefore yields

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_\rho^{\mathrm{B}}\Big(f(x), g\big(\rho x + \sqrt{1-\rho^2}\, y\big)\Big) d\gamma(x) d\gamma(y) \leq\, J_\rho^{\mathrm{B}}\bigg(\int_{\mathbb{R}^n} f\, d\gamma, \int_{\mathbb{R}^n} g\, d\gamma\bigg) \tag{6.1}$$

for every measurable functions $f, g : \mathbb{R}^n \to [0,1]$. Since $J_\rho^{\mathrm{B}}(0,0) = J_\rho^{\mathrm{B}}(1,0) = J_\rho^{\mathrm{B}}(0,1) = 0$ and $J_\rho^{\mathrm{B}}(1,1) = 1$, the application of (6.1) to $f = \mathbb{1}_A$ and $g = \mathbb{1}_B$ for Borel sets A, B in

$\mathbb{R}^n$ and the very definition of J_ρ^{B} yield a semigroup proof of Borell's noise stability theorem [60, 61]. This result was initially established via symmetrization with respect to the Gaussian measure introduced by A. Ehrhard [24, 41] (along the rearrangement ideas in Euclidean and spherical spaces [4, 28, 70, 71], see also [16, 30, 54]).

Theorem 6.1 (Borell's noise stability theorem). *For Borel sets $A, B \subset \mathbb{R}^n$, and for every $t \geq 0$,*

$$\int_{\mathbb{R}^n} \mathbb{1}_A P_t(\mathbb{1}_B) d\gamma \leq \int_{\mathbb{R}^n} \mathbb{1}_H P_t(\mathbb{1}_K) d\gamma$$

where $H = \{x_1 \leq a\}$, $K = \{x_1 \leq b\}$ are parallel half-spaces with respectively the same Gaussian measures $\gamma(H) = \gamma(A)$ and $\gamma(K) = \gamma(B)$.

Theorem 6.1 thus expresses that half-spaces are the most noise stable in the sense that they maximize $\int_{\mathbb{R}^n} \mathbb{1}_A P_t(\mathbb{1}_A) d\gamma$ over all Borel sets A in $\mathbb{R}^n$. The new semigroup proof by E. Mossel and J. Neeman [60, 61] was motivated by the equality case and the study of the deficit (see below). It is also connected to the discrete version on the cube $\{-1, +1\}^n$ and the "Majority is Stablest" theorem of [62] in the context of hardness of approximation for Max-Cut in Boolean analysis. While established first via an invariance principle on the basis of Theorem 6.1, a recent purely discrete proof is emphasized in [40].

Classical arguments providing (small time) heat flow descriptions of surface measures may be used to recover the standard Gaussian isoperimetric inequality from Theorem 6.1 [51]. Indeed, it holds true that

$$\gamma^+(A) \geq \limsup_{t \to 0} \sqrt{\frac{\pi}{t}} \left[\gamma(A) - \int_{\mathbb{R}^n} \mathbb{1}_A P_t(\mathbb{1}_A) d\gamma \right]$$

with equality on half-spaces, so that together with Theorem 6.1, $\gamma^+(A) \geq \gamma^+(H)$ if H is a half-space with $\gamma(A) = \gamma(H)$. Besides, a suitable limiting procedure, replacing (f, g) by $(\varepsilon f, \delta g)$ as $\varepsilon, \delta \to 0$, shows that (6.1) contains the hypercontractivity inequality (5.4) (cf. [53]).

Recent investigations study bounds on the deficit in the noise stability Theorem 6.1 and the Gaussian isoperimetric inequality (3.2). While semigroup tools may be used to some extent [60, 61], R. Eldan [43] achieved a complete picture with wider and more refined stochastic calculus tools (improving in particular upon former mass transportation arguments [35]). He showed that, up to a logarithmic factor, given $t > 0$ and a Borel set A, there exists a half-space H with $\gamma(H) = \gamma(A)$ such that

$$\int_{\mathbb{R}^n} \mathbb{1}_H P_t(\mathbb{1}_H) d\gamma - \int_{\mathbb{R}^n} \mathbb{1}_A P_t(\mathbb{1}_A) d\gamma \geq C\big(\gamma(A), t\big)\, \gamma(A \Delta H)^2$$

(and similarly for the isoperimetric deficit), independently of the dimension.

Multidimensional extensions of Theorem 6.1 on the basis of (5.8) are discussed in [48, 63], with connections with the classical Slepian inequality (cf. [53]).

Theorem 6.2 (Multidimensional Borell theorem). *Let $X = (X_1, \ldots, X_m)$ be a centered Gaussian vector in $\mathbb{R}^m$ with (non-degenerate) covariance matrix M such that $M_{k\ell} \geq 0$ for all $k, \ell = 1, \ldots, m$. Then, for any Borel sets $B_1, \ldots, B_m$ in $\mathbb{R}$,*

$$\mathbb{P}(X_1 \in B_1, \ldots, X_m \in B_m) \leq \mathbb{P}(X_1 \leq b_1, \ldots, X_m \leq b_m)$$

where $\mathbb{P}(X_k \in B_k) = \Phi(b_k/\sigma_k)$, $\sigma_k = \sqrt{M_{kk}}$, $k = 1, \ldots, m$.

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