Heat Flows, Geometric and Functional Inequalities

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heat flow and semigroup interpolations

Duhamel formula (19th century)

pde, probability, dynamics

major tool in

geometric and functional analysis

last decades developments

dynamical proofs of Euclidean and Riemannian

geometric, functional and transport inequalities

heat kernel bounds, spectral analysis,

integral inequalities,

measure concentration, statistical mechanics,

optimal transport

geometric aspects

gradient bounds

convexity

Ricci curvature lower bounds

D. Bakry and M. Émery (1985)

hypercontractive diffusions

selected topics

- logarithmic Sobolev form of the Li-Yau parabolic inequality
- (Gaussian) isoperimetric-type inequalities
- transport and Harnack inequalities
- Brascamp-Lieb inequalities and noise stability

joint with D. Bakry

"Analysis and geometry of Markov diffusion operators"

D. Bakry, I. Gentil, M. L. (2014)

selected topics

- logarithmic Sobolev form of the Li-Yau parabolic inequality
- (Gaussian) isoperimetric-type inequalities
- transport and Harnack inequalities
- Brascamp-Lieb inequalities and noise stability

to start with: a basic model example

the logarithmic Sobolev inequality

classical logarithmic Sobolev inequality

L. Gross (1975)

γ standard Gaussian (probability) measure on $\mathbb{R}^n$

$$d\gamma(x) = e^{-|x|^2/2} \frac{dx}{(2\pi)^{n/2}}$$

$f > 0$ smooth, $\int_{\mathbb{R}^n} f \, d\gamma = 1$

$$\int_{\mathbb{R}^n} f \log f \, d\gamma \leq \frac{1}{2} \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f} \, d\gamma$$

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(relative) entropy $$\int_{\mathbb{R}^n} f \log f \, d\gamma \leq \frac{1}{2} \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f} \, d\gamma$$

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$f > 0$ smooth, $\int_{\mathbb{R}^n} f \, d\gamma = 1$

(relative) entropy $\displaystyle\int_{\mathbb{R}^n} f \log f \, d\gamma \leq \frac{1}{2} \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f} \, d\gamma$ Fisher information

classical logarithmic Sobolev inequality

$$\int_{\mathbb{R}^n} f \log f \, d\gamma \,\leq\, \frac{1}{2} \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f} \, d\gamma$$

$f \,\to\, f^2$ $$\int_{\mathbb{R}^n} f^2 \log f^2 \, d\gamma \,\leq\, 2 \int_{\mathbb{R}^n} |\nabla f|^2 d\gamma$$

Wiener analysis, hypercontractivity,

convergence to equilibrium of kinetic equations

and Markov chains, measure concentration,

information theory, optimal transport (C. Villani)

Perelman's Ricci flow

heat flow interpolation

D. Bakry, M. Émery (1985)

simplest proof ?

$(P_t)_{t \geq 0}$ heat semigroup on $\mathbb{R}^n$

$$P_t f(x) = \int_{\mathbb{R}^n} f(y)\, e^{-|x-y|^2/4t} \frac{dy}{(4\pi t)^{n/2}} = \int_{\mathbb{R}^n} f\big(x + \sqrt{2t}\, y\big)\, d\gamma(y)$$

$$u(x,t) = P_t f(x), \quad t \geq 0, \ x \in \mathbb{R}^n$$

solves the heat equation $\quad \partial_t u = \Delta u, \qquad u(x,0) = f(x)$

$f > 0, \quad t > 0,$ at any point: $\qquad P_t(f \log f) - P_t f \log P_t f$

$$P_t f(x) = \int_{\mathbb{R}^n} f\big(x + \sqrt{2t}\, y\big)\, d\gamma(y)$$

$$t = \frac{1}{2} \quad (x = 0): \qquad P_t \to \gamma$$

$$P_t(f \log f) - P_t f \log P_t f = \int_{\mathbb{R}^n} f \log f \, d\gamma \qquad \Big(\int_{\mathbb{R}^n} f \, d\gamma = 1 \Big)$$

$$P_t(f \log f) - P_t f \log P_t f \;=\; \int_0^t \frac{d}{ds} P_s\big(P_{t-s} f \log P_{t-s} f\big)\, ds$$

heat equation $\qquad \partial_t P_t \;=\; \Delta P_t$

chain rule (both in time and space)

$$\frac{d}{ds} P_s\big(P_{t-s} f \log P_{t-s} f\big) \;=\; P_s\bigg(\frac{|\nabla P_{t-s} f|^2}{P_{t-s} f}\bigg)$$

$$P_t(f \log f) - P_t f \log P_t f \;=\; \int_0^t P_s\Big(\frac{|\nabla P_{t-s} f|^2}{P_{t-s} f}\Big)\, ds$$

gradient commutation $\qquad \nabla P_u f \;=\; P_u(\nabla f) \qquad$ (curvature)

$$|\nabla P_u f|^2 \;\leq\; \Big[P_u\big(|\nabla f|\big)\Big]^2 \;\leq\; P_u\Big(\frac{|\nabla f|^2}{f}\Big) P_u f$$

$$u \;=\; t - s$$

$$P_t(f \log f) - P_t f \log P_t f \;\leq\; \int_0^t P_s\Big(P_{t-s}\Big(\frac{|\nabla f|^2}{f}\Big)\Big)\, ds \;=\; t\, P_t\Big(\frac{|\nabla f|^2}{f}\Big)$$

classical logarithmic Sobolev inequality

L. Gross (1975)

γ standard Gaussian (probability) measure on $\mathbb{R}^n$

$$d\gamma(x) = e^{-|x|^2/2} \frac{dx}{(2\pi)^{n/2}}$$

$f > 0$ smooth, $\int_{\mathbb{R}^n} f \, d\gamma = 1$

$$\int_{\mathbb{R}^n} f \log f \, d\gamma \leq \frac{1}{2} \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f} \, d\gamma$$

$$P_t(f\log f) - P_t f \log P_t f \;=\; \int_0^t P_s\Big(\frac{|\nabla P_{t-s} f|^2}{P_{t-s} f}\Big)\, ds$$

gradient commutation $\qquad \nabla P_u f \;=\; P_u(\nabla f) \qquad$ (curvature)

$$|\nabla P_u f|^2 \;\leq\; \Big[P_u\big(|\nabla f|\big)\Big]^2 \;\leq\; P_u\Big(\frac{|\nabla f|^2}{f}\Big) P_u f$$

$$u \;=\; t - s$$

$$P_t(f\log f) - P_t f \log P_t f \;\leq\; \int_0^t P_s\Big(P_{t-s}\Big(\frac{|\nabla f|^2}{f}\Big)\Big)\, ds \;=\; t\, P_t\Big(\frac{|\nabla f|^2}{f}\Big)$$

$$P_t(f \log f) - P_t f \log P_t f \;=\; \int_0^t \phi(s)\, ds$$

$$\phi(s) \;=\; P_s\left(\frac{|\nabla P_{t-s} f|^2}{P_{t-s} f}\right)$$

gradient commutation $\qquad \nabla P_u f \;=\; P_u(\nabla f)$

ϕ increasing: $\qquad s \leq t, \qquad \phi(s) \;\leq\; \phi(t) \;=\; P_t\Big(\frac{|\nabla f|^2}{f}\Big)$

differentiate ϕ, $\qquad \phi' \;\geq\; 0$?

differentiate ϕ, $\qquad \phi' \geq 0$?

$$\phi(s) = P_s\left(\frac{|\nabla P_{t-s}f|^2}{P_{t-s}f}\right) = P_s\big(P_{t-s}f\, |\nabla \log P_{t-s}f|^2\big)$$

$$\phi'(s) = 2\, P_s\big(P_{t-s}f\, \Gamma_2\,(\log P_{t-s}f)\big)$$

Γ_2 Bakry-Émery operator

$$\Gamma_2(h) = \frac{1}{2}\,\Delta\big(|\nabla h|^2\big) - \nabla h \cdot \nabla(\Delta h)$$

$$= \big|\mathrm{Hess}(h)\big|^2 + \mathrm{Ric}_g(\nabla h, \nabla h)$$

on $\mathbb{R}^n$ or (M, g) n-dimensional Riemannian manifold

Bochner's formula

(M, g) non-negative Ricci curvature $\Gamma_2(h) \geq 0$, $\phi' \geq 0$

$$P_t(f \log f) - P_t f \log P_t f = \int_0^t \phi(s)\,ds \leq t\,P_t\Big(\frac{|\nabla f|^2}{f}\Big)$$

$(P_t)_{t \geq 0}$ heat semigroup on (M, g)

heat flow interpolation under $\Gamma_2 \geq 0$

heat kernel inequalities, functional inequalities

(Sobolev, logarithmic Sobolev, Poincaré, Nash...)

under geometric (Ricci) curvature conditions

more generally under $\mathrm{Ric_g} \geq K, \quad K \in \mathbb{R}$

equivalent gradient bounds (commutation)

$$|\nabla P_t f| \leq e^{-Kt} P_t\big(|\nabla f|\big)$$

reverse logarithmic Sobolev inequality

heat kernel inequality

$$P_t(f \log f) - P_t f \log P_t f \;\leq\; t\, P_t\Big(\frac{|\nabla f|^2}{f}\Big)$$

reverse form

$$t\, \frac{|\nabla P_t f|^2}{P_t f} \;\leq\; P_t(f \log f) - P_t f \log P_t f$$

gradient bounds

(both equivalent to $\Gamma_2 \geq 0$ as $t \to 0$)

dimensional improvement

$$P_t(f \log f) - P_t f \log P_t f \;=\; \int_0^t \phi(s)\, ds$$

$$\phi'(s) \;=\; 2\, P_s\big(P_{t-s} f\, \Gamma_2\, (\log P_{t-s} f)\big)$$

actually under $\mathrm{Ric}_g \geq 0$

$$\Gamma_2(h) \;=\; \big|\mathrm{Hess}(h)\big|^2 + \mathrm{Ric}_g(\nabla h, \nabla h) \;\geq\; \big|\mathrm{Hess}(h)\big|^2 \;\geq\; \frac{1}{n}\, (\Delta h)^2$$

$$\phi'(s) \;\geq\; \frac{2}{n}\, P_s\big(P_{t-s} f\, [\Delta \log P_{t-s} f\,]^2\big)$$

dimensional logarithmic Sobolev inequality

$$P_t(f \log f) - P_t f \log P_t f \ \leq \ t \, \Delta P_t f + \frac{n}{2} \, P_t f \log \left(1 - \frac{2t}{n} \, \frac{P_t(f \, \Delta \log f)}{P_t f} \right)$$

$$(\log(1 + x) \ \leq \ x \quad \text{or} \quad n \to \infty)$$

reverse form

$$P_t(f \log f) - P_t f \log P_t f \ \geq \ t \, \Delta P_t f - \frac{n}{2} \, P_t f \log \left(1 + \frac{2t}{n} \, \Delta(\log P_t f) \right)$$

reverse form

$$P_t(f \log f) - P_t f \log P_t f \ \geq \ t\, \Delta P_t f - \frac{n}{2}\, P_t f \log \Big(1 + \frac{2t}{n}\, \Delta(\log P_t f) \Big)$$

in particular $\quad 1 + \frac{2t}{n}\, \Delta(\log P_t f) \ > \ 0$

$$f : M \to \mathbb{R}, \quad f > 0, \quad \mathrm{Ric_g} \ \geq \ 0, \quad t > 0$$

$$\frac{|\nabla P_t f|^2}{(P_t f)^2} - \frac{\Delta P_t f}{P_t f} \ \leq \ \frac{n}{2t}$$

Li-Yau (1986) parabolic inequality

maximum principle

D. Bakry, M. L. (2006)

Li-Yau (1986) parabolic inequality

$$\frac{|\nabla P_t f|^2}{(P_t f)^2} - \frac{\Delta P_t f}{P_t f} \leq \frac{n}{2t}$$

$$(\mathrm{Ric}_g \geq 0)$$

Harnack inequalities

$$P_t f(x) \leq P_{t+s} f(y) \Big(\frac{t+s}{t} \Big)^{n/2} e^{d(x,y)^2/4s}$$

$d(x, y)$ Riemannian distance

heat kernel bounds

(M, g) weighted Riemannian manifold

$d\mu = e^{-V} dx, \qquad V$ smooth potential on M

$\mathrm{L} = \Delta - \nabla V \cdot \nabla, \qquad (P_t)_{t \geq 0}$ semigroup

non-negative curvature-dimension $CD(0, N)$

$$\Gamma_2 = \mathrm{Ric}_g + \mathrm{Hess}(V) \geq \frac{1}{N} (\mathrm{L}h)^2$$

curvature-dimension $CD(K, N), \quad K \in \mathbb{R}, \quad N \geq 1$

$$\Gamma_2(h) \geq K\, |\nabla h|^2 + \frac{1}{N} (\mathrm{L}h)^2$$

Markov Triple (E, μ, Γ)

semigroup $(P_t)_{t \geq 0}$, generator L, invariant measure μ

carré du champ operator Γ

$$\Gamma(f,g) = \frac{1}{2}\,\mathrm{L}(f\,g) - f\,\mathrm{L}g - g\,\mathrm{L}f, \qquad (f,g) \in \mathcal{A} \times \mathcal{A}$$

Bakry-Émery Γ_2 operator

$$\Gamma_2(f,g) = \frac{1}{2}\,\mathrm{L}\Gamma(f,g) - \Gamma(f, \mathrm{L}g) - \Gamma(g, \mathrm{L}f)$$

curvature-dimension $CD(K,N)$

$$\Gamma_2(h,h) \geq K\,\Gamma(h,h) + \frac{1}{N}\,(\mathrm{L}h)^2, \qquad h \in \mathcal{A}$$

- logarithmic Sobolev form of the Li-Yau parabolic inequality
- (Gaussian) isoperimetric-type inequalities
- transport and Harnack inequalities
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isoperimetric inequality for the standard Gaussian measure

$$d\gamma(x) = e^{-|x|^2/2} \frac{dx}{(2\pi)^{n/2}} \qquad \text{on} \quad \mathbb{R}^n$$

C. Borell (1975), V. Sudakov, B. Tsirelson (1974)

half-spaces extremal sets for the isoperimetric problem

$$H = \big\{ x \in \mathbb{R}^n ; \langle x, u \rangle \leq a \big\}, \quad |u| = 1, \quad a \in \mathbb{R}$$

if $\quad A \subset \mathbb{R}^n, \quad \gamma(A) = \gamma(H) \quad$ then $\quad \gamma^+(A) \geq \gamma^+(H)$

$$\gamma^+(A) = \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon} \big[\gamma(A_\varepsilon) - \gamma(A) \big]$$

$$\mathrm{I}(v) \,=\, \inf\big\{\gamma^+(A)\,;\ \gamma(A) = v\big\}, \quad A \subset \mathbb{R}^n, \quad v \in (0,1)$$

half-spaces: one-dimensional

$$\mathrm{I} \,=\, \varphi \circ \Phi^{-1}$$

$$\Phi(s) \,=\, \int_{-\infty}^{s} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}\,, \qquad \varphi \,=\, \Phi'$$

isoperimetric inequality $\qquad A \subset \mathbb{R}^n, \qquad \mathrm{I}\big(\gamma(A)\big) \,\leq\, \gamma^+(A)$

heat flow and isoperimetric-type inequalities

$\Gamma_2 \geq 0$ $\quad (CD(0,\infty)$ non-negative curvature)

M (weighted) Riemannian manifold

$f : M \to [0,1]$ smooth

$$\mathrm{I}(P_t f) \leq P_t\Big(\sqrt{\mathrm{I}^2(f) + 2t\,|\nabla f|^2}\,\Big)$$

(stronger than logarithmic Sobolev inequality, $\quad \varepsilon f,\ \varepsilon \to 0$)

D. Bakry, M. L. (1996)

heat flow and isoperimetric-type inequalities

$f : M \to [0,1]$ smooth

$$\mathrm{I}\,(P_t f) \leq P_t\Big(\sqrt{\mathrm{I}^2(f) + 2t\,|\nabla f|^2}\,\Big)$$

$$M = \mathbb{R}^n \qquad t = \tfrac{1}{2}: \quad P_t \to \gamma$$

Bobkov's inequality $\qquad \mathrm{I}\bigg(\int_{\mathbb{R}^n} f\,d\gamma\bigg) \leq \int_{\mathbb{R}^n} \sqrt{\mathrm{I}^2(f) + |\nabla f|^2}\,d\gamma$

$$f = \mathbb{1}_A \qquad (\mathrm{I}(0) = \mathrm{I}(1) = 0) \qquad \mathrm{I}\big(\gamma(A)\big) \leq \gamma^+(A)$$

(spherical) isoperimetric-type inequalities

similar under $CD(K, \infty)$ on (E, μ, Γ) Markov triple

comparison to the Gaussian isoperimetric profile

open problem:

heat flow proof of isoperimetry on the sphere on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$

spherical caps are the isoperimetric extremizers

Lévy-Gromov (1980) theorem

(M, g), μ normalized volume element, $CD(n-1, n)$

isoperimetric profile $\mathrm{I}_\mu \geq \mathrm{I}_{\mathbb{S}^n}$

$$f : M \to [0,1] \quad \text{smooth}, \qquad \Gamma_2 \geq 0$$

$$\big[\, \mathrm{I}\,(P_t f)\big]^2 - \big[P_t\big(\,\mathrm{I}\,(f)\big)\big]^2 \geq 2t \, |\nabla P_t f|^2$$

$$\Phi^{-1} \circ P_t f \qquad \frac{1}{\sqrt{2t}}\text{- Lipschitz}$$

$$\Phi^{-1} \circ P_t f(x) \leq \Phi^{-1} \circ P_t f(y) + \frac{d(x,y)}{\sqrt{2t}}$$

sharp gradient bounds, Harnack-type inequalities

new isoperimetric Harnack inequality

$\Gamma_2 \geq 0$ $\quad$ ($CD(0,\infty)$ non-negative curvature)

$$P_t(\mathbb{1}_A)(x) \leq P_t(\mathbb{1}_{A_{d(x,y)}})(y)$$

$d(x,y)$ distance from x to y $\qquad A_\varepsilon = \{z \in M\,; d(z,A) < \varepsilon\}$

similar under $CD(K,\infty)$, $\quad K \in \mathbb{R}$

on (E,μ,Γ) Markov triple

D. Bakry, I. Gentil, M. L. (2013)

Wang's Harnack inequality (1997)

$\Gamma_2 \geq 0$ $\quad$ ($CD(0,\infty)$ non-negative curvature)

$$\big(P_t f\big)^2(x) \leq P_t\big(f^2\big)(y)\, e^{d(x,y)^2/2t}$$

(infinite dimensional version of the Li-Yau Harnack inequality)

variant: log-Harnack inequality

$$P_t(\log f)(x) \leq \log P_t f(y) + \frac{d(x,y)^2}{4t}$$

links to optimal transport inequalities

F. Otto, C. Villani (2000)

M (weighted) Riemannian manifold $CD(0,\infty)$

$$d\nu = f\,d\mu, \quad f > 0, \quad \int_M f\,d\mu = 1$$

$$\int_M f \log f \,d\mu \,\leq\, W_2(\nu,\mu)\bigg(\int_M \frac{|\nabla f|^2}{f}\,d\mu\bigg)^{1/2}$$

Kantorovich-Wasserstein distance

$$W_2(\nu,\mu)^2 \,=\, \inf_{\pi|_\nu^\mu} \int_M \int_M d(x,y)^2 d\pi(x,y)$$

Kantorovich-Rubinstein duality

$$\frac{1}{2} W_2(\nu,\mu)^2 = \sup_{\varphi} \left(\int_M Q_1\varphi \, d\mu - \int_M \varphi \, d\nu \right)$$

Hofp-Lax infimum convolution semigroup

$$Q_s f(x) = \inf_{y \in M} \left[f(y) + \frac{d(x,y)^2}{2s} \right], \quad x \in M, \ s > 0$$

log-Harnack inequality $\qquad P_t(\log f) \le Q_{2t}(\log P_t f)$

$$\int_M P_t f \log P_t f \, d\mu \le \frac{1}{4t} W_2^2(\nu,\mu), \qquad d\nu = f \, d\mu$$

S. Bobkov, I. Gentil, M. L. (2001)

$(P_t)_{t \geq 0}$ heat semigroup $\qquad (Q_s)_{s \geq 0}$ Hopf-Lax semigroup

$$P_t(Q_s f) \leq Q_s(P_t f)$$

heat flow contraction in Wasserstein distance

$$W_2(\mu_t, \nu_t) \leq W_2(\mu_0, \nu_0)$$

$$d\mu_t = P_t f \, d\mu \qquad d\nu_t = P_t g \, d\mu$$

M. K. von Renesse, K. T. Sturm (2005),

F. Otto, M. Westdickenberg (2006), K. Kuwada (2010)

- logarithmic Sobolev form of the Li-Yau parabolic inequality

- (Gaussian) isoperimetric-type inequalities

- transport and Harnack inequalities

- Brascamp-Lieb inequalities and noise stability

γ standard Gaussian (probability) measure on $\mathbb{R}^n$

$$d\gamma(x) = e^{-|x|^2/2} \frac{dx}{(2\pi)^{n/2}}$$

$$T_\rho f(x) = \int_{\mathbb{R}^n} f\big(\rho x + \sqrt{1-\rho^2}\, y\big) d\gamma(y), \qquad \rho \in (0,1)$$

$$T_\rho f = P_t f, \qquad \rho = e^{-t}$$

$(P_t)_{t \geq 0}$ Ornstein-Uhlenbeck semigroup

generator $\mathrm{L} = \Delta - x \cdot \nabla$ invariant measure γ

E. Mossel, J. Neeman (2012)

$$\rho \in (0,1), \quad J : \mathbb{R}^2 \to \mathbb{R} \ \text{ smooth}, \quad f, g : \mathbb{R}^n \to \mathbb{R}$$

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} J\Big(f(x), g\big(\rho x + \sqrt{1-\rho^2}\, y\big)\Big) d\gamma(x) d\gamma(y) \leq J\Big(\int_{\mathbb{R}^n} f\, d\gamma, \int_{\mathbb{R}^n} g\, d\gamma \Big)$$

if and only if

$$\begin{pmatrix} \partial_{11} J & \rho\, \partial_{12} J \\ \rho\, \partial_{12} J & \partial_{22} J \end{pmatrix} \leq 0$$

(J ρ-concave)

$(P_t)_{t \geq 0}$ Ornstein-Uhlenbeck semigroup

$$\psi(t) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} J\Big(P_t f(x), P_t g\big(\rho x + \sqrt{1-\rho^2}\, y\big)\Big) d\gamma(x) d\gamma(y), \quad t \geq 0$$

$$\psi'(t) =$$

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Big[(-\partial_{11} J)|\nabla P_t f|^2 + (-\partial_{22} J)|\nabla P_t g|^2 - 2\,\rho\,\partial_{12} J\, \nabla P_t f \cdot \nabla P_t g\Big] d\gamma\, d\gamma$$

J ρ-concave $\quad \Longrightarrow \quad \psi'(t) \geq 0$

$$\psi(0) \leq \psi(\infty)$$

first example of J- function

$$J(u,v) \,=\, J_\rho^H(u,v) \,=\, u^\alpha v^\beta, \quad u,v > 0, \quad \alpha,\beta \in [0,1]$$

ρ-concave if $\qquad \rho^2 \alpha\beta \,\leq\, (\alpha - 1)(\beta - 1)$

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f^\alpha(x) g^\beta\big(\rho\, x + \sqrt{1-\rho^2}\, y\big)\, d\gamma(x) d\gamma(y)$$

$$\leq \, \bigg(\int_{\mathbb{R}^n} f\, d\gamma \bigg)^\alpha \bigg(\int_{\mathbb{R}^n} g\, d\gamma \bigg)^\beta$$

duality $\qquad \| T_\rho\, g\|_q \,\leq\, \|g\|_p \qquad \rho^2 \,\geq\, \dfrac{q-1}{p-1} \quad (1 < p < q < \infty)$

Nelson's hypercontractivity (1966-73)

$$f_k : \mathbb{R}^p \to \mathbb{R}, \qquad k = 1, \ldots, m$$

$$A_k \quad p \times n \quad \text{matrix}, \quad A_k\, {}^t\!A_k = \mathrm{Id}_{\mathbb{R}^p}$$

$$\Gamma_{k\ell} = A_\ell\, {}^t\!A_k \quad (p \times p \text{ matrix}), \qquad k, \ell = 1, \ldots, m$$

$$\text{smooth} \quad J : \mathbb{R}^m \to \mathbb{R}$$

$$\int_{\mathbb{R}^n} J(f_1 \circ A_1, \ldots, f_m \circ A_m)\, d\gamma \leq J\bigg(\int_{\mathbb{R}^n} f_1 \circ A_1 d\gamma, \ldots, \int_{\mathbb{R}^n} f_m \circ A_m d\gamma \bigg)$$

$$\text{if and only if} \qquad \sum_{k,\ell=1}^{m} \partial_{k\ell} J \, \langle \Gamma_{k\ell} v_k, v_\ell \rangle \leq 0 \qquad \text{for all} \quad v_k \in \mathbb{R}^p$$

K. Ball (1989)

decomposition of the identity

$$\sum_{k=1}^{m} c_k A_k \otimes A_k = \mathrm{Id}_{\mathbb{R}^n}, \quad c_k \geq 0, \quad A_k \text{ unit vectors in } \mathbb{R}^n$$

$$f_k : \mathbb{R} \to \mathbb{R}_+, \quad k = 1, \ldots, m$$

$$J(u_1, \ldots, u_m) = u_1^{c_1} \cdots u_m^{c_m}$$

$$\int_{\mathbb{R}^n} \prod_{k=1}^{m} f_k^{c_k}\big(\langle A_k, x\rangle\big)\, dx \leq \prod_{k=1}^{m} \left(\int_{\mathbb{R}} f_k \, dx \right)^{c_k}$$

Brascamp-Lieb inequalities

H. Brascamp, E. Lieb (1976)

multilinear inequalities

maximize $$\int_{\mathbb{R}^n} \prod_{k=1}^{m} f_k^{c_k}\big(\langle A_k, x\rangle\big)\,dx$$

maximizers: Gaussian kernels

heat flow methodology

E. Carlen, E. Lieb, M. Loss (2004)

J. Bennett, A. Carbery, M. Christ, T. Tao (2008)

second example of J-function

E. Mossel, J. Neeman (2012)

$$J(u,v) \,=\, J_\rho^B(u,v)$$

$$J_\rho^B(u,v) \,=\, \mathbb{P}\Big(X_1 \le \Phi^{-1}(u), \rho\, X_1 + \sqrt{1-\rho^2}\, Y_1 \le \Phi^{-1}(v)\Big), \quad u,v \in [0,1]$$

$X_1, Y_1 \sim \mathcal{N}(0,1)$ independent, distribution Φ

$$\Phi(s) \,=\, \int_{-\infty}^{s} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}$$

J_ρ^B ρ-concave

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_\rho^B\Big(f(x), g\big(\rho\, x + \sqrt{1-\rho^2}\, y\big)\Big)\, d\gamma(x) d\gamma(y)$$

$$\leq \; J_\rho^B\bigg(\int_{\mathbb{R}^n} f\, d\gamma, \int_{\mathbb{R}^n} g\, d\gamma\bigg)$$

$$f = \mathbb{1}_A \qquad g = \mathbb{1}_B, \qquad A, B \subset \mathbb{R}^n$$

$$\mathbb{P}\big(X \in A, \rho\, X + \sqrt{1-\rho^2}\, Y \in B\big) \leq \mathbb{P}\big(X \in H, \rho\, X + \sqrt{1-\rho^2}\, Y \in K\big)$$

X, Y independent with distribution γ in $\mathbb{R}^n$

H, K parallel half-spaces $\gamma(H) = \gamma(A), \quad \gamma(K) = \gamma(B)$

C. Borell (1985) Gaussian rearrangements noise stability of $A \subset \mathbb{R}^n$

$$\mathcal{S}_\rho(A) = \mathbb{P}\big(X \in A, \rho X + \sqrt{1-\rho^2}\, Y \in A\big)$$

$$\mathcal{S}_\rho(A) \leq \mathcal{S}_\rho(H), \qquad \gamma(A) = \gamma(H)$$

half-spaces are the most noise stable

Max-Cut and Majority is Stablest

E. Mossel, R. O'Donnell, K. Oleszkiewicz (2010)

from noise stability to isoperimetry

$$\mathcal{S}_\rho(A) = \mathbb{P}\big(X \in A, \rho X + \sqrt{1-\rho^2}\, Y \in A\big)$$

Borell's theorem $\quad \mathcal{S}_\rho(A) \leq \mathcal{S}_\rho(H) \quad (\gamma(A) = \gamma(H)$ half-space)

heat flow description of the perimeter

$$\gamma^+(A) \geq \limsup_{\rho \to 1} \sqrt{\frac{\pi}{\log \frac{1}{\rho}}}\, \big[\gamma(A) - \mathcal{S}_\rho(A)\big]$$

equality when $A = H$ half-space

Gaussian isoperimetry $\quad \gamma^+(A) \geq \gamma^+(H)$

log-concave mesures $d\mu = e^{-V}dx, \qquad \mathrm{Hess}(V) \geq K > 0$

(E, μ, Γ) Markov Triple, $CD(K, \infty), \quad K > 0$

deficit (Gaussian noise stability and isoperimetry)

R. Eldan (2013) (stochastic calculus)

developments in Boolean analysis

discrete cube and structures (Majority is Stablest)

A. De, E. Mossel, J. Neeman (2013)

Thank you for your attention