Markov operators, classical orthogonal polynomial ensembles, and random matrices

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recent study of

random matrix and random growth models

new asymptotics

common, non central, rate $(\mathrm{mean})^{1/3}$

universal limiting **Tracy-Widom** distribution

random matrices, longest increasing subsequence,

random growth models, last passage percolation...

P. Forrester, C. Tracy, H. Widom, J. Baik, P. Deift, K. Johansson

invariant random matrix models

$$P(dX) = \frac{1}{Z} \exp\Big(- \mathrm{Tr}\big(v(X)\big)\Big) dX, \quad v : \mathbb{R} \to \mathbb{R}$$

$X = X^N$ real symmetric or Hermitian $N \times N$ matrix

$v(x) = \beta\, x^2/4$ Gaussian Orthogonal $(\beta = 1)$ Unitary $(\beta = 2)$ Ensembles

joint law of the eigenvalues $(\lambda_1^N \leq \cdots \leq \lambda_N^N)$ of X^N

$$P^N(dx) = \frac{1}{Z} \big|\Delta_N(x)\big|^\beta \prod_{i=1}^N d\mu(x_i), \quad x = (x_1, \ldots, x_N) \in \mathbb{R}^N$$

$$\Delta_N(x) = \prod_{i<j}(x_i - x_j), \quad d\mu(x) = e^{-v(x)} dx$$

random growth models

directed last passage percolation

$w_{ij}, \quad 1 \le i,j \le N,$ iid geometric random variables

$$W_N = \max_{\pi} \sum_{(i,j)\in\pi} w_{ij}$$

π up/right paths from $(1,1)$ to (N,N)

K. Johansson (2000)

$$P(W_N \le t) = \int_{\{\max x_i \le t+N-1\}} \frac{1}{Z}\,\Delta_N(x)^2 \prod_{i=1}^{N} d\mu(x_i)$$

$\mu(\{x\}) = (1-q)q^x, \; x \in \mathbb{N}$ geometric distribution law of $(\lambda_1^N, \ldots, \lambda_N^N)$: $P^N(dx) = \frac{1}{Z} \left| \Delta_N(x) \right|^\beta \prod_{i=1}^N d\mu(x_i)$

global regime : spectral measure

spectral measure : $\widehat{\lambda}_i^N = \lambda_i^N$ suitably normalized

$$\widehat{\mu}^N = \frac{1}{N} \sum_{i=1}^N \delta_{\widehat{\lambda}_i^N} \ \rightarrow \ \text{equilibrium measure}$$

minimizer of $2 \int v \, d\nu - \beta \iint \log |x - y| \, d\nu(x) d\nu(y)$

weighted logarithmic potential theory

law of $(\lambda_1^N, \ldots, \lambda_N^N)$: $P^N(dx) = \frac{1}{Z} \left|\Delta_N(x)\right|^\beta \prod_{i=1}^N d\mu(x_i)$

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global regime : spectral measure

large deviation asymptotics (every β)

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global regime : spectral measure

$$P^N(dx) = \frac{1}{Z} \exp\bigg(-\sum_{i=1}^N v(x_i) + \beta \sum_{i<j} \log|x_i - x_j| \bigg) dx$$

spectral measure : $\quad \widehat{\lambda}_i^N = \lambda_i^N$ suitably normalized

$$\widehat{\mu}^N = \frac{1}{N} \sum_{i=1}^N \delta_{\widehat{\lambda}_i^N} \; \to \text{ equilibrium measure}$$

minimizer of $\quad 2\int v\,d\nu - \beta \iint \log|x-y|\,d\nu(x)d\nu(y)$

weighted logarithmic potential theory

local regime : individual behavior (spacings, extreme values)

$\beta = 2$ **orthogonal polynomial method**

$P_\ell,\ \ell \in \mathbb{N}$ orthogonal polynomials for μ

marginals of P^N : determinants of the kernel

$$K_N(x,y) = \sum_{\ell=0}^{N-1} P_\ell(x) P_\ell(y)$$

orthogonal polynomial ensemble

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$$\Delta_N(x) = C_n \det\big(P_{\ell-1}(x_k)\big)_{1 \leq k, \ell \leq N}$$

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P^N determinantal random point field structure

marginals of P^N : determinants of the kernel

$$K_N(x,y) = \sum_{\ell=0}^{N-1} P_\ell(x) P_\ell(y)$$

orthogonal polynomial ensemble

(common) **asymptotics of orthogonal polynomials**

universal local regime : spacings and edge behavior

top edge behavior

largest eigenvalue (particle) fluctuates at the rate $(\mathrm{mean})^{1/3}$

limiting **Tracy-Widom** distribution F_{TW}

C. Tracy, H. Widom, P. Deift, X. Zhou, J. Baik, K. Johansson,

T. Kriecherbauer, K. McLaughlin, P. Miller, S. Venakides, M. Vanlessen,

D. Gioev, P. Bleher, A. Its, A. Kuijlaars, M. Shcherbina, L. Pastur...

example : **Gaussian Unitary Ensemble (GUE)**

$$d\mu(x) = e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}\,, \qquad K_N \quad \text{Hermite kernel,} \qquad \widehat{\lambda}_i^N = \lambda_i^N / \sqrt{N}$$

spectral measure

$$\widehat{\mu}^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\widehat{\lambda}_i^N} \;\to\; \frac{1}{2\pi} \sqrt{4 - x^2}\, dx \quad \text{on} \quad (-2, +2)$$

Wigner semi-circle law

$$N^{1/6} \big[\lambda_N^N - 2\sqrt{N} \,\big] \;\to\; F_{\mathrm{TW}}$$ **Tracy-Widom** distribution

$$F_{\mathrm{TW}}(s) = \exp\bigg(- \int_s^\infty (x - s) u(x)^2 dx \bigg), \quad s \in \mathbb{R}$$

$u'' = 2u^3 + xu$ Painlevé II equation

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largest eigenvalue λ_N^N

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Wigner semi-circle law

largest eigenvalue $\lambda_N^N / \sqrt{N} \rightarrow 2$

$N^{1/6} \big[\lambda_N^N - 2\sqrt{N} \big] \rightarrow F_{\mathrm{TW}}$ **Tracy-Widom** distribution

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example : **last passage percolation** $\quad W_N = \max_{\pi} \sum_{(i,j)\in\pi} w_{ij}$

μ geometric, $\quad K_N$ Meixner kernel, $\quad \widehat{\lambda}_i^N = \lambda_i^N / N$

spectral measure

$$\widehat{\mu}^N = \frac{1}{N} \sum_{i=1}^N \delta_{\widehat{\lambda}_i^N} \ \to \ \text{equilibrium measure} \quad \text{on} \quad (a, b)$$

$$N^{-1/3} \big[W_N - bN \big] \ \to \ F_{\mathrm{TW}} \quad \textbf{Tracy-Widom} \ \text{distribution}$$

K. Johansson (2000)

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largest "particle" $\lambda_N^N \sim W_N$

$N^{-1/3}\big[W_N - bN\big] \ \to \ F_{\mathrm{TW}}$ **Tracy-Widom** distribution

K. Johansson (2000)

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largest "particle" $\quad \lambda_N^N \sim W_N / N \ \to \ b$

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K. Johansson (2000)

Markov operator tools

for **classical** orthogonal polynomial ensembles

of random matrix and random growth models

non asymptotic results

- **spectral description** (universal arcsine law)
- **recurrence equations for moments** (map enumeration)
- **recurrence equations for real symmetric models** (Gaussian Orthogonal Ensemble)
- **non asymptotic tail inequalities for largest eigenvalues** (optimal rate)

classical orthogonal polynomial ensembles

Hermite, Laguerre, Jacobi, Charlier, Meixner, Krawtchouk, Hahn

differential equations and operators

mean spectral measure $\quad \mu^N = E\big(\frac{1}{N}\sum_{i=1}\delta_{\lambda_i^N}\big)$

$$\langle f, \mu^N\rangle = \int f(x)\,\frac{1}{N}\,K_N(x,x)\,d\mu(x) = \int f\,\frac{1}{N}\sum_{\ell=0}^{N-1} P_\ell^2\,d\mu$$

example : Gaussian Unitary Ensemble (GUE)

$P_\ell,\ \ell \in \mathbb{N}$ $\quad$ Hermite polynomials for $\quad d\mu(x) = e^{-x^2/2}\frac{dx}{\sqrt{2\pi}}$

normalized in $\quad L^2(\mu)$

$$\langle f, \mu^N \rangle = \int f \, \frac{1}{N} \sum_{\ell=0}^{N-1} P_\ell^2 \, d\mu$$

preliminary investigation : measure $P_N^2 \, d\mu$

Laplace transform : $\varphi(t) = \int e^{tx} P_N^2 \, d\mu, \quad t \in \mathbb{R}$

Hermite operator : $Lf = f'' - x f'$

Gaussian integration by parts : $\int f(-Lg) d\mu = \int f' g' \, d\mu$

$-LP_N = N \, P_N$ eigenvectors

Laplace transform : $\varphi(t) = \int e^{tx} P_N^2 \, d\mu$

second order differential equation

$$t\varphi'' + \varphi' - t(t^2 + 4N + 2)\varphi = 0$$

under the GUE scaling : $t \mapsto t/\sqrt{N}$ $\quad (\widehat{\lambda}_i^N = \lambda_i^N/\sqrt{N})$

limiting differential equation

$$t\,\Phi'' + \Phi' - 4t\,\Phi = 0$$

Φ Laplace transform of the **arcsine law**

$$\frac{dx}{\pi\sqrt{4 - x^2}} \quad \text{on} \quad (-2, +2)$$

common behavior, with the limiting **arcsine law**

for the classical orthogonal polynomials

of (normalized) measures $P_N^2 \, d\mu$

continuous variable : Hermite, Laguerre, Jacobi

discrete variable : Charlier, Meixner, Krawtchouk, Hahn

varying parameters (in N)

compact case : **A. Maté, P. Nevai, V. Totik (1985)**

spectral measure : averaging procedure

GUE $$E\big(\langle f, \widehat{\mu}^N\rangle\big) = \frac{1}{N}\sum_{\ell=0}^{N-1}\int f\left(\sqrt{\frac{\ell}{N}}\cdot\frac{x}{\sqrt{\ell}}\right) P_\ell^2 \, d\mu$$

$$\widehat{\mu}^N = \frac{1}{N}\sum_{i=1}^{N}\delta_{\widehat{\lambda}_i^N} \;\to\; \sqrt{U}\,\xi$$

U uniform, ξ arcsine, independent

$\sqrt{U}\,\xi$ semi-circle law on $(-2, +2)$

Meixner example (μ geometric) : equilibrium measure example : **last passage percolation** $W_N = \max_{\pi} \sum_{(i,j)\in\pi} w_{ij}$

μ geometric, K_N Meixner kernel, $\widehat{\lambda}_i^N = \lambda_i^N / N$

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largest "particle" $\lambda_N^N \sim W_N / N \ \to \ b$

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K. Johansson (2000)

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Meixner example (μ geometric) : equilibrium measure

$$\frac{1}{1-q}\Big(\sqrt{qU(1+U)}\,\xi + \big[U + q(1+U)\big]\Big)$$

moment recursion formula

$$E\left(e^{t\,\mathrm{Tr}(X^N)}\right) = \psi(t) = \int e^{tx} \sum_{\ell=0}^{N-1} P_\ell^2 \, d\mu, \quad t \in \mathbb{R}$$

differential equation

GUE $\quad t\psi'' + 3\psi' - t(t^2 + 4N)\psi = 0$

U. Haagerup, S. Thorbjørnsen (2003)

recurrence equation on moments

$$E\Big(\mathrm{Tr}\left((X^N)^p\right)\Big) = \int x^p \sum_{\ell=0}^{N-1} P_\ell^2 \, d\mu, \quad p \in \mathbb{N}$$

$$a_p^N = E\Big(\mathrm{Tr}\,\big((X^N)^{2p}\big)\Big), \quad X^N \text{ GUE}$$

three term recurrence equation

$$(p+1)a_p^N = (4p-2)Na_{p-1}^N + (p-1)(2p-1)(2p-3)a_{p-2}^N$$

J. Harer, D. Zagier (1986)

map enumeration (Wick products)

$$a_p^N = \sum_{g \geq 0} \varepsilon_g(p)\, N^{p+1-2g}$$

genus series (oriented case)

$\varepsilon_0(p)$ Catalan numbers

Markov operator technology

similar recursion formulas

for the **classical** orthogonal polynomial ensembles

continuous variable : Hermite, Laguerre, Jacobi

discrete variables : Charlier, Meixner, Krawtchouk, Hahn

explicit expressions for the (factorial) moments

Gaussian Orthogonal Ensemble $\beta = 1$

$$P^N(dx) = \frac{1}{Z} \, |\Delta_N(x)| \prod_{i=1}^{N} d\mu(x_i), \quad x = (x_1, \ldots, x_N) \in \mathbb{R}^N$$

$$d\mu(x) = e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}$$

Pfaffian instead of determinant

limiting spectral distribution : **Wigner** semi-circle law

Tracy-Widom edge asymptotics

moment recursion formula ?

mean spectral measure

$$E\Big(\mathrm{Tr}\big(f(X^N)\big)\Big) = \int f(x)\,\mu^N_{\mathrm{GOE}}(x) d\mu(x)$$

$P_\ell, \ \ell \in \mathbb{N}$ Hermite polynomials

$$\mu^N_{\mathrm{GOE}} = \sum_{\ell=0}^{N-1} P_\ell^2 + \frac{e^{x^2/4} P_{N-1}}{\int e^{x^2/4} P_{N-1}\, d\mu}\, \mathbf{1}_{N \text{ odd}}$$

$$+\sqrt{\frac{\pi N}{8}}\, e^{x^2/4} P_{N-1} \int \mathrm{sgn}\,(x-y)\, e^{x^2/4}(y) P_N(y) d\mu(y)$$

Markov operator tools : differential equation

on the Laplace transform of the spectral measure

moment recursion formula

$$b_p^N = E\Big(\mathrm{Tr}\left((X^N)^{2p}\right)\Big), \quad p \in \mathbb{N}$$

recursion formula coupled with the moments a_p^N of the GUE

$$\begin{aligned} b_p^N &= (4N-2)b_{p-1}^N + 4(2p-2)(2p-3)b_{p-2}^N \\ &\quad + a_p^N - (4N-3)a_{p-1}^N - (2p-2)(2p-3)a_{p-2}^N \end{aligned}$$

effective values

GUE

$$
\begin{aligned}
a_0^N &= N \\
a_1^N &= N^2 \\
a_2^N &= 2N^3 + N \\
a_3^N &= 5N^4 + 10N^2 \\
a_4^N &= 14N^5 + 70N^3 + 21N
\end{aligned}
$$

GOE

$$
\begin{aligned}
b_0^N &= N \\
b_1^N &= N^2 + N \\
b_2^N &= 2N^3 + 5N^2 + 5N \\
b_3^N &= 5N^4 + 22N^3 + 52N^2 + 41N \\
b_4^N &= 14N^5 + 93N^4 + 374N^3 + 690N^2 + 509N
\end{aligned}
$$

$$b_p^N = E\Big(\mathrm{Tr}\,\big((X^N)^{2p}\big)\Big), \quad X^N \text{ GOE}$$

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b_p^N **five term** recurrence equation

closed form : **I. Goulden, D. Jackson (1997)**

map enumeration : unoriented case

duality with Symplectic Ensemble $(\beta = 4)$

recursion formulas lead to

sharp moment bounds

Gaussian Unitary Ensemble (GUE)

three term recurrence equation

$$E\Big(\mathrm{Tr}\big((X^N)^{2p}\big)\Big) \leq C\,(4N)^p\, e^{Cp^3/N^2}, \quad p^3 \geq N^2$$

similar bounds for

classical orthogonal polynomial ensembles $(\beta = 2)$

Gaussian Orthogonal Ensemble (GOE) $(\beta = 1)$

non asymptotic small deviation inequalities

$$P\big(\lambda_N^N \leq 2\sqrt{N} + s\,N^{-1/6}\big) \ \to \ F_{\mathrm{TW}}(s), \quad s \in \mathbb{R}$$

$$C^{-1}\,e^{-C\,s^{3/2}} \leq 1 - F_{\mathrm{TW}}(s) \leq C\,e^{-s^{3/2}/C} \quad (s \to \infty)$$

moment bounds (GUE, GOE)

$$P\big(\lambda_N^N \geq 2\sqrt{N} + \varepsilon\big) \leq C\,e^{-N^{1/4}\varepsilon^{3/2}/C}, \quad 0 < \varepsilon \leq \sqrt{N}$$

fits the **Tracy-Widom** asymptotics $\quad (\varepsilon = s\,N^{-1/6})$

deviation inequality on last passage percolation $\quad W_N$

K. Johansson (2000)

moment comparison

Wigner matrices (independent entries)

example : sign matrix $\quad Y^N = Y_{ij}^N = \pm 1$

asymptotic result : **Y. Sinai, A. Soshnikov (1998-99)**

$$P\big(\lambda_N^N(Y) \leq 2\sqrt{N} + s\,N^{-1/6}\big) \;\to\; F_{\mathrm{TW}}(s), \quad s \in \mathbb{R}$$

$$E\Big(\mathrm{Tr}\big((Y^N)^{2p}\big)\Big) \leq E\Big(\mathrm{Tr}\big((X^N)^{2p}\big)\Big), \quad X^N \text{ GOE}$$

$$P\big(\lambda_N^N(Y) \geq 2\sqrt{N} + \varepsilon\big) \leq C\,e^{-N^{1/4}\varepsilon^{3/2}/C}, \quad 0 < \varepsilon \leq \sqrt{N}$$