Concentration Inequalities for Random Matrices

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exponential tail inequalities

classical theme in probability and statistics

quantify the asymptotic statements

central limit theorems

large deviation principles

classical exponential inequalities

sum of independent random variables

$$S_n = \frac{1}{\sqrt{n}} (X_1 + \cdots + X_n)$$

$0 \leq X_i \leq 1$ independent

$$\mathbb{P}\big(S_n \geq \mathbb{E}(S_n) + t\big) \leq e^{-t^2/2}, \quad t \geq 0$$

Hoeffding's inequality

same as for X_i standard Gaussian

central limit theorem

measure concentration ideas

asymptotic geometric analysis

V. Milman (1970)

$$S_n = \frac{1}{\sqrt{n}} (X_1 + \cdots + X_n)$$

$F(X) = F(X_1, \ldots, X_n), \quad F : \mathbb{R}^n \to \mathbb{R}$ Lipschitz

Gaussian sample

independent random variables

concentration inequalities

$$S_n = \frac{1}{\sqrt{n}}\,(X_1 + \cdots + X_n)$$

$F(X) = F(X_1, \ldots, X_n), \quad F : \mathbb{R}^n \to \mathbb{R}$ 1-Lipschitz

$X_1, \ldots, X_n$ independenty standard Gaussian

$$\mathbb{P}\Big(F(X) \geq \mathbb{E}\big(F(X)\big) + t\Big) \leq e^{-t^2/2}, \quad t \geq 0$$

$0 \leq X_i \leq 1$ independent, F 1-Lipschitz and convex

$$\mathbb{P}\Big(F(X) \geq \mathbb{E}\big(F(X)\big) + t\Big) \leq 2\,e^{-t^2/4}, \quad t \geq 0$$

M. Talagrand (1995)

empirical processes

$X_1, \ldots, X_n$ independent with values in $(S, \mathcal{S})$

$\mathcal{F}$ collection of functions $f : S \to [0,1]$

$$Z = \sup_{f \in \mathcal{F}} \sum_{i=1}^{n} f(X_i)$$

Z Lipschitz and convex

concentration inequalities on

$$\mathbb{P}\Big(\big| Z - \mathbb{E}(Z) \big| \geq t \Big), \quad t \geq 0$$

$$Z = \sup_{f \in \mathcal{F}} \sum_{i=1}^{n} f(X_i)$$

$$|f| \leq 1, \quad \mathbb{E}(f(X_i)) = 0, \quad f \in \mathcal{F}$$

$$\mathbb{P}\big(|Z - M| \geq t\big) \leq C \exp\left(- \frac{t}{C} \log\left(1 + \frac{t}{\sigma^2 + M}\right)\right), \quad t \geq 0$$

$C > 0$ numerical constant, M mean or median of Z

$\sigma^2 = \sup_{f \in \mathcal{F}} \sum_{i=1}^{n} \mathbb{E}(f^2(X_i))$

M. Talagrand (1996)

P. Massart (2000)

S. Boucheron, G. Lugosi, P. Massart (2005)

P.-M. Samson (2000) (dependence)

concentration inequalities

numerous applications

- geometric functional analysis
- discrete and combinatorial probability
- empirical processes
- statistical mechanics
- random matrix theory

concentration inequalities

numerous applications

- geometric functional analysis
- discrete and combinatorial probability
- empirical processes
- statistical mechanics
- **random matrix theory**

recent studies of

random matrix and random growth models

new asymptotics

common, non-central, rate $(\mathrm{mean})^{1/3}$

universal limiting **Tracy-Widom** distribution

random matrices, longest increasing subsequence,

random growth models, last passage percolation...

sample covariance matrices

multivariate statistical inference

principal component analysis

population $(Y_1, \ldots, Y_N)$

Y_j vectors (column) in $\mathbb{R}^M$ (characters)

$Y = (Y_1, \ldots, Y_N)$ $M \times N$ matrix

sample covariance matrix $Y \, Y^t$ $(M \times M)$

(independent) **Gaussian** Y_j : **Wishart matrix models** is $Y\,Y^t$ a good approximation of the

population covariance matrix

$$\mathbb{E}(Y\,Y^t) \ ?$$

M finite

$$\frac{1}{N}\,Y\,Y^t \to \mathbb{E}(Y\,Y^t) \qquad N \to \infty$$

M infinite ?

$$M = M(N) \to \infty \qquad N \to \infty$$

$$\frac{M}{N} \sim \rho \in (0, \infty) \qquad N \to \infty$$

sample covariance matrices

$Y = (Y_1, \ldots, Y_N)$ $\quad M \times N$ matrix

$$Y = (Y_{ij})_{1 \leq i \leq M, 1 \leq j \leq N}$$

Y_{ij} independent identically distributed

(real or complex)

$$\mathbb{E}(Y_{ij}) = 0, \quad \mathbb{E}(Y_{ij}^2) = 1$$

Wishart model : Y_j standard Gaussian in $\mathbb{R}^M$

numerous extensions

sample covariance matrices

$$Y = (Y_1, \ldots, Y_N) \qquad M \times N \quad \text{matrix}$$

$$Y = (Y_{ij})_{1 \leq i \leq M, 1 \leq j \leq N} \quad \text{iid} \qquad \mathbb{E}(Y_{ij}) = 0, \ \mathbb{E}(Y_{ij}^2) = 1$$

center of interest : **eigenvalues** $0 \leq \lambda_1^N \leq \cdots \leq \lambda_M^N$

of $Y\,Y^t$ ($M \times M$ non-negative symmetric matrix)

$\sqrt{\lambda_k^N}$ singular values of Y

$\widehat{\lambda}_k^N = \dfrac{\lambda_k^N}{N}$ eigenvalues of $\dfrac{1}{N}\,Y\,Y^t$

spectral measure $\displaystyle \frac{1}{M} \sum_{k=1}^{M} \delta_{\widehat{\lambda}_k^N}$

asymptotics $M = M(N) \sim \rho\, N \qquad N \to \infty$

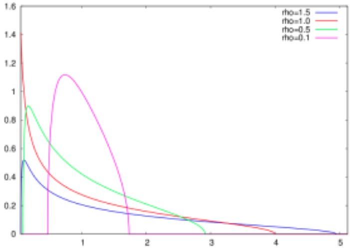
Marchenko-Pastur theorem (1967)

asymptotic behavior of the spectral measure $\left(\widehat{\lambda}_k^N = \lambda_k^N / N\right)$

$$\frac{1}{M}\sum_{k=1}^{M} \delta_{\widehat{\lambda}_k^N} \to \nu \quad \text{Marchenko-Pastur distribution}$$

$$d\nu(x) = \left(1 - \frac{1}{\rho}\right)_+ \delta_0 + \frac{1}{\rho\, 2\pi x}\sqrt{(b-x)(x-a)}\,\mathbf{1}_{[a,b]}dx$$

$$a = a(\rho) = \left(1 - \sqrt{\rho}\,\right)^2 \qquad b = b(\rho) = \left(1 + \sqrt{\rho}\,\right)^2$$



Marchenko-Pastur theorem

$$\frac{1}{M}\sum_{k=1}^{M}\delta_{\widehat{\lambda}_k^N} \;\to\; \nu \quad \text{on} \quad \big(a(\rho), b(\rho)\big) \qquad M \sim \rho\, N$$

global regime

large deviation asymptotics of the spectral measure

fluctuations of the spectral measure

$$\sum_{k=1}^{M}\Big[f\big(\widehat{\lambda}_k^N\big) - \int_{\mathbb{R}} f\, d\nu\Big] \;\to\; G \quad \text{Gaussian variable}$$

$f : \mathbb{R} \to \mathbb{R}$ smooth

Marchenko-Pastur theorem

$$\frac{1}{M}\sum_{k=1}^{M}\delta_{\widehat{\lambda}_k^N} \;\to\; \nu \quad \text{on} \quad \big(a(\rho), b(\rho)\big) \qquad M \sim \rho\, N$$

local regime

behavior of the individual eigenvalues

spacings (bulk behavior)

extremal eigenvalues (edge behavior)

extremal eigenvalues

largest eigenvalue $\lambda_M^N = \max_{1 \le k \le M} \lambda_k^N$

$$\widehat{\lambda}_M^N = \frac{\lambda_M^N}{N} \;\to\; b(\rho) = \left(1 + \sqrt{\rho}\,\right)^2 \qquad M \sim \rho\, N$$

Marchenko-Pastur theorem (1967)

asymptotic behavior of the spectral measure $\left(\widehat{\lambda}_k^N = \lambda_k^N / N\right)$

$$\frac{1}{M}\sum_{k=1}^{M}\delta_{\widehat{\lambda}_k^N} \;\to\; \nu$$ **Marchenko-Pastur** distribution

$$d\nu(x) = \left(1-\frac{1}{\rho}\right)_+ \delta_0 + \frac{1}{\rho\, 2\pi x}\sqrt{(b-x)(x-a)}\,\mathbf{1}_{[a,b]}dx$$

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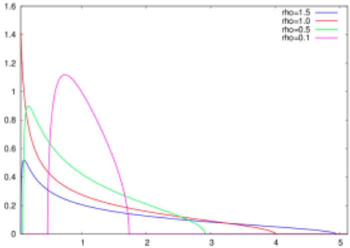

extremal eigenvalues

largest eigenvalue $\lambda_M^N = \max_{1 \le k \le M} \lambda_k^N$

$$\widehat{\lambda}_M^N = \frac{\lambda_M^N}{N} \;\to\; b(\rho) = \big(1 + \sqrt{\rho}\,\big)^2 \qquad M \sim \rho\, N$$

fluctuations around $b(\rho)$

complex **or** **real Gaussian (Wishart matrices)**

F_{TW} **C. Tracy, H. Widom (1994)** distribution

K. Johansson (2000), I. Johnstone (2001)

extremal eigenvalues

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fluctuations around $b(\rho)$

complex or real Gaussian (Wishart matrices)

$$M^{2/3}\big[\,\widehat{\lambda}_M^N - b(\rho)\big] \ \to \ C(\rho)\, F_{\mathrm{TW}}$$

F_{TW} **C. Tracy, H. Widom (1994)** distribution

K. Johansson (2000), I. Johnstone (2001)

extremal eigenvalues

largest eigenvalue $\lambda_M^N = \max_{1 \leq k \leq M} \lambda_k^N$

$$\widehat{\lambda}_M^N = \frac{\lambda_M^N}{N} \ \rightarrow \ b(\rho) = \big(1 + \sqrt{\rho}\,\big)^2 \qquad M \sim \rho N$$

fluctuations around $b(\rho)$

complex or real Gaussian (Wishart matrices)

$$M^{2/3} N^{-1} \big[\lambda_M^N - b(\rho) N\big] \ \rightarrow \ C(\rho)\, F_{\mathrm{TW}}$$

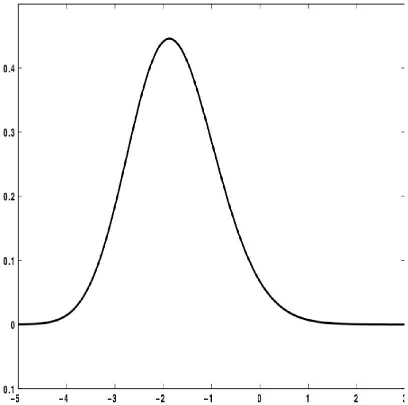
F_{TW} **C. Tracy, H. Widom (1994)** distribution

K. Johansson (2000), I. Johnstone (2001)

F_{TW} **C. Tracy, H. Widom (1994)** distribution

(complex) $$F_{\mathrm{TW}}(s) = \exp\left(- \int_s^\infty (x-s)u(x)^2 dx \right), \quad s \in \mathbb{R}$$

$$u'' = 2u^3 + xu$$ Painlevé II equation

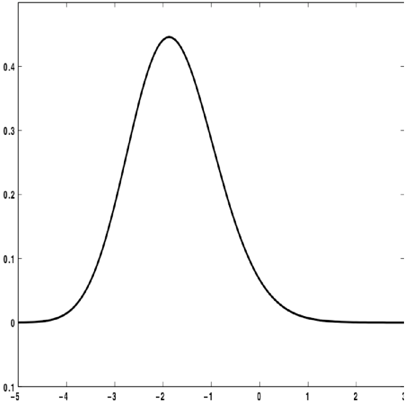


density

mean $\simeq -1.77$

$$F_{\mathrm{TW}}(s) \sim e^{-s^3/12} \quad \text{as} \quad s \to -\infty$$

$$1 - F_{\mathrm{TW}}(s) \sim e^{-4s^{3/2}/3} \quad \text{as} \quad s \to +\infty$$



density (similar for real case)

extremal eigenvalues

largest eigenvalue $\lambda_M^N = \max_{1 \leq k \leq M} \lambda_k^N$

$$\widehat{\lambda}_M^N = \frac{\lambda_M^N}{N} \ \rightarrow \ b(\rho) = \big(1 + \sqrt{\rho}\,\big)^2 \qquad M \sim \rho\, N$$

fluctuations around $b(\rho)$

complex or real Gaussian (Wishart matrices)

$$M^{2/3} \big[\, \widehat{\lambda}_M^N - b(\rho) \big] \ \rightarrow \ C(\rho)\, F_{\mathrm{TW}}$$

F_{TW} **C. Tracy, H. Widom (1994)** distribution

K. Johansson (2000), I. Johnstone (2001)

Gaussian (Wishart matrices)

completely solvable models

determinantal structure

orthogonal polynomial analysis

asymptotics of Laguerre orthogonal polynomials

C. Tracy, H. Widom (1994)

K. Johansson (2000), I. Johnstone (2001)

extension to non-Gaussian matrices

A. Soshnikov (2001-02)

moment method $\mathbb{E}\big(\mathrm{Tr}\big((YY^t)^p\big)\big)$

L. Erdös, H.-T. Yau (2009-12) (and collaborators)

local Marchenko-Pastur law

T. Tao, V. Vu (2010-11)

Lindeberg comparison method

symmetric matrices

(brief) survey of recent approaches to

non-asymptotic exponential inequalities

quantify the limit theorems

spectral measure

extremal eigenvalues

catch the **new rate** $(\text{mean})^{1/3}$

from the Gaussian case to non-Gaussian models

two main questions and objectives

tail inequalities for the spectral measure

$$\mathbb{P}\left(\sum_{k=1}^{M} f(\widehat{\lambda}_k^N) \geq t\right)$$

Marchenko-Pastur theorem

$$\frac{1}{M}\sum_{k=1}^{M}\delta_{\widehat{\lambda}_k^N} \;\to\; \nu \quad \text{on} \quad \big(a(\rho), b(\rho)\big) \qquad M \sim \rho\, N$$

global regime

large deviation asymptotics of the spectral measure

fluctuations of the spectral measure

$$\sum_{k=1}^{M}\Big[f\big(\widehat{\lambda}_k^N\big) - \int_{\mathbb{R}} f\, d\nu\Big] \;\to\; G \quad \text{Gaussian variable}$$

$f : \mathbb{R} \to \mathbb{R}$ smooth

two main questions and objectives

tail inequalities for the spectral measure

$$\mathbb{P}\Big(\sum_{k=1}^{M} f(\widehat{\lambda}_k^N) \geq t \Big)$$

tail inequalities for the extremal eigenvalues

$$\mathbb{P}\big(\widehat{\lambda}_M^N \geq b(\rho) + \varepsilon \big)$$

extremal eigenvalues

largest eigenvalue $\lambda_M^N = \max_{1 \leq k \leq M} \lambda_k^N$

$$\widehat{\lambda}_M^N = \frac{\lambda_M^N}{N} \;\to\; b(\rho) = \left(1 + \sqrt{\rho}\,\right)^2 \qquad M \sim \rho\, N$$

fluctuations around $b(\rho)$

complex **or** **real Gaussian (Wishart matrices)**

$$M^{2/3} \left[\, \widehat{\lambda}_M^N - b(\rho)\right] \;\to\; C(\rho)\, F_{\mathrm{TW}}$$

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K. Johansson (2000), I. Johnstone (2001)

two main questions and objectives

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$$\mathbb{P}\Big(\sum_{k=1}^{M} f(\widehat{\lambda}_k^N) \geq t\Big)$$

tail inequalities for the extremal eigenvalues

$$\mathbb{P}\big(\widehat{\lambda}_M^N \geq b(\rho) + \varepsilon\big)$$

Wishart matrices

more general covariance matrices

measure concentration tool

$$F = F(Y\,Y^t) = F(Y_{ij})$$

satisfactory for the global regime

less satisfactory for the local regime

specific functionals

eigenvalue counting function

extreme eigenvalues

two main questions and objectives

tail inequalities for the spectral measure

$$\mathbb{P}\Big(\sum_{k=1}^{M} f(\widehat{\lambda}_k^N) \geq t\Big)$$

tail inequalities for the extremal eigenvalues

$$\mathbb{P}\big(\widehat{\lambda}_M^N \geq b(\rho) + \varepsilon\big)$$

Wishart matrices

more general covariance matrices

tail inequalities for the spectral measure

A. Guionnet, O. Zeitouni (2000)

measure concentration tool

$f : \mathbb{R} \to \mathbb{R}$ smooth (Lipschitz)

$X = (X_{ij})_{1 \leq i,j \leq M}$ $M \times M$ symmetric matrix

eigenvalues $\lambda_1 \leq \cdots \leq \lambda_M$

$$F : X \to \mathrm{Tr}\, f(X) = \sum_{k=1}^{M} f(\lambda_k)$$ Lipschitz

with respect to the Euclidean structure on $M \times M$ matrices

convex if f is convex

concentration inequalities

$$S_n = \frac{1}{\sqrt{n}} (X_1 + \cdots + X_n)$$

$F(X) = F(X_1, \ldots, X_n), \quad F : \mathbb{R}^n \to \mathbb{R} \quad 1$-Lipschitz

$X_1, \ldots, X_n$ independenty standard Gaussian

$$\mathbb{P}\Big(F(X) \geq \mathbb{E}\big(F(X)\big) + t\Big) \leq e^{-t^2/2}, \quad t \geq 0$$

$0 \leq X_i \leq 1$ independent, F 1-Lipschitz and convex

$$\mathbb{P}\Big(F(X) \geq \mathbb{E}\big(F(X)\big) + t\Big) \leq 2\, e^{-t^2/4}, \quad t \geq 0$$

M. Talagrand (1995)

tail inequalities **for the** **spectral measure**

Gaussian entries Y_{ij}

$f : \mathbb{R} \to \mathbb{R}$ such that $f(x^2)$ 1-Lipschitz

$$\mathbb{P}\bigg(\sum_{k=1}^{M} \Big[f(\widehat{\lambda}_k^N) - \mathbb{E}\big(f(\widehat{\lambda}_k^N)\big) \Big] \geq t \bigg) \leq C(\rho)\, e^{-t^2/C(\rho)}, \quad t \geq 0$$

compactly supported entries Y_{ij}

$f : \mathbb{R} \to \mathbb{R}$ such that $f(x^2)$ 1-Lipschitz and convex

Marchenko-Pastur theorem

$$\frac{1}{M}\sum_{k=1}^{M}\delta_{\widehat{\lambda}_k^N} \ \to \ \nu \quad \text{on} \quad \big(a(\rho), b(\rho)\big) \qquad M \sim \rho\, N$$

global regime

large deviation asymptotics of the spectral measure

fluctuations of the spectral measure

$$\sum_{k=1}^{M}\Big[f\big(\widehat{\lambda}_k^N\big) - \int_{\mathbb{R}} f\, d\nu\Big] \ \to \ G \quad \text{Gaussian variable}$$

$f : \mathbb{R} \to \mathbb{R}$ smooth

non-Lipschitz functions f

typically $f = \mathbf{1}_I, \quad I \subset \mathbb{R}$ interval

$$\sum_{k=1}^{M} f\big(\widehat{\lambda}_k^N\big) = \#\big\{\widehat{\lambda}_k^N \in I\big\} = \mathcal{N}_I$$ counting function

Wishart matrices (determinantal structure)

I interval in (a, b)

$$\frac{1}{\sqrt{\log M}}\big[\mathcal{N}_I - \mathbb{E}(\mathcal{N}_I)\big] \ \to \ G$$ Gaussian variable

exponential tail inequalities

$$\mathbb{P}\big(\mathcal{N}_I - \mathbb{E}(\mathcal{N}_I) \geq t\big) \leq C\, e^{-ct \log(1+t/\log M)}, \quad t \geq 0$$

$$\mathrm{Var}\big(\mathcal{N}_I\big) = O(\log M)$$

non-Gaussian covariance matrices

comparison with Wishart model

partial results

localization results **L. Erdös, H.-T. Yau (2009-12)**

Lindeberg comparison method **T. Tao, V. Vu (2010-11)**

$$\mathrm{Var}\big(\mathcal{N}_I\big) = O(\log M)$$

S. Dallaporta, V. Vu (2011)

$$\mathbb{P}\big(\mathcal{N}_I - \mathbb{E}(\mathcal{N}_I) \geq t\big) \leq C\, e^{-ct^{\delta}}, \quad t \geq C \log M,\ 0 < \delta \leq 1$$

T. Tao, V. Vu (2012)

non-Lipschitz functions f

typically $f = \mathbf{1}_I, \quad I \subset \mathbb{R}$ interval

$$\sum_{k=1}^{M} f\big(\widehat{\lambda}_k^N\big) = \#\big\{\widehat{\lambda}_k^N \in I\big\} = \mathcal{N}_I \qquad \text{counting function}$$

Wishart matrices (determinantal structure)

I interval in (a, b)

$$\frac{1}{\sqrt{\log M}}\,\big[\mathcal{N}_I - \mathbb{E}(\mathcal{N}_I)\big] \;\to\; G \quad \text{Gaussian variable}$$

exponential tail inequalities

$$\mathbb{P}\big(\mathcal{N}_I - \mathbb{E}(\mathcal{N}_I) \geq t\big) \leq C\, e^{-ct\log(1+t/\log M)}, \quad t \geq 0$$

$$\mathrm{Var}\big(\mathcal{N}_I\big) = O(\log M)$$

two main questions and objectives

tail inequalities for the spectral measure

$$\mathbb{P}\Big(\sum_{k=1}^{M} f(\widehat{\lambda}_k^N) \geq t \Big)$$

tail inequalities for the extremal eigenvalues

$$\mathbb{P}\big(\widehat{\lambda}_M^N \geq b(\rho) + \varepsilon \big)$$

Wishart matrices

more general covariance matrices

two main questions and objectives

tail inequalities for the spectral measure

$$\mathbb{P}\Big(\sum_{k=1}^{M} f(\widehat{\lambda}_k^N) \geq t \Big)$$

tail inequalities for the extremal eigenvalues

$$\mathbb{P}\big(\widehat{\lambda}_M^N \geq b(\rho) + \varepsilon \big)$$

Wishart matrices

more general covariance matrices

tail inequalities for the **extremal eigenvalues**

fluctuations of the largest eigenvalue

$$M^{2/3}\left[\,\widehat{\lambda}_M^N - b(\rho)\right] \to C(\rho)\,F_{\mathrm{TW}} \qquad M \sim \rho\,N$$

extremal eigenvalues

largest eigenvalue $\lambda_M^N = \max_{1 \leq k \leq M} \lambda_k^N$

$$\widehat{\lambda}_M^N = \frac{\lambda_M^N}{N} \;\to\; b(\rho) = \big(1 + \sqrt{\rho}\,\big)^2 \qquad M \sim \rho\, N$$

fluctuations around $b(\rho)$

complex or real Gaussian (Wishart matrices)

$$M^{2/3}\big[\,\widehat{\lambda}_M^N - b(\rho)\big] \;\to\; C(\rho)\, F_{\mathrm{TW}}$$

F_{TW} **C. Tracy, H. Widom (1994)** distribution

K. Johansson (2000), I. Johnstone (2001)

tail inequalities for the **extremal eigenvalues**

fluctuations of the largest eigenvalue

$$M^{2/3}\left[\,\widehat{\lambda}_M^N - b(\rho)\right] \;\to\; C(\rho)\,F_{\mathrm{TW}} \qquad M \sim \rho\,N$$

finite M inequalities

at the $(\mathrm{mean})^{1/3}$ rate

reflecting the tails of F_{TW}

bounds on $\mathrm{Var}(\,\widehat{\lambda}_M^N)$

measure concentration tool

(Gaussian) Wishart matrix $\quad Y\,Y^t$

$$\lambda_M^N = \max_{1 \leq k \leq M} \lambda_k^N = \sup_{|v|=1} |Y\,v|^2$$

$s_M^N = \sqrt{\lambda_M^N}$ Lipschitz of the Gaussian entries $\quad Y_{ij}$

Gaussian concentration

$$\mathbb{P}\Big(\widehat{s}_M^N \geq \mathbb{E}(\widehat{s}_M^N) + t \Big) \leq e^{-M\,t^2/C}, \quad t \geq 0$$

$$\mathbb{E}(\widehat{s}_M^N) \sim \sqrt{b(\rho)}$$

correct large deviation bounds $\quad (t \geq 1)$

measure concentration tool

(Gaussian) Wishart matrix $Y\,Y^t$

$$\lambda_M^N = \max_{1 \leq k \leq M} \lambda_k^N = \sup_{|v|=1} |Y\,v|^2$$

$s_M^N = \sqrt{\lambda_M^N}$ Lipschitz of the Gaussian entries Y_{ij}

Gaussian concentration

$$\mathbb{P}\Big(\widehat{s}_M^N \geq \mathbb{E}(\widehat{s}_M^N) + t \Big) \leq e^{-M\,t^2/C}, \quad t \geq 0$$

$$\mathbb{E}(\widehat{s}_M^N) \sim \sqrt{b(\rho)}$$

does not fit the small deviation regime $t = s\,M^{-2/3}$

extreme eigenvalues

alternate tools

Riemann-Hilbert analysis (Wishart matrices)

tri-diagonal representations (Wishart and β-ensembles)

moment methods (Wishart and non-Gaussian matrices)

extreme eigenvalues

alternate tools

Riemann-Hilbert analysis (Wishart matrices)

tri-diagonal representations (Wishart and β-ensembles)

moment methods (Wishart and non-Gaussian matrices)

$$M^{2/3}\big[\,\widehat{\lambda}_M^N - b(\rho)\big] \,\to\, C(\rho)\,F_{\mathrm{TW}}$$

$$\mathbb{P}\Big(\widehat{\lambda}_M^N \leq b(\rho) + s\,M^{-2/3}\Big) \,\to\, F_{\mathrm{TW}}(C\,s)$$

bounds for Wishart matrices

tri-diagonal representation

B. Rider, M. L. (2010)

$$\mathbb{P}\big(\,\widehat{\lambda}_M^N \geq b(\rho) + \epsilon\big) \,\leq\, C\,e^{-M\varepsilon^{3/2}/C}, \qquad 0 < \varepsilon \leq 1$$

$$\mathbb{P}\big(\,\widehat{\lambda}_M^N \leq b(\rho) - \epsilon\big) \,\leq\, C\,e^{-M\varepsilon^{3}/C}, \qquad 0 < \varepsilon \leq b(\rho)$$

$$\mathbb{P}\Big(\widehat{\lambda}_M^N \leq b(\rho) + s\,M^{-2/3}\Big) \;\to\; F_{\mathrm{TW}}(C\,s)$$

bounds for Wishart matrices

$$\mathbb{P}\big(\,\widehat{\lambda}_M^N \geq b(\rho) + \epsilon\big) \;\leq\; C\,e^{-M\varepsilon^{3/2}/C}, \qquad 0 < \varepsilon \leq 1$$

$$\mathbb{P}\big(\,\widehat{\lambda}_M^N \leq b(\rho) - \epsilon\big) \;\leq\; C\,e^{-M\varepsilon^{3}/C}, \qquad 0 < \varepsilon \leq b(\rho)$$

fit the **Tracy-Widom** asymptotics $\quad (\varepsilon = s\,M^{-2/3})$

$$1 - F_{\mathrm{TW}}(s) \sim e^{-s^{3/2}/C} \quad (s \to +\infty)$$

$$F_{\mathrm{TW}}(s) \sim e^{-s^{3}/C} \quad (s \to -\infty)$$

$$\mathrm{Var}\big(\,\widehat{\lambda}_M^N\big) = O\Big(\frac{1}{M^{4/3}}\Big)$$

$$M^{2/3}\left[\,\widehat{\lambda}_M^N - b(\rho)\right] \;\to\; C(\rho)\,F_{\mathrm{TW}}$$

$$b(\rho) = \left(1+\sqrt{\rho}\right)^2$$

$$\widehat{\lambda}_M^N = \lambda_M^N/N, \qquad M = M(N) \sim \rho\,N$$

$$\frac{(\sqrt{MN})^{1/3}}{(\sqrt{M}+\sqrt{N})^{4/3}}\left(\lambda_M^N - (\sqrt{M}+\sqrt{N})^2\right) \;\to\; F_{\mathrm{TW}}$$

$$N+1 \geq M \qquad 0 < \varepsilon \leq 1$$

$$\mathbb{P}\left(\lambda_M^N \geq (\sqrt{M}+\sqrt{N})^2(1+\varepsilon)\right) \;\leq\; C\,e^{-\sqrt{MN}\,\varepsilon^{3/2}\left(\frac{1}{\sqrt{\varepsilon}}\wedge\left(\frac{M}{N}\right)^{1/4}\right)/C}$$

$$\mathbb{P}\left(\lambda_M^N \leq (\sqrt{M}+\sqrt{N})^2(1-\varepsilon)\right) \;\leq\; C\,e^{-MN\,\varepsilon^{3}\left(\frac{1}{\varepsilon}\wedge\left(\frac{M}{N}\right)^{1/2}\right)/C}$$

bi and tri-diagonal representation

$$B = \begin{pmatrix} \chi_N & 0 & 0 & \cdots & \cdots & 0 \\ \widetilde{\chi}_{(M-1)} & \chi_{N-1} & 0 & 0 & \cdots & \vdots \\ 0 & \widetilde{\chi}_{(M-2)} & \chi_{N-3} & 0 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & \ddots & \widetilde{\chi}_2 & \chi_{N-M+2} & 0 \\ 0 & \cdots & \cdots & 0 & \widetilde{\chi}_1 & \chi_{N-M+1} \end{pmatrix}$$

$\chi_{(N-1)}, \ldots, \chi_1, \quad \widetilde{\chi}_{(M-1)}, \ldots, \widetilde{\chi}_1$ independent chi-variables

$B\,B^t$ same spectrum as $Y\,Y^t$ (Wishart)

H. Trotter (1984), A. Edelman, I. Dimitriu (2002)

extension to β-ensembles

bounds for non-Gaussian entries

moment method $\mathbb{E}\big(\mathrm{Tr}\big((YY^t)^p\big)\big)$

O. Feldheim, S. Sodin (2010)

largest eigenvalue (symmetric, subGaussian entries)

$$\mathbb{P}\big(\widehat{\lambda}_M^N \geq b(\rho) + \varepsilon\big) \leq C\, e^{-M\,\varepsilon^{3/2}/C}, \quad 0 < \varepsilon \leq 1$$

below the mean ?

necessary for variance bounds

variance level

$$\mathrm{Var}(\widehat{\lambda}_M^N) = O\Big(\frac{1}{M^{4/3}}\Big)$$

S. Dallaporta (2012)

comparison with Wishart model

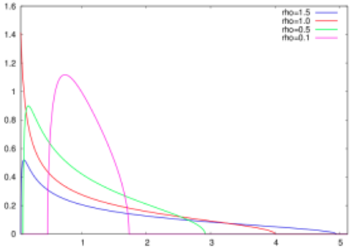
localization results **L. Erdös, H.-T. Yau (2009-12)**

Lindeberg comparison method **T. Tao, V. Vu (2010-11)**

smallest eigenvalue

soft edge $M = M(N) \sim \rho N, \quad \rho < 1$

$$a(\rho) = \left(1 - \sqrt{\rho}\right)^2$$



$$\mathbb{P}\left(\widehat{\lambda}_1^N \leq a(\rho) - \varepsilon\right) \leq C\, e^{-M\,\varepsilon^{3/2}/C}, \quad 0 < \varepsilon \leq 1$$

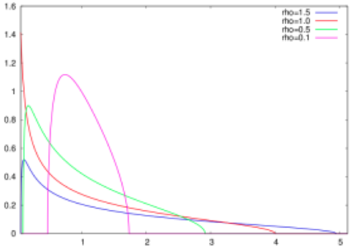
$$\mathbb{P}\left(\widehat{\lambda}_1^N \geq a(\rho) + \varepsilon\right) \leq C\, e^{-M\,\varepsilon^{3}/C}, \quad 0 < \varepsilon \leq a(\rho)$$

Wishart matrices **B. Rider, M. L. (2010)**

smallest eigenvalue

hard edge $M = N, \quad \rho = 1$

$$a(\rho) = \left(1 - \sqrt{\rho}\,\right)^2 = 0$$



$$\mathbb{P}\left(\widehat{\lambda}_1^N \leq \frac{\varepsilon}{N^2}\right) \leq C\sqrt{\varepsilon} + C\,e^{-cN}$$

large families of covariance matrices

M. Rudelson, R. Vershynin (2008-10)