Chaos of a Markov operator and the fourth moment condition

M. Ledoux

Institut de Mathématiques de Toulouse, France

fourth-moment theorem

D. Nualart, G. Peccati (2005)

condition for a Wiener chaos

to be close to Gaussian

multiple Wiener integrals

simplified (finite-dimensional) model

Wiener (Gaussian) chaos of order k

$$F = F(x) = \sum_{i_1,\ldots,i_k=1}^{N} a_{i_1,\ldots,i_k} x_{i_1} \cdots x_{i_k}, \qquad x = (x_1, \ldots, x_N) \in \mathbb{R}^N$$

multi-linear form

$a_{i_1,\ldots,i_k} \in \mathbb{R}$ symmetric, vanishing on diagonals

$$\int_{\mathbb{R}^N} F^2 d\gamma_N = 1 \qquad \left(\int_{\mathbb{R}^N} F \, d\gamma_N = 0\right)$$

$d\gamma_N(x) = e^{-|x|^2/2} \dfrac{dx}{(2\pi)^n}$ standard Gaussian measure on $\mathbb{R}^N$

multiple Wiener integrals

simplified (finite-dimensional) model

Wiener (Gaussian) chaos of order k

$$F = F(X) = \sum_{i_1,\ldots,i_k=1}^{N} a_{i_1,\ldots,i_k} X_{i_1} \cdots X_{i_k}$$

$X_1, \ldots, X_N$ independent standard normal

$a_{i_1,\ldots,i_k} \in \mathbb{R}$ symmetric, vanishing on diagonals

$$\mathbb{E}\big(F(X)^2\big) = 1 \qquad \big(\mathbb{E}\big(F(X)\big) = 0\big)$$

$F = F_n, \quad n \in \mathbb{N} \qquad k\text{-chaos} \quad (k \ \text{fixed})$

$$N = N_n \to \infty$$

$$\int_{\mathbb{R}^{N^n}} F_n^2 \, d\gamma_{N_n} = 1 \qquad (\text{or} \ \to 1)$$

Theorem by **D. Nualart, G. Peccati (2005)**

distribution of F_n converges to γ_1 (standard normal on $\mathbb{R}$)

if and only if

$$\int_{\mathbb{R}^{N_n}} F_n^4 \, d\gamma_{N_n} \to 3 \qquad \left(= \int_{\mathbb{R}} x^4 d\gamma_1 \right)$$

striking reduction of the moment method

original proof

stochastic calculus

Wiener chaos (multiple Wiener integrals)

$$F = \int_{[0,1]^k} f(t_1, \ldots, t_k) dB_{t_1} \cdots dB_{t_k}$$

$f \in \mathrm{L}^2([0,1]^k; \mathbb{R})$ symmetric

$$F = I_k(f)$$

main tool : multiplication formula

$$I_k(f)\, I_\ell(g) = \sum_{r=0}^{k\wedge\ell} r!\binom{k}{r}\binom{\ell}{r}\, I_{k+\ell-2}(f\,\widetilde{\otimes}_r g)$$

main tool : multiplication formula

$$H_k \, H_\ell = \sum_{r=0}^{k \wedge \ell} r! \binom{k}{r} \binom{\ell}{r} H_{k+\ell-2r}$$

H_k Hermite polynomials

main tool : multiplication formula

$$I_k(f)\, I_\ell(g) \,=\, \sum_{r=0}^{k\wedge\ell} r! \binom{k}{r} \binom{\ell}{r} I_{k+\ell-2}(f \widetilde{\otimes}_r g)$$

contraction

$$f \otimes_r g = \int_{[0,1]^r} f(t_1, \ldots, t_{k-r}, s_1, \ldots, s_r) \times$$

$$g(t_{k-r+1}, \ldots, t_{k+\ell-2r}, s_1, \ldots, s_r) ds_1 \cdots ds_r$$

$f \,\widetilde{\otimes}_r\, g$ symmetrized

$\int_{\mathbb{R}^{N_n}} F_n^4 \, d\gamma_{N_n} \,\to\, 3$ implies

$$\big\| f_n \,\widetilde{\otimes}_p\, f_n \big\|_2 \,\to\, 0, \qquad p = 1, \ldots, k-1$$

combinatorial arguments

D. Nualart, G. Peccati (2005)

$I_k(f_n) = W_{T_n}$ time change $T_n \to 1$

I. Nourdin, G. Peccati (2009)

moments $\mathbb{E}\big(I_k(f_n)^q\big) \to \int_{\mathbb{R}} x^q d\gamma_1$

D. Nualart, S. Ortiz-Latorre (2008)

stochastic calculus (Malliavin)

further equivalence

$\mathrm{Var}_{\gamma_{N_n}}\big(|\nabla F_n|^2\big) \to 0$

I. Nourdin, J. Rosinski (2012)

covariance criterion

first objectives

Gaussian k-chaos

$$F = F(x) = \sum_{i_1,\ldots,i_k=1}^{N} a_{i_1,\ldots,i_k} \, x_{i_1} \cdots x_{i_k}$$

understand what is used on F

why a fourth moment condition $\int_{\mathbb{R}^N} F^4 d\gamma_N \sim 3$

connection with $\mathrm{Var}_{\gamma_N}\big(|\nabla F|^2\big)$

Gaussian k-chaos

$$F = F(x) = \sum_{i_1,\ldots,i_k=1}^{N} a_{i_1,\ldots,i_k} \, x_{i_1} \cdots x_{i_k}$$

first feature : **eigenfunction,** eigenvalue k

$$-\mathrm{L}F = k\,F$$

$\mathrm{L} = \Delta - x \cdot \nabla$ Ornstein-Uhlenbeck operator on $\mathbb{R}^N$

$$\mathrm{L}\,x_1 = -x_1, \qquad \mathrm{L}\,x_1 x_2 = -2\,x_1 x_2$$

invariant (reversible) measure $\gamma_N = \gamma$

integration by parts

$$\int_{\mathbb{R}^N} f(-\mathrm{L}g)d\gamma = \int_{\mathbb{R}^N} \nabla f \cdot \nabla g \, d\gamma$$

connection with $\mathrm{Var}_{\gamma_N}(|\nabla F|^2)$

$$-\mathrm{L}F = k\,F$$

integration by parts

$$k\int_{\mathbb{R}^N} F^4 d\gamma = \int_{\mathbb{R}^N} F^3(-\mathrm{L}F)d\gamma = 3\int_{\mathbb{R}^N} F^2|\nabla F|^2 d\gamma$$

similarly $$k\int_{\mathbb{R}^N} F^2 d\gamma = \int_{\mathbb{R}^N} F\,(-\mathrm{L}F)d\gamma = \int_{\mathbb{R}^N} |\nabla F|^2 d\gamma$$

normalization $\int_{\mathbb{R}^N} F^2 d\gamma = 1$

$$\int_{\mathbb{R}^N} |\nabla F|^2 d\gamma = k$$

$$\mathrm{Var}_\gamma(|\nabla F|^2) = \int_{\mathbb{R}^N} \big(|\nabla F|^2 - k\big)^2 d\gamma$$

$$k \int_{\mathbb{R}^N} F^4 d\gamma = \int_{\mathbb{R}^N} F^3(-\mathrm{L}F) d\gamma = 3 \int_{\mathbb{R}^N} F^2 |\nabla F|^2 d\gamma$$

$$\int_{\mathbb{R}^N} |\nabla F|^2 d\gamma = k$$

$$k \left(\frac{1}{3} \int_{\mathbb{R}^N} F^4 d\gamma - 1 \right) = \int_{\mathbb{R}^N} F^2 \big(|\nabla F|^2 - k \big) d\gamma$$

$$\int_{\mathbb{R}^N} F^4 d\gamma \sim 3 \quad \Longrightarrow \quad |\nabla F|^2 \sim k$$

$$\mathrm{Var}_\gamma \big(|\nabla F|^2 \big) = \int_{\mathbb{R}^N} \big(|\nabla F|^2 - k \big)^2 d\gamma$$

technical task

from $\int_{\mathbb{R}^N} F^2 \big(|\nabla F|^2 - k \big) d\gamma$ to $\int_{\mathbb{R}^N} \big(|\nabla F|^2 - k \big)^2 d\gamma$

main step

if $$\int_{\mathbb{R}^N} F^4 d\gamma \sim 3$$

then $$\mathrm{Var}_\gamma\big(|\nabla F|^2\big) = \int_{\mathbb{R}^N} \big(|\nabla F|^2 - k\big)^2 d\gamma \sim 0$$

if $|\nabla F|^2 \sim k$ $\quad \big(\mathrm{Var}_\gamma\big(|\nabla F|^2\big) \sim 0\big)$

then the distribution of F

is approximatively Gaussian

second (main) step

if $$\int_{\mathbb{R}^N} F^4 d\gamma \sim 3$$

then $$\mathrm{Var}_\gamma\big(|\nabla F|^2\big) = \int_{\mathbb{R}^N} \big(|\nabla F|^2 - k\big)^2 d\gamma \sim 0$$

first step

if $|\nabla F|^2 \sim k \qquad \big(\mathrm{Var}_\gamma\big(|\nabla F|^2\big) \sim 0\big)$

then the distribution of F

is approximatively Gaussian

first step

Ornstein-Uhlenbeck operator on $\mathbb{R}^N$

$$\mathrm{L} = \Delta - x \cdot \nabla$$

$F : \mathbb{R}^N \to \mathbb{R}$ Gaussian chaos

F eigenfunction of L

$$-\mathrm{L}F = \lambda F \qquad (\lambda > 0)$$

$\varphi : \mathbb{R} \to \mathbb{R}$ smooth

chain rule formula for L (Laplacian)

$$\mathrm{L}(\varphi \circ F) = \varphi'(F)\,\mathrm{L}F + \varphi''(F)\,|\nabla F|^2 = -\lambda F \varphi'(F) + \varphi''(F)\,|\nabla F|^2$$

$$\mathrm{L}(\varphi \circ F) = -\lambda F \varphi'(F) + \varphi''(F)\,|\nabla F|^2$$

if $|\nabla F|^2 = \lambda$ then

$$\mathrm{L}(\varphi \circ F) = -\lambda F \varphi'(F) + \lambda \varphi''(F)$$

$$\mathrm{L}(\varphi \circ F) = \lambda\,(\mathcal{L}\varphi)(F)$$

$\mathcal{L}\psi = \psi'' - x\psi'$ on $\mathbb{R}$ (one-dimensional O-U operator)

$\gamma_{\# F}$ distribution of $F : \mathbb{R}^N \to \mathbb{R}$ under γ

$$0 = \int_{\mathbb{R}^N} \mathrm{L}(\varphi \circ F) d\gamma = \lambda \int_{\mathbb{R}} \mathcal{L}\varphi \, d\gamma_{\# F}$$

$\gamma_{\# F}$ invariant measure of $\mathcal{L}$

$$\gamma_{\# F} = \gamma_1$$

Stein's method

quantify the preceding

I. Nourdin, G. Peccati (2009)

$$-\mathrm{L}\, F = \lambda\, F$$

$\gamma_{\# F}$ distribution of $F : \mathbb{R}^N \to \mathbb{R}$ under γ

$$\left| \int_{\mathbb{R}} \varphi \, d\gamma_{\# F} - \int_{\mathbb{R}} \varphi \, d\gamma_1 \right| \leq \frac{C_\varphi}{\lambda} \, \mathrm{Var}_\gamma \big(|\nabla F|^2 \big)^{1/2}$$

sufficiently many smooth $\varphi : \mathbb{R} \to \mathbb{R}$

if $\mathrm{Var}_\gamma \big(|\nabla F|^2 \big) \sim 0$

then distribution $\gamma_{\# F}$ close to Gaussian γ_1

second (main) step

when does

$$\int_{\mathbb{R}^N} F^4 d\gamma \sim 3$$

imply that

$$\mathrm{Var}_\gamma\big(|\nabla F|^2\big) \sim 0\,?$$

is it enough to use

$$-\mathrm{L}\,F = k\,F\,?$$

more information is needed

convenient framework

Γ calculus

Markov operator L on state space E

μ invariant symmetric probability measure

Γ (bilinear) operator

$$\Gamma(f,g) = \frac{1}{2}\big[\mathrm{L}(fg) - f\,\mathrm{L}g - g\,\mathrm{L}f\big]$$

$f, g : E \to \mathbb{R}$ in some nice algebra $\mathcal{A}$

integration by parts

$$\int_E f(-\mathrm{L}g)d\mu = \int_E \Gamma(f,g)d\mu$$

example

Ornstein-Uhlenbeck operator on $E = \mathbb{R}^N$

$$\mathrm{L} = \Delta - x \cdot \nabla$$

invariant measure $\mu = \gamma$

γ standard Gaussian measure on $\mathbb{R}^N$

$$\Gamma(f, g) = \nabla f \cdot \nabla g$$

$$\int_{\mathbb{R}^N} f(-\mathrm{L}g) d\gamma = \int_{\mathbb{R}^N} \nabla f \cdot \nabla g \, d\gamma$$

iterated gradients

$$\Gamma_m, \quad m \geq 2$$

$$\Gamma(f,g) \,=\, \frac{1}{2}\left[\mathrm{L}(fg) - f\,\mathrm{L}g - g\,\mathrm{L}f\right]$$

$$\Gamma_2(f,g) \,=\, \frac{1}{2}\left[\mathrm{L}\,\Gamma(f,g) - \Gamma(f,\mathrm{L}g) - \Gamma(g,\mathrm{L}f)\right]$$

D. Bakry, M. Émery (1985)

Γ_2 operator (criterion) : Bochner's formula

$$\Gamma_m(f,g) \,=\, \frac{1}{2}\left[\mathrm{L}\,\Gamma_{m-1}(f,g) - \Gamma_{m-1}(f,\mathrm{L}g) - \Gamma_{m-1}(g,\mathrm{L}f)\right]$$

$$\Gamma_0(f,f) = f^2, \qquad \Gamma_1 = \Gamma$$

$$\Gamma_m(f) = \Gamma_m(f,f)$$

example

Ornstein-Uhlenbeck operator on $E = \mathbb{R}^N$

$$\mathrm{L} = \Delta - x \cdot \nabla$$

$$\Gamma(f,f) = \Gamma(f) = \Gamma_1(f) = |\nabla f|^2$$

$$\Gamma_2(f) = |\nabla^{\otimes 2} f|^2 + |\nabla f|^2$$

$$\Gamma_3(f) = |\nabla^{\otimes 3} f|^2 + 3\,|\nabla^{\otimes 2} f|^2 + |\nabla f|^2$$

$$|\nabla^{\otimes 2} f|^2 = \Gamma_2(f) - \Gamma_1(f)$$

$$|\nabla^{\otimes 3} f|^2 = \Gamma_3(f) - 3\,\Gamma_2(f) + 2\,\Gamma_1(f)$$

additional datum

Gaussian chaos

$$F = F(x) = \sum_{i_1,\dots,i_k=1}^{N} a_{i_1,\dots,i_k}\, x_{i_1} \cdots x_{i_k}$$

$$-\mathrm{L}F = k\,F$$

$$|\nabla^{\otimes k+1} f|^2 = 0$$

$$(|\nabla^{\otimes k} f|^2 = \text{ constant})$$

intrinsic description in terms of the Γ_m

L (pure) point spectrum

$$S = \{0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots\}$$

polynomials

$$Q_k(X) = \prod_{i=0}^{k-1} (X - \lambda_i) = \sum_{i=1}^{k} q_i \, X^i$$

$$Q_k(\Gamma) = \sum_{i=1}^{k} q_i \, \Gamma_i$$

bilinear forms on $\mathcal{A} \times \mathcal{A}$

$$Q_k(\Gamma)(f, f) = Q_k(\Gamma)(f)$$

example

Ornstein-Uhlenbeck operator on $E = \mathbb{R}^N$

$$\mathrm{L} = \Delta - x \cdot \nabla$$

spectrum $S = \mathbb{N}$

$$Q_2(X) = X(X - \lambda_1) = X^2 - X$$

$$|\nabla^{\otimes 2} f|^2 = \Gamma_2(f) - \Gamma_1(f) = Q_2(\Gamma)(f)$$

$$Q_3(X) = X(X - \lambda_1)(X - \lambda_2) = X^3 - 3X^2 + 2X$$

$$|\nabla^{\otimes 3} f|^2 = \Gamma_3(f) - 3\,\Gamma_2(f) + 2\,\Gamma_1(f) = Q_3(\Gamma)(f)$$

$$|\nabla^{\otimes k} f|^2 = Q_k(\Gamma)(f)$$

Definition *An eigenfunction* F *of* $-\mathrm{L}$ *with eigenvalue* λ_k

$$-\mathrm{L}F = \lambda_k F$$

is said to be a chaos of degree $k \geq 1$ *relative to* $S = (\lambda_n)_{n \in \mathbb{N}}$ *if*

$$Q_{k+1}(\Gamma)(F) = 0$$

(μ-almost everywhere)

F **chaos eigenfunction** (with eigenvalue λ_k)

Wiener (Gaussian) chaos

$$F = F(x) = \sum_{i_1, \ldots, i_k = 1}^{N} a_{i_1, \ldots, i_k} \, x_{i_1} \cdots x_{i_k}$$

F k-chaos eigenfunction

$$Q_{k+1}(X) = \prod_{i=0}^{k} (X - \lambda_i) = \sum_{i=1}^{k+1} q_i X^i$$

further polynomials

$$R_{k+1}(X) = \frac{1}{X^2}\left[Q_{k+1}(X) - q_1 X\right] = \sum_{i=2}^{k+1} q_i X^{i-2}$$

$$T_{k+1}(X) = R_{k+1}(X + \lambda_k) - R_{k+1}(\lambda_k)$$

for example

$$Q_2(X) = X^2 - \lambda_1 X, \qquad R_2 \equiv 1, \qquad T_2 \equiv 0$$

$$Q_3(X) = X^3 - (\lambda_1 + \lambda_2)X^2 + \lambda_1 \lambda_2 X$$

$$R_3(X) = X - (\lambda_1 + \lambda_2), \qquad T_3(X) = X$$

main statement

$$R_{k+1}(X) \,=\, \frac{1}{X^2}\big[Q_{k+1}(X) - q_1\,X\big] \,=\, \sum_{i=2}^{k+1} q_i\,X^{i-2}$$

$$T_{k+1}(X) \,=\, R_{k+1}(X+\lambda_k) - R_{k+1}(\lambda_k)$$

$$\pi_k \,=\, \lambda_1 \cdots \lambda_k, \quad k \geq 1 \qquad (\pi_0 = 1)$$

Theorem *Let* F *be a* *k-chaos eigenfunction with eigenvalue* λ_k. *Set* $\Gamma = \Gamma(F)$. *Then*

$$\pi_{k-1} \int_E \Gamma^2 d\mu \,=\, \pi_k \int_E F^2\,\Gamma\,d\mu + (-1)^k \int_E \Gamma\; T_{k+1}\big(\tfrac{\mathrm{L}}{2}\big)\,\Gamma\,d\mu$$

$$\pi_{k-1} \int_E \Gamma^2 d\mu \,=\, \pi_k \int_E F^2 \, \Gamma \, d\mu + (-1)^k \int_E \Gamma \, T_{k+1}\big(\tfrac{\mathrm{L}}{2}\big) \, \Gamma \, d\mu$$

Corollary *Recall the spectrum* $S = (\lambda_n)_{n \in \mathbb{N}}$ *of* $-\mathrm{L}$. *If*

$$(-1)^k \, T_{k+1}\big(-\tfrac{\lambda_n}{2}\big) \,\leq\, 0 \quad \textit{for every } \, n \in \mathbb{N}$$

then

$$\int_E \Gamma^2 d\mu \,\leq\, \lambda_k \int_E F^2 \, \Gamma \, d\mu$$

F normalized in $\mathrm{L}^2(\mu)$

$$\int_E \Gamma \, d\mu \,=\, \int_E F(-\mathrm{L}F) d\mu \,=\, \lambda_k \int_E F^2 d\mu \,=\, \lambda_k$$

$$\mathrm{Var}_\mu(\Gamma) \,\leq\, \lambda_k \bigg(\int_E F^2 \, \Gamma \, d\mu - \lambda_k \bigg)$$

$$k \int_{\mathbb{R}^N} F^4 d\gamma = \int_{\mathbb{R}^N} F^3(-\mathrm{L}F) d\gamma = 3 \int_{\mathbb{R}^N} F^2 |\nabla F|^2 d\gamma$$

$$\int_{\mathbb{R}^N} |\nabla F|^2 d\gamma = k$$

$$k\left(\frac{1}{3} \int_{\mathbb{R}^N} F^4 d\gamma - 1\right) = \int_{\mathbb{R}^N} F^2 \big(|\nabla F|^2 - k\big) d\gamma$$

$$\int_{\mathbb{R}^N} F^4 d\gamma \sim 3 \quad \Longrightarrow \quad |\nabla F|^2 \sim k$$

$$\mathrm{Var}_\gamma\big(|\nabla F|^2\big) = \int_{\mathbb{R}^N} \big(|\nabla F|^2 - k\big)^2 d\gamma$$

technical task

from $\int_{\mathbb{R}^N} F^2 \big(|\nabla F|^2 - k\big) d\gamma$ to $\int_{\mathbb{R}^N} \big(|\nabla F|^2 - k\big)^2 d\gamma$

$$\mathrm{Var}_\mu(\Gamma) \leq \lambda_k \Big(\int_E F^2 \, \Gamma \, d\mu - \lambda_k \Big)$$

recall : integration by parts $\int_E F^2 d\mu = 1$

$$\lambda_k \Big(\frac{1}{3} \int_E F^4 d\mu - 1 \Big) = \int_E F^2 \, \Gamma \, d\mu - \lambda_k$$

$$\mathrm{Var}_\mu(\Gamma) \leq \lambda_k^2 \Big(\frac{1}{3} \int_E F^4 d\mu - 1 \Big)$$

if $\int_E F^4 d\mu = 3$ then $\Gamma = \Gamma(F)$ constant

Stein's method (first step)

distribution of F is Gaussian

Nualart-Peccati theorem

main statement

$$R_{k+1}(X) \,=\, \frac{1}{X^2}\big[Q_{k+1}(X) - q_1\,X\big] \,=\, \sum_{i=2}^{k+1} q_i\,X^{i-2}$$

$$T_{k+1}(X) \,=\, R_{k+1}(X+\lambda_k) - R_{k+1}(\lambda_k)$$

$$\pi_k \,=\, \lambda_1 \cdots \lambda_k, \quad k \geq 1 \qquad (\pi_0 = 1)$$

Theorem *Let* F *be a* k*-chaos eigenfunction with eigenvalue* λ_k*. Set* $\Gamma = \Gamma(F)$*. Then*

$$\pi_{k-1} \int_E \Gamma^2 d\mu \,=\, \pi_k \int_E F^2\,\Gamma\,d\mu + (-1)^k \int_E \Gamma\; T_{k+1}\big(\tfrac{\mathrm{L}}{2}\big)\,\Gamma\,d\mu$$

key argument of the **proof**

F eigenfunction of $-\mathrm{L}$ eigenvalue λ, $\quad -\mathrm{L}F = \lambda F$

$$\Gamma_m = \Gamma_m(F), \quad m \geq 1$$

$$\Gamma_m(F) \,=\, \tfrac{1}{2}\,\mathrm{L}\,\Gamma_{m-1}(F) - \Gamma_{m-1}(F, \mathrm{L}F)$$

$$\Gamma_m \,=\, \tfrac{1}{2}\,\mathrm{L}\,\Gamma_{m-1} + \lambda\,\Gamma_{m-1}$$

consequences

$$\Gamma_m \,=\, \big(\tfrac{1}{2}\,\mathrm{L} + \lambda\,\mathrm{Id}\big)^{m-1}\Gamma$$

$$\int_E F^2\,\Gamma_m d\mu \,=\, \int_E \Gamma_0\,\Gamma_m d\mu \,=\, \int_E \Gamma\,\Gamma_{m-1} d\mu$$

proof when $k = 1$

$$Q_2(X) = X^2 - \lambda_1 X, \qquad R_2 \equiv 1, \qquad T_2 \equiv 0, \qquad \pi_1 = \lambda_1$$

F 1-chaos

$$-\mathrm{L}F = \lambda_1 F$$

$$Q_2(\Gamma) = \Gamma_2 - \lambda_1 \Gamma = 0$$

multiply by F^2, integrate

$$\int_E F^2 \, \Gamma_2 \, d\mu = \lambda_1 \int_E F^2 \, \Gamma \, d\mu$$

key argument of the **proof**

F eigenfunction of $-\mathrm{L}$ eigenvalue λ, $\quad -\mathrm{L}F = \lambda F$

$$\Gamma_m = \Gamma_m(F), \quad m \geq 1$$

$$\Gamma_m(F) = \frac{1}{2}\,\mathrm{L}\,\Gamma_{m-1}(F) - \Gamma_{m-1}(F, \mathrm{L}F)$$

$$\Gamma_m = \frac{1}{2}\,\mathrm{L}\,\Gamma_{m-1} + \lambda\,\Gamma_{m-1}$$

consequences

$$\Gamma_m = \Big(\frac{1}{2}\,\mathrm{L} + \lambda\,\mathrm{Id}\Big)^{m-1}\Gamma$$

$$\int_E F^2\,\Gamma_m d\mu = \int_E \Gamma_0\,\Gamma_m d\mu = \int_E \Gamma\,\Gamma_{m-1} d\mu$$

proof when $k = 1$

$$Q_2(X) = X^2 - \lambda_1 X, \qquad R_2 \equiv 1, \qquad T_2 \equiv 0, \qquad \pi_1 = \lambda_1$$

F 1-chaos

$$-\mathrm{L}F = \lambda_1 F$$

$$Q_2(\Gamma) = \Gamma_2 - \lambda_1 \Gamma = 0$$

multiply by F^2, integrate

$$\int_E F^2 \, \Gamma_2 \, d\mu = \lambda_1 \int_E F^2 \, \Gamma \, d\mu$$

$$\pi_{k-1} \int_E \Gamma^2 d\mu = \pi_k \int_E F^2 \, \Gamma \, d\mu + (-1)^k \int_E \Gamma \, T_{k+1}\big(\tfrac{\mathrm{L}}{2}\big) \, \Gamma \, d\mu$$

proof when $k = 1$

$$Q_2(X) = X^2 - \lambda_1 X, \qquad R_2 \equiv 1, \qquad T_2 \equiv 0, \qquad \pi_1 = \lambda_1$$

F 1-chaos

$$-\mathrm{L}F = \lambda_1 F$$

$$Q_2(\Gamma) = \Gamma_2 - \lambda_1 \Gamma = 0$$

multiply by F^2, integrate

$$\int_E \Gamma^2 d\mu = \lambda_1 \int_E F^2 \, \Gamma \, d\mu$$

$$\pi_{k-1} \int_E \Gamma^2 d\mu = \pi_k \int_E F^2 \, \Gamma \, d\mu + (-1)^k \int_E \Gamma \, T_{k+1}\big(\tfrac{\mathrm{L}}{2}\big) \, \Gamma \, d\mu$$

proof when $k = 2$

$$Q_3(X) = X^3 - (\lambda_1 + \lambda_2)\, X^2 + \lambda_1\lambda_2 X$$

$$R_3(X) = X - (\lambda_1 + \lambda_2), \qquad T_3(X) = X$$

$$\pi_2 = \lambda_2\lambda_1, \quad \pi_1 = \lambda_1$$

F 2-chaos

$$-\mathrm{L}F \;=\; \lambda_2 F$$

$$Q_3(\Gamma) \;=\; \Gamma_3 - (\lambda_1 + \lambda_2)\, \Gamma_2 + \lambda_1\lambda_2\, \Gamma \;=\; 0$$

multiply by F^2, integrate

$$\int_E \Gamma\, \Gamma_2\, d\mu - (\lambda_1 + \lambda_2) \int_E \Gamma^2 d\mu + \lambda_1\lambda_2 \int_E F^2\, \Gamma\, d\mu \;=\; 0$$

$$\int_E \Gamma\,\Gamma_2\,d\mu - (\lambda_1 + \lambda_2)\int_E \Gamma^2 d\mu + \lambda_1\lambda_2 \int_E F^2\,\Gamma\,d\mu \;=\; 0$$

$$\Gamma_2 \;=\; \tfrac{1}{2}\,\mathrm{L}\,\Gamma + \lambda_2\,\Gamma$$

$$\frac{1}{2}\int_E \Gamma\,\mathrm{L}\,\Gamma\,d\mu - \lambda_1 \int_E \Gamma^2 d\mu + \lambda_1\lambda_2 \int_E F^2\,\Gamma\,d\mu \;=\; 0$$

$$T_3(X) = X$$

$$\pi_{k-1}\int_E \Gamma^2 d\mu \;=\; \pi_k \int_E F^2\,\Gamma\,d\mu + (-1)^k \int_E \Gamma\; T_{k+1}\big(\tfrac{\mathrm{L}}{2}\big)\,\Gamma\,d\mu$$

$$\pi_{k-1} \int_E \Gamma^2 d\mu \, = \, \pi_k \int_E F^2 \, \Gamma \, d\mu + (-1)^k \int_E \Gamma \, T_{k+1}\big(\tfrac{\mathrm{L}}{2}\big) \, \Gamma \, d\mu$$

Corollary *Recall the spectrum* $S = (\lambda_n)_{n \in \mathbb{N}}$ *of* $-\mathrm{L}$. *If*

$$(-1)^k \, T_{k+1}\big(- \tfrac{\lambda_n}{2} \big) \, \leq \, 0 \quad \textit{for every } \; n \in \mathbb{N}$$

then

$$\int_E \Gamma^2 d\mu \, \leq \, \lambda_k \int_E F^2 \, \Gamma \, d\mu$$

$$\mathrm{Var}_\mu(\Gamma) \, \leq \, \lambda_k \bigg(\int_E F^2 \, \Gamma \, d\mu - \lambda_k \bigg)$$

spectral condition

$$(-1)^k \, T_{k+1}\big(- \tfrac{\lambda_n}{2} \big) \, \leq \, 0, \qquad n \in \mathbb{N}$$

Theorem *The spectral condition*

$$(-1)^k \, T_{k+1}\big(- \tfrac{\lambda_n}{2} \big) \, \leq \, 0, \qquad n \in \mathbb{N}$$

is satisfied when

$$S \, = \, (\lambda_n)_{n \in \mathbb{N}} \, = \, \mathbb{N}$$

Wiener (Gaussian) chaos

Nualart-Peccati theorem

spectral condition

$$(-1)^k T_{k+1}\big(-\tfrac{\lambda_n}{2}\big) \le 0, \qquad n \in \mathbb{N}$$

$$S = (\lambda_n)_{n\in\mathbb{N}} = \mathbb{N}$$

$$Q_{k+1}(X) = \prod_{i=0}^{k}(X-i)$$

$$\pi_k = k!$$

$$\left(\frac{n}{2}-k\right)^{-2}\left[\prod_{i=0}^{k}\left(\frac{n}{2}-i\right) - k!\left(\frac{n}{2}-k\right)\right] \ge (k-1)!$$

elementary exercise

extensions

infinite dimensional Wiener chaos

abstract Markov chaos

continuous and discrete (cube, Poisson $S = (\lambda_n)_{n \in \mathbb{N}} = \mathbb{N}$)

convergence to other distributions (gamma)

I. Nourdin, G. Peccati (2009)

Wigner chaos (free probability) ?

T. Kemp, I. Nourdin, G. Peccati, R. Speicher (2012)

convergence to gamma distributions

Theorem F *k-chaos with eigenvalue* λ_k *such that* $\int_E F^2 d\mu = p > 0$. *Set* $\Gamma = \Gamma(F)$. *Under the spectral condition*

$$(-1)^k T_{k+1}\Big(-\frac{\lambda_n}{2}\Big) \leq 0, \qquad n \in \mathbb{N}$$

it holds

$$\mathrm{Var}_\mu(\Gamma - \lambda_k F) \leq \lambda_k \int_E F^2 \,\Gamma\, d\mu + A_k \int_E F\,\Gamma\, d\mu - pB_k - p^2\lambda_k^2$$

where

$$A_k = \frac{2(-1)^k \lambda_k}{\pi_{k-1}} R_{k+1}\Big(\frac{\lambda_k}{2}\Big) \qquad \text{and} \qquad B_k = \frac{(-1)^k \lambda_k^2}{\pi_{k-1}} R_{k+1}\Big(\frac{\lambda_k}{2}\Big)$$

$$S = \mathbb{N}$$

k even

$$\frac{3}{k^2}\,\mathrm{Var}_\mu(\Gamma - k\,F) \leq \int_E F^4 d\mu - 6\int_E F^3 d\mu + 6p - 3p^2$$

$(F_n)_{n\in\mathbb{N}}$ sequence of k-chaos $\int_E F_n^2\,d\mu = p$

if $\int_E F_n^4 d\mu - 6\int_E F_n^3 d\mu + 6p - 3p^2 \to 0$

then $(F_n + p)_{n\in\mathbb{N}}$ converges in distribution

to gamma distribution p

I. Nourdin, G. Peccati (2009)