# *Exponential tail inequalities for eigenvalues of random matrices*

M. Ledoux

Institut de Mathématiques de Toulouse, France

**exponential tail inequalities**

classical theme in probability and statistics

quantify the asymptotic statements

central limit theorems

large deviation principles

classical exponential inequalities

sum of independent random variables

$$S_n = \frac{1}{\sqrt{n}}\,(X_1 + \cdots + X_n)$$

$0 \leq X_i \leq 1$ independent

$$\mathbb{P}\big(S_n \geq \mathbb{E}(S_n) + t\big) \leq e^{-t^2/2}, \quad t \geq 0$$

Hoeffding's inequality

same as for $X_i$ standard Gaussian

central limit theorem

## measure concentration ideas

asymptotic geometric analysis

**V. D. Milman (1970)**

$$S_n = \frac{1}{\sqrt{n}} (X_1 + \cdots + X_n)$$

$F(X) = F(X_1, \ldots, X_n), \quad F : \mathbb{R}^n \to \mathbb{R}$ Lipschitz

Gaussian sample

independent random variables **(M. Talagrand 1995)**

## concentration inequalities

$$S_n = \frac{1}{\sqrt{n}}\,(X_1 + \cdots + X_n)$$

$F(X) = F(X_1, \ldots, X_n), \quad F : \mathbb{R}^n \to \mathbb{R}$ 1-Lipschitz

$X_1, \ldots, X_n$ independenty standard Gaussian

$$\mathbb{P}\Big(F(X) \geq \mathbb{E}(F(X)) + t\Big) \leq e^{-t^2/2}, \quad t \geq 0$$

$0 \leq X_i \leq 1$ independent, $F$ 1-Lipschitz and convex

$$\mathbb{P}\Big(F(X) \geq \mathbb{E}(F(X)) + t\Big) \leq 2\,e^{-t^2/4}, \quad t \geq 0$$

**M. Talagrand (1995)**

concentration inequalities

**numerous applications**

- geometric functional analysis
- discrete and combinatorial probability
- empirical processes
- statistical mechanics
- random matrix theory

recent studies of

**random matrix and random growth models**

new asymptotics

common, non central, rate $(\mathrm{mean})^{1/3}$

universal limiting **Tracy-Widom** distribution

random matrices, longest increasing subsequence,

random growth models, last passage percolation...

## random matrix models

Wigner matrix

$X^N = (X_{ij}^N)_{1 \leq i,j \leq N}$ symmetric $N \times N$ matrix

$X_{ij}^N$, $i \leq j$, independent identically distributed

$$\mathbb{E}(X_{ij}^N) = 0, \quad \mathbb{E}\big((X_{ij}^N)^2\big) = 1$$

eigenvalues $\lambda_1^N \leq \cdots \leq \lambda_N^N$ of $X^N$

asymptotics of the eigenvalues as the size $N \to \infty$

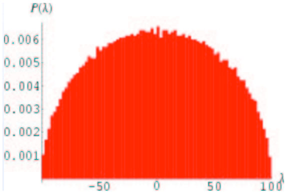
scaling $\widehat{X}^N = X^N / \sqrt{N}$

## Wigner's theorem (1955)

asymptotic behavior of the spectral measure $(\widehat{\lambda}_k^N = \lambda_k^N/\sqrt{N})$

$$\frac{1}{N}\sum_{k=1}^{N}\delta_{\widehat{\lambda}_k^N} \;\to\; d\nu(x) = \frac{1}{2\pi}\sqrt{4-x^2}\,dx \quad \text{on} \quad (-2,+2)$$

# Wigner's semi-circle law



## Wigner's theorem

$$\frac{1}{N}\sum_{k=1}^{N}\delta_{\widehat{\lambda}_k^N} \;\to\; d\nu(x) = \frac{1}{2\pi}\sqrt{4-x^2}\,dx \quad \text{semi-circle law}$$

**global regime**

large deviation asymptotics of the spectral measure

fluctuations of the spectral measure

$$\sum_{k=1}^{N}\Big[f\big(\widehat{\lambda}_k^N\big) - \int f\,d\nu\Big] \;\to\; G \quad \text{Gaussian variable}$$

$$f:\mathbb{R}\to\mathbb{R} \quad \text{smooth}$$

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**local regime**

behavior of the individual eigenvalues

spacings (bulk behavior)

extremal eigenvalues (edge behavior)

## extremal eigenvalues

largest eigenvalue $\lambda_N^N = \max_{1 \leq k \leq N} \lambda_k^N$

$$\widehat{\lambda}_N^N = \lambda_N^N / \sqrt{N}$$

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$$\widehat{\lambda}_N^N = \lambda_N^N / \sqrt{N} \ \rightarrow \ 2$$

**fluctuations** around $2$

Gaussian Orthogonal or Unitary Ensemble (GOE, GUE)

$N^{2/3} \big[ \widehat{\lambda}_N^N - 2 \big] \ \rightarrow \ F_{\rm TW}$ **C. Tracy, H. Widom (1994)**

rate $\ ({\rm mean})^{1/3}: \ N^{1/6} \big[ \lambda_N^N - 2\sqrt{N} \, \big]$

GUE $\quad F_{\rm TW}(s) = \exp \bigg( - \int_s^\infty (x-s) u(x)^2 dx \bigg), \quad s \in \mathbb{R}$

$u'' = 2u^3 + xu$ Painlevé II equation (similar for GOE)

Gaussian Orthogonal or Unitary Ensemble (GOE, GUE)

independent entries $X_{ij}^N$ real or complex Gaussian

$$P(dX) = \frac{1}{Z} \exp\Big( - \mathrm{Tr}\big(\beta X^2/4\big)\Big) dX$$

$dX$ Lebesgue measure on $N \times N$ matrices

orthogonal $(\beta = 1)$ or unitary conjugation $(\beta = 2)$

joint law of the eigenvalues $(\lambda_1^N \leq \cdots \leq \lambda_N^N)$ of $X^N$

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$$P^N(dx) = \frac{1}{Z} \left|\Delta_N(x)\right|^\beta \prod_{k=1}^N e^{-\beta x_k^2/4}\, dx_k$$

$$x = (x_1, \ldots, x_N) \in \mathbb{R}^N$$

$$\Delta_N(x) = \prod_{k<\ell}(x_k - x_\ell)$$ Vandermonde determinant

$\beta = 1:$ real $\qquad$ $\beta = 2:$ complex

large deviation principles (Laplace methods)

fluctuations of the spectral measure

law of $(\lambda_1^N, \ldots, \lambda_N^N)$ : $P^N(dx) = \frac{1}{Z} \left|\Delta_N(x)\right|^\beta \prod_{k=1}^{N} e^{-\beta x_k^2/4}\, dx_k$

fluctuations of extremal eigenvalues

$$N^{2/3}\big[\,\widehat{\lambda}_N^N - 2\,\big] \;\to\; F_{\mathrm{TW}}$$ **Tracy-Widom** distribution

orthogonal polynomial method $(\beta = 2)$

marginals of $(\lambda_1^N, \ldots, \lambda_N^N)$ in terms of the kernel

$$K_N(x,y) = \sum_{\ell=0}^{N-1} P_\ell(x) P_\ell(y)$$

$P_\ell$ Hermite orthogonal polynomials for $d\mu(x) = e^{-\beta x^2/4}\, \frac{dx}{Z}$

orthogonal polynomial asymptotics

## **fluctuations** of **extremal eigenvalues**

recent years : extensions to (**non Gaussian**) Wigner matrices

comparison with the Gaussian case

$N^{2/3}\big[\,\widehat{\lambda}_N^N - 2\,\big] \;\to\; F_{\mathrm{TW}}$ **Tracy-Widom** distribution

**A. Soshnikov (1999)** combinatorial moment method

$$\mathbb{E}\bigg(\sum_{k=1}^{N}(\lambda_k^N)^p\bigg) = \mathbb{E}\Big(\mathrm{Tr}\big((X^N)^p\big)\Big), \quad p = O(N^{2/3})$$

**T. Tao, V. Vu (2009)** Lindeberg method survey of recent approaches to

**non asymptotic exponential inequalities**

quantify the limit theorems

spectral measure

extremal eigenvalues

**catch** the **new rate** $(\mathrm{mean})^{1/3}$

## two main questions and objectives

tail inequalities for the spectral measure

$$\mathbb{P}\Big(\sum_{k=1}^{N} f(\widehat{\lambda}_k^N) \geq t\Big)$$

## Wigner's theorem

$$\frac{1}{N}\sum_{k=1}^{N}\delta_{\widehat{\lambda}_k^N} \;\to\; d\nu(x) = \frac{1}{2\pi}\sqrt{4-x^2}\,dx \quad \text{semi-circle law}$$

**global regime**

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$$\sum_{k=1}^{N}\Big[f\big(\widehat{\lambda}_k^N\big) - \int f\,d\nu\Big] \;\to\; G \quad \text{Gaussian variable}$$

$$f:\mathbb{R}\to\mathbb{R} \quad \text{smooth}$$

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$$\mathbb{P}\bigg( \sum_{k=1}^{N} f(\widehat{\lambda}_k^N) \geq t \bigg)$$

tail inequalities for the extremal eigenvalues

$$\mathbb{P}\big(\widehat{\lambda}_N^N \geq 2 + \varepsilon\big), \quad \mathbb{P}\big(\widehat{\lambda}_N^N \leq 2 - \varepsilon\big)$$

**extremal eigenvalues**

largest eigenvalue $\lambda_N^N = \max_{1\leq k\leq N} \lambda_k^N$

$$\widehat{\lambda}_N^N = \lambda_N^N / \sqrt{N} \;\to\; 2$$

**fluctuations** around $2$

Gaussian Orthogonal or Unitary Ensemble (GOE, GUE)

$N^{2/3}\big[\,\widehat{\lambda}_N^N - 2\,\big] \;\to\; F_{\mathrm{TW}}$ **Tracy-Widom** distribution

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Gaussian and more general Wigner matrices

measure concentration tool : $F = F(X^N) = F(X_{ij}^N)$

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**tail inequalities** for the **spectral measure**

**A. Guionnet, O. Zeitouni (2000)**

$f$ smooth    measure concentration

$f : \mathbb{R} \to \mathbb{R}$ Lipschitz

$$F : X^N \to \mathrm{Tr}\, f(\widehat{X}^N) = \sum_{k=1}^{N} f(\widehat{\lambda}_k^N) \quad \text{Lipschitz}$$

with respect to the Euclidean structure on $N \times N$ matrices

convex if $f$ is convex

## concentration inequalities

$$S_n = \frac{1}{\sqrt{n}}\,(X_1 + \cdots + X_n)$$

$F(X) = F(X_1, \ldots, X_n)$, $\quad F : \mathbb{R}^n \to \mathbb{R}$ $\quad$ 1-Lipschitz

$X_1, \ldots, X_n$ independenty standard Gaussian

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**M. Talagrand (1995)**

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$$\mathbb{P}\bigg( \sum_{k=1}^{N} \Big[ f(\widehat{\lambda}_k^N) - \mathbb{E}\big(f(\widehat{\lambda}_i^N)\big) \Big] \geq t \bigg) \leq C\, e^{-t^2/C}, \quad t \geq 0$$

## Wigner's theorem (1955)

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$$X^N \ \to \ \mathrm{Tr}\, f\big(\widehat{X}^N\big) = \sum_{k=1}^{N} f\big(\widehat{\lambda}_k^N\big)$$ Lipschitz

with respect to the Euclidean structure on $N \times N$ matrices

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$$\mathbb{P}\bigg( \sum_{k=1}^{N} \Big[ f(\widehat{\lambda}_k^N) - \mathbb{E}\big(f(\widehat{\lambda}_i^N)\big) \Big] \geq t \bigg) \leq C\, e^{-t^2/C}, \quad t \geq 0$$

**non Lipschitz** functions $f$

typically $f = \mathbf{1}_A$, $A$ interval

$$\sum_{k=1}^{N} f\big(\widehat{\lambda}_k^N\big) = \#\,\big\{\widehat{\lambda}_k^N \in A\big\}$$ counting function

suitable $X^N = (X_{ij}^N)$, $A$ interval in $(-2, +2)$

$$\frac{1}{\sqrt{\log N}} \sum_{k=1}^{N} \Big[ f\big(\widehat{\lambda}_k^N\big) - \mathbb{E}\big(f\big(\widehat{\lambda}_k^N\big)\big) \Big] \;\to\; G$$ Gaussian variable

exponential tail inequalities (GUE)

$$\mathbb{P}\bigg( \sum_{k=1}^{N} \Big[ f(\widehat{\lambda}_k^N) - \mathbb{E}\big(f(\widehat{\lambda}_k^N)\big) \Big] \geq t \bigg) \leq C\, e^{-ct \log(1+t/\log N)}, \quad t \geq 0$$

non Gaussian Wigner matrices ?

partial results (Stieltjes transform)

control of the Kolmogorov distance in Wigner's theorem

**F. Götze, A. Tikhomirov**

crucial tool in the recent solution **(2009)**

of the (bulk) spacings for Wigner matrices

**L. Erdös, S. Péché, J. Ramirez, B. Schlein, H.-T. Yau**

**V. Vu, T. Tao**

$$\mathbb{P}\Big(\#\left\{\widehat{\lambda}_k^N \in A\right\} \geq t|A|N\Big) \leq C\, e^{-t|A|N/C}, \quad t \geq C$$

Wigner's law at small scales $|A| \sim \frac{1}{N}$

## **two main questions** and **objectives**

**tail inequalities for the spectral measure**

$$\mathbb{P}\bigg(\sum_{k=1}^{N} f(\widehat{\lambda}_k^N) \geq t\bigg)$$

tail inequalities for the extremal eigenvalues

$$\mathbb{P}(\widehat{\lambda}_N^N \geq 2 + \varepsilon), \quad \mathbb{P}(\widehat{\lambda}_N^N \leq 2 - \varepsilon)$$

Gaussian and more general Wigner matrices

measure concentration tool : $\quad F = F(X^N) = F(X_{ij}^N)$

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## **tail inequalities** for the **extremal eigenvalues**

fluctuations of the largest eigenvalue

$$N^{2/3}\big[\,\widehat{\lambda}_N^N - 2\,\big] \ \to \ F_{\mathrm{TW}} \quad \textbf{Tracy-Widom} \ \text{distribution}$$

$$\mathbb{P}\big(\,\widehat{\lambda}_N^N \leq 2 + s\,N^{-2/3}\big) \ \to \ F_{\mathrm{TW}}(s), \quad s \in \mathbb{R}$$

**finite** $N$ inequalities

at the $(\mathrm{mean})^{1/3}$ rate

reflecting the tails of $F_{\mathrm{TW}}$

bounds on $\mathrm{Var}(\widehat{\lambda}_N^N)$

$N^{2/3}\big[\,\widehat{\lambda}_N^N - 2\,\big] \ \to \ F_{\mathrm{TW}}$ **Tracy-Widom** distribution

GUE $\quad F_{\mathrm{TW}}(s) = \exp\left( - \int_s^\infty (x-s)u(x)^2 dx \right), \quad s \in \mathbb{R}$

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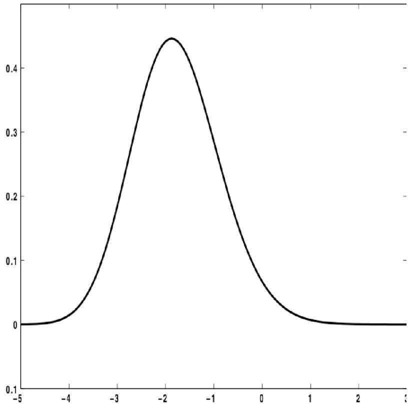
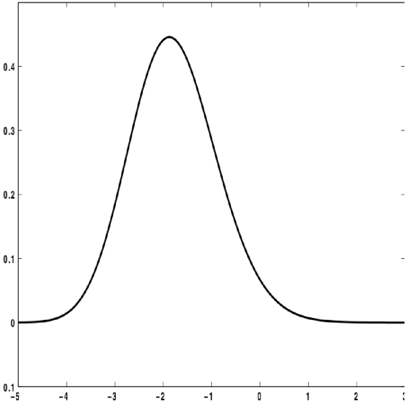



density

$$\text{mean} \simeq -1.77$$

$$F_{\mathrm{TW}}(s) \sim e^{-s^3/12} \quad \text{as} \quad s \to -\infty$$

$$1 - F_{\mathrm{TW}}(s) \sim e^{-4s^{3/2}/3} \quad \text{as} \quad s \to +\infty$$




density (similar for GOE)

## non asymptotic small deviation inequalities

$$\mathbb{P}\big(\,\widehat{\lambda}_N^N \leq 2 + s\,N^{-2/3}\big) \;\to\; F_{\mathrm{TW}}(s), \quad s \in \mathbb{R}$$

## expected bounds

$$\mathbb{P}\big(\,\widehat{\lambda}_N^N \geq 2 + \varepsilon\big)$$

## non asymptotic small deviation inequalities

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$$1 - F_{\rm TW}(s) \sim e^{-s^{3/2}/C} \quad (s \to +\infty)$$

## expected bounds

$$\mathbb{P}\big(\,\widehat{\lambda}_N^N \geq 2 + \varepsilon\big) \leq C\,e^{-N\,\varepsilon^{3/2}/C}, \quad 0 < \varepsilon \leq 1$$

fit the **Tracy-Widom** asymptotics $\quad (\varepsilon = s\,N^{-2/3})$

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fit the **Tracy-Widom** asymptotics $\quad (\varepsilon = s\,N^{-2/3})$

general **concentration** inequalities **useless**

Gaussian matrix

$$\lambda_N^N = \sup_{|v|=1} \langle X^N v, v \rangle$$

$\lambda_N^N$ Lipschitz of the Gaussian entries $X_{ij}^N$

Gaussian concentration

$$\mathbb{P}\big( \widehat{\lambda}_N^N \geq \mathbb{E}(\widehat{\lambda}_N^N) + t \big) \leq e^{-N\, t^2/C}, \quad t \geq 0$$

$$\mathbb{E}(\widehat{\lambda}_N^N) \sim 2$$

**does not fit** the (small deviation) regime $t = s\, N^{-2/3}$

correct large deviation bounds $(t \geq 1)$

## several (specific) approaches

variety of techniques

to the non asymptotic inequalities

fragmentary results : geometric percolation model

orthogonal polynomial bounds, Riemann-Hilbert methods (GUE, GOE)

- moment recurrence equations (GUE, GOE, invariant ensembles)
- combinatorial moment method (Wigner, sample covariance matrices)
- tridiagonal matrices ($\beta$-ensembles)

## moment recurrence equations

Gaussian Unitary Ensemble (GUE)

**Harer-Zagier (1986)** (map enumeration)

$$a_p^N = \mathbb{E}\Big(\mathrm{Tr}\big((X^N)^{2p}\big)\Big) = \mathbb{E}\bigg(\sum_{k=1}^{N}(\lambda_k^N)^{2p}\bigg), \quad p \in \mathbb{N}$$

three term recurrence equation

$$(p+1)a_p^N = (4p-2)Na_{p-1}^N + (p-1)(2p-1)(2p-3)a_{p-2}^N$$

$$a_p^N = E\Big(\mathrm{Tr}\big((X^N)^{2p}\big)\Big) \leq C\,(4N)^p\, e^{Cp^3/N^2}, \quad p^3 \geq N^2$$

$$\mathbb{P}\big(\,\widehat{\lambda}_N^N \geq 2+\varepsilon\big) \leq C\, e^{-N\varepsilon^{3/2}/C}, \quad 0 < \varepsilon \leq 1$$

extends to various **unitary invariant models**

$$P(dX) = \frac{1}{Z} \exp\Big( - \mathrm{Tr}\big(v(X)\big)\Big) dX, \quad v : \mathbb{R} \to \mathbb{R}$$

$d\mu = e^{-v} \frac{dx}{Z}$ classical orthogonal polynomials

**continuous :** Hermite, Laguerre, Jacobi

**discrete :** Charlier, Meixner, Krawtchouk, Hahn

oriented percolation model

length of the longest increasing subsequence

**real** Gaussian Orthogonal Ensemble (GOE)

five term recurrence equation

## combinatorial moment method

Wigner matrices (proof of Wigner's theorem)

$$\mathbb{E}\Big(\sum_{k=1}^{N}(\lambda_k^N)^p\Big) = \mathbb{E}\Big(\mathrm{Tr}\big((X^N)^p\big)\Big)$$

asymptotic results : **A. Soshnikov (1999)**

**O. Feldheim, S. Sodin (2009)**

non asymptotic moment inequalities

$$\mathbb{P}\big(\,\widehat{\lambda}_N^N \geq 2 + \varepsilon\big) \leq C\, e^{-N\,\varepsilon^{3/2}/C}, \quad 0 < \varepsilon \leq 1$$

entries $X_{ij}^N$ symmetric, subGaussian

extension to sample covariance matrices

## sample covariance matrices

multivariate statistical inference

principal component analysis

population $(Y_1, \ldots, Y_N)$

$Y_j$ vector in $\mathbb{R}^M$ (characters)

sample covariance matrix $Y^* Y$

$$M \sim \kappa N, \quad N \to \infty$$

(Gaussian) Wishart matrix models

## sample covariance matrices

$$Y = Y^{M,N} = (Y_{ij})_{1 \leq i \leq M, 1 \leq j \leq N} \quad M \times N \quad \text{matrix} \quad (M \geq N)$$

$Y_{ij}$ independent identically distributed

$$\mathbb{E}(Y_{ij}) = 0, \quad \mathbb{E}\big((Y_{ij})^2\big) = 1$$

eigenvalues $0 \leq \lambda_1^N \leq \cdots \leq \lambda_N^N$ of $Y^* Y$

asymptotic spectral measure $M \sim \kappa N, \quad \kappa \geq 1$

$$\widehat{\lambda}_k^N = \lambda_k^N / N$$

$$\frac{1}{N} \sum_{k=1}^{N} \delta_{\widehat{\lambda}_k^N} \to \rho \quad \text{on} \quad \big((\sqrt{\kappa} - 1)^2, (\sqrt{\kappa} + 1)^2\big)$$

$\rho$ Marchenko-Pastur distribution

# Tracy-Widom theorem for the largest eigenvalue

**O. Feldheim, S. Sodin (2009)**

non asymptotic moment inequalities

largest eigenvalue (symmetric, subGaussian entries)

$$\mathbb{P}\big(\,\widehat{\lambda}_N^N \geq (\sqrt{\kappa} + 1)^2 + \varepsilon\big) \leq C\, e^{-N\,\varepsilon^{3/2}/C}, \quad 0 < \varepsilon \leq 1$$

## sample covariance matrices

$Y = Y^{M,N} = (Y_{ij})_{1 \leq i \leq M, 1 \leq j \leq N}$ $\quad M \times N$ matrix $\quad (M \geq N)$

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asymptotic spectral measure $M \sim \kappa N, \quad \kappa \geq 1$

$$\widehat{\lambda}_k^N = \lambda_k^N / N$$

$$\frac{1}{N} \sum_{k=1}^{N} \delta_{\widehat{\lambda}_k^N} \to \rho \quad \text{on} \quad \big((\sqrt{\kappa} - 1)^2, (\sqrt{\kappa} + 1)^2\big)$$

$\rho$ **Marchenko-Pastur** distribution

# Tracy-Widom theorem for the largest eigenvalue

**O. Feldheim and S. Sodin (2009)**

non asymptotic moment inequalities

largest eigenvalue (symmetric, subGaussian entries)

$$\mathbb{P}\big( \widehat{\lambda}_N^N \geq (\sqrt{\kappa} + 1)^2 + \varepsilon \big) \leq C\, e^{-N\, \varepsilon^{3/2}/C}, \quad 0 < \varepsilon \leq 1$$

smallest eigenvalue

at the soft edge $M \sim \kappa\, N, \quad \kappa > 1$

$$\mathbb{P}\big( \widehat{\lambda}_1^N \leq (\sqrt{\kappa} - 1)^2 - \varepsilon \big) \leq C\, e^{-N\, \varepsilon^{3/2}/C}, \quad 0 < \varepsilon \leq 1$$

**tridiagonal models** ($\beta$-ensembles)

GOE, GUE, Wishart models

joint law of the eigenvalues $(\lambda_1^N \leq \cdots \leq \lambda_N^N)$ of $X^N$

$$P^N(dx) = \frac{1}{Z} \, |\Delta_N(x)|^\beta \prod_{k=1}^N e^{-\beta x_k^2/4} \, dx_k$$

$\beta = 1:$ GOE $\beta = 2:$ GUE

tridiagonal representation

$$\begin{pmatrix} g_1 & \chi_{N-1} & 0 & \cdots & \cdots & 0 \\ \chi_{N-1} & g_2 & \chi_{N-2} & 0 & \cdots & \vdots \\ 0 & \chi_{N-2} & g_3 & \chi_{N-3} & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & \ddots & \chi_2 & g_{N-1} & \chi_1 \\ 0 & \cdots & \cdots & 0 & \chi_1 & g_N \end{pmatrix}$$

$g_1, \ldots, g_N$ independent standard normal

$\chi_{N-1}, \ldots, \chi_1$ independent chi-variables

**same** GOE eigenvalues

**H. Trotter (1984), A. Edelman, I. Dimitriu (2002)**

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$\beta = 1$ : GOE $\quad \beta = 2$ : GUE

## tridiagonal representation

$$\begin{pmatrix} g_1 & \chi_{(N-1)\beta} & 0 & \cdots & \cdots & 0 \\ \chi_{(N-1)\beta} & g_2 & \chi_{(N-2)\beta} & 0 & \cdots & \vdots \\ 0 & \chi_{(N-2)\beta} & g_3 & \chi_{(N-3)\beta} & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & \ddots & \chi_{2\beta} & g_{N-1} & \chi_\beta \\ 0 & \cdots & \cdots & 0 & \chi_\beta & g_N \end{pmatrix}$$

$g_1, \ldots, g_N$ independent standard normal

$\chi_{(N-1)\beta}, \ldots, \chi_\beta$ independent chi-variables

**same** GOE eigenvalues

**H. Trotter (1984), A. Edelman, I. Dimitriu (2002)**

similar for the $\beta$-ensembles , Wishart models

## A. Ramirez, B. Rider, B. Virag (2007)

new proof of **Tracy-Widom** theorem (every $\beta$)

new description of the **Tracy-Widom** distribution

$$TW_\beta = \sup_f \left\{ \frac{2}{\sqrt{\beta}} \int_0^\infty f^2(x)\, dB(x) - \int_0^\infty \left[ f'(x)^2 + x\, f^2(x) \right] dx \right\}$$

$$f(0) = 0, \quad \int_0^\infty f^2(x) dx = 1$$

$$\int_0^\infty [f'(x)^2 + x\, f^2(x)] dx < \infty$$

Painlevé equation ?

**bounds** on the **largest eigenvalue** $\lambda_N^N = \max_{1 \leq k \leq N} \lambda_k^N$

tridiagonal representation

$$\begin{pmatrix} g_1 & \chi_{N-1} & 0 & \cdots & \cdots & 0 \\ \chi_{N-1} & g_2 & \chi_{N-2} & 0 & \cdots & \vdots \\ 0 & \chi_{N-2} & g_3 & \chi_{N-3} & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & \ddots & \chi_2 & g_{N-1} & \chi_1 \\ 0 & \cdots & \cdots & 0 & \chi_1 & g_N \end{pmatrix}$$

$g_1, \ldots, g_N$ independent standard normal

$\chi_{N-1}, \ldots, \chi_1$ independent chi-variables

**bounds** on the **largest eigenvalue** $\lambda_N^N = \max_{1 \leq k \leq N} \lambda_k^N$

$$\lambda_N^N = \sup_{|v|=1} \left( \sum_{i=1}^{N} g_i v_i^2 + 2 \sum_{i=1}^{N-1} \chi_{N-i} v_i v_{i+1} \right)$$

explicit computations

$$\mathbb{P}\big(\, \widehat{\lambda}_N^N \geq 2 + \varepsilon \big) \leq C\, e^{-N\, \varepsilon^{3/2}/C}, \quad 0 < \varepsilon \leq 1$$

$$\mathbb{P}\big(\, \widehat{\lambda}_N^N \leq 2 - \varepsilon \big) \leq C\, e^{-N^2\, \varepsilon^{3}/C}, \quad 0 < \varepsilon \leq 1$$

$\beta$-ensembles (GOE, GUE, Wishart models)

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explicit computations

$$\mathbb{P}\big( \widehat{\lambda}_N^N \geq 2 + \varepsilon \big) \geq \frac{1}{C} e^{-C N \varepsilon^{3/2}}, \quad 0 < \varepsilon \leq 1$$

$$\mathbb{P}\big( \widehat{\lambda}_N^N \leq 2 - \varepsilon \big) \geq \frac{1}{C} e^{-C N^2 \varepsilon^3}, \quad 0 < \varepsilon \leq 1$$

$\beta$-ensembles (GOE, GUE, Wishart models)

$$\mathbb{P}\big(\,\widehat{\lambda}_N^N \geq 2 + \varepsilon\big) \leq C\, e^{-N\,\varepsilon^{3/2}/C}, \quad 0 < \varepsilon \leq 1$$

$$\mathbb{P}\big(\,\widehat{\lambda}_N^N \leq 2 - \varepsilon\big) \leq C\, e^{-N^2\,\varepsilon^{3}/C}, \quad 0 < \varepsilon \leq 1$$

**finite $N$ variance bounds**

$$\mathrm{Var}(\widehat{\lambda}_N^N) \leq C\, N^{-4/3}$$

$N^{2/3}\big[\,\widehat{\lambda}_N^N - 2\,\big] \;\to\; F_{\mathrm{TW}}$ **Tracy-Widom** distribution

$\beta$-ensembles (GOE, GUE, Wishart models)

open for general Wigner matrices