# *The concentration of measure phenomenon*

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## concentration of measure phenomenon

### first historical observations

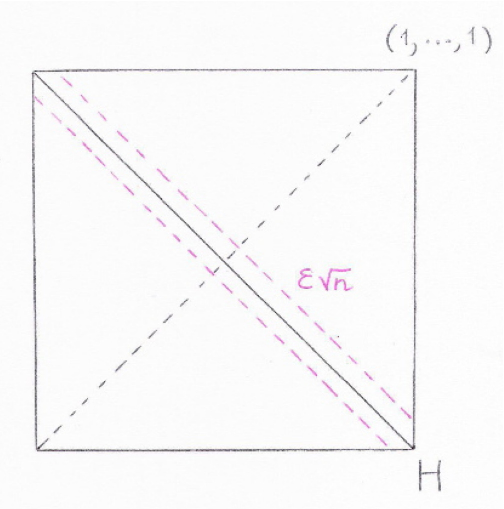
**E. Borel 1914**

(geometric) law of large numbers

$\mu^n$ uniform on $[-1,+1]^n$

$H = (1,\ldots,1)^\perp$ hyperplan

$$\mu^n\big(d(\cdot,H) \leq \varepsilon\sqrt{n}\,\big) \to 1$$




$$\mu^n\big(d(\cdot, H) \le \varepsilon\sqrt{n}\big) \to 1$$

## relevant observable

$$f : (x_1, \ldots, x_n) \mapsto \frac{1}{n} \sum_{i=1}^{n} x_i$$

$$\mathrm{Ker}\,(f) = H = (1, \ldots, 1)^{\perp}$$

law of large numbers $\qquad \displaystyle\frac{1}{n} \sum_{i=1}^{n} x_i \to 0 \qquad \mu^n$-almost surely

$$\mu^n\big(d(\cdot, H) \leq \varepsilon\sqrt{n}\,\big) = \mu^n\left(\left|\frac{1}{n} \sum_{i=1}^{n} x_i\right| \leq \varepsilon\right) \to 1$$

$$\mu^n\big(d(\cdot, H) \leq \varepsilon\sqrt{n}\,\big) \geq 1 - C\, e^{-n\,\varepsilon^2/C}$$

## P. Lévy 1919

$\mu$ normalized uniform measure on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ standard sphere

$F : \mathbb{S}^n \to \mathbb{R}$ continuous

$$\mu\big(|F - m| < \omega(\eta)\big) \geq 1 - 2\, e^{-(n-1)\, \eta^2/2}$$

$m$ median of $F$ for $\mu$

$\omega(\eta)$ modulus of continuity of $F$

for $n$ large, functions with small oscillations

are almost constant

**high dimensional effect**

## **real birth** of **measure concentration**

new proof by **V. Milman 1970**

**Dvoretzky's** theorem on spherical sections of convex bodies

for all $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$

every convex body $K \subset \mathbb{R}^n$ (unit ball for $\|\cdot\|$)

there exist $F$ subspace of $\mathbb{R}^n$, $\dim(F) \geq \delta(\varepsilon) \log n$

and $\mathcal{E} \subset F$ ellipsoid

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$K \cap F$

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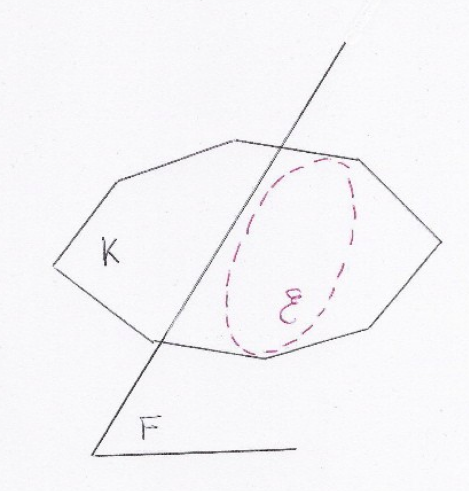
$$(1-\varepsilon)\,\mathcal{E} \subset K \cap F \subset (1+\varepsilon)\,\mathcal{E}$$

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there exist $F$ subspace of $\mathbb{R}^n$, $\dim(F) \geq \delta(\varepsilon) \log n$

and $\mathcal{E} \subset F$ ellipsoid

$$(1 - \varepsilon)\,\mathcal{E} \,\subset\, K \cap F \,\subset\, (1 + \varepsilon)\,\mathcal{E}$$

$K = [-1, +1]^n$ cube (sup-norm)

$\dim(F) \geq \delta(\varepsilon) \log n$ optimal

$$K = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n ; \sum_{i=1}^{n} |x_i| \leq 1 \right\} \qquad (\ell^1\text{-norm})$$

$\dim(F) \geq \delta(\varepsilon)\, n$ proportional

**Milman's (Dvoretzky)** idea :

find $F$ **at random**

concentration of spherical measures in high dimension

**Lévy's inequality**

$$\mu\big(|F-m|<\omega(\eta)\big)\ \geq\ 1-2\,e^{-(n-1)\,\eta^2/2}$$

$$F:\mathbb{S}^n\to\mathbb{R},\quad F(x)=\|x\|$$

$\|\cdot\|$ gauge of $K$

**most** sections are spherical

new proof by **V. Milman 1970**

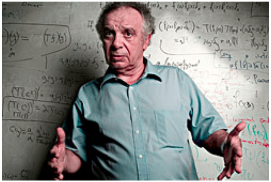
**asymptotic geometric analysis**

"The concentration of measure phenomenon, ubiquitous
ins probability theory and statistical mechanics,
was brought to geometry (starting from Banach spaces)
by Vitali Milman, following the earlier work by Paul Lévy"

**(M. Gromov 1999)**

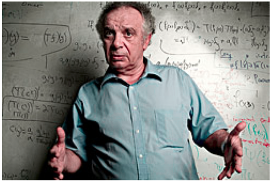
"The idea of concentration of measure,
which was discovered by Vitali Milman,
is arguably one of the great ideas of analysis in our times"

**(M. Talagrand 1996)**



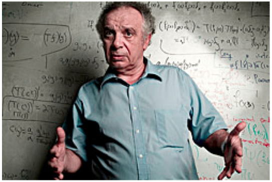
# Vitali Milman

## David Milman (Krein-Milman's theorem)



## Vitali Milman , Pierre Milman

**David Milman (Krein-Milman's theorem)**



**Vitali Milman, Pierre Milman**

**Emanuel Milman**

new proof by **V. Milman 1970**

main tool : **Lévy's inequality**

$\mu$ normalized uniform measure on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ standard sphere

$F : \mathbb{S}^n \to \mathbb{R}$ continuous

$$\mu\big(|F - m| < \omega(\eta)\big) \ \geq \ 1 - 2\, e^{-(n-1)\,\eta^2/2}$$

$m$ median of $F$ for $\mu$

$\omega(\eta)$ modulus of continuity of $F$

origin : **isoperimetric inequalities**

# isoperimetric inequalities

Queen Dido

**isoperimetric inequality** on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$

**E. Schmidt 1948**

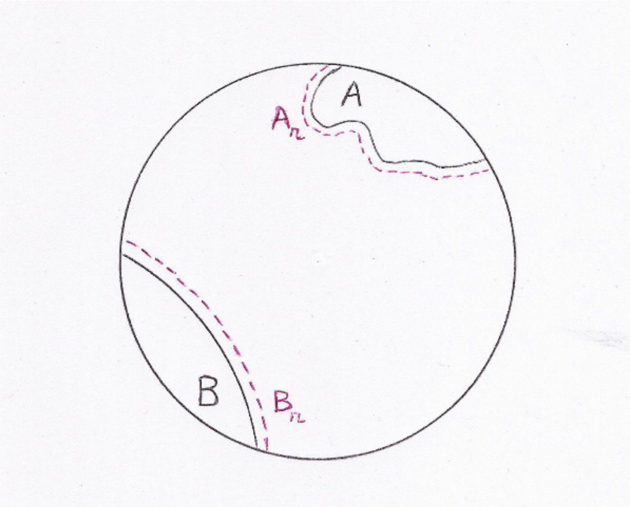
extremal sets are geodesic balls

integrated form

if $\mathrm{vol}\,(A) \geq \mathrm{vol}\,(B)$, $B$ geodesic ball

then $\mathrm{vol}\,(A_r) \geq \mathrm{vol}\,(B_r)$, $r > 0$

$$A_r = \{x \in \mathbb{S}^n ; d(x, A) < r\}$$




$$\mathrm{vol}\,(A_r) \;\geq\; \mathrm{vol}\,(B_r)$$

**isoperimetric inequality** on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$

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if $\mathrm{vol}\,(A) \ge \mathrm{vol}\,(B), \quad B$ geodesic ball

then $\mathrm{vol}\,(A_r) \ge \mathrm{vol}\,(B_r), \quad r > 0$

$$A_r = \big\{x \in \mathbb{S}^n ; d(x, A) < r\big\}$$

infinitesimal form

$r \to 0 \qquad \mathrm{vol}\,(\partial A) \ge \mathrm{vol}\,(\partial B)$

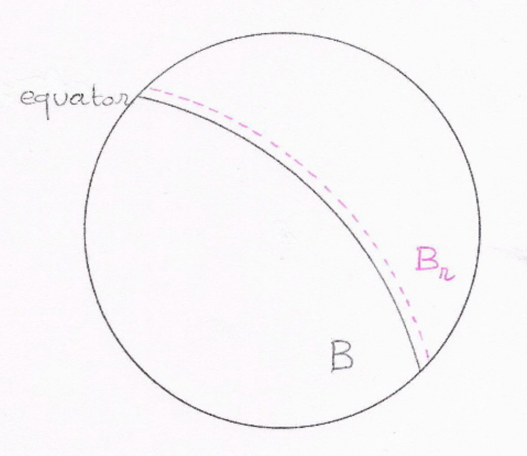
## spherical concentration

normalized volume $\mu(\cdot) = \frac{\mathrm{vol}\,(\cdot)}{\mathrm{vol}\,(\mathbb{S}^n)}$

if $\mu(A) \geq \frac{1}{2} \quad = (\mu(B),$ $B$ half-sphere)

then, for all $r > 0$,

$$\mu(A_r) \geq \mu(B_r)$$




$$\mu(B_r) \geq 1 - e^{-(n-1)\, r^2/2}$$

## spherical concentration

normalized volume $\quad \mu(\cdot) = \dfrac{\mathrm{vol}\,(\cdot)}{\mathrm{vol}\,(\mathbb{S}^n)}$

if $\quad \mu(A) \geq \dfrac{1}{2} \quad = (\mu(B), \quad B$ half-sphere)

then, for all $\quad r > 0,$

$$\mu(A_r) \geq \mu(B_r) \geq 1 - e^{-(n-1)\, r^2/2}$$

**Lévy's inequality**

$$r \sim \frac{1}{\sqrt{n}} \qquad (n \to \infty)$$

$$\mu(A_r) \approx 1$$

another example : **Gaussian concentration**

limit of spherical concentration

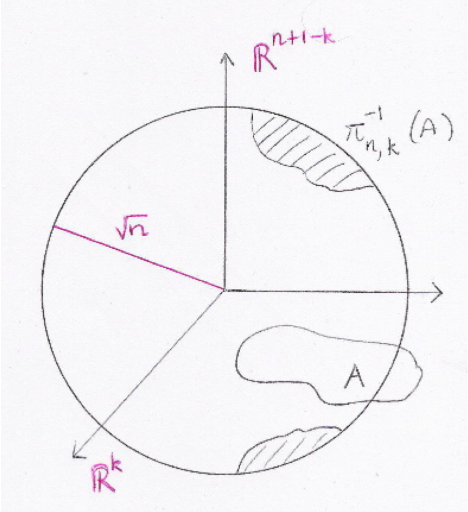
**Poincaré's** lemma

$$\pi_{n,k} : \mathbb{R}^{n+1} \to \mathbb{R}^k$$

$\mu$ uniform (normalized) on $\mathbb{S}^n(\sqrt{n}\,)$

if $A \subset \mathbb{R}^k$

$$\mu\Big(\pi_{n,k}^{-1}(A) \cap \mathbb{S}^n(\sqrt{n}\,)\Big)$$




$$\mu\left(\pi_{n,k}^{-1}(A) \cap \mathbb{S}^n(\sqrt{n})\right) \to \gamma(A)$$

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if $A \subset \mathbb{R}^k$

$$\mu\Big(\pi_{n,k}^{-1}(A) \cap \mathbb{S}^n(\sqrt{n}\,)\Big) \to \gamma(A)$$

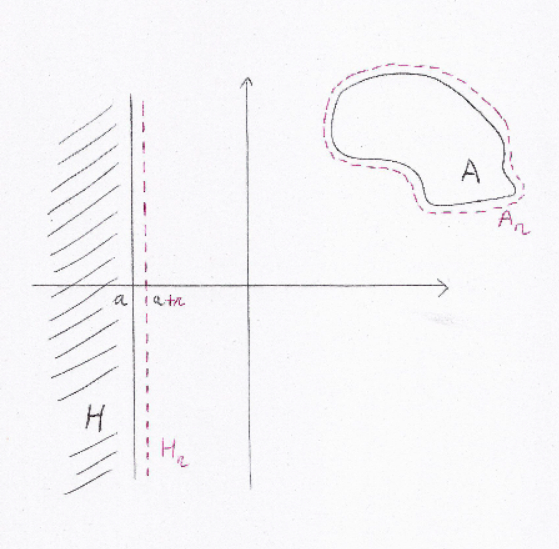
$$d\gamma(x) = e^{-|x|^2/2}\,\frac{dx}{(2\pi)^{k/2}}$$

## Gaussian isoperimetric inequality (in $\mathbb{R}^k$)

$$d\gamma(x) = e^{-|x|^2/2} \, \frac{dx}{(2\pi)^{k/2}}$$

extremal sets are half-spaces $H = \{x \in \mathbb{R}^k; \, x_1 \leq a\}$

(balls with centers at infinity)




$$\gamma(A) \geq \gamma(H) \quad \Longrightarrow \quad \gamma(A_r) \geq \gamma(H_r)$$

## **Gaussian isoperimetric inequality** (in $\mathbb{R}^k$)

$$d\gamma(x) = e^{-|x|^2/2} \frac{dx}{(2\pi)^{k/2}}$$

extremal sets are half-spaces $H = \{x \in \mathbb{R}^k ; \, x_1 \leq a\}$

(balls with centers at infinity)

if $\gamma(A) \geq \gamma(H) = \Phi(a) = \int_{-\infty}^{a} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}$

then $\gamma(A_r) \geq \gamma(H_r) = \Phi(a + r), \quad r > 0$

$A_r$ Euclidean neighbourhood

**C. Borell, V. Sudakov - B. Tsirelson 1974**

## Gaussian concentration

if $\gamma(A) \geq \frac{1}{2} = \Phi(0)$

$$\Phi(t) = \int_{-\infty}^{t} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}$$

then $\gamma(A_r) \geq \Phi(0 + r) \geq 1 - e^{-r^2/2}, \quad r > 0$

$r = 5$ or $10$, $\quad \gamma(A_r) \approx 1$

**independent** of the dimension $k$

extension to Wiener space

## framework for measure concentration

metric measure space $(X, d, \mu)$

$(X, d)$ metric space

$\mu$ Borel measure on $X$, $\mu(X) = 1$

## concentration function

$$\alpha_\mu(r) = \alpha_{(X,d,\mu)}(r) = \sup\left\{1 - \mu(A_r);\, \mu(A) \geq \tfrac{1}{2}\right\}, \quad r > 0$$

$$A_r = \left\{x \in X; d(x, A) < r\right\}$$

$\mu$ uniform on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ : $\alpha_\mu(r) \leq e^{-(n-1)\, r^2/2}$

$\gamma$ standard Gaussian on $\mathbb{R}^k$ : $\alpha_\gamma(r) \leq e^{-r^2/2}$

**equivalent formulation** on **functions** (**Lévy's inequality**)

$F : X \to \mathbb{R}$ 1-Lipschitz

$m$ median of $F$ for $\mu$

$$\mu(F \leq m),\ \mu(F \geq m) \geq \frac{1}{2}$$

$$A = \{F \leq m\} \quad \Longrightarrow \quad A_r \subset \{F < m + r\}$$

$$\mu(F < m + r) \geq 1 - \alpha_\mu(r), \quad r > 0$$

$$\mu\big(|F - m| < r\big) \geq 1 - 2\,\alpha_\mu(r), \quad r > 0$$

**deviation inequality**

$$\mu(F \geq m + r) \leq \alpha_\mu(r), \quad r > 0$$

**median $\leftrightarrow$ mean**

## dual description : observable diameter (M. Gromov)

$$\kappa > 0 \qquad (\kappa = 10^{-10})$$

$$\mathrm{PartDiam}_\mu(X, d)$$

$$= \inf \big\{ D \geq 0, \, \exists \, A \subset X, \, \mathrm{Diam}\,(A) \leq D, \, \mu(A) \geq 1 - \kappa \big\}$$

$$\mathrm{ObsDiam}_\mu(X, d) \, = \sup_{F \ 1-\mathrm{Lip}} \mathrm{PartDiam}_{\mu_F}(\mathbb{R})$$

$$F : \mu \, \to \, \mu_F$$

$$\mathrm{ObsDiam}_\mu(X,d) = \sup_{F \text{ 1-Lip}} \mathrm{PartDiam}_{\mu_F}(\mathbb{R})$$

$\mu$ state on the configuration space $(X,d)$

$F : X \to \mathbb{R}$ observable

yielding the tomographic image $\mu \to \mu_F$ on $\mathbb{R}$

$$\mu_F\big(]m-r, m+r[\big) = \mu\big(|F-m| < r\big) \geq 1 - 2\,\alpha_\mu(r) \sim 1 - \kappa$$

$$\mathrm{ObsDiam}_\mu(X,d) \sim \alpha_\mu^{-1}(\kappa)$$

$$\mathrm{ObsDiam}_\mu(\mathbb{S}^n) = O\Big(\frac{1}{\sqrt{n}}\Big), \qquad \mathrm{ObsDiam}_\gamma(\mathbb{R}^k) = O(1)$$

## measure concentration property

less restrictive than isoperimetry $(r \to 0)$

easier to establish, widely shared

## variety of examples and tools

- spectral methods
- probabilistic and combinatorial tools
- product measures
- geometric, functional, transportation inequalities

## illustration

**simple (functional) proof** of Gaussian concentration $\alpha_\gamma(r) \leq e^{-r^2/2}$

$$d\gamma(x) = e^{-|x|^2/2} \frac{dx}{(2\pi)^{k/2}}$$

$(P_t)_{t \geq 0}$ heat semigroup on $\mathbb{R}^k$,

$\partial_t P_t f \big|_{t=0} = \Delta f, \quad P_0 f = f, \quad \Delta$ Laplace operator

$F : \mathbb{R}^k \to \mathbb{R}$ Lipschitz, $\qquad \|F\|_{\mathrm{Lip}} = L$

$t > 0$, at any point

$$\phi(s) = P_s\Big(e^{P_{t-s}F}\Big), \quad 0 \leq s \leq t$$

interpolation $\qquad \phi(t) = P_t(e^F) \qquad \phi(0) = e^{P_t F}$

$$\phi(s) = P_s\Big(e^{P_{t-s}F}\Big), \quad 0 \leq s \leq t$$

$$\phi'(s) = P_s\Big(\Delta e^{P_{t-s}F} - \Delta P_{t-s}F\, e^{P_{t-s}F}\Big)$$

$$= P_s\Big(|\nabla P_{t-s}F|^2\, e^{P_{t-s}F}\Big)$$

$F$ Lipschitz (smooth), $\quad \|F\|_{\mathrm{Lip}} = L, \quad \|\nabla F\|_\infty \leq L$

$$\big|\nabla P_{t-s}F\big|^2 \leq L^2$$

$$\phi'(s) \leq L^2\, P_s\Big(e^{P_{t-s}F}\Big) = L^2\, \phi(s), \quad 0 \leq s \leq t$$

$$\phi'(s) \le L^2 \phi(s), \quad 0 \le s \le t$$

integrate

$$t = \tfrac{1}{2} : \quad P_t \to \gamma$$

$$\int_{\mathbb{R}^k} e^F \, d\gamma \le e^{\int_{\mathbb{R}^k} F d\gamma + L^2/2}$$

Chebyschev's inequality

$$\gamma\big(F \ge \textstyle\int_{\mathbb{R}^k} F \, d\gamma + r\big) \le e^{-r^2/2L^2}, \quad r > 0$$

Gaussian concentration $\alpha_\gamma(r) \le e^{-r^2/2}$

$$\phi'(s) \,\leq\, L^2\,\phi(s), \quad 0 \leq s \leq t$$

integrate

$$\phi(t) \,\leq\, \phi(0)\, e^{L^2 t}$$

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integrate

$$P_t(e^F) = \phi(t) \le \phi(0)\, e^{L^2 t} = e^{P_t F + L^2 t}$$

$$t = \tfrac{1}{2}: \quad P_t \to \gamma$$

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Gaussian concentration $\alpha_\gamma(r) \le e^{-r^2/2}$

**same** methodology **D. Bakry, M. Émery 1985**

$$d\mu = e^{-V} dx, \qquad V'' \geq c > 0$$

$$d\mu(x) = \exp\left(-\sum_{i=1}^{k} v(x_i)\right) \frac{dx}{Z}$$

$v : \mathbb{R} \to \mathbb{R}$ strictly convex at infinity

Gaussian concentration

$$\alpha_\mu(r) \leq C\, e^{-r^2/C}$$

no isoperimetry in general

a first **application** (of Gaussian concentration)

the **Johnson-Lindenstrauss 1984** **flatening lemma**

(metric geometry, theoretical computer science)

$N$ points $p_1, \ldots, p_N$ in $\mathbb{R}^n$

$\varepsilon > 0$

there exists $\pi : \mathbb{R}^n \to \mathbb{R}^k$ linear

$$k \geq \frac{c}{\varepsilon^2} \log N$$

$$(1-\varepsilon)|p_i - p_j| \leq \big|\pi(p_i) - \pi(p_j)\big| \leq (1+\varepsilon)|p_i - p_j|$$

$\pi$ **quasi-isometry**

find $\pi$ **at random**

$$\mathbb{R}^{nk} \ni X = (X_{ij})_{1 \le i \le k, 1 \le j \le n}, \qquad u \in \mathbb{R}^n$$

$$F(X) = |Xu|, \quad |u|\text{-Lipschitz}$$

$$\int_{\mathbb{R}^{nk}} F \, d\gamma \sim \sqrt{k}\,|u|$$

$$\gamma\Big(\big|Xu - \sqrt{k}\,|u|\big| \ge r\,|u|\Big) \le 2\,e^{-r^2/2}$$

$$r = \varepsilon\sqrt{k}, \quad \ell \text{ points } \; u_1, \dots, u_\ell \qquad (u = p_i - p_j)$$

$$\gamma\bigg(\bigcup_{i=1}^{\ell}\Big\{\frac{|Xu_i|}{\sqrt{k}} \notin \big[(1-\varepsilon)|u_i|, (1+\varepsilon)|u_i|\big]\Big\}\bigg) \le 2\,\ell\,e^{-\varepsilon^2 k/2} < 1$$

$$k \sim \frac{1}{\varepsilon^2}\log \ell, \qquad \ell \sim N^2$$

further concentration examples

**discrete cube**

$$X = \{0, 1\}^n$$

$d(x, y) \,=\, \#\big\{1 \leq i \leq n;\ x_i \neq y_i\big\}$ Hamming metric

$\mu$ uniform

$$\alpha_\mu(r) \,\leq\, e^{-r^2/2n}$$

$$\mathrm{ObsDiam}_\mu\big(\{0, 1\}^n)\big) \,=\, O\big(\sqrt{n}\,\big)$$

## symmetric group

$S^n$ symmetric group over $n$ objects

$$d(x,y) = \#\big\{1 \le i \le n;\, x(i) \neq y(i)\big\}, \quad x, y \in S^n$$

$$\mu \quad \text{uniform} \quad \mu\big(\{x\}\big) = \frac{1}{n!}$$

$$\alpha_\mu(r) \le C\, e^{-r^2/Cn}$$

$$\mathrm{ObsDiam}_\mu(S^n) = O\big(\sqrt{n}\,\big)$$

**B. Maurey 1979**

**discrete cube**

$$X = \{0, 1\}^n$$

$$d(x, y) = \# \left\{1 \leq i \leq n;\ x_i \neq y_i\right\}$$ Hamming metric

$\mu$ uniform

$$\alpha_\mu(r) \leq e^{-r^2/2n}$$

same **on any** product space

$$X = X_1 \times \cdots \times X_n$$

$$d(x, y) = \# \left\{1 \leq i \leq n;\ x_i \neq y_i\right\}$$ Hamming metric

$$\mu = \mu_1 \otimes \cdots \otimes \mu_n$$

## discrete cube

$$X = \{0,1\}^n$$

$$d(x,y) = \#\left\{1 \leq i \leq n;\ x_i \neq y_i\right\}$$ Hamming metric

$\mu$ uniform : product measure

$$\alpha_\mu(r) \leq e^{-r^2/2n}$$

same **on any** product space

$$X = X_1 \times \cdots \times X_n$$

$$d(x,y) = \#\left\{1 \leq i \leq n;\ x_i \neq y_i\right\}$$ Hamming metric

$$\mu = \mu_1 \otimes \cdots \otimes \mu_n$$

principle of proof

$$n = 1$$

$F$ $L$-Lipschitz, $\int_{X_1} F \, d\mu_1 = 0$

Jensen $$\int_{X_1} e^F \, d\mu_1 \, \leq \, \int_{X_1 \times X_1} e^{F(x) - F(y)} d\mu_1(x) d\mu_1(y)$$

$$\big| F(x) - F(y) \big| \leq L \, d(x,y) = L \, \mathbf{1}_{x \neq y}$$

$$\int_{X_1 \times X_1} e^{F(x) - F(y)} d\mu_1(x) d\mu_1(y) \, \leq \, e^{L^2}$$

induction over $n$

symmetric group : martingale arguments

$$X \,=\, X_1 \times \cdots \times X_n$$

$$d(x,y) \,=\, \#\big\{1 \le i \le n; x_i \neq y_i\big\}$$

$$\mu \,=\, \mu_1 \otimes \cdots \otimes \mu_n$$

$$\alpha_\mu(r) \,\le\, e^{-r^2/2n}$$

**weighted** Hamming metric

$$d_w(x,y) \,=\, \sum_{i=1}^n w_i \, \mathbf{1}_{x_i \neq y_i}, \quad w_i \ge 0$$

$$\alpha_\mu(r) \,\le\, e^{-r^2/2|w|^2}, \qquad |w|^2 = \textstyle\sum_{i=1}^n w_i^2$$

$$\mu(A) \ge \frac{1}{2}, \qquad \mu\big(d_w(x,A) \ge r\big) \,\le\, e^{-r^2/2|w|^2}$$

## Talagrand's inequality (1995)

$$\mu(A) \geq \frac{1}{2},$$

**induction** over the dimension $n$

application

$$\mu = \mu_1 \otimes \cdots \otimes \mu_n \quad \text{on} \quad [0,1]^n$$

$F : \mathbb{R}^n \to \mathbb{R}$ 1-Lipschitz , **convex**

$m$ median

$$\mu\big(F \geq m + r\big) \leq 2\,e^{-r^2/4}, \quad r > 0$$

**same** as for Gaussian

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**same** as for Gaussian

## **Talagrand's inequality (1995)**

$$\mu(A) \geq \frac{1}{2}, \quad \mu\Big( \sup_{|w|=1} d_w(x, A) \geq r \Big) \leq 2\, e^{-r^2/4}$$

**induction** over the dimension $n$

application

$$\mu = \mu_1 \otimes \cdots \otimes \mu_n \quad \text{on} \quad [0,1]^n$$

$F : \mathbb{R}^n \to \mathbb{R}$ 1-Lipschitz , **convex**

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## illustration

central limit theorem

$X_1, \dots, X_n$ independent identically distributed

$$0 \leq X_i \leq 1$$

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i - \sqrt{n}\,\mathbb{E}(X_1) \sim G$$

$G$ Gaussian

any $F : \mathbb{R}^n \to \mathbb{R}$ 1-Lipschitz convex

$$F(X_1, \dots, X_n)$$

$$\mathbb{P}\Big(F(X_1, \dots, X_n) - \mathbb{E}(F) \geq r\Big) \leq C\, e^{-r^2/C}$$

**empirical processes**

$X_1, \ldots, X_n$ independent identically distributed in $(S, \mathcal{S})$

$\mathcal{F}$ collection of functions $f : S \to [0,1]$

$$Z = \sup_{f \in \mathcal{F}} \sum_{i=1}^{n} f(X_i)$$

$Z$ Lipschitz and convex

**concentration inequalities** on

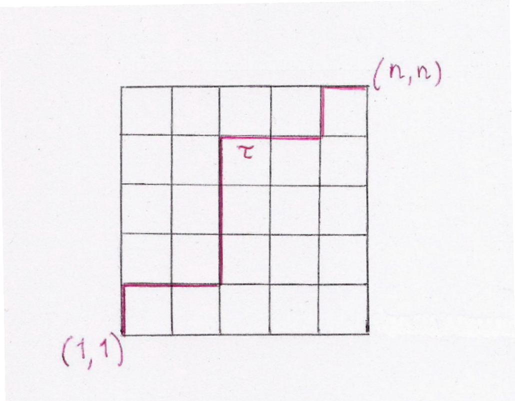
$$\mathbb{P}\Big( \big| Z - \mathbb{E}(Z) \big| \geq r \Big), \quad r > 0$$

# oriented last passage percolation

$w_{ij}, \quad 1 \leq i, j \leq n, \quad$ iid random variables

$$W_n = \max_{\tau} \sum_{(i,j) \in \tau} w_{ij}$$

$\tau \quad$ up/right paths from $\quad (1,1) \quad$ to $\quad (n,n)$



$$W_n = \max_{\tau} \sum_{(i,j)\in\tau} w_{ij}$$

$W_n$ $2n$-Lipschitz (in the $w_{ij}$'s)

$w_{ij}$ Gaussian or Bernoulli

$$\mathbb{P}\Big( W_n \geq \mathbb{E}(W_n) + r \Big) \leq C\, e^{-r^2/Cn}$$

Gaussian behaviour $\mathrm{Var}(W_n) = O(n)$

**however !** ($w_{ij}$ exponential)

$$\mathrm{Var}(W_n) = O(n^{2/3})$$

**new challenges** of measure concentration