Measure concentration, functional inequalities, and curvature of metric measure spaces

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circle of ideas

between analysis, geometry and probability theory

concentration of measure phenomenon

dimension free

geometric, functional,

measure/information theoretic inequalities

optimal transportation and evolution equations

(Ricci) curvature bounds on metric measure spaces

concentration of measure phenomenon

Vitali Milman (1970)

Dvoretzky's theorem on spherical sections

of convex bodies in high dimension

Milman's idea : find the Euclidean section at random

using a concentration (isoperimetric) property

of spherical measures in high dimension **(P. Lévy)**

if $\mu(A) \geq \frac{1}{2} = \mu(B)$ (B half-sphere of $\mathbb{S}^n$)

$$\mu(A_r) \geq \mu(B_r) \geq 1 - e^{-(n-1)r^2/2}, \quad r > 0$$

$$r \sim \frac{1}{\sqrt{n}} \quad (n \to \infty), \qquad \mu(A_r) \approx 1$$

metric measure spaces

metric measure space (X, d, μ)

(X, d) metric space

μ Borel measure on X, $\mu(X) = 1$

concentration function

$$\alpha_\mu(r) = \alpha_{(X,d,\mu)}(r) = \sup\big\{1 - \mu(A_r); \mu(A) \geq 1/2\big\}, \quad r > 0$$

μ uniform on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$: $\alpha_\mu(r) \leq e^{-(n-1)r^2/2}, \quad r > 0$

$F : X \to \mathbb{R}$ 1- Lipschitz, $m = m_F$ median of F

$$\mu\big(\{|F - m| < r\}\big) \geq 1 - 2\,\alpha_\mu(r), \quad r > 0$$

dual description : observable diameter (M. Gromov)

$$\mathrm{PartDiam}_\mu(X, d)$$

$$= \inf \big\{ D \geq 0, \exists\, A \subset X, \mathrm{Diam}\,(A) \leq D, \mu(A) \geq 1 - \kappa \big\}, \quad \kappa > 0$$

$$\mathrm{ObsDiam}_\mu(X, d) \;=\; \sup_{F \ 1-\mathrm{Lip}} \; \mathrm{PartDiam}_{\mu_F}(\mathbb{R})$$

$$F : \mu \to \mu_F$$

$$\mathrm{ObsDiam}_\mu(\mathbb{S}^n) \;=\; O\Big(\frac{1}{\sqrt{n}}\Big)$$

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$$F : \mu \to \mu_F$$

$$\mathrm{ObsDiam}_\mu(X,d) \sim \alpha_\mu^{-1}(\kappa)$$

$$\mathrm{ObsDiam}_\mu(\mathbb{S}^n) = O\Big(\frac{1}{\sqrt{n}}\Big)$$

measure concentration property

less restrictive than isoperimetry, widely shared

variety of examples and tools

- spectral methods **(M. Gromov, V. Milman 1983)**
- probabilistic and combinatorial tools
- product measures **(M. Talagrand 1995)**
- geometric, functional, measure/information theoretic inequalities

measure concentration property

less restrictive than isoperimetry, widely shared

recent work by

Emanuel Milman (2007-2008)

under curvature lower bounds (Riemannian manifold)

log-concave measure

reverse way

from concentration **to** isoperimetry

geometric measure theory, semigroups

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dimension free concentration inequalities

model : Gaussian measure

$$d\gamma(x) = e^{-|x|^2/2} \, \frac{dx}{(2\pi)^{k/2}} \quad \text{on} \quad (\mathbb{R}^k, |\cdot|)$$

concentration property

$$R \sim \sqrt{n}$$

μ uniform on $\mathbb{S}^n_{\sqrt{n}} \quad \to \quad \gamma$ Gaussian

Gauss space : curvature 1 dimension ∞

dimension free concentration inequalities

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$$\mathbb{S}^n : \quad \alpha_\mu(r) \leq e^{-(n-1)r^2/2}$$

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concentration property

$$\mathbb{S}^n_R : \quad \alpha_\mu(r) \leq e^{-(n-1)r^2/2R^2}$$

$$R \sim \sqrt{n}$$

μ uniform on $\mathbb{S}^n_{\sqrt{n}} \quad \to \quad \gamma$ Gaussian

Gauss space : curvature 1 dimension ∞

Gaussian concentration $\alpha_\gamma(r) \le e^{-r^2/2}$

if $A \subset \mathbb{R}^k, \ \gamma(A) \ge \frac{1}{2}$

then $\gamma(A_r) \ge 1 - e^{-r^2/2}, \quad r > 0$

$$\mathrm{ObsDiam}_\gamma(\mathbb{R}^k) = O(1)$$

C. Borell, A. Ibragimov, V. Sudakov, B. Tsirel'son (1974-75)

independent of the dimension

infinite dimensional analysis (Wiener space)

triple description of **Gaussian concentration** $\alpha_\gamma(r) \leq e^{-r^2/2}$

- **geometric**
- **functional**
- **measure/information theoretic**

notion of curvature bound in metric measure spaces

complementary to PDE and calculus of variations viewpoint

geometric description : Brunn-Minkowski inequality

Prékopa-Leindler theorem

$$\theta \in [0,1], \quad u, v, w : \mathbb{R}^n \to \mathbb{R}_+$$

if $\quad w\big(\theta x + (1-\theta)y\big) \geq u(x)^\theta v(y)^{1-\theta}, \quad x, y \in \mathbb{R}^n$

then $\quad \int w\, dx \geq \Big(\int u\, dx\Big)^\theta \Big(\int v\, dx\Big)^{1-\theta}$

$u = \chi_A, \ \ v = \chi_B$ multiplicative form of Brunn-Minkowski

$$\mathrm{vol}_n\big(\theta A + (1-\theta)B\big) \geq \mathrm{vol}_n(A)^\theta \mathrm{vol}_n(B)^{1-\theta}$$

$$dx \;\to\; d\gamma(x) = e^{-|x|^2/2}\,\frac{dx}{(2\pi)^{n/2}}$$

then $$\int w\,d\gamma \;\geq\; \Big(\int u\,d\gamma\Big)^{\theta}\Big(\int v\,d\gamma\Big)^{1-\theta}$$

concentration : $\theta = 1/2, \quad w \equiv 1, \quad v = \chi_A, \quad u = e^{d(\cdot,A)^2/4}$

$$\int e^{d(\cdot,A)^2/4}d\gamma \;\leq\; \frac{1}{\gamma(A)}$$

$$\gamma(A_r) \;\geq\; 1 - 2\,e^{-r^2/4}, \quad \gamma(A) \;\geq\; \frac{1}{2}$$

extension : $d\mu = e^{-V}dx, \quad V'' \geq c > 0$

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if $\quad w\big(\theta x + (1-\theta)y\big) \;\geq\; e^{-\theta(1-\theta)|x-y|^2/2} u(x)^\theta v(y)^{1-\theta}, \quad x, y \in \mathbb{R}^n$

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extension : $\quad d\mu = e^{-V} dx, \quad V'' \geq c > 0$

functional description : logarithmic Sobolev inequality

$$f \geq 0 \quad \text{smooth,} \int f \, d\gamma = 1$$

$$\int f \log f \, d\gamma \, \leq \, \frac{1}{2} \int \frac{|\nabla f|^2}{f} \, d\gamma$$

$$\int f \log f \, d\gamma \, = \, H(f \, d\gamma \,|\, d\gamma) \qquad \text{entropy}$$

$$\int \frac{|\nabla f|^2}{f} \, d\gamma \qquad \text{Fisher information}$$

$$f \to f^2 : \qquad \int f^2 \log f^2 \, d\gamma \, \leq \, 2 \int |\nabla f|^2 \, d\gamma$$

A. Stam (1959), L. Gross (1975)

concentration via the logarithmic Sobolev inequality **(I. Herbst)**

$F : \mathbb{R}^n \to \mathbb{R}$ 1-Lipschitz, $\int F \, d\gamma = 0$

$f = e^{\lambda F} / \int e^{\lambda F} d\gamma, \quad \lambda \in \mathbb{R}$

differential inequality on $\int e^{\lambda F} d\gamma$

$$\int e^{\lambda F} d\gamma \leq e^{\lambda^2/2}, \quad \lambda \in \mathbb{R}$$

$$\gamma\big(\{F < r\}\big) \geq 1 - e^{-r^2/2}, \quad r > 0$$

measure/information theoretic description :

transportation cost inequality

ν probability measure on $\mathbb{R}^n$, $\nu \ll \gamma$

$$W_2(\nu,\gamma)^2 \leq H(\nu \,|\, \gamma)$$

$$H(\nu \,|\, \gamma) = \int \log \frac{d\nu}{d\gamma}\, d\nu$$

relative entropy

$$W_2(\nu,\gamma)^2 = \inf_{\nu \leftarrow \pi \rightarrow \gamma} \iint \frac{1}{2}\, |x-y|^2 d\pi(x,y)$$

Kantorovich-Rubinstein-Wasserstein distance

M. Talagrand (1996)

concentration via the transportation cost inequality **(K. Marton)**

$$A, B \subset \mathbb{R}^n, \quad d(A,B) \geq r > 0$$

$$\gamma_A = \gamma(\cdot\,|A), \quad \gamma_B = \gamma(\cdot\,|B)$$

$$W_2(\gamma_A, \gamma_B) \leq \left(\log \frac{1}{\gamma(A)}\right)^{1/2} + \left(\log \frac{1}{\gamma(B)}\right)^{1/2}$$

$$\frac{r}{\sqrt{2}} \leq W_2(\gamma_A, \gamma_B) \leq \left(\log \frac{1}{\gamma(A)}\right)^{1/2} + \left(\log \frac{1}{\gamma(B)}\right)^{1/2}$$

$\gamma(A) \geq 1/2, \quad B =$ complement of A_r

$$\gamma(A_r) \geq 1 - e^{-r^2/4}, \quad r \geq r_0$$

Prékopa-Leindler inequality

logarithmic Sobolev inequality

transportation cost inequality

Prékopa-Leindler inequality

$$\int w \, d\gamma \ \geq \ \Big(\int u \, d\gamma \Big)^{\theta} \Big(\int v \, d\gamma \Big)^{1-\theta}$$

logarithmic Sobolev inequality

$$\int f \log f \, d\gamma \ \leq \ \frac{C}{2} \int \frac{|\nabla f|^2}{f} \, d\gamma$$

transportation cost inequality

$$W_2(\nu, \gamma)^2 \ \leq \ C \, H(\nu \,|\, \gamma)$$

Prékopa-Leindler inequality

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μ probability measure on $\mathbb{R}^n$

or more general spaces

hierarchy

Prékopa-Leindler inequality

$$\Downarrow$$

logarithmic Sobolev inequality

$$\Downarrow$$

transportation cost inequality

logarithmic Sobolev inequality

$\Downarrow$

transportation cost inequality

F. Otto, C. Villani (1999)

PDE and transportation methods

hypercontractivity of Hamilton-Jacobi equations

P. Cattiaux, A. Guillin (2004), N. Gozlan, P.-M. Samson (2009)

transportation cost inequality is **equivalent**

to logarithmic Sobolev inequality for semi-convex functions

hierarchy

Prékopa-Leindler inequality

$\Downarrow$

logarithmic Sobolev inequality

$\Downarrow$

transportation cost inequality

$\Downarrow$

dimension free Gaussian concentration

stability by products

stability by products

μ satisfies Prékopa-Leindler,

logarithmic Sobolev, or transportation cost inequality

then $\mu^{\otimes k}$ (Euclidean structure)

also satisfies these inequalities (with the same constant)

N . Gozlan (2008) large deviations techniques

product stable Gaussian concentration

equivalent to transportation cost inequality

$$W_2(\nu, \mu)^2 \leq C\, H(\nu \,|\, \mu)$$

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$$\alpha_\mu(r) \leq e^{-r^2/C}$$

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hierarchy

Prékopa-Leindler inequality

$\Downarrow$

logarithmic Sobolev inequality

$\Downarrow$

transportation cost inequality

$\Updownarrow$

dimension free measure concentration

tools to establish these inequalities

parametrisation methods

heat kernel parametrisation

logarithmic Sobolev inequality $\int f \log f \, d\gamma \leq \frac{1}{2} \int \frac{|\nabla f|^2}{f} \, d\gamma$

generator $Lf = \Delta f - x \cdot \nabla f, \quad P_t = e^{tL}$ semigroup

invariant and symmetric for γ (Gaussian measure)

$$\frac{d}{dt} \int P_t f \log P_t f \, d\gamma = -\frac{1}{2} \int \frac{|\nabla P_t f|^2}{P_t f} \, d\gamma$$

commutation $|\nabla P_t f| \leq e^{-t} P_t(|\nabla f|)$

equivalent to a curvature condition

extensions

$\diamond$ $d\mu = e^{-V} dx$ on $\mathbb{R}^n$, $V'' \geq c > 0$

$\diamond$ Riemannian manifolds $\mathrm{Ric} \geq c > 0$

Bochner's formula

$$\frac{1}{2}\,\Delta\big(|\nabla f|^2\big) - \nabla f \cdot \nabla(\Delta f) \;=\; \mathrm{Ric}(\nabla f, \nabla f) + \|\mathrm{Hess}\,(f)\|^2 \;\geq\; c\,|\nabla f|^2$$

equivalent to $|\nabla P_t f| \;\leq\; e^{-ct} P_t\big(|\nabla f|\big)$ **D. Bakry (1986)**

(Gauss space : curvature 1)

$\diamond$ manifolds with weights

$$d\mu = e^{-V} dx, \quad \mathrm{Ric} + \mathrm{Hess}(V) \geq c > 0$$

extensions

◇ second order differential operators

D. Bakry, M. Emery (1985)

Γ_2 curvature principle

analogue of Bochner's formula for Markov operator

diffusion processes, statistical mechanics,

geometric functional inequalities

Markov chains, discrete structures

Y. Ollivier (2008)

parametrisation by optimal transportation

Riemannian geometry of $(\mathcal{P}_2, W_2)$ **F. Otto (2001), C. Villani (2005)**

(Brunn-Minkowski, transportation cost inequalities)

ν probability measure on $\mathbb{R}^n$

$T : \gamma \to \nu$, γ Gaussian measure

triangular transportation **H. Knothe (1957)**

optimal : $W_2(\gamma, \nu)^2 = \int \frac{1}{2} \left| x - T(x) \right|^2 d\gamma(x)$

Y. Brenier, S. T. Rachev - L. Rüschendorf (1990) $T = \nabla \phi$, ϕ convex

manifold case **R. McCann (1995)**

parametrisation $\quad T_\theta = (1-\theta)\,\mathrm{Id} + \theta T, \quad \theta \in [0,1]$

$$(T_0\gamma = \gamma, \quad T_1\gamma = T\gamma = \nu)$$

$$T_\theta : \gamma \to f_\theta \, d\gamma$$

Monge-Ampère equation

$$e^{-|x|^2/2} = f_\theta \circ T_\theta \, e^{-|T_\theta|^2/2} \det\big((1-\theta)\,\mathrm{Id} + \theta\phi''\big)$$

ϕ'' symmetric positive definite

non-smooth analysis, PDE methods

mass transportation method

- **F. Barthe (1998) :** geometric Brascamp-Lieb inequalities, inverse forms

- **D. Cordero-Erausquin, R. McCann, M. Schmuckenschläger (2001, 2006) :** extension of Prékopa-Leindler inequality to manifolds, **J. Lott - C. Villani, K. Th. Sturm (2006-09) :** notion of Ricci curvature bound in metric measure spaces

- **D. Cordero-Erausquin (2002) :** transportation cost and functional inequalities (logarithmic Sobolev...), **D. Cordero-Erausquin, B. Nazaret, C. Villani (2004) :** optimal classical Sobolev inequalities

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Prékopa-Leindler theorem on a Riemannian manifold X

D. Cordero-Erausquin, R. McCann, M. Schmuckenschläger (2001, 2006)

$$d\mu = e^{-V} dx, \quad \mathrm{Ric} + \mathrm{Hess}(V) \geq c$$

if $\quad w\big(z_\theta(x,y)\big) \ \geq \ e^{-c\theta(1-\theta)d(x,y)^2/2} u(x)^\theta v(y)^{1-\theta}, \quad x, y \in X$

for every $z_\theta(x,y)$ *theta*-barycenter of x, y

then $$\int w \, d\mu \ \geq \ \Big(\int u \, d\mu \Big)^\theta \Big(\int v \, d\mu \Big)^{1-\theta}$$

characterizes curvature $\quad \mathrm{Ric} + \mathrm{Hess}(V) \geq c$

K. Bacher (2008), E. Hillion (2009)

optimal parametrisation and entropy

J. Lott - C. Villani, K. Th. Sturm (2006-09)

μ_0, μ_1 probability measures on $\mathbb{R}^n$, $T : \mu_0 \to \mu_1$ optimal

$T_\theta = (1-\theta)\,\mathrm{Id} + \theta\, T, \; \theta \in [0,1]$ geodesic in $(\mathcal{P}_2, W_2)$

reference measure $d\mu = e^{-V} dx$ on $\mathbb{R}^n$, $V'' \geq c$

c-convexity property of entropy along geodesic $\mu_\theta = T_\theta(\mu_0)$

H relative entropy, W_2 Wasserstein distance

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J. Lott - C. Villani, K. Th. Sturm (2006-09)

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$T_\theta = (1-\theta)\,\mathrm{Id} + \theta\, T$, $\theta \in [0,1]$ geodesic in $(\mathcal{P}_2, W_2)$

reference measure $d\mu = e^{-V} dx$ on $\mathbb{R}^n$, $V'' \geq c$

c-convexity property of entropy along geodesic $\mu_\theta = T_\theta(\mu_0)$

$$c = 0 \qquad H(\mu_\theta \,|\, \mu) \leq (1-\theta) H(\mu_0 \,|\, \mu) + \theta H(\mu_1 \,|\, \mu)$$

H relative entropy, W_2 Wasserstein distance

optimal parametrisation and entropy

J. Lott - C. Villani, K. Th. Sturm (2006-09)

μ_0, μ_1 probability measures on $\mathbb{R}^n$, $T : \mu_0 \to \mu_1$ optimal

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$$H(\mu_\theta \,|\, \mu) \leq (1-\theta) H(\mu_0 \,|\, \mu) + \theta H(\mu_1 \,|\, \mu) - c\,\theta(1-\theta) W_2(\mu_0, \mu_1)^2$$

H relative entropy, W_2 Wasserstein distance

c-convexity property of entropy along geodesic $\mu_\theta = T_\theta(\mu_0)$

$$H(\mu_\theta \,|\, \mu) \;\leq\; (1-\theta) H(\mu_0 \,|\, \mu) + \theta H(\mu_1 \,|\, \mu) - c\, \theta (1-\theta) W_2(\mu_0, \mu_1)^2$$

characterizes $V'' \geq c$

reference measure $d\mu = e^{-V} dx$

extends to (weighted) manifolds

characterizes $\mathrm{Ric} + \mathrm{Hess}(V) \geq c$

K. Th. Sturm (2005)

notion of Ricci curvature bound

in a metric measure space (length space) (X, d, μ)

$(\mu_\theta)_{\theta \in [0,1]}$ geodesic in $(\mathcal{P}_2(X), W_2)$ connecting μ_0, μ_1

definition of lower bound on curvature

postulate that entropy is c-convex along one geodesic $(\mu_\theta)_{\theta \in [0,1]}$

$$H(\mu_\theta \,|\, \mu) \leq (1-\theta) H(\mu_0 \,|\, \mu) + \theta H(\mu_1 \,|\, \mu) - c\,\theta(1-\theta) W_2(\mu_0, \mu_1)^2$$

H relative entropy, W_2 Wasserstein distance

J. Lott - C. Villani, K.-Th. Sturm (2006-09)

definition of lower bound on curvature

in metric measure space

$$H(\mu_\theta \,|\, \mu) \leq (1-\theta)H(\mu_0 \,|\, \mu) + \theta H(\mu_1 \,|\, \mu) - c\,\theta(1-\theta)W_2(\mu_0, \mu_1)^2$$

◇ generalizes Ricci curvature in Riemannian manifolds

◇ allows for geometric and functional inequalities

◇ main result : stability of the definition by Gromov-Hausdorff limit

analysis on singular spaces (limits of Riemannian manifolds)