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BOOK REVIEW

Dominique Bakry, Ivan Gentil, Michel Ledoux: "Analysis and Geometry of Markov Diffusion Operators"

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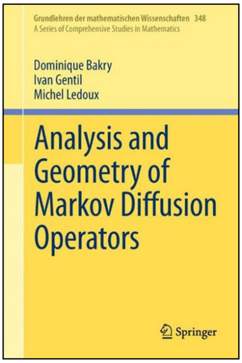

Everything is already in the title: analysis, geometry, but also probability (Markov and diffusion), and functional analysis (operators). The crossroads here is the notion of Markov semigroup of operators, namely a family of linear operators $(P_t)_{t>0}$ acting on bounded measurable functions on a measure space $(E, \mathcal{F}, \mu)$, such that $P_t \circ P_s = P_{t+s}$ for all $t, s > 0$, P_t is positivity-preserving, $P_t 1 = 1$ for all $t > 0$, and P_t converges to identity in a strong L^2 sense as t goes to zero. In other words, $(P_t)_{t>0}$ describes a continuous, positivity-preserving, and mass conservative evolution. More often than not, the operators P_t are in addition symmetric on L^2. It is known since [5] that this simple axiomatic hides a superb analytic tool.

It is typical of this book that this definition is introduced for the first time on p. 12, after many introductory comments. The full range of definitions of the associated objects is given on pp. 53–56. This is because: (a) everything in this book starts from examples (b) the difficulties hanging around the definitions of the objects, including the basic ones, are described with an extreme care.

Of course this style of exposition reflects the taste, choices, personalities of the authors. It also stems from the very nature and ambition of the subject: the game is to encapsulate in a single, abstract, and soft framework an awful lot of different situations. But in order to stay as close as possible to real life, and to describe accurately the various species under scrutiny, some very precise assumptions have to be made.

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A basic notion is the one of infinitesimal generator, a densely defined operator L that is nothing but the right derivative of $t \to P_t$ at zero. One may write $P_t = e^{tL}$. The Dirichlet form is then defined by $\mathcal{E}(f,g) = -\langle Lf, g\rangle$, $\langle .,.\rangle$ being the scalar product in L^2, and the carré du champ operator by

$$\Gamma(f,g) := \frac{1}{2}\big[L(fg) - fLg - gLf\big]. \tag{1}$$

Note that

$$\mathcal{E}(f,g) = \int_E \Gamma(f,g).$$

Without assuming any differential structure on the state space E, $\Gamma(f,g)$ provides the analogue of $\nabla f.\nabla g$ in the case where $L = \Delta$, the Laplacian on the Euclidean space or on a Riemannian manifold. Note that the theory of Dirichlet forms is in itself beautiful, simple and very efficient, see [2].

Why is this about probability? Because of the one-to-one correspondence between Markov semigroups and Markov processes.

Why is this about geometry? Because adding a time dimension, in other words looking at the way a state space reacts to an evolution, leads to a very efficient description of its geometry (see [1, 3]). Also because this book revolves around a functional analytic point of view on curvature, seen as an obstruction to commutation between gradient and Laplace operator.

Why is this about analysis? Because of the techniques employed. The book indeed begins with the classical mock definition of analysis "How far can you go with the Cauchy-Schwarz inequality and integration by parts"? in fact, a more accurate description could be: "How far can you go by simply integrating (geometric differential inequalities along time)"? Beyond that, many analytic tools and notions are central here. To name a few, Sobolev inequalities, Poincaré inequalities, or (sub)-ellipticity.

The structure of the book has four levels: three parts (on semigroups, on functional inequalities, on varia), three chapters in each part, a number of sections and subsections (only sections appear in the table of contents). Part I is about the basic objects. Chapter 1 introduces the concepts : Markov semigroups, invariant measures, kernel representation, infinitesimal generator, carré du champ operator.

Formula (1) already shows that the existence of a sufficiently rich algebra for the pointwise product $\mathcal{A}$ inside the domain $\mathcal{D}(L)$ of the generator L is necessary for the theory to make sense. This is explored at length. The connection with diffusion processes on the one hand, differential operators on the other hand, is exposed in detail. Only after a survey of a wealth of examples, the list of necessary and possible hypotheses is presented. The basic operations one can perform on a Markov semigroup: change of coordinates, time change, products, adding a vector field or a potential, projection, subordination, are explained and illustrated by examples. Finally, the curvature-dimension condition is introduced. In a similar way as one defines Γ from L and the pointwise product, define the iterated gradient Γ_2 from L and Γ:

$$\Gamma_2(f,g) := \frac{1}{2}\big[L\Gamma(f,g) - \Gamma(f,Lg) - \Gamma(Lf,g)\big].$$

One can then say that the space has curvature bounded below by $\rho \in \mathbb{R}$ and dimension bounded above by $n \in [1, \infty]$ if

$$\Gamma_2(f, f) \geq \rho \Gamma(f, f) + \frac{1}{n}(Lf)^2 \tag{2}$$

for all suitable f.

When E is a Riemannian manifold and L its Laplace-Beltrami operator, this condition indeed holds if and only if the Ricci curvature is bounded below by ρ and the topological dimension is not larger than n.

Chapter 2 introduces examples: heat semigroups on spaces with constant (null, positive and negative) curvature, heat semigroups on the half-line and on the bounded interval, Ornstein-Uhlenbeck, Jacobi and Laguerre semigroups.

Finally, the general framework is set up, under the name "Markov triple": a state space, an invariant measure, a "carré du champ". This is as simple as that, but it only appears on p. 121 and all the traps and subtleties are exposed during a little more than fifty pages.

In Part II, functional inequalities : Poincaré, Sobolev, logarithmic Sobolev inequalities (one chapter for each) enter the play. The general idea is that they involve the Dirichlet form on the right-hand side, and some expression related to the size and/or oscillation of the function on the left-hand side. In the Poincaré inequality, the Dirichlet form simply dominates the variance. Another way to see this phenomenon is to say that the generator admits a spectral gap. This is typical of compact situations or of measures with exponential tail. In the logarithmic Sobolev inequality, the Dirichlet form plus the L^2-norm dominates the entropy. This is typical of infinite-dimensional situations. A beautiful theorem due to Gross (to whom the book is understandably dedicated) says that logarithmic Sobolev inequalities characterize hypercontractivity of the semigroup, i.e. the fact that P_t maps L^p into L^q, $q > p$, for some time $t = t(p, q)$. Finally, Sobolev or Nash type inequalities are typical of finite-dimensional situations, and Varopoulos (resp. Carlen-Kusuoka-Stroock) have shown that they imply ultracontractivity, that is the fact that P_t maps L^1 into L^∞, for all $t > 0$, the norm decaying like a negative power of time related to the dimension. The main results say that (2) with $\rho > 0$ and $n = +\infty$ implies a Poincaré inequality and a logarithmic Sobolev inequality, and that (2) with $\rho > 0$ and $2 < n < +\infty$ implies a Sobolev inequality. The proofs of these results are quite simple: this is where one integrates simple differential inequalities. This only means the theory is well-adapted and effective. Its value lies in the variety of examples treated, and in the wealth of operations one can perform on the objects.

Note that deliberate choices are made here: more flexible forms of the inequalities could be considered (see for instance [4]). More importantly, the beauty of the connection between curvature and functional inequalities of the above type and mapping properties of the semigroup (in other words, heat kernel estimates) is that a local geometric assumption yields a global analytic conclusion. The corresponding weakness of this theory is that it does not like perturbations. One may be interested in more direct and more stable connections between geometry and analysis (see for instance [1] or [3]).

Part III is devoted to generalizations and reinterpretations of functional inequalities: energy-entropy inequalities (governing ultracontractivity with decays other than polynomial), set theoretic formulations of functional inequalities (with two sister theories, one from outside: capacity, one from inside: isoperimetry), and optimal transportation.

This book is not exhaustive, but it is extremely rich. It is more original and inspirational than a treatise. One can use it and benefit from it in many ways: as a reference book, as an inspiration source, by focusing on a property or on an example.

A huge effort has been put into bibliography and historical notes. Needless to say, the set of inaccuracies is not empty.

The book is accessible: the prerequisites are well-explained in three appendices, devoted to semigroups of operators on Banach spaces, stochastic calculus, differential and Riemannian geometry. But in the end, it connects to a wide mathematical landscape.

Many things are not in this book, even on some subjects to which the authors have contributed: pointwise estimates of heat kernels, especially lower bounds, Riesz transforms, heat semigroups on fractals or on forms. But so many things are there, including a beautiful and unexpected ending: the recipe of the chicken "Gaston Gérard".

From the beginning to the end, this book definitely has a strong personality and a characteristic taste. I would discuss many of its choices. But I would urge anybody who wants to explore analytic, probabilistic or geometric properties of Markov semigroups to have a look at it first.

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