How does the heat equation explore geometric and functional inequalities?

M. Ledoux

University of Toulouse, France

& Institut Universitaire de France

developments of the last decades

fruitful interactions

between heat flows

and geometric and functional inequalities

heat flow monotonicity

multiple integral inequalities,

Sobolev and isoperimetric-type inequalities,

Ricci flows, geometric analysis

developments of the last decades

fruitful interactions

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heat flow monotonicity

D. Bakry, M. Émery (1985)

E. Carlen, E. Lieb, M. Loss (2004)

J. Bennett, A. Carbery, M. Christ, T. Tao (2008)

developments of the last decades

fruitful interactions

between heat flows

and geometric and functional inequalities

parallel/interactive investigations

along optimal transport

F. Otto, C. Villani

developments of the last decades

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Sobolev and isoperimetric-type inequalities,

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heat flow monotonicity

heat kernel/semigroup/equation

various settings

basics on heat kernel

heat semigroup, heat equation in $\mathbb{R}^n$

standard heat kernel on $\mathbb{R}^n$

$$h_t(x) = \frac{1}{(4\pi t)^{n/2}}\, e^{-|x|^2/4t}, \quad t > 0,\ x \in \mathbb{R}^n$$

solution of $\partial_t h_t = \Delta h_t$

heat semigroup $\varphi : \mathbb{R}^n \to \mathbb{R}$

$$H_t\varphi(x) = \varphi * h_t(x) = \int_{\mathbb{R}^n} \varphi(y)\, e^{-|x-y|^2/4t}\, \frac{dy}{(4\pi t)^{n/2}}, \quad t > 0,\ x \in \mathbb{R}^n$$

$$H_s \circ H_t = H_{s+t}, \quad H_0 = \mathrm{Id}$$

probabilistic interpretation

$$H_t\varphi(x) = \mathbb{E}\big(\varphi(x + B_{2t})\big), \quad (B_t)_{t\geq 0}$$ Brownian motion

$$H_t\varphi(x) \, = \, \varphi * h_t(x) \, = \, \int_{\mathbb{R}^n} \varphi(y)\, e^{-|x-y|^2/4t} \, \frac{dy}{(4\pi t)^{n/2}} \,, \quad t > 0, \; x \in \mathbb{R}^n$$

$$H_s \circ H_t \, = \, H_{s+t}, \quad H_0 \, = \, \mathrm{Id}$$

for every $\varphi : \mathbb{R}^n \to \mathbb{R}$ bounded and continuous

$$u(t,x) \, = \, H_t\varphi(x), \quad t \geq 0, \; x \in \mathbb{R}^n$$

solution of (heat equation)

$$\partial_t u \, = \, \Delta u \quad \text{on} \quad]0,\infty[\, \times \mathbb{R}^n$$

with initial condition $u(0,\cdot) = \varphi$

(Δ generator of $(H_t)_{t \geq 0}$, $H_t \, = \, e^{t\Delta}$)

huge topic: mathematics, physics, mechanics, biology...

domains on $\mathbb{R}^n$ (Dirichlet, Neumann, mixture)

heat semigroup on a Riemannian manifold (M, g)

generator Laplace-Beltrami operator Δ

geometric structures

with drift $\mathrm{L} = \Delta - \langle \nabla V, \nabla \rangle$

$V : \mathbb{R}^n \to \mathbb{R}$ smooth, $e^{-V}dx$ invariant measure

$P_t = e^{t\,\mathrm{L}}$, $t \geq 0$,

$u = P_t f$ solves $\partial_t u = \mathrm{L}u$

example: $V(x) = \frac{|x|^2}{2}$ Ornstein-Uhlenbeck semigroup partial differential equations

dual formulation of

$$\partial_t u = \mathrm{L} u, \qquad \mathrm{L} = \Delta - \langle \nabla V, \nabla \rangle$$

Fokker-Planck equation

$$\partial_t p = \Delta p + \nabla \cdot (p \nabla V)$$

probability densities $p = p(t,x)$ $(= u\, e^{-V})$

(with respect to Lebesgue measure)

developments of the last decades

fruitful interactions

between heat flow approach

and geometric and functional inequalities

heat flow monotonicity

multiple integral inequalities,

Sobolev and isoperimetric-type inequalities

illustration in a (very!) elementary sample example sample example

$$\int_{\mathbb{R}^n} f^\theta g^{1-\theta} dx \;\leq\; \Big(\int_{\mathbb{R}^n} f\, dx \Big)^\theta \Big(\int_{\mathbb{R}^n} g\, dx \Big)^{1-\theta}$$

$$f, g : \mathbb{R}^n \to \mathbb{R}_+, \quad \theta \in [0,1]$$

show that, for every $t > 0$,

$$H_t\big(f^\theta g^{1-\theta}\big) \;\leq\; (H_t f)^\theta (H_t g)^{1-\theta}$$

$$\lim_{t \to \infty} (4\pi t)^{n/2} H_t \varphi \;=\; \int_{\mathbb{R}^n} \varphi\, dx$$

inequality between functions, at every point (omitted)

$$H_t\big(f^\theta g^{1-\theta}\big) \;\leq\; (H_t f)^\theta (H_t g)^{1-\theta}$$

interpolation along the heat semigroup (Duhamel's principle)

$$\Lambda(s) \;=\; H_s\Big(\big(H_{t-s} f\big)^\theta \big(H_{t-s}\, g\big)^{1-\theta}\Big), \quad s \in [0, t]$$

decreasing

$$H_t\big(f^\theta g^{1-\theta}\big) \;=\; \Lambda(t) \;\leq\; \Lambda(0) \;=\; (H_t f)^\theta (H_t\, g)^{1-\theta}$$

take derivative!

$$\Lambda'(s) \;\leq\; 0\,?$$

$$\Lambda(s) \,=\, H_s\Big(\big(H_{t-s} f\big)^{\theta}\big(H_{t-s}\, g\big)^{1-\theta}\Big), \quad s \in [0, t]$$

$$F \,=\, \log H_{t-s} f, \quad e^F \,=\, H_{t-s} f, \qquad G \,=\, \log H_{t-s}\, g, \quad e^G \,=\, H_{t-s}\, g$$

$$\Lambda(s) \,=\, H_s\big(e^{\theta F} e^{(1-\theta)G}\big) \,=\, H_s(e^K)$$

$$K \,=\, \theta F + (1-\theta)G$$

heat equation for F

$$\partial_s F \,=\, \frac{\partial_s H_{t-s} f}{H_{t-s} f} \,=\, -\frac{\Delta H_{t-s} f}{H_{t-s} f} \,=\, -\,e^{-F}\Delta(e^F)$$

similarly for G

$$F \,=\, \log H_{t-s} f, \quad G \,=\, \log H_{t-s}\, g, \quad K = \theta F + (1-\theta) G$$

$$\Lambda(s) = H_s(e^K)$$

chain rule

$$\Lambda'(s) \,=\, \partial_s H_s(e^K) + H_s\big(\partial_s(e^K)\big)$$

heat equation

$$\partial_s H_s(e^K) = \Delta H_s(e^K) = H_s \Delta(e^K)$$

$$\partial_s(e^K) \,=\, e^K \partial_s K \,=\, -\, e^K \big[\theta\, e^{-F} \Delta(e^F) + (1-\theta) e^{-G} \Delta(e^G)\big]$$

$$\Lambda'(s) \,=\, H_s\Big(e^K \Big[e^{-K}\Delta(e^K) - \big[\theta\, e^{-F}\Delta(e^F) + (1-\theta)e^{-G}\Delta(e^G)\big]\Big]\Big)$$

$$\Lambda'(s) = H_s\Big(e^K\Big[e^{-K}\Delta(e^K) - \big[\theta\, e^{-F}\Delta(e^F) + (1-\theta)e^{-G}\Delta(e^G)\big]\Big]\Big)$$

derivation in space $\quad e^{-F}\Delta(e^F) = \Delta F + |\nabla F|^2$

similarly for G and $K = \theta F + (1-\theta)G$

$$\Lambda'(s) = H_s\Big(e^K\big[|\nabla K|^2 - \theta|\nabla F|^2 - (1-\theta)|\nabla G|^2\big]\Big)$$

$$\nabla K = \theta\nabla F + (1-\theta)\nabla G$$

(quadratic) convexity $\quad \Lambda'(s) \le 0$

important aspect of the heat flow proof

reduction to a quadratic inequality

(for any $\theta \in [0,1]$)

proof of Hölder by Cauchy-Schwarz!

introduction of geometric features

H. Brascamp, E. Lieb (1976)

K. Ball (1989)

$u_1, \ldots, u_m$ unit vectors in $\mathbb{R}^n$

decomposition of the identity

$$\sum_{k=1}^{m} c_k u_k \otimes u_k = \mathrm{Id}_{\mathbb{R}^n}$$

$$0 \leq c_k \leq 1, \quad k = 1, \ldots, m$$

for all non-negative functions $f_k : \mathbb{R} \to \mathbb{R}, \quad k = 1, \ldots, m$

$$\int_{\mathbb{R}^n} \prod_{k=1}^{m} f_k^{c_k}\big(\langle u_k, x\rangle\big)dx \leq \prod_{k=1}^{m} \left(\int_{\mathbb{R}} f_k \, dx \right)^{c_k}$$

$$\int_{\mathbb{R}^n} \prod_{k=1}^{m} f_k^{c_k}(\langle u_k, x\rangle)dx \leq \prod_{k=1}^{m} \left(\int_{\mathbb{R}} f_k \, dx \right)^{c_k}$$

decomposition of the identity $\sum_{k=1}^m c_k u_k \otimes u_k = \mathrm{Id}_{\mathbb{R}^n}$

$$\sum_{k=1}^m c_k = n$$

improvement of Hölder's inequality

in the directions u_k, $k = 1, \ldots, m$

example: decomposition of the identity in $\mathbb{R}^2$

along the cubic roots of unity

$$\int_{\mathbb{R}^2} f_1^{2/3}(x) f_2^{2/3}\big(-\tfrac{1}{2}x + \tfrac{\sqrt{3}}{2}y\big) f_3^{2/3}\big(-\tfrac{1}{2}x - \tfrac{\sqrt{3}}{2}y\big) dxdy \ \leq \ \prod_{k=1}^{3} \bigg(\int_{\mathbb{R}} f_k \, dx \bigg)^{2/3}$$

best constants in Young's convolution inequality

Shannon's inequality in information theory

hypercontractivity

multidimensional versions

H. Brascamp, E. Lieb (1976) rearrangements

F. Barthe (1998) optimal transport

heat flow monotonicity

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geometric and combinatorial analysis of extremal functions

sphere, symmetric spaces, Lie groups

discrete models, symmetric group

standard Gaussian measure on $\mathbb{R}^n$

$$d\gamma(x) = d\gamma_n(x) = e^{-|x|^2/2} \frac{dx}{(2\pi)^{n/2}}$$

(product of $d\gamma_1$ on $\mathbb{R}$)

$$\sum_{k=1}^{m} c_k u_k \otimes u_k = \mathrm{Id}_{\mathbb{R}^n}$$

$$0 \le c_k \le 1, \quad k = 1, \ldots, m$$

for all non-negative functions $f_k : \mathbb{R} \to \mathbb{R}, \quad k = 1, \ldots, m$

$$\int_{\mathbb{R}^n} \prod_{k=1}^{m} f_k^{c_k}\big(\langle u_k, x\rangle\big) d\gamma_n(x) \le \prod_{k=1}^{m} \left(\int_{\mathbb{R}} f_k \, d\gamma_1 \right)^{c_k}$$

$$n = 2, \ m = 2, \qquad u_1 = (1, 0), \ u_2 = \big(\rho, \sqrt{1-\rho^2}\,\big), \quad \rho \in \,]0, 1[$$

(sub-) decomposition of identity $\quad c_1 u_1 \otimes u_1 + c_2 u_2 \otimes u_2 \leq \mathrm{Id}_{\mathbb{R}^2}$

$$c_1, c_2 \in \,]0, 1[\,, \quad \rho^2 c_1 c_2 = (1 - c_1)(1 - c_2)$$

$f_1, f_2 : \mathbb{R} \to \mathbb{R}$ non-negative

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f_1^{c_1}(x_1) f_2^{c_2}\big(\rho x_1 + \sqrt{1-\rho^2}\, x_2\big) d\gamma(x_1) d\gamma(x_2)$$

$$\leq \left(\int_{\mathbb{R}} f_1 \, d\gamma \right)^{c_1} \left(\int_{\mathbb{R}} f_2 \, d\gamma \right)^{c_2}$$

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$f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$ non-negative

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f_1^{c_1}(x_1) f_2^{c_2}\big(\rho x_1 + \sqrt{1-\rho^2}\,x_2\big) d\gamma(x_1) d\gamma(x_2) \leq \left(\int_{\mathbb{R}^n} f_1\,d\gamma\right)^{c_1}\left(\int_{\mathbb{R}^n} f_2\,d\gamma\right)^{c_2}$$

$$T_\rho \varphi(x) = \int_{\mathbb{R}^n} \varphi\big(\rho x + \sqrt{1-\rho^2}\, y\big)\, d\gamma(y), \quad x \in \mathbb{R}^n$$

T_ρ contraction in $\mathrm{L}^p(\gamma), \quad 1 \leq p \leq \infty$

$P_t = T_{e^{-t}},\ t \geq 0$ Ornstein-Uhlenbeck semigroup huge topic: mathematics, physics, mechanics, biology...

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$$c_1, c_2 \in \,]0, 1[\,, \quad \rho^2 c_1 c_2 \ = \ (1 - c_1)(1 - c_2)$$

$f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$ non-negative

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f_1^{c_1}(x_1) f_2^{c_2}\big(\rho x_1 + \sqrt{1-\rho^2}\, x_2\big) d\gamma(x_1) d\gamma(x_2)$$

$$\leq \ \bigg(\int_{\mathbb{R}^n} f_1 \, d\gamma \bigg)^{c_1} \bigg(\int_{\mathbb{R}^n} f_2 \, d\gamma \bigg)^{c_2}$$

$$T_\rho \varphi(x) = \int_{\mathbb{R}^n} \varphi\big(\rho x + \sqrt{1-\rho^2}\, y\big)\, d\gamma(y), \quad x \in \mathbb{R}^n$$

T_ρ contraction in $\mathrm{L}^p(\gamma)$, $\quad 1 \leq p \leq \infty$

$P_t = T_{e^{-t}},\ t \geq 0$ Ornstein-Uhlenbeck semigroup

$$\int_{\mathbb{R}^n} f_1^{c_1} T_\rho(f_2^{c_2})\, d\gamma \leq \bigg(\int_{\mathbb{R}^n} f_1\, d\gamma\bigg)^{c_1} \bigg(\int_{\mathbb{R}^n} f_2\, d\gamma\bigg)^{c_2}$$

$$p_1 = \frac{1}{c_1}, \quad p_2 = \frac{1}{c_2}, \qquad f_i \to f_i^{p_i}$$

duality

E. Nelson (1966)

$$\|T_\rho f\|_{p_1'} \leq \|f\|_{p_2}$$

$$1 < p_2 < p_1' < \infty, \qquad \frac{1}{\rho^2} = \frac{p_1' - 1}{p_2 - 1}$$

stronger than contractivity: $\mathrm{L}^{p_1'}(\gamma) \subset \mathrm{L}^{p_2}(\gamma)$

quantum field theory

smoothing property

Sobolev-type inequalities

L. Gross (1975)

hypercontractivity

$$\|T_\rho f\|_{p_1'} \leq \|f\|_{p_2}, \qquad p_1' = 1 + \frac{1}{\rho^2}(p_2 - 1)$$

derivative in $\rho \quad (\to 1)$

equivalent to logarithmic Sobolev inequality

$$\int_{\mathbb{R}^n} f^2 \log f^2 d\gamma \leq 2 \int_{\mathbb{R}^n} |\nabla f|^2 d\gamma, \qquad \int_{\mathbb{R}^n} f^2 d\gamma = 1$$

proof by Ornstein-Uhlenbeck semigroup D. Bakry, M. Émery (1985) logarithmic Sobolev inequality

$$\int_{\mathbb{R}^n} f^2 \log f^2 d\gamma \,\leq\, 2 \int_{\mathbb{R}^n} |\nabla f|^2 d\gamma, \qquad \int_{\mathbb{R}^n} f^2 d\gamma = 1$$

classical Sobolev inequality in $\mathbb{R}^n$ $(n \geq 3)$

$$\|f\|^2_{\frac{2n}{n-2}} \,\leq\, C_n \, \|\nabla f\|_2^2 \,=\, C_n \int_{\mathbb{R}^n} |\nabla f|^2 dx, \qquad f \in C_c^1(\mathbb{R}^n)$$

logarithmic Sobolev inequality: independent of dimension

infinite dimensional analysis

$$\|f\|^2_{\frac{2n}{n-2}} \leq C_n \|\nabla f\|_2^2 = C_n \int_{\mathbb{R}^n} |\nabla f|^2 dx$$

L^1 Sobolev inequality in $\mathbb{R}^n$

$$n\omega_n^{1/n} \|f\|_{\frac{n}{n-1}} \leq \|\nabla f\|_1 = \int_{\mathbb{R}^n} |\nabla f| \, dx$$

root of the full scale of Sobolev inequalities

$$f \to f^\alpha$$

$$\|f\|_{\frac{pn}{n-p}} \leq C_{n,p} \|\nabla f\|_p, \quad 1 \leq p < n$$

$$n\omega_n^{1/n}\|f\|_{\frac{n}{n-1}} \le \|\nabla f\|_1 = \int_{\mathbb{R}^n} |\nabla f|\, dx$$

$$f = \mathbb{1}_A$$

$$n\omega_n^{1/n}\mathrm{vol}_n(A)^{(n-1)/n} \le \mathrm{vol}_{n-1}(\partial A)$$

isoperimetric inequality

$A \subset \mathbb{R}^n$ same volume as a ball B

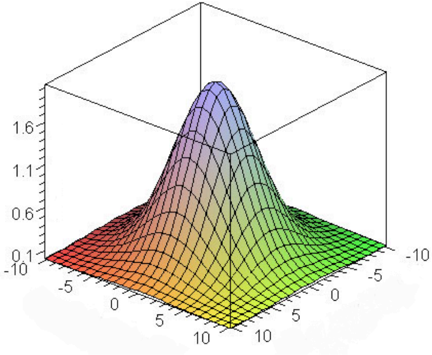
$$\mathrm{vol}_n(A) = \mathrm{vol}_n(B) = \omega_n r^n$$

$$\mathrm{vol}_{n-1}(\partial B) = n\omega_n r^{n-1}$$

$$\mathrm{vol}_{n-1}(\partial B) = n\omega_n^{1/n}\mathrm{vol}_n(A)^{(n-1)/n} \le \mathrm{vol}_{n-1}(\partial A)$$

standard Gaussian measure on $\mathbb{R}^n$

$$d\gamma(x) = e^{-|x|^2/2} \frac{dx}{(2\pi)^{n/2}}$$



standard Gaussian measure on $\mathbb{R}^n$

$$d\gamma(x) = e^{-|x|^2/2} \frac{dx}{(2\pi)^{n/2}}$$

boundary measure

$$\gamma(\partial A) = \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[\gamma(A_\varepsilon) - \gamma(A) \right]$$

A_ε neighbourhood of A

what are the extremal sets

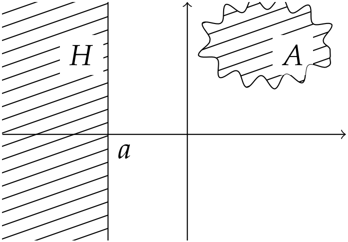
of the isoperimetric problem for γ?

half-spaces

$$H = \{x \in \mathbb{R}^n; \langle x, u \rangle \le a\}, \quad u \in \mathbb{R}^n \text{ unit vector}, \quad a \in \mathbb{R}$$

Gaussian isoperimetric inequality

if $\gamma(A) = \gamma(H)$ then $\gamma(\partial A) \ge \gamma(\partial H)$

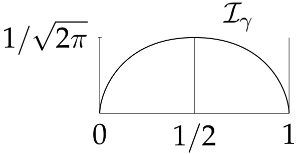


$$\mathcal{I}_\gamma(v) \;=\; \inf\big\{\gamma(\partial A); \gamma(A) = v\big\}, \quad v \in \,]0,1[$$

$$\mathcal{I}_\gamma \;=\; \Phi' \circ \Phi^{-1}$$

(dimension one) $\quad H \;=\; \big\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n;\ x_1 \leq a\big\}$

$$\gamma(H) \;=\; \int_{-\infty}^{a} e^{-x^2/2}\,\frac{dx}{\sqrt{2\pi}} \;=\; \Phi(a), \qquad \gamma(\partial H) \;=\; \Phi'(a) \;=\; \frac{e^{-a^2/2}}{\sqrt{2\pi}}$$



heat flow monotonicity

$$\Downarrow$$

hypercontractivity

$$\Updownarrow$$

logarithmic Sobolev inequalities

$$\Uparrow$$

Gaussian isoperimetry

$$\Uparrow ?$$

heat flow monotonicity

C. Borell (1985)

$$T_\rho \varphi(x) = \int_{\mathbb{R}^n} \varphi\big(\rho x + \sqrt{1-\rho^2}\, y\big) d\gamma(y), \quad x \in \mathbb{R}^n$$

$P_t = T_{e^{-t}},\ t \geq 0$ Ornstein-Uhlenbeck semigroup

A, B Borel sets in $\mathbb{R}^n$, $\rho \in [0,1]$

$$\int_{\mathbb{R}^n} \mathbb{1}_A\, T_\rho(\mathbb{1}_B) d\gamma \leq \int_{\mathbb{R}^n} \mathbb{1}_H\, T_\rho(\mathbb{1}_K) d\gamma$$

H, K half-spaces, $\gamma(H) = \gamma(A)$, $\gamma(K) = \gamma(B)$

$$\int_{\mathbb{R}^n} \mathbb{1}_A \, T_\rho(\mathbb{1}_B) d\gamma \ \leq \ \int_{\mathbb{R}^n} \mathbb{1}_H \, T_\rho(\mathbb{1}_K) d\gamma$$

$$\gamma(\partial A) \ \geq \ \limsup_{\rho \to 1} \sqrt{\frac{\pi}{\log \frac{1}{\rho}}} \left[\gamma(A) - \int_{\mathbb{R}^n} \mathbb{1}_A \, T_\rho(\mathbb{1}_A) d\gamma \right]$$

with equality for half-spaces

if $\gamma(A) = \gamma(H)$ then $\gamma(\partial A) \geq \gamma(\partial H)$

Gaussian isoperimetric inequality

C. Borell (1985)

$$T_\rho \varphi(x) = \int_{\mathbb{R}^n} \varphi\big(\rho x + \sqrt{1-\rho^2}\, y\big) d\gamma(y), \quad x \in \mathbb{R}^n$$

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H, K half-spaces, $\gamma(H) = \gamma(A),\ \gamma(K) = \gamma(B)$

rearrangement techniques A. Ehrhard (1983)

E. Mossel, J. Neeman (2015)

J_ρ function

$$\int_{\mathbb{R}^n} \mathbb{1}_A \, T_\rho(\mathbb{1}_B) d\gamma \le \int_{\mathbb{R}^n} \mathbb{1}_H \, T_\rho(\mathbb{1}_K) d\gamma = J_\rho(u, v), \;\; u = \gamma(H), v = \gamma(K)$$

$J_\rho : [0,1]^2 \to [0,1]$ explicit

$$J_\rho(1,1) = 1, \qquad J_\rho(1,0) = J_\rho(0,1) = J_\rho(0,0) = 0$$

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} J_\rho\Big(\mathbb{1}_A(x), \mathbb{1}_B\big(\rho x + \sqrt{1-\rho^2}\, y\big)\Big) d\gamma(x) d\gamma(y)$$
$$\le J_\rho\bigg(\int_{\mathbb{R}^n} \mathbb{1}_A \, d\gamma, \int_{\mathbb{R}^n} \mathbb{1}_B \, d\gamma\bigg)$$

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_\rho\big(\mathbb{1}_A(x), \mathbb{1}_B\big(\rho x + \sqrt{1-\rho^2}\, y\big)\big)\, d\gamma(x)\, d\gamma(y)$$
$$\leq J_\rho\bigg(\int_{\mathbb{R}^n} \mathbb{1}_A\, d\gamma, \int_{\mathbb{R}^n} \mathbb{1}_B\, d\gamma\bigg)$$

replace $\mathbb{1}_A$ by $P_t \mathbb{1}_A$, $\mathbb{1}_B$ by $P_t \mathbb{1}_B$, $t \geq 0$

$P_t = T_{e^{-t}}$, $t \geq 0$ Ornstein-Uhlenbeck semigroup

generator $\mathrm{L} = \Delta - \langle \nabla V, \nabla \rangle$, $V(x) = \frac{1}{2}\,|x|^2$

run the flow between $t = 0$ and $t = \infty$

variations in $t \geq 0$

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_\rho\big(\mathbb{1}_A(x), \mathbb{1}_B\big(\rho x + \sqrt{1-\rho^2}\, y\big)\big)\,d\gamma(x)\,d\gamma(y) \le J_\rho\left(\int_{\mathbb{R}^n} \mathbb{1}_A\,d\gamma, \int_{\mathbb{R}^n} \mathbb{1}_B\,d\gamma\right)$$

provided

$$\begin{pmatrix} \partial_{11}J_\rho & \rho\,\partial_{12}J_\rho \\ \rho\,\partial_{12}J_\rho & \partial_{22}J_\rho \end{pmatrix}$$

is negative definite

hypercontractivity $\quad J^{\mathrm{H}}(u,v) = u^{c_1}v^{c_2}, \quad \rho^2 c_1 c_2 = (1-c_1)(1-c_2)$

Bellman functions P. Ivanisvili, A. Volberg (2015)

C. Borell (1985)

$$T_\rho \varphi(x) = \int_{\mathbb{R}^n} \varphi\big(\rho x + \sqrt{1-\rho^2}\, y\big) d\gamma(y), \quad x \in \mathbb{R}^n$$

$P_t = T_{e^{-t}},\ t \geq 0$ Ornstein-Uhlenbeck semigroup

A, B Borel sets in $\mathbb{R}^n$, $\rho \in [0,1]$

$$\int_{\mathbb{R}^n} \mathbb{1}_A \, T_\rho(\mathbb{1}_B) d\gamma \leq \int_{\mathbb{R}^n} \mathbb{1}_H \, T_\rho(\mathbb{1}_K) d\gamma$$

H, K half-spaces, $\gamma(H) = \gamma(A)$, $\gamma(K) = \gamma(B)$

isoperimetric comparison theorem

$d\mu = e^{-V}dx$ probability, $V : \mathbb{R}^n \to \mathbb{R}$ smooth

$$\mathrm{Hess}(V) \geq \rho\,\mathrm{Id}$$

comparison of isoperimetric profile

$$\mathcal{I}_\mu \geq \frac{1}{\sqrt{\rho}}\,\mathcal{I}_\gamma$$

infinite dimensional analogue of the Lévy-Gromov theorem

Riemannian manifold (M^n, g), $\mathrm{Ric} \geq n-1$

$$\mathcal{I}_{\mathrm{vol}_g} \geq \mathcal{I}_{\mathbb{S}^n}$$

heat flow proof of classical isoperimetric inequality in $\mathbb{R}^n$?

on the sphere $\mathbb{S}^n$?

mass transportation

H. Knothe (1957), M. Gromov (1980),

B. Klartag (2017), F. Cavaletti, A. Mondino (2017)

metric measure space

synthetic Ricci curvature lower bounds

J. Lott, C. Villani, K.-Th. Sturm (2005-10) heat flow proof of Borell's theorem

motivation: noise stability

theoretical computer science

Boolean analysis on $\{0,1\}^n$

social choice, cryptography,

complexity, learning theory,

hardness of approximation, random graphs...

Gaussian setting: probabilistic interpretation

$X : \Omega \to \mathbb{R}^n$ with distribution γ

$$X^\rho = \rho X + \sqrt{1-\rho^2}\, Y$$

Y independent copy (noise)

correlation $\mathbb{E}(X \otimes X^\rho) = \rho \, \mathrm{Id}$

$f, g : \mathbb{R}^n \to \mathbb{R}$ non-negative

$$\mathbb{E}\big(f(X)g(X^\rho)\big) = \int_{\mathbb{R}^n} f \, T_\rho g \, d\gamma$$

A Borel set in $\mathbb{R}^n$

$$\mathcal{S}_\rho(A) = \mathbb{P}(X \in A, X^\rho \in A)$$

noise stability of A (sensitivity)

stablest sets?

$$\mathcal{S}_\rho(A) = \mathbb{E}\big(\mathbb{1}_A(X)\mathbb{1}_A(X^\rho)\big) = \int_{\mathbb{R}^n} \mathbb{1}_A \, T_\rho(\mathbb{1}_A) d\gamma$$

Borell's theorem

$$\int_{\mathbb{R}^n} \mathbb{1}_A \, T_\rho(\mathbb{1}_B) d\gamma \leq \int_{\mathbb{R}^n} \mathbb{1}_H \, T_\rho(\mathbb{1}_K) d\gamma$$

half-spaces are the most stable

$\{-1, +1\}^n$ discrete cube

$X = (X_1, \ldots, X_n)$ uniform on $\{-1, +1\}^n$

X^ρ uniform and ρ-correlated

$$X^\rho = (X_1^\rho, \ldots, X_n^\rho)$$

Y independent copy

$X_i^\rho = X_i$ with probability ρ and $X_i^\rho = Y_i$ with probability $1 - \rho$

correlation $\mathbb{E}(X \otimes X^\rho) = \rho \, \mathbf{Id}$

$X = (X_1, \ldots, X_n)$ uniform on $\{-1, +1\}^n$

X^ρ uniform and ρ-correlated

$A \subset \{-1, +1\}^n$

$\mathcal{S}_\rho(A) = \mathbb{P}(X \in A, X^\rho \in A)$

noise stability of A (sensitivity)

stablest sets?

$\mathbb{P}(X \in A) = \frac{1}{2}$

$$\mathcal{S}_\rho(A) \;=\; \mathbb{P}(X \in A, X^\rho \in A)$$

$$M_n \;=\; \left\{ x = (x_1, \ldots, x_n) \in \{-1, +1\}^n ;\ \mathrm{sign}\left(\sum_{i=1}^n x_i \right) = + \right\}$$

majority (set)

$$\lim_{n\to\infty} \mathcal{S}_\rho(M_n) \;=\; \frac{1}{2} - \frac{1}{2\pi} \arccos(\rho)$$

W. Sheppard 99

$$\mathcal{S}_\rho(A) \;=\; \mathbb{P}(X \in A, X^\rho \in A)$$

$$M_n \;=\; \left\{ x = (x_1, \dots, x_n) \in \{-1,+1\}^n;\ \mathrm{sign}\left(\sum_{i=1}^n x_i\right) = + \right\}$$

majority (set)

$$\lim_{n\to\infty} \mathcal{S}_\rho(M_n) \;=\; \frac{1}{2} - \frac{1}{2\pi}\arccos(\rho)$$

W. Sheppard... 1899

$$\lim_{n\to\infty} \mathcal{S}_\rho(M_n) \;=\; \frac{1}{2} - \frac{1}{2\pi}\arccos(\rho) \;=\; J_\rho\big(\tfrac{1}{2}, \tfrac{1}{2}\big)$$

central limit theorem

$$X = (X_1, \ldots, X_n) \text{ uniform on } \{-1, +1\}^n$$

X^ρ uniform and ρ-correlated

$$\mathcal{S}_\rho(A) = \mathbb{P}(X \in A, X^\rho \in A) \qquad \big(\mathbb{P}(X \in A) = \tfrac{1}{2}\big)$$

is it true that

$$\mathcal{S}_\rho(A) \leq J_\rho\big(\tfrac{1}{2}, \tfrac{1}{2}\big) \ ?$$

no, dictator $D = \{x_1 = +1\}$

$$\mathcal{S}_\rho(D) = \mathbb{P}(X \in D, X^\rho \in D) = \frac{1}{4}(1 + \rho) > J_\rho\big(\tfrac{1}{2}, \tfrac{1}{2}\big)$$

dictators have a notable coordinate

influence of $i = 1, \ldots, n$ on $A \subset \{-1, +1\}^n$

$$I_i(A) \;=\; \mathbb{P}\big(X \in A, \tau_i(X) \notin A\big)$$

$$\tau_i(x) \;=\; (x_1, \ldots, -x_i, \ldots, x_n)$$

dictator: big influences $\frac{1}{2}$

majority: small influences $\frac{1}{\sqrt{n}}$ $(\to\; 0)$

E. Mossel, R. O'Donnell, K. Oleszkiewicz (2010)

Majority is Stablest among sets with small influences

for $\varepsilon > 0$,

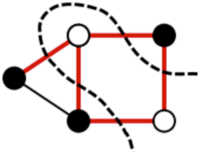
if $I_i(A) \leq \eta(\rho, \varepsilon)$, $i = 1, \ldots, n$, then

$$\mathcal{S}_\rho(A) \leq J_\rho\big(\tfrac{1}{2}, \tfrac{1}{2}\big) + \varepsilon$$

approximation of the Gaussian model

by central limit theorem for multilinear forms

Max-Cut problem in graph theory



NP-complete

optimality of the proportion of algorithmic approximation

under the unique game conjecture of S. Khot

polynomial algorithm of proportion given by $J_\rho\big(\frac{1}{2}, \frac{1}{2}\big)$

simplified proof of Majority is Stablest

A. De, E. Mossel, J. Neeman (2016)

discrete version of Borell's inequality

as a four-point inequality

(analogue of two-point inequality of hypercontractivity

Bonami-Beckner inequality)

symmetric group, graphs...

Thank you for your attention