Logarithmic Sobolev Inequalities

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logarithmic Sobolev inequalities

what they are, some history

analytic, geometric, optimal transportation proofs

last decade developments

at the interface between

analysis, probability, geometry

what are

logarithmic Sobolev inequalities ?

Sobolev inequalities

$f : \mathbb{R}^m \to \mathbb{R}$ smooth, compactly supported

$$\left(\int_{\mathbb{R}^m} |f|^p dx \right)^{2/p} \leq C_m \int_{\mathbb{R}^m} |\nabla f|^2 dx$$

$$p = \frac{2m}{m-2} \;\; (> 2) \qquad (m \geq 3)$$

sharp constant $\qquad C_m = \dfrac{1}{\pi m(m-2)} \left(\dfrac{\Gamma(m)}{\Gamma(\frac{m}{2})} \right)^{2/m}$

$$\left(\int_{\mathbb{R}^m} |f|^p dx\right)^{2/p} \le C_m \int_{\mathbb{R}^m} |\nabla f|^2 dx$$

$$\frac{2}{p} \log\left(\int_{\mathbb{R}^m} |f|^p dx\right) \le \log\left(C_m \int_{\mathbb{R}^m} |\nabla f|^2 dx\right)$$

assume $\int_{\mathbb{R}^m} f^2 dx = 1$

Jensen's inequality for $f^2 dx$

$$\log\left(\int_{\mathbb{R}^m} |f|^p dx\right) = \log\left(\int_{\mathbb{R}^m} |f|^{p-2} f^2 dx\right) \ge \int_{\mathbb{R}^m} \log\left(|f|^{p-2}\right) f^2 dx$$

$$\frac{p-2}{p} \int_{\mathbb{R}^m} f^2 \log f^2 dx \le \log\left(C_m \int_{\mathbb{R}^m} |\nabla f|^2 dx\right)$$

$$\frac{p-2}{p}\int_{\mathbb{R}^m} f^2 \log f^2 dx \leq \log\left(C_m \int_{\mathbb{R}^m} |\nabla f|^2 dx\right), \quad \int_{\mathbb{R}^m} f^2 dx = 1$$

form of logarithmic Sobolev inequality

formally come back to Sobolev (worse constants)

issue on sharp constants

$f : \mathbb{R}^n \to \mathbb{R}$ smooth, $\int_{\mathbb{R}^n} f^2 dx = 1$

$$f^{\otimes kn} : \mathbb{R}^{kn} \to \mathbb{R}, \quad m = kn, \quad k \to \infty$$

$$\int_{\mathbb{R}^n} f^2 \log f^2 dx \leq \frac{n}{2} \log\left(\frac{2}{n\pi e}\int_{\mathbb{R}^n} |\nabla f|^2 \, dx\right), \quad \int_{\mathbb{R}^n} f^2 dx = 1$$

sharp (Euclidean) **logarithmic Sobolev inequality**

used by **G. Perelman (2002)**

(Euclidean) **logarithmic Sobolev inequality**

$$\int_{\mathbb{R}^n} f^2 \log f^2 dx \;\leq\; \frac{n}{2} \log \left(\frac{2}{n\pi e} \int_{\mathbb{R}^n} |\nabla f|^2 \, dx \right), \quad \int_{\mathbb{R}^n} f^2 dx = 1$$

$$dx \;\to\; d\mu(x) = e^{-|x|^2/2} \, \frac{dx}{(2\pi)^{n/2}}$$

μ standard Gaussian probability measure on $\mathbb{R}^n$

change f^2 into $f^2 \, e^{-|x|^2/2}$

$f : \mathbb{R}^n \to \mathbb{R}$ smooth, $\int_{\mathbb{R}^n} f^2 d\mu = 1$

$$\int_{\mathbb{R}^n} f^2 \log f^2 d\mu \;\leq\; 2 \int_{\mathbb{R}^n} |\nabla f|^2 d\mu$$

(Gaussian) **logarithmic Sobolev inequality**

$$\int_{\mathbb{R}^n} f^2 \log f^2 d\mu \, \leq \, 2 \int_{\mathbb{R}^n} |\nabla f|^2 \, d\mu, \quad \int_{\mathbb{R}^n} f^2 d\mu = 1$$

$$d\mu(x) = e^{-|x|^2/2} \, \frac{dx}{(2\pi)^{n/2}}$$

Sobolev type inequality (for μ)

constant is sharp

constant independent of n (stability by product)

extension to infinite dimensional Wiener space

Gibbs measures, models from statistical mechanics

(Gaussian) **logarithmic Sobolev inequality**

$$\int_{\mathbb{R}^n} f^2 \log f^2 d\mu \;\leq\; 2 \int_{\mathbb{R}^n} |\nabla f|^2 \, d\mu, \quad \int_{\mathbb{R}^n} f^2 d\mu = 1$$

$$d\mu(x) = e^{-|x|^2/2} \, \frac{dx}{(2\pi)^{n/2}}$$

different forms

Sobolev type inequality

information theory

PDE formulation

information theory description

$$f \to \sqrt{f}, \quad f > 0, \quad \int_{\mathbb{R}^n} f \, d\mu = 1$$

$$\int_{\mathbb{R}^n} f \log f \, d\mu \leq \frac{1}{2} \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f} \, d\mu$$

$d\nu = f \, d\mu$ probability

$\int_{\mathbb{R}^n} f \log f \, d\mu = H(\nu \,|\, \mu)$ (relative) **entropy**

$\int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f} \, d\mu = I(\nu \,|\, \mu)$ (relative) **Fisher information**

entropy $H(\nu \,|\, \mu) \leq \frac{1}{2} I(\nu \,|\, \mu)$ Fisher information

PDE description

f function $\rightarrow$ probability (Lebesgue) density ρ

$$\rho_\infty = \frac{e^{-|x|^2/2}}{(2\pi)^{n/2}}$$

$$\int_{\mathbb{R}^n} f\,d\mu = 1, \quad \rho = f\,\rho_\infty, \quad \int_{\mathbb{R}^n} \rho\,dx = 1$$

logarithmic Sobolev inequality

$$\int_{\mathbb{R}^n} \rho \log\left(\frac{\rho}{\rho_\infty}\right) dx = H(\rho\,|\,\rho_\infty) \le \frac{1}{2}\, I(\rho\,|\,\rho_\infty) = 2\int_{\mathbb{R}^n} \left|\nabla\left(\sqrt{\frac{\rho}{\rho_\infty}}\right)\right|^2 \rho_\infty\,dx$$

another formulation of the Euclidean logarithmic Sobolev inequality

trend to **equilibrium**

$\rho > 0$ smooth, $\int_{\mathbb{R}^n} \rho \, dx = 1$

$$H\big(\rho \,|\, \rho_\infty\big) \,\leq\, \frac{1}{2} \, I\big(\rho \,|\, \rho_\infty\big)$$

$$V(x) = \tfrac{|x|^2}{2}$$

linear Fokker-Planck equation $\quad \dfrac{\partial \rho}{\partial t} \,=\, \nabla \cdot \big[\rho \, \nabla (\log \rho + V)\big]$

Boltzmann H-theorem

$$\frac{d}{dt} \, H\big(\rho_t \,|\, \rho_\infty\big) \,=\, - \, I\big(\rho_t \,|\, \rho_\infty\big)$$

$$\rho_t \,\to\, \rho_\infty = \tfrac{e^{-V}}{Z}$$

$$H\big(\rho_t \,|\, \rho_\infty\big) \,\leq\, e^{-2t} \, H\big(\rho_0 \,|\, \rho_\infty\big)$$

history

(Euclidean) **logarithmic Sobolev inequality**

$$\int_{\mathbb{R}^n} f^2 \log f^2 \, dx \leq \frac{n}{2} \log\left(\frac{2}{n\pi e} \int_{\mathbb{R}^n} |\nabla f|^2 \, dx\right), \quad \int_{\mathbb{R}^n} f^2 dx = 1$$

(Gaussian) **logarithmic Sobolev inequality**

$$\int_{\mathbb{R}^n} f^2 \log f^2 d\mu \leq 2 \int_{\mathbb{R}^n} |\nabla f|^2 d\mu, \quad \int_{\mathbb{R}^n} f^2 d\mu = 1$$

$$d\mu(x) = e^{-|x|^2/2} \, \frac{dx}{(2\pi)^{n/2}}$$

logarithmic Sobolev inequalities

various origins

Boltzmann H-theorem

mathematical physics

quantum field theory (hypercontractivity)

information theory

L. Gross

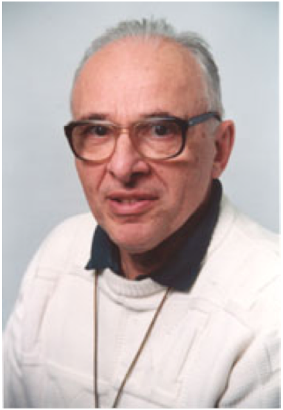
Logarithmic Sobolev inequalities

Amer. J. Math. 97, 1061-1083 (1975)

P. Federbush

A partially alternate derivation of a result of Nelson

J. Math. Phys. 10, 50-52 (1969)



Shannon-Stam entropy power inequality

$$e^{\frac{2}{n}H(\rho * h)} \geq e^{\frac{2}{n}H(\rho)} + e^{\frac{2}{n}H(h)}$$

$$H(\rho) = -\int_{\mathbb{R}^n} \rho \log \rho \, dx, \quad \rho > 0, \quad \int_{\mathbb{R}^n} \rho \, dx = 1$$

$h = h_\varepsilon$ Gaussian kernel, $\varepsilon \to 0$

$$e^{-\frac{2}{n}H(\rho)} \leq \frac{1}{2n\pi e} \int_{\mathbb{R}^n} \frac{|\nabla \rho|^2}{\rho} \, dx$$

$$\int_{\mathbb{R}^n} \rho \log \rho \, dx \leq \frac{n}{2} \log \left(\frac{1}{2n\pi e} \int_{\mathbb{R}^n} \frac{|\nabla \rho|^2}{\rho} \, dx \right)$$

$(\rho \to f^2)$ (Euclidean) **logarithmic Sobolev inequality**

A. Stam (1959)

(Gaussian) **logarithmic Sobolev inequality**

$f : \mathbb{R}^n \to \mathbb{R}$ smooth, $\int_{\mathbb{R}^n} f^2 d\mu = 1$, $d\mu(x) = e^{-|x|^2/2} \frac{dx}{(2\pi)^{n/2}}$

$$\int_{\mathbb{R}^n} f^2 \log f^2 d\mu \,\leq\, 2 \int_{\mathbb{R}^n} |\nabla f|^2 d\mu$$

at least 15 different proofs

two-point space (central limit theorem)

hypercontractivity

analytic semigroup theory

geometric convexity

optimal transportation

(Gaussian) **logarithmic Sobolev inequality**

$f : \mathbb{R}^n \to \mathbb{R}$ smooth, $\int_{\mathbb{R}^n} f^2 d\mu = 1, \quad d\mu(x) = e^{-|x|^2/2} \frac{dx}{(2\pi)^{n/2}}$

$$\int_{\mathbb{R}^n} f^2 \log f^2 d\mu \, \leq \, 2 \int_{\mathbb{R}^n} |\nabla f|^2 d\mu$$

at least 15 different proofs

two-point space (central limit theorem)

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two-point space

$$f : \{-1,+1\} \to \mathbb{R}, \quad \int_{\{-1,+1\}} f^2 d\nu = 1, \quad \nu(-1) = \nu(+1) = \tfrac{1}{2}$$

$$\int_{\{-1,+1\}} f^2 \log f^2 d\nu \ \leq \ \frac{1}{2} \int_{\{-1,+1\}} |Df|^2 d\nu$$

$$Df = f(+1) - f(-1)$$

$$f(-1) = \alpha, \quad f(+1) = \beta, \quad \tfrac{\alpha^2}{2} + \tfrac{\beta^2}{2} = 1$$

$$\alpha^2 \log \alpha^2 + \beta^2 \log \beta^2 \ \leq \ (\alpha - \beta)^2$$

(not so easy) exercise

L. Gross (1975)

$$\int_{\{-1,+1\}} f^2 \log f^2 d\nu \;\leq\; \frac{1}{2} \int_{\{-1,+1\}} |Df|^2 d\nu$$

tensorization

$$\int_{\{-1,+1\}^n} f^2 \log f^2 d\nu^{\otimes n} \;\leq\; \frac{1}{2} \int_{\{-1,+1\}^n} \sum_{i=1}^{n} |D_i f|^2 d\nu^{\otimes n}$$

central limit theorem

$\nu^{\otimes n} \;\to\; \mu$ Gaussian measure

$$\int_{\mathbb{R}^n} f^2 \log f^2 d\mu \;\leq\; 2 \int_{\mathbb{R}^n} |\nabla f|^2 d\mu, \quad \int_{\mathbb{R}^n} f^2 d\mu = 1$$

hypercontractivity

two-point space

$$f : \{-1, +1\} \to \mathbb{R}, \quad f(x) = a + bx$$

$$P_t f(x) = a + b e^{-t} x, \quad t \geq 0$$

$(P_t)_{t \geq 0}$ semigroup of contractions on $\mathrm{L}^p(\nu)$

$$1 < p < q < \infty, \quad e^{2t} \geq \frac{q-1}{p-1}$$

$$\|P_t f\|_q \leq \|f\|_p$$

$$\Big(\frac{1}{2}\,|a + b e^{-t}|^q + \frac{1}{2}\,|a - b e^{-t}|^q\Big)^{1/q} \leq \Big(\frac{1}{2}\,|a + b|^p + \frac{1}{2}\,|a - b|^p\Big)^{1/p}$$

A. Bonami (1970), W. Beckner (1975)

two-point space $\rightarrow$ Gaussian

$$d\mu(x) = e^{-|x|^2/2} \, \frac{dx}{(2\pi)^{n/2}}$$

$L = \Delta - x \cdot \nabla$ Ornstein-Uhlenbeck operator (Fokker-Planck)

μ invariant measure, $P_t = e^{tL}$ semigroup (contractions on $\mathrm{L}^p(\mu)$)

hypercontractivity property

$$1 < p < q < \infty, \quad e^{2t} \geq \frac{q-1}{p-1}$$

$$\|P_t f\|_q \leq \|f\|_p$$

E. Nelson (1966-73)

quantum field theory

L. Gross (1975)

logarithmic Sobolev inequality

equivalent

hypercontractivity

(general context of Markov operators)

$$\|P_t f\|_q \leq \|f\|_p$$

$$q = q(t) = 1 + e^{2t}(p-1), \quad t \geq 0$$

$$\frac{d}{dt} \|P_t f\|_{q(t)} \leq 0$$

$$\int_{\mathbb{R}^n} f^2 \log f^2 \, d\mu \leq 2 \int_{\mathbb{R}^n} f(-Lf) \, d\mu = 2 \int_{\mathbb{R}^n} |\nabla f|^2 \, d\mu$$

three proofs of the

logarithmic Sobolev inequality

analytic : parametrisation by heat kernels

geometric : Brunn-Minkowski inequality

measure theoretic : parametrisation by optimal transport

interface of analysis, probability and geometry

analytic proof (semigroup)

D. Bakry, M. Emery (1985)

(the ?) simplest one (according to **L. Gross 2010**)

$f > 0$ smooth, $d\mu(x) = e^{-|x|^2/2} \frac{dx}{(2\pi)^{n/2}}$

$(P_t)_{t \geq 0}$ heat semigroup, generator Δ

$$P_t f(x) = \int_{\mathbb{R}^n} f(y)\, e^{-|x-y|^2/4t} \, \frac{dy}{(4\pi t)^{n/2}}\,, \quad t > 0,\ x \in \mathbb{R}^n$$

$$t = \frac{1}{2} \quad (x = 0) \; : \quad P_t \to \mu$$

analytic proof (semigroup)

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(the ?) simplest one (according to **L. Gross 2010**)

$f > 0$ smooth, $\quad d\mu(x) = e^{-|x|^2/2} \frac{dx}{(2\pi)^{n/2}}$

$$\int_{\mathbb{R}^n} f \log f \, d\mu \leq \frac{1}{2} \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f} \, d\mu, \quad \int_{\mathbb{R}^n} f \, d\mu = 1$$

$(P_t)_{t \geq 0}$ heat semigroup, generator Δ

$$P_t f(x) = \int_{\mathbb{R}^n} f(y) \, e^{-|x-y|^2/4t} \, \frac{dy}{(4\pi t)^{n/2}}, \quad t > 0, \ x \in \mathbb{R}^n$$

$$t = \frac{1}{2} \quad (x = 0) : \quad P_t \to \mu$$

analytic proof (semigroup)

D. Bakry, M. Emery (1985)

(the ?) simplest one   (according to **L. Gross 2010**)

$f > 0$ smooth, $d\mu(x) = e^{-|x|^2/2} \, \frac{dx}{(2\pi)^{n/2}}$

$$\int_{\mathbb{R}^n} f \log f \, d\mu - \int_{\mathbb{R}^n} f \, d\mu \, \log \left(\int_{\mathbb{R}^n} f \, d\mu \right) \leq \frac{1}{2} \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f} \, d\mu$$

$(P_t)_{t \geq 0}$ heat semigroup, generator Δ

$$P_t f(x) = \int_{\mathbb{R}^n} f(y) \, e^{-|x-y|^2/4t} \, \frac{dy}{(4\pi t)^{n/2}} \, , \quad t > 0, \; x \in \mathbb{R}^n$$

$$t = \frac{1}{2} \quad (x = 0) \; : \quad P_t \to \mu$$

$f > 0$ smooth, $t > 0$, at any point

$$P_t(f \log f) - P_t f \log P_t f = \int_0^t \frac{d}{ds} P_s\big(P_{t-s} f \log P_{t-s} f\big)\, ds$$

$$\frac{d}{ds} P_s\big(P_{t-s} f \log P_{t-s} f\big)$$

$$= P_s\Big(\Delta\big(P_{s-t} f \log P_{t-s} f\big) - \Delta P_{t-s} f \log P_{t-s} f - \Delta P_{t-s} f\Big)$$

$$= P_s\Big(\frac{|\nabla P_{t-s} f|^2}{P_{t-s} f}\Big)$$

$f > 0$ smooth, $t > 0$, at any point

$$P_t(f \log f) - P_t f \log P_t f \; = \; \int_0^t \frac{d}{ds} P_s\big(P_{t-s} f \log P_{t-s} f\big)\, ds$$

$$= \; \int_0^t P_s\Big(\frac{|\nabla P_{t-s} f|^2}{P_{t-s} f}\Big)\, ds$$

$$\nabla P_u f \; = \; P_u(\nabla f)$$

$$|\nabla P_u f|^2 \; \leq \; \Big[P_u\big(|\nabla f|\big)\Big]^2 \; \leq \; P_u\Big(\frac{|\nabla f|^2}{f}\Big)\, P_u f$$

$$u = t - s$$

$$\frac{|\nabla P_{t-s} f|^2}{P_{t-s} f} \; \leq \; P_{t-s}\Big(\frac{|\nabla f|^2}{f}\Big)$$

$$P_t(f \log f) - P_t f \log P_t f \; \leq \; \int_0^t P_s\Big(P_{t-s}\Big(\frac{|\nabla f|^2}{f}\Big)\Big)\, ds \; = \; t\, P_t\Big(\frac{|\nabla f|^2}{f}\Big)$$

same proof

$d\mu = e^{-V} dx$ probability measure

$V : \mathbb{R}^n \to \mathbb{R}$ smooth

$$\nabla\nabla V \ \geq \ c > 0$$

$$\int_{\mathbb{R}^n} f^2 \log f^2 d\mu \ \leq \ \frac{2}{c} \int_{\mathbb{R}^n} |\nabla f|^2 d\mu, \quad \int_{\mathbb{R}^n} f^2 d\mu = 1$$

weighted Riemannian manifold

$$\mathrm{Ric} + \nabla\nabla V \ \geq \ c > 0$$

D. Bakry, M. Emery (1985)

geometric (convexity) proof

Brunn-Minkowski-Lusternik inequality

A, B compact subsets of $\mathbb{R}^n$

$$\mathrm{vol}_n(A+B)^{1/n} \geq \mathrm{vol}_n(A)^{1/n} + \mathrm{vol}_n(B)^{1/n}$$

$$A + B = \{x + y; x \in A, y \in B\}$$

isoperimetric inequality

$$B = B(0, \varepsilon), \quad \varepsilon \to 0$$

Brunn-Minkowski : functional form

Prékopa-Leindler (1971) theorem

$$\theta \in [0,1], \quad u, v, w \geq 0 \quad \text{on} \quad \mathbb{R}^n$$

if $\quad w\big(\theta x + (1-\theta)y\big) \;\geq\; u(x)^\theta v(y)^{1-\theta}, \quad x, y \in \mathbb{R}^n$

then $\quad \displaystyle\int_{\mathbb{R}^n} w\,dx \;\geq\; \Big(\int_{\mathbb{R}^n} u\,dx\Big)^\theta \Big(\int_{\mathbb{R}^n} v\,dx\Big)^{1-\theta}$

$$u = \chi_A,\ v = \chi_B$$

(equivalent, dimension free) multiplicative form of Brunn-Minkowski

$$\mathrm{vol}_n\big(\theta A + (1-\theta)B\big) \;\geq\; \mathrm{vol}_n(A)^\theta \mathrm{vol}_n(B)^{1-\theta}$$

Brunn-Minkowski : functional form

Prékopa-Leindler (1971) theorem

$$\theta \in [0,1], \quad u, v, w \geq 0 \quad \text{on} \quad \mathbb{R}^n$$

if $\quad w\big(\theta x + (1-\theta)y\big) \, \geq \, u(x)^\theta v(y)^{1-\theta}, \quad x, y \in \mathbb{R}^n$

then $\quad \displaystyle\int_{\mathbb{R}^n} w \, dx \, \geq \, \Big(\int_{\mathbb{R}^n} u \, dx \Big)^\theta \Big(\int_{\mathbb{R}^n} v \, dx \Big)^{1-\theta}$

$$dx \, \to \, d\mu(x) = e^{-|x|^2/2} \, \frac{dx}{(2\pi)^{n/2}}$$

Brunn-Minkowski : functional form

Prékopa-Leindler (1971) theorem

$$dx \;\to\; d\mu(x) = e^{-|x|^2/2}\,\frac{dx}{(2\pi)^{n/2}}$$

$$f \;\to\; f\,e^{-|x|^2/2}$$

$\theta \in [0,1], \quad u, v, w \geq 0$ on $\mathbb{R}^n$

then $$\int_{\mathbb{R}^n} w\,d\mu \;\geq\; \Big(\int_{\mathbb{R}^n} u\,d\mu\Big)^{\theta}\Big(\int_{\mathbb{R}^n} v\,d\mu\Big)^{1-\theta}$$

Brunn-Minkowski : functional form

Prékopa-Leindler (1971) theorem

$$dx \;\to\; d\mu(x) = e^{-|x|^2/2} \, \frac{dx}{(2\pi)^{n/2}}$$

$$f \;\to\; f \, e^{-|x|^2/2}$$

$\theta \in [0,1], \quad u, v, w \geq 0$ on $\mathbb{R}^n$

if $\quad w\big(\theta x + (1-\theta)y\big) \;\geq\; u(x)^\theta v(y)^{1-\theta}, \quad x, y \in \mathbb{R}^n$

then $\quad \displaystyle\int_{\mathbb{R}^n} w \, d\mu \;\geq\; \bigg(\int_{\mathbb{R}^n} u \, d\mu\bigg)^\theta \bigg(\int_{\mathbb{R}^n} v \, d\mu\bigg)^{1-\theta}$

Brunn-Minkowski : functional form

Prékopa-Leindler (1971) theorem

$$dx \;\to\; d\mu(x) = e^{-|x|^2/2}\,\frac{dx}{(2\pi)^{n/2}}$$

$$f \;\to\; f\,e^{-|x|^2/2}$$

$\theta \in [0,1], \quad u, v, w \geq 0$ on $\mathbb{R}^n$

if $\quad w\big(\theta x + (1-\theta)y\big) \;\geq\; u(x)^\theta v(y)^{1-\theta}\, e^{-\theta(1-\theta)|x-y|^2/2}, \quad x, y \in \mathbb{R}^n$

then $\quad \displaystyle\int_{\mathbb{R}^n} w\,d\mu \;\geq\; \left(\int_{\mathbb{R}^n} u\,d\mu\right)^\theta \left(\int_{\mathbb{R}^n} v\,d\mu\right)^{1-\theta}$

$f : \mathbb{R}^n \to \mathbb{R}$ bounded, $\theta \in (0,1)$

$$w(z) = e^{f(z)}$$

$$v(y) = 1$$

$$u(x) = e^{g(x)} \qquad g \;??$$

$$w\big(\theta x + (1-\theta)y\big) \geq u(x)^{\theta} v(y)^{1-\theta} e^{-\theta(1-\theta)|x-y|^2/2}, \quad x, y \in \mathbb{R}^n$$

$$f\big(\theta x + (1-\theta)y\big) \geq \theta\, g(x) - \tfrac{\theta(1-\theta)}{2}\, |x-y|^2$$

$$f\big(\theta x + (1-\theta)y\big) \;\geq\; \theta\, g(x) - \tfrac{\theta(1-\theta)}{2}\,|x-y|^2$$

$$g(x) \;=\; \frac{1}{\theta}\, Q_{(1-\theta)/\theta} f(x)$$

$$Q_t f(x) \;=\; \inf_{y\in\mathbb{R}^n} \Big\{ f(y) + \frac{1}{2t}\,|x-y|^2 \Big\}, \quad t>0,\; x\in\mathbb{R}^n$$

infimum-convolution with quadratic cost

Prékopa-Leindler theorem

$$w(z) \,=\, e^{f(z)}$$

$$v(y) \,=\, 1$$

$$u(x) \,=\, e^{\frac{1}{\theta} Q_{(1-\theta)/\theta} f(x)}$$

$$w\big(\theta x + (1-\theta)y\big) \,\geq\, u(x)^{\theta} v(y)^{1-\theta} \, e^{-\theta(1-\theta)|x-y|^2/2}, \quad x, y \in \mathbb{R}^n$$

$$\int_{\mathbb{R}^n} e^f \, d\mu \,\geq\, \bigg(\int_{\mathbb{R}^n} e^{\frac{1}{\theta} Q_{(1-\theta)/\theta} f} \, d\mu \bigg)^{\theta}$$

$$\frac{1}{\theta} = 1 + t$$

$$\int_{\mathbb{R}^n} e^f \, d\mu \,\geq\, \bigg(\int_{\mathbb{R}^n} e^{(1+t) Q_t f} \, d\mu \bigg)^{1/(1+t)}, \quad t > 0$$

$$\| e^{Q_t f} \|_{1+t} \leq \| e^f \|_1$$

$$Q_t f(x) = \inf_{y \in \mathbb{R}^n} \left\{ f(y) + \tfrac{1}{2t} |x-y|^2 \right\}, \quad t > 0, \ x \in \mathbb{R}^n$$

Hopf-Lax representation of Hamilton-Jacobi solutions

$$\partial_t Q_t f_{|t=0} = -\frac{1}{2} |\nabla f|^2$$

differentiate at $t = 0$

$$\int_{\mathbb{R}^n} f \, e^f d\mu \leq \frac{1}{2} \int_{\mathbb{R}^n} e^f |\nabla f|^2 d\mu, \quad \int_{\mathbb{R}^n} e^f d\mu = 1$$

$f \to \log f^2$ logarithmic Sobolev inequality

same proof

$d\mu = e^{-V} dx$ probability measure

$V : \mathbb{R}^n \to \mathbb{R}$ smooth

$$\nabla\nabla V \geq c > 0$$

$$\| e^{Q_t f} \|_{1+t} \leq \| e^f \|_1$$

$$\int_{\mathbb{R}^n} f^2 \log f^2 \, d\mu \leq \frac{2}{c} \int_{\mathbb{R}^n} |\nabla f|^2 \, d\mu, \quad \int_{\mathbb{R}^n} f^2 \, d\mu = 1$$

weighted Riemannian manifold

$$\mathrm{Ric} + \nabla\nabla V \geq c > 0$$

D. Cordero-Erausquin, R. McCann, M. Schmuckenschläger (2006)

$$\| e^{Q_t f} \|_{1+t} \leq \| e^f \|_1$$

analogue of hypercontractivity

equivalent to logarithmic Sobolev inequality

$$a \to 0$$

$$\int_{\mathbb{R}^n} e^{Q_t f} d\mu \leq e^{\int_{\mathbb{R}^n} f \, d\mu}$$

$f : \mathbb{R}^n \to \mathbb{R}$ measurable bounded

$$Q_t f(x) = \inf_{y \in \mathbb{R}^n} \left\{ f(y) + \frac{1}{2t} |x-y|^2 \right\}, \quad t > 0, \ x \in \mathbb{R}^n$$

$$\| e^{Q_t f} \|_{a+t} \leq \| e^f \|_a , \quad a > 0$$

analogue of hypercontractivity

equivalent to logarithmic Sobolev inequality

$$a \to 0$$

$$\int_{\mathbb{R}^n} e^{Q_t f} d\mu \leq e^{\int_{\mathbb{R}^n} f \, d\mu}$$

$f : \mathbb{R}^n \to \mathbb{R}$ measurable bounded

$$Q_t f(x) = \inf_{y \in \mathbb{R}^n} \left\{ f(y) + \frac{1}{2t} |x - y|^2 \right\}, \quad t > 0, \ x \in \mathbb{R}^n$$

$$\int_{\mathbb{R}^n} e^{Q_t f} d\mu \, \leq \, e^{\int_{\mathbb{R}^n} f \, d\mu}$$

dual form of **transportation cost inequality**

Wasserstein distance $\qquad W_2(\nu, \mu)^2 \, \leq \, 2 \, H(\nu \,|\, \mu) \qquad$ relative entropy

$$H(\nu \,|\, \mu) \, = \, \int_{\mathbb{R}^n} \log \frac{d\nu}{d\mu} \, d\nu, \quad \nu \ll \mu$$

relative entropy

$$W_2(\nu, \mu)^2 \, = \, \inf_{\nu \leftarrow \pi \rightarrow \mu} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |x - y|^2 d\pi(x, y)$$

Kantorovich-Rubinstein-Wasserstein distance

$$W_2(\nu, \mu)^2 \, = \, \sup \left\{ \int_{\mathbb{R}^n} Q_1 f \, d\nu - \int_{\mathbb{R}^n} f \, d\mu \right\}$$

$$W_2(\nu, \mu)^2 \ \leq \ 2\, H(\nu \,|\, \mu), \qquad \nu << \mu$$

$$d\mu(x) = e^{-|x|^2/2} \, \frac{dx}{(2\pi)^{n/2}}$$

M. Talagrand (1996)

F. Otto, C. Villani (2000)

consequence of logarithmic Sobolev inequality

general μ

optimal transportation framework

$$\big\| \, e^{Q_t f} \big\|_{a+t} \ \leq \ \big\| e^f \big\|_a \, , \quad a > 0$$

$$a \ \to \ 0$$

parametrisation proof by optimal transportation

μ, ν probability measures on $\mathbb{R}^n$ smooth densities

$$T : \mu \to \nu$$

optimal : $\quad W_2(\mu, \nu)^2 = \int_{\mathbb{R}^n} \big| x - T(x) \big|^2 d\mu(x)$

$$T = \nabla \phi, \quad \phi \quad \text{convex}$$

Y. Brenier, S. T. Rachev - L. Rüschendorf (1990)

manifold case **R. McCann (1995)**

transportation proof of the

logarithmic Sobolev inequality

$$d\mu(x) = e^{-|x|^2/2} \, \frac{dx}{(2\pi)^{n/2}}$$

$$f > 0, \quad \int_{\mathbb{R}^n} f \, d\mu = 1, \quad d\nu = f \, d\mu$$

Brenier map : $\quad T : f\,\mu \to \mu$

$$\int_{\mathbb{R}^n} b \circ T \, f \, d\mu = \int_{\mathbb{R}^n} b \, d\mu$$

$$T = \nabla\phi = x + \nabla\psi, \quad \phi \ \text{ convex}$$

Monge-Ampère equation

$$f(x) \, e^{-|x|^2/2} = e^{-|T(x)|^2/2} \det\big(\mathrm{Id} + \nabla\nabla\psi(x)\big)$$

$$f(x)\, e^{-|x|^2/2} \,=\, e^{-|T(x)|^2/2} \det\bigl(\mathrm{Id} + \nabla\nabla\psi(x)\bigr)$$

$$\log f \,=\, \frac{1}{2}\left[|x|^2 - |T|^2\right] + \log\det\left(\mathrm{Id} + \nabla\nabla\psi\right)$$

integrate with respect to $f\, d\mu$

$$\leq\, \frac{1}{2}\int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f}\, d\mu$$

logarithmic Sobolev inequality

$$f(x)\, e^{-|x|^2/2} \;=\; e^{-|T(x)|^2/2} \det\big(\mathrm{Id} + \nabla\nabla\psi(x)\big)$$

$$\log f \;=\; \frac{1}{2}\left[|x|^2 - |T|^2\right] + \log\det\left(\mathrm{Id} + \nabla\nabla\psi\right)$$

$$\;=\; -x \cdot \nabla\psi - \frac{1}{2}\,|\nabla\psi|^2 + \log\det\left(\mathrm{Id} + \nabla\nabla\psi\right)$$

integrate with respect to $f\, d\mu$

$$\leq\; \frac{1}{2}\int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f}\, d\mu$$

logarithmic Sobolev inequality

$$f(x)\, e^{-|x|^2/2} \,=\, e^{-|T(x)|^2/2} \det\big(\mathrm{Id} + \nabla\nabla\psi(x)\big)$$

$$\log f \,=\, \frac{1}{2}\left[|x|^2 - |T|^2\right] + \log\det\big(\mathrm{Id} + \nabla\nabla\psi\big)$$

$$\leq\, -x\cdot\nabla\psi - \frac{1}{2}\,|\nabla\psi|^2 + \Delta\psi$$

integrate with respect to $f\,d\mu$

$$\leq\, \frac{1}{2}\int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f}\, d\mu$$

logarithmic Sobolev inequality

$$f(x)\, e^{-|x|^2/2} \;=\; e^{-|T(x)|^2/2}\, \det\big(\mathrm{Id} + \nabla\nabla\psi(x)\big)$$

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$$\log f \;=\; \frac{1}{2}\left[|x|^2 - |T|^2\right] + \log\det\big(\mathrm{Id} + \nabla\nabla\psi\big)$$

$$\leq\; L\,\psi - \frac{1}{2}\,|\nabla\psi|^2$$

integrate with respect to $f\, d\mu$

$$\leq\; \frac{1}{2}\int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f}\, d\mu$$

logarithmic Sobolev inequality

$$f(x)\, e^{-|x|^2/2} \;=\; e^{-|T(x)|^2/2} \det\big(\mathrm{Id} + \nabla\nabla\psi(x)\big)$$

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$$\leq\; L\,\psi - \frac{1}{2}\,|\nabla\psi|^2$$

integrate with respect to $f\,d\mu$

$$\int_{\mathbb{R}^n} f \log f\, d\mu \;\leq\; \int_{\mathbb{R}^n} L\,\psi\, f\, d\mu - \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\psi|^2 f\, d\mu$$

$$\leq\; \frac{1}{2}\int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f}\, d\mu$$

logarithmic Sobolev inequality

$$f(x)\, e^{-|x|^2/2} \;=\; e^{-|T(x)|^2/2} \det\bigl(\mathrm{Id} + \nabla\nabla\psi(x)\bigr)$$

$$\log f \;=\; \frac{1}{2}\left[|x|^2 - |T|^2\right] + \log\det\left(\mathrm{Id} + \nabla\nabla\psi\right)$$

$$\leq\; L\psi - \frac{1}{2}\,|\nabla\psi|^2$$

integrate with respect to $f\,d\mu$

$$\int_{\mathbb{R}^n} f \log f \, d\mu \;\leq\; -\int_{\mathbb{R}^n} \nabla\psi \cdot \nabla f \, d\mu - \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\psi|^2 f \, d\mu$$

$$\leq\; \frac{1}{2}\int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f}\, d\mu$$

logarithmic Sobolev inequality

same proof

$d\mu = e^{-V} dx$ probability measure

$V : \mathbb{R}^n \to \mathbb{R}$ smooth

$$\nabla\nabla V \geq c > 0$$

$$\int_{\mathbb{R}^n} f^2 \log f^2 d\mu \leq \frac{2}{c} \int_{\mathbb{R}^n} |\nabla f|^2 d\mu, \quad \int_{\mathbb{R}^n} f^2 d\mu = 1$$

weighted Riemannian manifold

$$\mathrm{Ric} + \nabla\nabla V \geq c > 0$$

D. Cordero-Erausquin (2002)

general parametrisation

$$T_\theta = (1-\theta)\,\mathrm{Id} + \theta\, T, \quad \theta \in [0,1]$$

$$(\,T_0\mu = \mu, \quad T_1\mu = T\mu = \nu)$$

$$T_\theta : \mu \to f_\theta \, d\mu$$

$$d\mu = e^{-V} dx$$

Monge-Ampère equation

$$e^{-V} = f_\theta \circ T_\theta \, e^{-V \circ T_\theta} \det\big((1-\theta)\,\mathrm{Id} + \theta\, \nabla\nabla\phi\big)$$

$\nabla\nabla\phi$ symmetric positive definite

non-smooth analysis, PDE methods

optimal parametrisation and entropy

J. Lott - C. Villani, K.-Th. Sturm (2006-10)

Ricci curvature lower bounds in metric measure space

Riemannian geometry of $(\mathcal{P}_2, W_2)$

$(\mathcal{P}_2, W_2)$ probability measures (second moment)

F. Otto (2001), C. Villani (2005)

μ_0, μ_1 probability measures on $\mathbb{R}^n$

$T : \mu_0 \to \mu_1$ optimal

$T_\theta = (1-\theta)\,\mathrm{Id} + \theta\, T, \ \ \theta \in [0,1]$ geodesic in $(\mathcal{P}_2, W_2)$

reference measure $d\mu = e^{-V} dx$ on $\mathbb{R}^n, \ \ \nabla\nabla V \geq c, \quad c \in \mathbb{R}$

c-convexity property of entropy along geodesic $\mu_\theta = T_\theta(\mu_0) = f_\theta\, \mu$

H relative entropy, W_2 Wasserstein distance

R. McCann (1995) **displacement convexity**

μ_0, μ_1 probability measures on $\mathbb{R}^n$

$T : \mu_0 \to \mu_1$ optimal

$T_\theta = (1-\theta)\,\mathrm{Id} + \theta\, T, \ \ \theta \in [0,1]$ geodesic in $(\mathcal{P}_2, W_2)$

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reference measure $d\mu = e^{-V} dx$ on $\mathbb{R}^n, \ \ \nabla\nabla V \geq c, \quad c \in \mathbb{R}$

c-convexity property of entropy along geodesic $\mu_\theta = T_\theta(\mu_0) = f_\theta\, \mu$

$$c = 0 \qquad H(\mu_\theta \,|\, \mu) \,\leq\, (1-\theta) H(\mu_0 \,|\, \mu) + \theta H(\mu_1 \,|\, \mu)$$

H relative entropy, W_2 Wasserstein distance

R. McCann (1995) **displacement convexity**

μ_0, μ_1 probability measures on $\mathbb{R}^n$

$T : \mu_0 \to \mu_1$ optimal

$T_\theta = (1-\theta)\,\mathrm{Id} + \theta T, \ \ \theta \in [0,1]$ geodesic in $(\mathcal{P}_2, W_2)$

reference measure $d\mu = e^{-V} dx$ on $\mathbb{R}^n, \ \ \nabla\nabla V \geq c, \quad c \in \mathbb{R}$

c-convexity property of entropy along geodesic $\mu_\theta = T_\theta(\mu_0) = f_\theta \, \mu$

$$H(\mu_\theta \,|\, \mu) \leq (1-\theta)H(\mu_0 \,|\, \mu) + \theta H(\mu_1 \,|\, \mu) - c\,\theta(1-\theta)W_2(\mu_0, \mu_1)^2$$

H relative entropy, W_2 Wasserstein distance

R. McCann (1995) **displacement convexity**

c-convexity property of entropy along geodesic $\mu_\theta = T_\theta(\mu_0)$

$$H(\mu_\theta \,|\, \mu) \,\leq\, (1-\theta) H(\mu_0 \,|\, \mu) + \theta H(\mu_1 \,|\, \mu) - c\,\theta(1-\theta) W_2(\mu_0, \mu_1)^2$$

characterizes $\nabla\nabla V \geq c$

reference measure $d\mu = e^{-V} dx$

extends to weighted manifolds

characterizes $\mathrm{Ric} + \nabla\nabla V \geq c$

M. von Renesse, K.-Th. Sturm (2005)

notion of Ricci curvature bound

in a metric measure space (length space) (X, d, μ)

$(\mu_\theta)_{\theta\in[0,1]}$ geodesic in $(\mathcal{P}_2(X), W_2)$ connecting μ_0, μ_1

definition of lower bound on curvature

postulate that entropy is c-convex along one geodesic $(\mu_\theta)_{\theta\in[0,1]}$

$$H(\mu_\theta \,|\, \mu) \leq (1-\theta)H(\mu_0 \,|\, \mu) + \theta H(\mu_1 \,|\, \mu) - c\,\theta(1-\theta)W_2(\mu_0, \mu_1)^2$$

H relative entropy, W_2 Wasserstein distance

J. Lott - C. Villani, K.-Th. Sturm (2006-09)

definition of lower bound on curvature

in metric measure space

$$H(\mu_\theta \,|\, \mu) \,\leq\, (1-\theta)H(\mu_0 \,|\, \mu) + \theta H(\mu_1 \,|\, \mu) - c\,\theta(1-\theta)W_2(\mu_0,\mu_1)^2$$

- $\diamond$ generalizes Ricci curvature in Riemannian manifolds
- $\diamond$ allows for geometric and functional inequalities
- $\diamond$ main result : stability by Gromov-Hausdorff limit
- $\diamond$ analysis on singular spaces (limits of Riemannian manifolds)

mass transportation method

- **F. Barthe (1998) :** geometric Brascamp-Lieb inequalities, inverse forms

- **D. Cordero-Erausquin, R. McCann, M. Schmuckenschläger (2001, 2006) :** extension of Prékopa-Leindler theorem to manifolds, **J. Lott - C. Villani, K.-Th. Sturm (2006-10) :** notion of Ricci curvature bound in metric measure spaces

- **D. Cordero-Erausquin (2002) :** transportation cost and functional inequalities (logarithmic Sobolev...), **D. Cordero-Erausquin, B. Nazaret, C. Villani (2004) :** optimal classical Sobolev inequalities