JOURNAL OF FUNCTIONAL ANALYSIS 105, 444–465 (1992)

On an Integral Criterion for Hypercontractivity of Diffusion Semigroups and Extremal Functions

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Communicated by the Editors

Received May 21, 1990; revised March 18, 1991

In the line of investigation of the works by D. Bakry and M. Emery (Lecture Notes in Math., Vol. 1123, pp. 175–206, Springer-Verlag, New York/Berlin, 1985) and O. S. Rothaus (*J. Funct. Anal.* **42** (1981), 102–109; *J. Funct. Anal.* **65** (1986), 358–367), we study an integral inequality behind the "Γ_2 criterion" of D. Bakry and M. Emery (see previous reference) and its applications to hypercontractivity of diffusion semigroups. With, in particular, a short proof of the hypercontractivity property of the Ornstein–Uhlenbeck semigroup, our exposition unifies in a simple way several previous results, interpolating smoothly from the spectral gap inequalities to logarithmic Sobolev inequalities and even true Sobolev inequalities. We examine simultaneously the extremal functions for hypercontractivity and logarithmic Sobolev inequalities of the Ornstein–Uhlenbeck semigroup and heat semigroup on spheres.

To introduce to this paper, we would like to start with a short and simple proof of the hypercontractivity property of the Ornstein–Uhlenbeck semigroup. This proof is inspired from the work [B-E1] by D. Bakry and M. Emery and the present paper may actually be considered as a re-reading of [B-E1]. On $\mathbb{R}^k$, let μ be the canonical Gaussian measure with density with respect to Lebesgue measure $(2\pi)^{-k/2}\exp(-|x|^2/2)$, where $|x|^2=\sum_{i=1}^k x_i^2$ for $x=(x_1, ..., x_k)$ a generic point in $\mathbb{R}^k$. For f in $L^1(\mu)$, and $t\geqslant 0$, set

$$P_t f(x)=\int f(e^{-t}x+(1-e^{-2t})^{1/2}y)\,d\mu(y),$$

which thus defines a Markovian semigroup of positive contractions in all $L^p(\mu)$ $(1\leqslant p\leqslant\infty)$ with μ as the invariant and symmetric measure (i.e., $\int fP_tg\,d\mu=\int gP_tf\,d\mu$), known as the Ornstein-Uhlenbeck semigroup, or

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0022-1236/92 $3.00

Hermite semigroup with respect to a Gaussian measure. Its generator L acts on each smooth function f on $\mathbb{R}^k$ as

$$Lf(x) = \sum_{i=1}^{k} \frac{\partial^2 f}{\partial x_i^2}(x) - \sum_{i=1}^{k} x_i \frac{\partial f}{\partial x_i}(x) = \Delta f(x) - x \cdot \nabla f(x)$$

(and is therefore sometimes abbreviated as $L = \Delta - x \cdot \nabla$). It is a Dirichlet form for the squared gradient with respect to μ (integration by parts) in the sense that for all smooth functions f,

$$\int f(-Lf)\, d\mu = \int |\nabla f|^2\, d\mu.$$

It has been shown by E. Nelson [N] that the Ornstein–Uhlenbeck semigroup $(P_t)_{t \geqslant 0}$ satisfies the following (dimension free) *hypercontractivity* property: whenever $1 < p < q < \infty$ and $t > 0$ satisfy

$$e^t \geqslant \left(\frac{q-1}{p-1}\right)^{1/2},$$

then, for all functions f in $L^p(\mu)$,

$$\|P_t f\|_q \leqslant \|f\|_p$$

(where $\|\cdot\|_p$ is the norm in $L^p(\mu)$). In other words, P_t maps L^p in L^q with *norm one.* Furthermore, the exponential functions $f(x) = \exp(a \cdot x)$, $a \in \mathbb{R}^k$, are easily seen to be extremal functions, that is they satisfy the equality $\|P_t f\|_q = \|f\|_p$ for all triples (p, q, t) with $1 < p < q < \infty, t > 0$, and $e^t = ((q-1)/(p-1))^{1/2}$.

This hypercontractivity property has been shown by L. Gross [G] to be equivalent to *logarithmic Sobolev inequalities.* Namely, for all smooth functions f on $(\mathbb{R}^k, \mu)$ (for example, such that their gradients are in $L^2(\mu)$), we have

$$\int f^2 \log |f|\, d\mu - \int f^2\, d\mu \log \left(\int f^2\, d\mu\right)^{1/2} \leqslant \int f(-Lf)\, d\mu = \int |\nabla f|^2\, d\mu.$$

Actually, by a change of variables, this inequality involves a whole family of logarithmic Sobolev inequalities: replacing f (positive) by $f^{p/2}$, $p \geqslant 1$, we get equivalently

$$\int f^p \log f\, d\mu - \int f^p\, d\mu \log \left(\int f^p\, d\mu\right)^{1/p}$$
$$\leqslant \frac{p}{2(p-1)} \int f^{p-1}(-Lf)\, d\mu = \frac{p}{2} \int f^{p-2}\, |\nabla f|^2\, d\mu \tag{1}$$

($(1/2) \int (-Lf) \log f\, d\mu$ when $p = 1$). This equivalence between hypercontractivity and logarithmic Sobolev inequalities, thus due to L. Gross [G], holds in rather a general setting as will be used implicitly in the rest of the paper. To briefly recall its proof [G, B-E1, D-S1], let $p>1$ and let f be smooth and positive. Consider then the function in $t\geqslant 0$, $\Phi(t)=\|P_t f\|_{q(t)}$ where $q(t)=1+(p-1)e^{2t}$. Under the hypercontractivity property, $\Phi(t)\leqslant\Phi(0)$ for every $t\geqslant 0$; hence $\Phi'(0)\leqslant 0$ and, by performing this differentiation, this relation is exactly (1). Conversely, (1) applied to $P_t f$ shows that $\Phi'(t)\leqslant 0$ for every t; therefore Φ is decreasing and $\Phi(t)\leqslant\Phi(0)$ which is hypercontractivity, at least for f smooth, but actually for all f by a standard approximation. The case $p=1$ may be obtained in the limit (or directly).

To announce a parallel that will be constant throughout this work, let us note that this differentiation argument is similar to the one used for the spectral gap property. In this Gaussian example, it is simply the equivalence between

$$\|P_t f\|_2\leqslant e^{-t}\,\|f\|_2$$

for all $t\geqslant 0$ and all f in $L^2(\mu)$ with mean zero, and the Poincaré type inequality

$$\int f^2\,d\mu\leqslant\int f(-Lf)\,d\mu=\int|\nabla f|^2\,d\mu$$

for all smooth f with mean zero. Accordingly, the proof simply uses differentiation of the function $\Psi(t)=e^t\,\|P_t f\|_2$.

To verify the hypercontractivity property, it is usually easier to work with the logarithmic Sobolev inequalities (1). The following simple and short argument is inspired from the work [B-E1] (see also [D-S1, p. 267]). Let f be a positive[2] (or even, to start with, such that $0<a\leqslant f\leqslant b<\infty$ for some constants a, b) smooth function on $\mathbb{R}^k$. The semigroup properties and integration by parts allow us to write that

$$\begin{aligned}
&\int f\log f\,d\mu-\int f\,d\mu\log\left(\int f\,d\mu\right)\\
&\quad=-\int_0^\infty\left(\frac{d}{dt}\int P_t f\log P_t f\,d\mu\right)dt\\
&\quad=-\int_0^\infty\left(\int LP_t f\log P_t f\,d\mu\right)dt\\
&\quad=\int_0^\infty\left(\int\nabla P_t f\cdot\nabla(\log P_t f)\,d\mu\right)dt\\
&\quad=\int_0^\infty\left(\int\frac{1}{P_t f}|\nabla P_t f|^2\,d\mu\right)dt.
\end{aligned}$$

[2] We agree in this paper that positive (for a function f) means strictly positive (i.e., $f>0$).

Now, if $F(t) = \int (1/P_t f)\, |\nabla P_t f|^2\, d\mu$, the integral representation of P_t and the Cauchy–Schwarz inequality on this representation show that, for every $t \geqslant 0$,

$$F(t) = e^{-2t} \sum_{i=1}^{k} \int \frac{1}{P_t f} \left(P_t \frac{\partial f}{\partial x_i} \right)^2 d\mu$$

$$\leqslant e^{-2t} \sum_{i=1}^{k} \int P_t \left(\frac{1}{f} \left(\frac{\partial f}{\partial x_i} \right)^2 \right) d\mu = e^{-2t} \int \frac{1}{f} |\nabla f|^2\, d\mu.$$

Combining with the preceding identity, this already yields (1) (for $p = 1$), and *hypercontractivity is thus established in this way.*

As an alternate argument, which will be the key argument in the more abstract setting next, one can also note from an elementary computation that $F'(t) = -2F(t) - 2e^{-4t} K(t)$ where

$$K(t) = \sum_{i,j=1}^{k} \int \frac{1}{(P_t f)^3} \left(P_t \frac{\partial f}{\partial x_i} P_t \frac{\partial f}{\partial x_j} - P_t f\, P_t \frac{\partial^2 f}{\partial x_i\, \partial x_j} \right)^2 d\mu$$

which is in particular always *positive*. Hence

$$\int_0^\infty F(t)\, dt = -\frac{1}{2} \int_0^\infty F'(t)\, dt - \int_0^\infty e^{-4t} K(t)\, dt,$$

that is,

$$\int f \log f\, d\mu - \int f\, d\mu \log \left(\int f\, d\mu \right)$$

$$= \frac{1}{2} \int \frac{1}{f} |\nabla f|^2\, d\mu - \int_0^\infty e^{-4t} K(t)\, dt. \tag{2}$$

The preceding simple argument has an interesting consequence to extremal functions for hypercontractivity and logarithmic Sobolev inequalities. Let us agree that a (positive) function f is an *extremal* or *saturating* function for hypercontractivity of the Ornstein–Uhlenbeck semigroup $(P_t)_{t \geqslant 0}$ if there exists a triple (p, q, t) with $1 < p < q < \infty$, $t > 0$, and $e^t = ((q-1)/(p-1))^{1/2}$ for which $\|P_t f\|_q = \|f\|_p$. Constants are of course extremal and, as we have seen, exponential functions are extremal (and for *all* admissible triples (p, q, t)). As a result, we observe that these are actually the only (smooth) ones. Indeed, coming back to the equivalence between hypercontractivity and logarithmic Sobolev inequalities, if f is a positive smooth extremal function for (p, q, t), the function $\Phi(s) = \|P_s f\|_{q(s)}$ is constant for $s \leqslant t$. Thus, differentiating at $s = 0$, f satisfies the *equality* in the logarithmic Sobolev inequality (1) (for the corresponding value of $p > 1$). By a change of variables, $h = f^{1/p}$ satisfies the equality in (1) for $p = 1$, that is,

$$\int h \log h \, d\mu - \int h \, d\mu \log\left(\int h \, d\mu\right) = \frac{1}{2} \int \frac{1}{h} |\nabla h|^2 \, d\mu.$$

For this function h, we can write as before

$$\int h \log h \, d\mu - \int h \, d\mu \log\left(\int h \, d\mu\right) = \int_0^\infty H(t) \, dt,$$

where $H(t) = \int (1/P_t h) |\nabla P_t h|^2 \, d\mu$. Then, the previous argument shows that, for almost every t, we must have *equality* in the Cauchy–Schwarz inequalities

$$\left(P_t \frac{\partial h}{\partial x_i}\right)^2 \leqslant P_t h \, P_t\left(\frac{1}{h}\left(\frac{\partial h}{\partial x_i}\right)^2\right), \qquad i = 1, ..., n.$$

Therefore, if h is, for example, C^1,

$$\frac{\partial h}{\partial x_i} = a_i h$$

for some real numbers a_i, $i = 1, ..., n$, and h is an exponential function. Alternatively, the error term $\int_0^\infty e^{-4t} K(t) \, dt$ in (2) only vanishes on exponential functions (see also [C]).

As we have seen, an extremal function for hypercontractivity saturates the corresponding logarithmic Sobolev inequality. However, there is *a priori* no reason that the converse need be true in general. Now, exponential functions saturate the Gaussian hypercontractivity inequalities so that we actually identified in the same way (smooth) extremal functions for hypercontractivity. That exponential functions are the saturating functions for hypercontractivity in Gauss space was established (and with no smoothness restriction) by H. J. Brascamp and E. H. Lieb [B-L] (see also [L2]). That they are the saturating functions for the logarithmic Sobolev inequality was shown by E. Carlen [C] who proved a sharpened version of this inequality with an error term vanishing only on exponential functions.

Summarizing the conclusions of the previous elementary arguments, we may state the following.

THEOREM 1. *The Ornstein–Uhlenbeck semigroup with respect to the Gaussian measure μ on $\mathbb{R}^k$ is hypercontractive, and a positive C^1 function f on $\mathbb{R}^k$ whose gradient is in $L^2(\mu)$ satisfies the equality in the logarithmic Sobolev inequality*

$$\int f^2 \log f \, d\mu - \int f^2 \, d\mu \log\left(\int f^2 \, d\mu\right)^{1/2} \leqslant \int |\nabla f|^2 \, d\mu$$

if and only if it is of the form $f(x) = c\exp(a \cdot x)$ *for some positive* c *and some* a *in* $\mathbb{R}^k$.

In the first section of this paper, we bring out from this introduction an inequality for the Ornstein–Uhlenbeck operator $\Delta - x \cdot \nabla$ which interpolates smoothly the Poincaré inequality and the logarithmic Sobolev inequality. We then apply in the second section the idea of this integral criterion to the heat semigroup on the sphere and we determine with it the extremal functions for the logarithmic Sobolev inequality. Finally, we include our investigation in the abstract diffusion setting of [B-E1] using their notion of curvature and dimension of a semigroup. In the Appendix, some recent developments of [Ba] are considered.

1. An Inequality for the Operator $\Delta - x \cdot \nabla$

The preceding approach to hypercontractivity seems to contain in its body a simple inequality (actually a family of inequalities), of possible further interest, which expresses in an unified way several properties of the Ornstein–Uhlenbeck semigroup $(P_t)_{t \geqslant 0}$ and its generator $L = \Delta - x \cdot \nabla$. This inequality is a consequence of the "Γ_2 principle" of [B-E1] as will be seen in Section 3. It is given by the following theorem. As above, μ is the canonical Gaussian measure on $\mathbb{R}^k$.

Theorem 2. *For* $s = 0$ *and* $s = 1$ (*and thus for every* s *in* $[0, 1]$), *and for every positive smooth function* f *on* $\mathbb{R}^k$,

$$\int |\nabla f|^2 \, d\mu \leqslant \int (Lf)^2 \, d\mu + s \int \frac{1}{f} Lf \, |\nabla f|^2 \, d\mu.$$

It is not possible that $s > 1$ in this inequality (even with the left hand side replaced by $\tau \int |\nabla f|^2 \, d\mu$ for some τ) as can be seen from the example of exponential functions.

Proof. It is elementary and is based on the identity

$$\frac{1}{2} L(|\nabla f|^2) - \nabla f \cdot \nabla(Lf) = |\nabla f|^2 + \sum_{i,j=1}^{k} \left(\frac{\partial^2 f}{\partial x_i \, \partial x_j} \right)^2$$

whose verification is immediate (start in dimension one). In particular therefore (see Section 3 for the relation to Γ_2),

$$\tfrac{1}{2} L(|\nabla f|^2) - \nabla f \cdot \nabla(Lf) \geqslant |\nabla f|^2 \tag{3}$$

that, after integration, already yields the inequality of the theorem for $s=0$. To reach the case $s=1$, and actually every s in $[0,1]$ in the same way, simply apply (3) to $\varphi(f)$ where φ on $\mathbb{R}_+$ is such that $\varphi'(u)=u^{-s}$, $u>0$. After simplification by f^{-2s}, we get

$$\frac{1}{2}L(|\nabla f|^2)-\nabla f\cdot\nabla(Lf)\geqslant|\nabla f|^2+s\frac{1}{f}\nabla f\cdot\nabla(|\nabla f|^2)-s^2\frac{1}{f^2}|\nabla f|^4.$$

Integrating by parts,

$$\int|\nabla f|^2\,d\mu\leqslant\int(Lf)^2\,d\mu+s\int\frac{1}{f}Lf\,|\nabla f|^2\,d\mu-s(1-s)\int\frac{1}{f^2}|\nabla f|^4\,d\mu,$$

an inequality stronger than the one of the theorem. The proof is complete.

The interesting point of this inequality is that it kind of interpolates between the Poincaré inequality (corresponding to $s=0$) and the logarithmic Sobolev inequality (corresponding to $s=1$). To explain this, we need simply repeat the argument developed in the introduction following [B-E1]. Let ψ on $\mathbb{R}_+$ be such that $\psi''(u)=u^r$, $u>0$, where $r=-2s/(1+s)$ $(0\leqslant s\leqslant 1)$. Let further f be positive and smooth on $\mathbb{R}^k$. We can write as before by the semigroup properties and integration by parts

$$\int\psi(f)\,d\mu-\psi\left(\int f\,d\mu\right)=\int_0^\infty F(t)\,dt,$$

where

$$F(t)=\int\psi''(P_tf)\,|\nabla P_tf|^2\,d\mu=\int(P_tf)^r\,|\nabla P_tf|^2\,d\mu.$$

The derivative F' of F is given by

$$F'(t)=r\int(P_tf)^{r-1}\,LP_tf\,|\nabla P_tf|^2\,d\mu+2\int(P_tf)^r\,\nabla P_tf\cdot\nabla(LP_tf)\,d\mu.$$

If we change f into $f^{(2+r)/2}=f^{1/(1+s)}$ in Theorem 2, and apply it to P_tf for every t, we see that it exactly expresses that $F'(t)\leqslant-2F(t)$ for every t. Hence

$$\int_0^\infty F(t)\,dt\leqslant-\frac{1}{2}\int_0^\infty F'(t)\,dt=\frac{1}{2}F(0),$$

and therefore

$$\int\psi(f)\,d\mu-\psi\left(\int f\,d\mu\right)\leqslant\frac{1}{2}\int f^r\,|\nabla f|^2\,d\mu.$$

This is already Poincaré's inequality for $r=0$ $(s=0)$ and one of the equivalent logarithmic Sobolev inequalities (1) for $r=-1$ $(s=1)$. Changing back f into $f^{2/(2+r)}=f^{1+s}$ if one wishes it, we also get the inequalities, $0\leqslant s<1$,

$$\frac{1}{1-s}\left[\int f^2\,d\mu-\left(\int f^{1+s}\,d\mu\right)^{2/1+s}\right]\leqslant\int|\nabla f|^2\,d\mu, \tag{4}$$

as well as the limiting case as s approaches 1,

$$\int f^2\log f\,d\mu-\int f^2\,d\mu\log\left(\int f^2\,d\mu\right)^{1/2}\leqslant\int|\nabla f|^2\,d\mu.$$

These inequalities (4) have been obtained recently by W. Beckner [Be1] and follow here rather easily from Theorem 2.

Actually, these inequalities may be given an even simpler proof following the approach of the introduction. Indeed, and letting $k=1$ to simplify the notations and the idea,

$$F(t)=\int(P_tf)^r\,|\nabla P_tf|^2\,d\mu=e^{-2t}\int(P_tf)^r(P_tf')^2\,d\mu.$$

By the Cauchy–Schwarz inequality and concavity $(0\leqslant -r\leqslant 1)$,

$$(P_tf')^2=(P_t(f^{r/2}f'\cdot f^{-r/2}))^2\leqslant P_t(f^rf'^2)\,P_t(f^{-r})\leqslant P_t(f^{-r}f'^2)(P_tf)^{-r}$$

so that $F(t)\leqslant e^{-2t}F(0)$ for every t which immediately gives the result.

2. The Heat Semigroup on a Compact Riemannian Manifold

A second basic example in the study of hypercontractivity (and Sobolev inequalities) is the heat semigroup on spheres. As is well known and familiar, this example is closely related and similar to the Ornstein–Uhlenbeck semigroup with respect to a Gaussian measure. To present this case, it is convenient to widen the investigation and to deal with the heat semigroup on a compact Riemannian manifold. Thus, let E be here a compact connected Riemannian manifold of dimension n, μ be the normalized Riemannian measure, and Δ be the Laplace–Beltrami operator on E. Consider then the heat semigroup $P_t=e^{-t\Delta}$, $t\geqslant 0$. O.S. Rothaus [R1] showed that the heat semigroup is hypercontractive; i.e., there exists $\rho>0$ such that whenever $1<p<q<\infty$ and $t>0$ satisfy

$$e^{\rho t}\geqslant\left(\frac{q-1}{p-1}\right)^{1/2},$$

then, for all functions f in $L^p(\mu)$,

$$\|P_t f\|_q \leqslant \|f\|_p$$

(where $\|\cdot\|_p$ is the norm in $L^p(\mu)$). The best possible value ρ_0 for ρ is called the *hypercontractive constant* of the semigroup $(P_t)_{t \geqslant 0}$ or of its generator Δ. Rothaus' proof uses extremals; a simple alternate proof (see [D-S1]) may be given using the classical Sobolev inequality and the spectral gap (see below).

As for the first non-trivial eigenvalue λ_1 of the Laplacian, the main question lies in interesting minorations of its hypercontractive constant ρ_0. The first eigenvalue or spectral gap λ_1 of the Laplacian is characterized by the Poincaré type inequality

$$\lambda_1 \int |f - \textstyle\int f \, d\mu|^2 \, d\mu \leqslant \int |\nabla f|^2 \, d\mu$$

holding for all smooth functions f on E, as well as equivalently (see below) by

$$\lambda_1 \int |\nabla f|^2 \, d\mu \leqslant \int (\Delta f)^2 \, d\mu.$$

It is actually on the idea of this dual description of λ_1 that several of the analogous developments for ρ_0 and Sobolev inequalities are based (Definition 3). From the general equivalence between hypercontractivity and logarithmic Sobolev inequalities sketched in the introduction, ρ_0 is characterized by the logarithmic Sobolev inequality

$$\rho_0 \left[\int f^2 \log |f| \, d\mu - \int f^2 \, d\mu \log \left(\left(\int f^2 \, d\mu \right)^{1/2} \right) \right] \leqslant \int |\nabla f|^2 \, d\mu$$

holding for all smooth functions f on E (or only positive ones) where ∇ is the gradient on E, as well as, after a change of variables, by the equivalent forms (1) (with ρ_0 as a multiplicative constant on the left). It might be interesting to already recall at this stage that we always have that $\rho_0 \leqslant \lambda_1$ as can be shown by applying the preceding logarithmic Sobolev inequality to $1 + \varepsilon f$ where f has mean zero and by letting ε go to 0 (cf. [R1]). In the case E is the Euclidean sphere S^n_r of dimension n and radius $r > 0$, it has been shown in [M-W] that $\rho_0 = n/r^2 = \lambda_1$. As we will see, this result has been extended in [B-E1, R2] to compact Riemannian manifolds with non-negative Ricci curvature.

We will try to investigate this setting along the idea just alluded to through the inequality of Theorem 2.

DEFINITION 3. For any real number s, let $\tau(s)$ be the largest τ such that

$$\tau \int |\nabla f|^2 \, d\mu \leqslant \int (\Delta f)^2 \, d\mu + s \int \frac{1}{f} \Delta f \, |\nabla f|^2 \, d\mu \tag{5}$$

for all positive smooth functions f on E ($\tau(s) = -\infty$ if no such τ exists).

We will be interested in this definition in values of $s \geqslant 0$ for which $\tau(s) > 0$. Starting with some elementary properties, let us first note that $\tau(0)$ can be identified with λ_1 (see below) and that $\tau(s) \leqslant \tau(0) = \lambda_1$ for every s as can be seen by applying (5) to $1 + \varepsilon f$ and by letting ε tend to zero. Furthermore, as an infimum on affine functionals, $\tau(\cdot)$ is easily seen to be a *concave* function. Besides $\tau(0) = \lambda_1$, we will see that $\tau(1) \leqslant \rho_0$. The range of interest for the values of s seems actually to be the interval $[0, 2n/(n-2) - 1]$ for $n \geqslant 3$, and every $s \geqslant 0$ for $n = 1$, 2, the value $2n/(n-2)$ being of course the best exponent in the Sobolev imbedding theorem in Riemannian manifolds [A]. These observations are clearly drawn from the following proposition whose proof simply reproduces, in this framework, the argument developed from Theorem 2 at the end of the preceding section.

PROPOSITION 4. *For all s and all positive smooth functions f on E,*

$$\frac{\tau(s)}{1-s} \left[\int f^2 \, d\mu - \left(\int f^{1+s} \, d\mu \right)^{2/1+s} \right] \leqslant \int |\nabla f|^2 \, d\mu \tag{6}$$

and

$$\tau(1) \left[\int f^2 \log f \, d\mu - \int f^2 \, d\mu \log \left(\int f^2 \, d\mu \right)^{1/2} \right] \leqslant \int |\nabla f|^2 \, d\mu$$

when $s = 1$.

Proof. We need simply sketch it. Given f positive and smooth on E, and ψ on $\mathbb{R}_+$ is such that $\psi''(u) = u^r$, $u > 0$, where $r = -2s/(1+s)$, we can write

$$\int \psi(f) \, d\mu - \psi \left(\int f \, d\mu \right) = \int_0^\infty F(t) \, dt,$$

where $F(t) = \int (P_t f)^r \, |\nabla P_t f|^2 \, d\mu$. After a change of variables, (5) indicates that, for every t, $F'(t) \leqslant -2\tau(s) \, F(t)$ and hence

$$2\tau(s) \left[\int \psi(f) \, d\mu - \psi \left(\int f \, d\mu \right) \right] \leqslant -\int_0^\infty F'(t) \, dt = F(0)$$

that yields the conclusion after another change of variables.

We see from this proposition how Definition 3 smoothly interpolates between a Poincaré type inequality ($s=0$), a logarithmic Sobolev inequality ($s=1$), and a true Sobolev inequality ($s>1$). As will be seen below with the example of the sphere, this approach further yields optimal constants in these inequalities. As announced, $\tau(1)\leqslant\rho_0$, and, after a simple change of variables, $\tau(1)$ is actually easily seen to be identical to the criterion put forward in [B-E1, Corollaire 1]. When the dimension n is larger than 3, inequality (5) with $\tau(s)>0$ cannot be satisfied with $1+s>2n/(n-2)$ since if not (6) would contradict the Sobolev imbedding theorem.

Simple lower bounds for the hypercontractive constant ρ_0 of the heat semigroup on a compact Riemannian manifold E, as well as Sobolev inequalities, may thus be easily drawn from the inequalities (4). Of course, the main question now lies in finding efficient lower bounds for $\tau(s)$ for as many as possible values of s, hopefully for all s of the interval $[0, 2n/(n-2)-1]$. However, while logarithmic Sobolev inequalities and Sobolev inequalities exist in any compact Riemannian manifold, it is not known whether the inequalities (5) are satisfied with $\tau(s)>0$ for the appropriate values of s. One may ask, for example, whether the Sobolev inequalities conversely imply these inequalities. In particular, it is an open question (raised in [B-E1]) to know whether $\tau(1)=\rho_0$. This is in contrast with the corresponding property for λ_1 for which one easily verifies that $\tau(0)=\lambda_1$: Proposition 4 shows that $\tau(0)\leqslant\lambda_1$, while conversely one may simply invoke the Cauchy–Schwarz inequality to see that

$$\begin{aligned}\int |\nabla f|^2\,d\mu &= \int f(-\Delta f)\,d\mu = \int \Big(f-\int f\,d\mu\Big)(-\Delta f)\,d\mu\\ &\leqslant\left(\int \Big|f-\int f\,d\mu\Big|^2 d\mu\right)^{1/2}\left(\int (\Delta f)^2\,d\mu\right)^{1/2}\\ &\leqslant\frac{1}{\sqrt{\lambda_1}}\left(\int |\nabla f|^2\,d\mu\right)^{1/2}\left(\int (\Delta f)^2\,d\mu\right)^{1/2}\\ &\leqslant\frac{1}{\lambda_1}\int (\Delta f)^2\,d\mu.\end{aligned}$$

As a very partial answer to these questions, we have only been able so far, following [B-E1, R2], to show (5) for some values of s under curvature assumptions on the manifold, with however in this case sharp lower bounds on $\tau(s)$. Even under these hypotheses, we moreover do not reach the full interval of interest. Recently, D. Bakry [Ba] developed an approach to reach this full interval via *weak* Sobolev inequalities which share the same dimension in the sense of [D, V] as the Sobolev inequalities (see the Appendix).

Denote by R the infimum of the Ricci curvature tensor over all unit tangent vectors. Recall that Bochner's formula (cf. [B-G-M]) implies that for all smooth functions f on E,

$$\frac{1}{2}\Delta(|\nabla f|^2)-\nabla f\cdot\nabla(\Delta f)\geqslant R|\nabla f|^2+\frac{1}{n}(\Delta f)^2. \tag{7}$$

We will agree that when $n=1$, then $R=0$ (the torus). To exploit this relation towards the inequalities (5), we may follow [R2] and the preceding section and apply it to $\varphi(f)$ where f is smooth and positive on E and where φ on $\mathbb{R}_+$ is such that $\varphi'(u)=u^{-q}$, $u>0$, q real. After simplification by f^{-2q}, we get

$$\begin{aligned}\frac{1}{2}\Delta(|\nabla f|^2)-\nabla f\cdot\nabla(\Delta f)\geqslant{}& R|\nabla f|^2+\frac{1}{n}(\Delta f)^2+q\frac{1}{f}\nabla f\cdot\nabla(|\nabla f|^2)\\&-\frac{2}{n}q\frac{1}{f}\Delta f\,|\nabla f|^2-\left(1-\frac{1}{n}\right)q^2\frac{1}{f^2}|\nabla f|^4,\end{aligned}$$

hence, by integration by parts,

$$\begin{aligned}R\int|\nabla f|^2\,d\mu\leqslant{}&\left(1-\frac{1}{n}\right)\int(\Delta f)^2\,d\mu+q\left(1+\frac{2}{n}\right)\int\frac{1}{f}\Delta f\,|\nabla f|^2\,d\mu\\&-q\left[1-q\left(1-\frac{1}{n}\right)\right]\int\frac{1}{f^2}|\nabla f|^4\,d\mu.\end{aligned}\tag{8}$$

Inspecting the values of q for which $q[1-q(1-1/n)]\geqslant 0$ we deduce the following consequence.

PROPOSITION 5. *Let E be a compact Riemannian manifold of dimension n and Ricci curvature $R>0$. Then, in Definition* 3,

$$\tau(s)\geqslant\frac{R}{1-1/n}>0$$

for every s in the interval $[0,(1+2/n)/(1-1/n)^2]$.

For $s=0$, the minoration of this proposition is simply the celebrated Lichnerowicz minoration of λ_1 [B-G-M]. Provided with this result, it would have been of course enough, by concavity, to prove the minoration of Proposition 5 for $s=(1+2/n)/(1-1/n)^2$. It turned out that it was just as easy to work with an arbitrary s.

If $E=S_r^n$, it is known (cf. [B-G-M]) that $R=(n-1)/r^2$ and $\lambda_1=R/(1-1/n)=n/r^2$. Hence, since $\tau(s)\leqslant\tau(0)=\lambda_1$, we conclude

$$\tau(s)=\lambda_1=\frac{n}{r^2}$$

for every s in the interval of the statement. In particular, since $\tau(1) \leqslant \rho_0 \leqslant \lambda_1$, $\rho_0 = \lambda_1 = n/r^2$. It might be worthwhile noting that Proposition 5 further implies the analogue of Obata's theorem (cf. [B-G-M]) for the hypercontractive constant ρ_0. Indeed, let E be a compact Riemannian manifold of dimension n and positive Ricci curvature R such that $\rho_0 = R/(1-1/n)$. By Proposition 5, $\tau(s) \geqslant \rho_0$ on an interval $[0, 1+\varepsilon]$ for some $\varepsilon > 0$ and $\tau(1) = \rho_0$. By concavity of $\tau(\cdot)$, it is then constant and equal to $\rho_0 = R/(1-1/n)$ on this interval. In particular, $\lambda_1 = R/(1-1/n)$ and E is therefore isometric to a sphere by Obata's theorem. Concerning the case $n=1$, we agreed that $R=0$ and (8) (with $q=1$, for example) then shows that $\int (1/f)\, \Delta f\, |\nabla f|^2\, d\mu \geqslant 0$, something that may of course be shown directly by a simple integration by parts. In this case therefore, $\tau(s) = \tau(0) = \lambda_1$ for every $s \geqslant 0$, and in particular $\rho_0 = \lambda_1$ [E-Y].

As a consequence of this discussion, and Proposition 4, if $E = S^n_r$, we established that for f positive and smooth on E,

$$\left(\int f^{1+s}\, d\mu\right)^{2/1+s} \leqslant \int f^2\, d\mu + (s-1)\frac{r^2}{n}\int |\nabla f|^2\, d\mu$$

for every s in the interval $[1, (1+2/n)/(1-1/n)^2]$ (every $s \geqslant 1$ if $n=1$). W. Beckner [Be2] recently showed, using Lieb's calculation of the best constant for a fractional integral inequality on $\mathbb{R}^n$ [L1], that this inequality holds up to $s \leqslant 2n/(n-2)-1$ $(n \geqslant 3)$. One might conjecture that this is thus also the case for the inequalities (5) with $\tau(s) = n/r^2$ which would yield similarly best constants in this Sobolev inequality (cf. [A, p. 50]).

To conclude this section, let us come back to the question of extremal functions for hypercontractivity in this setting. As in Gauss space, a saturating function for the hypercontractive inequalities satisfies the equality in the corresponding logarithmic Sobolev inequality. O. S. Rothaus [R1] has shown that for every $\rho > \rho_0$, there exists a positive smooth non-constant function f on E such that

$$\rho\left[\int f^2 \log f\, d\mu - \int f^2\, d\mu \log\left(\int f^2\, d\mu\right)^{1/2}\right] = \int |\nabla f|^2\, d\mu.$$

Moreover, when $\rho_0 < \lambda_1$, there exists such an extremal non-constant function even *for* $\rho = \rho_0$. His argument however does not cover examples where $\rho_0 = \lambda_1$ as spheres, and, as a result, we will actually observe that on a sphere S^n_r the infimum

$$\rho_0 = \inf\left\{\frac{\int |\nabla f|^2\, d\mu}{\int f^2 \log f\, d\mu}\right\}$$

over all positive smooth non-constant functions f with $\int f^2\,d\mu = 1$ is *not* attained. On the sphere S_r^n, we know indeed that $\rho_0 = n/r^2 = R/(1-1/n) = \tau(1)$. Then, consider a positive smooth, for example, C^2, function f which saturates the logarithmic Sobolev inequality for ρ_0 in the sense that

$$\int f \log f\,d\mu - \int f\,d\mu \log\left(\int f\,d\mu\right) = \frac{1}{2\rho_0}\int \frac{1}{f}|\nabla f|^2\,d\mu.$$

If we recall $F(t) = \int (1/P_t f)\,|\nabla P_t f|^2\,d\mu$, $t \geqslant 0$, we know that the left hand side of the preceding equality is precisely $\int_0^\infty F(t)\,dt$ and that $F'(t) \leqslant -2\tau(1)\,F(t)$. Since $\rho_0 = \tau(1)$, it follows in particular that $F'(0) = -2\tau(1)\,F(0)$ and, going back to the expression of F and F' (see the first section) and letting $h = f^2$, we must have that h satisfy (5) for $s = 1$ as an equality, i.e.,

$$\tau(1)\int |\nabla h|^2\,d\mu = \int (\Delta h)^2\,d\mu + \int \frac{1}{h}\,\Delta h\,|\nabla h|^2\,d\mu.$$

But if we compare this equality with (8) for the value of q for which $q(1+2/n) = (1-1/n)$ (which satisfies $q[(1-q(1-1/n)] > 0$), it follows necessarily that

$$\int \frac{1}{h^2}|\nabla h|^4\,d\mu \leqslant 0$$

and $h = f^2$ is thus constant. Summarizing some of our conclusions, we may state the following.

THEOREM 6. *The heat semigroup on S_r^n is hypercontractive with $\rho_0 = n/r^2$, and a positive C^2 function on S_r^n which satisfies the equality in the logarithmic Sobolev inequality*

$$\frac{n}{r^2}\left[\int f^2 \log f\,d\mu - \int f^2\,d\mu \log\left(\int f^2\,d\mu\right)^{1/2}\right] \leqslant \int |\nabla f|^2\,d\mu$$

can only be constant.

3. A GENERAL FRAMEWORK AND THE Γ_2 CRITERION

In this final part, we adopt the general diffusion semigroup setting developed by D. Bakry and M. Emery [B-E1] to focus on the "Γ_2 criterion" behind the integral inequalities studied in the previous sections. It will help us to interpret some of the conclusions we obtained so far. On extremality in particular, we will see how the concept of "dimension" of a semigroup influences the nature of the extremal functions, the Ornstein–Uhlenbeck semigroup being considered as *infinite dimensional.* To briefly recall the setting of [B-E1], let $(P_t)_{t \geqslant 0}$ be a Markovian semigroup on a probability space $(E, \mathscr{E}, \mu)$ with generator L. The semigroup $(P_t)_{t \geqslant 0}$ is assumed to be symmetric, invariant, and ergodic with respect to μ and L to be a diffusion; i.e., there exists an algebra $\mathscr{A}$ of bounded functions in the domain of L, stable by L and by the action of C^∞ functions φ such that $\varphi(0) = 0$, and L satisfies the change of variables formula

$$L\varphi(f) = \varphi'(f)\, Lf + \varphi''(f)\, \Gamma(f, f),$$

where $\Gamma(f, f) = (1/2)(Lf^2 - 2f\, Lf)$. The "squared gradient" Γ is always positive and we have the integration by parts formula

$$\int \Gamma(f, f)\, d\mu = \int f(-Lf)\, d\mu.$$

The main tool in the study by D. Bakry and M. Emery is the "iterated squared gradient" Γ_2 defined, for every f in $\mathscr{A}$, by

$$\Gamma_2(f, f) = \tfrac{1}{2}\, L\Gamma(f, f) - \Gamma(f, Lf).$$

We refer to [B-E1] for further and more precise details on these definitions.

The reader recognizes in these objects the natural extension of the two preceding examples. Namely for the heat semigroup on a compact Riemannian manifold E, $\Gamma(f, f) = |\nabla f|^2$ and Bochner's formula (7) reads as

$$\Gamma_2(f, f) \geqslant R\Gamma(f, f) + \frac{1}{n}(Lf)^2$$

for all smooth functions f (one might take as $\mathscr{A}$ the algebra of C^∞ functions on E). In the case of the Ornstein–Uhlenbeck semigroup, we find that $\Gamma(f, f) = |\nabla f|^2$ and

$$\Gamma_2(f, f) = \Gamma(f, f) + \sum_{i, j=1}^{k} \left(\frac{\partial^2 f}{\partial x_i\, \partial x_j} \right)^2.$$

In particular $\Gamma_2(f, f) \geqslant \Gamma(f, f)$, *that is the relation leading to Theorem* 2.

According to [B-E2], these examples invite us to set as a definition that, in the preceding abstract setting, the diffusion semigroup $(P_t)_{t \geqslant 0}$, or its generator L, is of *dimension* n $(\geqslant 1)$ and *curvature* R if, for every f in $\mathscr{A}$,

$$\Gamma_2(f, f) \geqslant R\Gamma(f, f) + \frac{1}{n}(Lf)^2. \tag{9}$$

Of course, a semigroup of curvature R and dimension n is also of curvature $R' \leqslant R$ and dimension $n' \geqslant n$. The Ornstein–Uhlenbeck semigroup (even when $k = 1$) is thus of dimension $n = \infty$ (and nothing better) and curvature $R = 1$. As in the classical case, we will agree that $R = 0$ if $n = 1$. We denote by λ_1 the analogue of the first non-trivial eigenvalue of the Laplacian, that is, the best constant λ for which

$$\lambda \int f^2 \, d\mu \leqslant \int \Gamma(f, f) \, d\mu$$

for every f with mean zero in $\mathscr{A}$. Similarly, the hypercontractive constant ρ_0 of the semigroup $(P_t)_{t \geqslant 0}$ is characterized (under a density property of $\mathscr{A}$ in all $L^p(\mu)$'s, see [B-E1]) by the logarithmic Sobolev inequality

$$\rho_0 \left[\int f^2 \log f \, d\mu - \int f^2 \, d\mu \log \left(\int f^2 \, d\mu \right)^{1/2} \right] \leqslant \int \Gamma(f, f) \, d\mu$$

(or the corresponding inequalities (1)).

We may actually complete these numbers with the set of inequalities of Definition 3 and thus denote by $\tau(s)$, for every s, the largest real number τ such that

$$\tau \int \Gamma(f, f) \, d\mu \leqslant \int (Lf)^2 \, d\mu + s \int \frac{1}{f} Lf \, \Gamma(f, f) \, d\mu \tag{10}$$

for all positive functions f in $\mathscr{A}$ ($\tau(s) = -\infty$ if no such τ exists). This definition leads as in the previous sections to the Sobolev inequalities

$$\frac{\tau(s)}{1-s} \left[\int f^2 \, d\mu - \left(\int f^{1+s} \, d\mu \right)^{2/1+s} \right] \leqslant \int \Gamma(f, f) \, d\mu, \tag{11}$$

for f positive in $\mathscr{A}$, as well as for $s = 1$ to

$$\tau(1) \left[\int f^2 \log f \, d\mu - \int f^2 \, d\mu \log \left(\int f^2 \, d\mu \right)^{1/2} \right] \leqslant \int \Gamma(f, f) \, d\mu.$$

In particular, $\tau(0) = \lambda_1$ and $\tau(1) \leqslant \rho_0$. We also have here that $\tau(s) \leqslant \tau(0) = \lambda_1$ for every s (that $\tau(0) \geqslant \lambda_1$ follows as in the previous section), and that $\tau(\cdot)$ is a concave function.

The study of semigroups of dimension n and curvature R (>0) is similar to what was developed in the examples of the preceding sections. Namely, by the change of variables formula (see [B-E1]), for f positive in $\mathscr{A}$ and φ on $\mathbb{R}_+$ such that $\varphi'(u) = u^{-q}$, $u > 0$, q real, we have

$$\int f^{2q}\left[\Gamma_2(\varphi(f),\varphi(f))-R\Gamma(\varphi(f),\varphi(f))-\frac{1}{n}(L\varphi(f))^2\right]d\mu$$
$$=\int\left[\Gamma_2(f,f)-R\Gamma(f,f)-\frac{1}{n}(Lf)^2\right]d\mu+q\left(1+\frac{2}{n}\right)\int\frac{1}{f}Lf\,\Gamma(f,f)\,d\mu$$
$$-q\left[1-q\left(1-\frac{1}{n}\right)\right]\int\frac{1}{f^2}\Gamma(f,f)^2\,d\mu. \tag{12}$$

Since the left hand side of this identity is always positive by (9), and since we clearly have that $\int\Gamma_2(f,f)\,d\mu=\int(Lf)^2\,d\mu$, we may first deduce from (12) the analogue of (8), namely

$$R\int\Gamma(f,f)\,d\mu\leqslant\left(1-\frac{1}{n}\right)\int(Lf)^2\,d\mu+q\left(1+\frac{2}{n}\right)\int\frac{1}{f}Lf\,\Gamma(f,f)\,d\mu$$
$$-q\left[1-q\left(1-\frac{1}{n}\right)\right]\int\frac{1}{f^2}\Gamma(f,f)^2\,d\mu. \tag{13}$$

Thus, we recover in this general setting that $\tau(s)\geqslant R/(1-1/n)$ for every s in the interval $[0,(1+2/n)/(1-1/n)^2]$ (for all $s\geqslant 0$ if $n=1$). In the example of the Ornstein-Uhlenbeck generator, $R=1$ and $n=\infty$, and the function $\tau(s)$ is therefore constant and equal to 1 on the interval $[0,1]$ and equal to $-\infty$ for $s>1$ (a true Sobolev inequality cannot hold for the Gauss measure [G]). In finite dimension, we may slightly improve the minoration of ρ_0 by use of $\tau(0)=\lambda_1$ and $\tau(s)$ for some $s>1$. Namely, by concavity, for every $s>1$,

$$\rho_0\geqslant\tau(1)\geqslant\frac{1}{s}\tau(s)+\left(1-\frac{1}{s}\right)\tau(0)$$

that yields

$$\rho_0\geqslant\frac{1-1/n}{1+2/n}R+\frac{4/n-1/n^2}{1+2/n}\lambda_1$$

(see also [D-S2]). This argument has to be put together with the proof of the existence of $\rho_0>0$ in a compact Riemannian manifold using the Sobolev imbedding and the spectral gap (cf. [D-S1, p. 248]). The preceding minoration is however weaker than the seemingly optimal one

$$\rho_0\geqslant\frac{1-1/n}{(1+1/n)^2}R+\frac{4/n}{(1+1/n)^2}\lambda_1$$

obtained by O. S. Rothaus [R2] in compact Riemannian manifolds. His argument relies on extremals and perhaps indicates a weakness in the approach developed here.

We conclude with extremal functions in this setting and their relation to the dimension of the semigroup. A semigroup $(P_t)_{t \geqslant 0}$ of dimension n and curvature R is hypercontractive for $\rho = R/(1-1/n)$ ($\rho = \lambda_1$ if $n = 1$). What are the extremal functions of the logarithmic Sobolev inequality for this value of ρ? If f is positive in $\mathscr{A}$ and satisfies the equality

$$\int f \log f \, d\mu - \int f \, d\mu \log\left(\int f \, d\mu\right) = \frac{1}{2\rho} \int \frac{1}{f} \Gamma(f, f) \, d\mu,$$

then, as in Section 2, f satisfies (10) with $s = 1$ and $\tau(1) = \rho$ as an equality also. Then, (12) yields

$$\begin{aligned} 0 \leqslant & -\left[\left(1 - \frac{1}{n}\right) - q\left(1 + \frac{2}{n}\right)\right] \int \frac{1}{f} Lf \, \Gamma(f, f) \, d\mu \\ & - q\left[1 - q\left(1 - \frac{1}{n}\right)\right] \int \frac{1}{f^2} \Gamma(f, f)^2 \, d\mu. \end{aligned}$$

In finite dimension, we can choose $q \neq 1$ such that $(1 - 1/n) - q(1 + 2/n) = 0$ and deduce that $\Gamma(f, f) = 0$. f is then constant since $\Gamma(f, Lf)^2 \leqslant \Gamma(f, f)\,\Gamma(Lf, Lf)$ (Γ is positive) from which one gets $\Gamma(f, Lf) = 0$ and

$$0 = \int \Gamma(f, Lf) \, d\mu = -\int (Lf)^2 \, d\mu;$$

the claim follows by ergodicity. When $n = \infty$, we choose $q = 1$ in (12) which then reads as

$$\int f^2 \left[\Gamma_2(\log f, \log f) - R\Gamma(\log f, \log f)\right] d\mu = 0$$

so that

$$\Gamma_2(\log f, \log f) = R\Gamma(\log f, \log f).$$

Applied to the preceding examples of the heat semigroup on spheres and the infinite dimensional Ornstein–Uhlenbeck semigroup, these observations thus appear as a kind of explanation for the different extremal functions of logarithmic Sobolev inequalities in these two examples.

Appendix

In the abstract diffusion setting of Section 3, D. Bakry [Ba] recently showed that the dimension-curvature condition (9) actually implies the Sobolev inequality

$$\left(\int f^p \, d\mu\right)^{2/p} \leqslant A \int f^2 \, d\mu + B \int \Gamma(f, f) \, d\mu, \tag{14}$$

for constants $A, B > 0$ and all positive f in $\mathscr{A}$ *with best exponent* $p = 2n/(n-2)$ $(n > 2)$, and not only what is obtained in (11) together with (13). His approach however goes through what he called *weak* Sobolev inequalities. To incorporate these inequalities in our framework, we may consider, for p and α such that $1/p = 1/2 - 1/n\alpha$ the family of inequalities

$$\left(\int f^p \, d\mu\right)^{2\alpha/p} \left(\int f^2 \, d\mu\right)^{1-\alpha} \leqslant A \int f^2 \, d\mu + B \int \Gamma(f, f) \, d\mu, \tag{15}$$

f positive in $\mathscr{A}$. In other words, for f positive in $\mathscr{A}$ with $\int f^2 \, d\mu = 1$,

$$\left(\int f^p \, d\mu\right)^{2\alpha/p} \leqslant A + B \int \Gamma(f, f) \, d\mu.$$

For $p = 2n/(n-2)$, we recognize the classical Sobolev inequality (14). When $p = 2$, the left hand side of (15) has to be understood in the limit as

$$\int f^2 \log f \, d\mu - \int f^2 \, d\mu \log \left(\int f^2 \, d\mu\right)^{1/2} \leqslant \frac{n}{4} \int f^2 \, d\mu \log \left(A + B \frac{\int \Gamma(f, f) \, d\mu}{\int f^2 \, d\mu}\right).$$

A form of this logarithmic Sobolev type inequality has been studied extensively in the book by E. B. Davies [D], and is called weak Sobolev inequality in [Ba] (as we will see, it is indeed more a Sobolev inequality than a logarithmic Sobolev inequality as studied in the previous sections). A forerunner of this inequality may be found in [W]. When $p = 1$, (15) is the Nash inequality studied in [C-K-S].

The natural interval for the values of p in (15) is the interval $[1, 2n/(n-2)]$. It is easy to see that the inequalities (15) are weaker as p decreases to 1. In particular, Nash's inequality seems to be the weakest of the family. Actually, these inequalities turn out to be all equivalent (possibly changing of course the constants). Indeed, as a result of the works [V] (see also [C-SC-V]), [D], [C-K-S] for the respective values $p = 2n/(n-2)$, $p = 2$, $p = 1$, these inequalities (15) are all equivalent to the behavior of the semigroup $(P_t)_{t \geqslant 0}$ as

$$\|P_t f\|_\infty \leqslant C t^{-n/2} \|f\|_1, \qquad 0 < t \leqslant 1, \tag{16}$$

for the same value of n. Therefore (15) has the same meaning for *one* or *all* p in $[1, 2n/(n-2)]$.

Now, if we want to investigate one of the inequalities (15) under the dimension-curvature assumption (9), we may trivially repeat the arguments developed previously. As before, we attempt to reach $A = 1$, and given f

positive in $\mathscr{A}$ with $\int f^2\,d\mu = 1$, we would like to establish, taking logarithms, that

$$\alpha \log\left(\int f^p\,d\mu\right)^{2/p} \leqslant \log\left(1 + B\int \Gamma(f,f)\,d\mu\right).$$

Change f into $f^{1/p}$ and write as in the previous sections

$$\alpha \log\left(\int f\,d\mu\right)^{2/p} = \alpha\frac{2}{p}\left(1-\frac{2}{p}\right)\int_0^\infty F(t)\,dt = \frac{4n}{p}\int_0^\infty F(t)\,dt, \tag{17}$$

where

$$F(t) = \left(\int (P_t f)^{2/p}\,d\mu\right)^{-1}\int (P_t f)^{(2/p)-2}\,\Gamma(P_t f, P_t f)\,d\mu.$$

One then looks for a criterion so that (17) is estimated by $\log(1 + Bp^{-2}F(0))$ (recall the change of variables). This is the case as soon as

$$\frac{4n}{p}F(t) \leqslant -(1 + Bp^{-2}F(t))^{-1}\,Bp^{-2}\,F'(t)$$

for every $t \geqslant 0$. Performing this differentiation, it is easily seen, as in the preceding sections, that this will be satisfied as soon as for every positive f in $\mathscr{A}$ with $\int f^2\,d\mu = 1$,

$$\sigma(p)\int \Gamma(f,f)\,d\mu \leqslant \int (Lf)^2\,d\mu + (p-1)\int \frac{1}{f}\,Lf\,\Gamma(f,f)\,d\mu$$
$$+\left[p-2-\frac{2p}{n}\right]\left(\int \Gamma(f,f)\,d\mu\right)^2 \tag{18}$$

for $\sigma(p) = 2p/nB$.

These inequalities (18) may of course be compared with the inequalities of Definition 3, or rather of (10). We realized in the preceding sections the difficult task in establishing (18) for $p = 2n/(n-2)$ (for which $p - 2 - 2p/n = 0$) under the dimension-curvature hypothesis (9). To be more specific, we showed under (9) (see (13)) a form of (18) but with n replaced by some $n' > n$ (and all p such that $p - 2 - 2p/n' \leqslant 0$). The main conclusion of [Ba] is that (9) actually implies (18) with the *same* n, *for* $p = 2$, and with $\sigma(p) = R$. By the preceding comments, we reach in this way the best exponent $p = 2n/(n-2)$ in (14). One may wonder which is the weakest, or at least the easiest to establish, among the inequalities (18) when p varies between $2n/(n-2)$ and 1. While the computations of [Ba] are rather tricky, one may hope for some simplifications when another value of p (in particular $p=1$) is considered.

ACKNOWLEDGMENTS

I am grateful to D. Bakry and M. Emery, and to Th. Coulhon, for helpful comments and remarks during the preparation of this paper, and to the referee for a judicious observation on extremal functions of logarithmic Sobolev inequalities.

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